

In some computations at the end we use the gfun-package.

It is just needed to guess simple functional equations for generating functions and manipulate their coefficient.

This is not needed in the remainder of this worksheet.

For the latest version see Salvy's homepage <http://perso.ens-lyon.fr/bruno.salvy/software/the-gfun-package/>.

For these computations gfun version 3.98 was used.

```
> #libname := "<path>\gfun.mla", libname;      #set the path to
gfun.mla
libname := "D:\\lib\\gfun.mla", "D:\\lib",
"/usr/people/mwallner/lib/gfun.mla", "/usr/people/mwallner/lib",
libname;
with(plots):with(gfun): gfun:-version;
libname := "D:\lib\gfun.mla", "D:\lib", "/usr/people/mwallner/lib/gfun.mla",
"/usr/people/mwallner/lib", "C:\Program Files\Maple 2020\lib"
```

3.98

(1)

Recurrences (code from the Maple session accompanying [Elvey Price, Fang, Wallner 2021]; see <https://dmg.tuwien.ac.at/mwallner/>)

Maximal number of computed terms

```
> NN:=100;
```

$NN := 100$

(1.1)

Relaxed binary trees

Compute the number up to size NN

```
> for n from 0 to NN do
  for m from 0 to NN do
    rr2[n,m] := 0:
  end:
end:

#initial conditions
for n from 0 to NN do
  rr2[n,0] := 1:
end:

for n from 1 to NN do
  for m from 1 to n do
    rr2[n,m] := rr2[n,m-1]+(m+1)*rr2[n-1,m];
  end do:
end do:
```

print the array

```
> for m from 7 to 0 by -1 do
  for n from 0 to 10 do
    printf("%10.0f ", rr2[n,m]);
```

```

end;
printf("\n");
end;
0      0      0      0      0
0      0      311250  6173791  86626584  1035808538
0      18628   311250  3683791  37236256  342795866
1363   18628   180854  1505041  11449719  82142074
1363   10450   69086   419917  2419473  13443760
728    3635    16836   74487   319888  1346395
220    723     2296    7143    21940   66843
31     63     127     255     511     1023
1      1      1      1      1      1
1      1      1      1      1      1

```

<https://oeis.org/????>

Number of deterministic completely defined initially connected acyclic automata with 2 inputs and n transient unlabeled states (and a unique absorbing state).

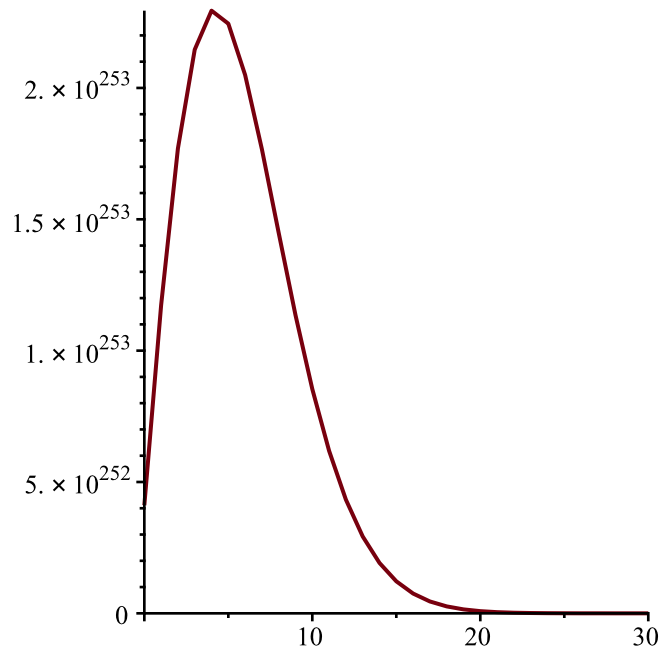
```
> seq(rr2[n,n],n=0..min(floor(NN),12));
```

1, 1, 3, 16, 127, 1363, 18628, 311250, 6173791, 142190703, 3737431895, 110577492346, 3641313700916 (1.1.1)

```

> N1 := 450;
plot([seq([i,evalf(rr2[N1+i,N1-i]/factorial(N1+i))],i=0..30)]);
NI := 450

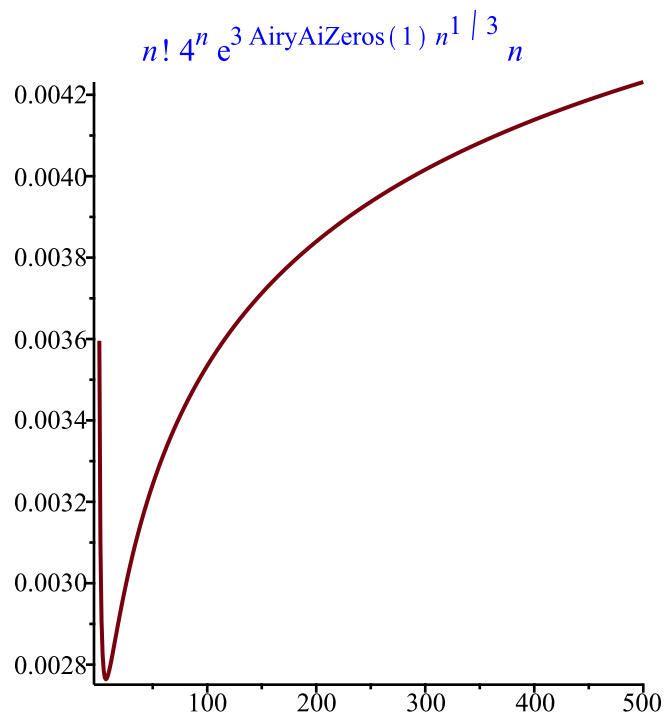
```



```

> n:='n':
factorial(n)*4^n*exp(3*AiryAiZeros(1)*n^(1/3))*n^(1);
plot([seq([i,(evalf(subs(n=i,%)))/rr2[i,i]],i=1..500)]);

```

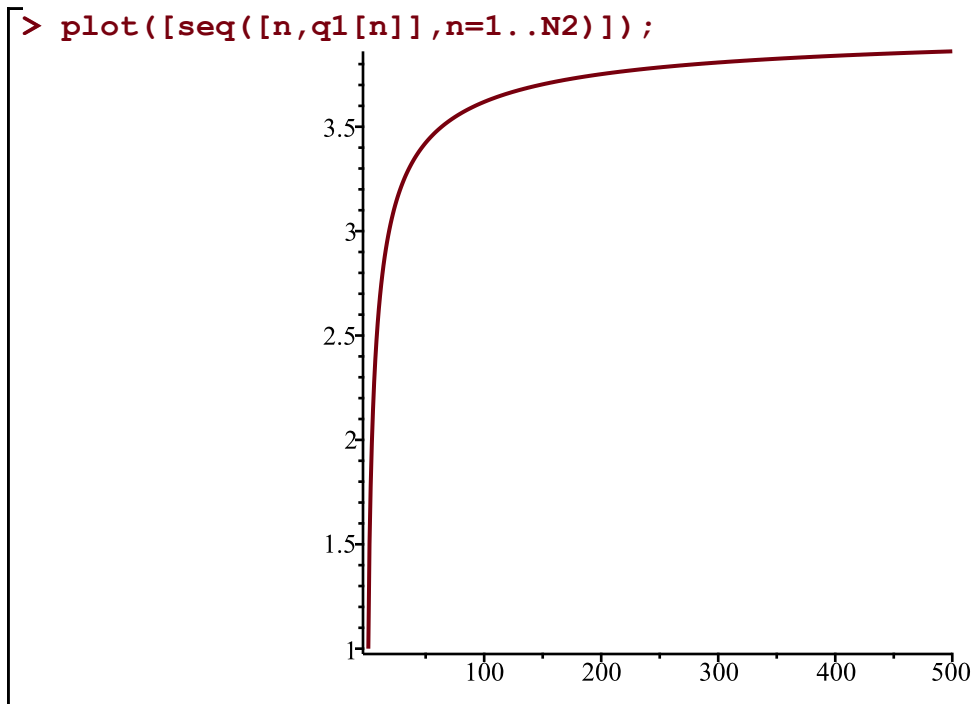


```

> N2 := floor(NN);
                                     N2 := 500
> for n from 0 to N2 do
    r2n[n] := rr2[n,n];
end:
> for n from 1 to N2 do
    q1[n] := r2n[n]/r2n[n-1]/n;
end:

```

(1.1.2)



```
> evalf(q1[N2]);
expgrowth2 := 4:
expgrowth2,evalf(%);
```

3.861137885

4, 4.

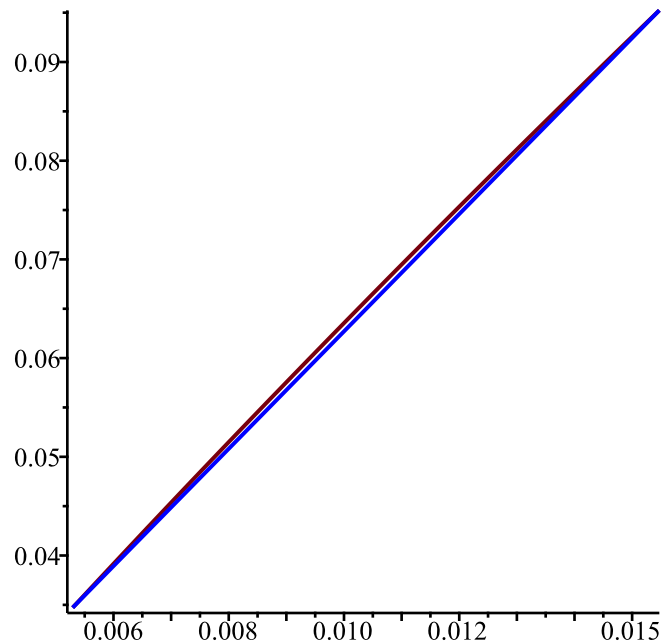
(1.1.3)

```
> for n from 1 to N2 do
  q2[n] := q1[n]/expgrowth2;
end:
```

we seem to have $\sigma=1/3$

```
> N0 := 100:
sigma := 1/3;
P1 := plot([seq([sigma*n^(sigma-1), 1-q2[n]], n=N0..N2)]):
P2 := plot([[sigma*N0^(sigma-1), 1-q2[N0]], [sigma*N2^(sigma-1), 1-
q2[N2]]], color=blue):
display(P1, P2);
```

$\sigma := \frac{1}{3}$



Relaxed ternary trees

Compute the number up to size NN

```
> for n from 0 to NN do
  for m from 0 to max(ceil(n/2), 30) do
    rr3[n,m] := 0:
  end:
end:
```

```
#initial conditions
for n from 0 to NN do
  rr3[n,0] := 1:
end:
```

```
for n from 1 to NN do
```

```

    for m from 1 to floor(n/2) do
      rr3[n,m] := rr3[n,m-1]+(m+1)*rr3[n-1,m];
    end do:
  end do:
print the array
> for m from 7 to 0 by -1 do
  for n from 0 to 10 do
    printf("%10.0f ",rr3[n,m]);
  end;
  printf("\n");
end;

```

| | | | | | | |
|----|-----|------|------|-------|--------|--------|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 408354 |
| 0 | 0 | 0 | 5711 | 56355 | 408354 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 139 | 1036 | 5711 | 27800 | 126579 | 0 |
| 36 | 0 | 0 | 0 | 0 | 0 | 7 |
| 15 | 139 | 480 | 1567 | 4956 | 15379 | 7 |
| 1 | 0 | 0 | 1 | 3 | 255 | 511 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

<https://oeis.org/A082162>

Number of deterministic completely defined initially connected acyclic automata with 3 inputs and n transient unlabeled states (and a unique absorbing state).

```
> seq(rr3[2*n,n],n=0..min(floor(NN/2),12));
```

1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275, (1.2.1)

183049273155939442, 86211400693272461866, 49662741844330581221741

from below

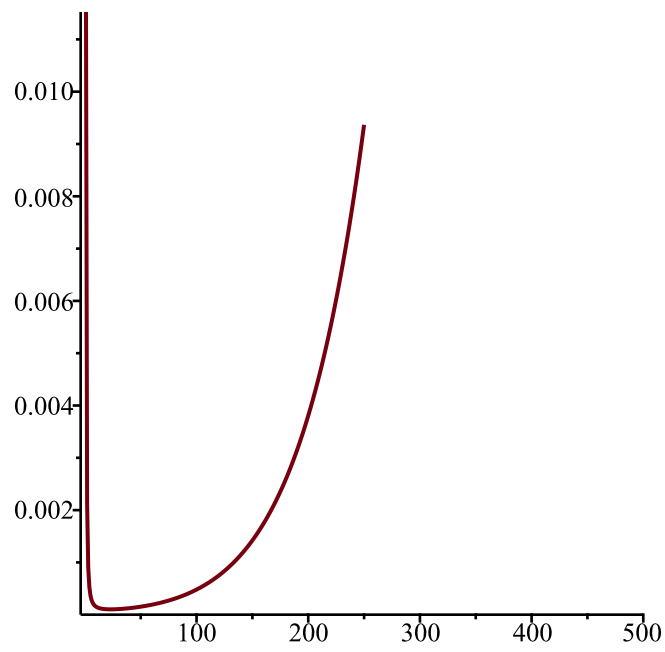
```
> n:='n':
mu := exp((3*AiryAiZeros(1)*2^(2/3)*K^(1/3))/(2*(K-1)^(1/3)));
```

$$\mu := e^{\frac{3 \operatorname{AiryAiZeros}(1) 2^{2/3} K^{1/3}}{2(K-1)^{1/3}}} \quad (1.2.2)$$

could this asymptotics work? just guess on the exponential growth, at the moment it still seems a bit weird

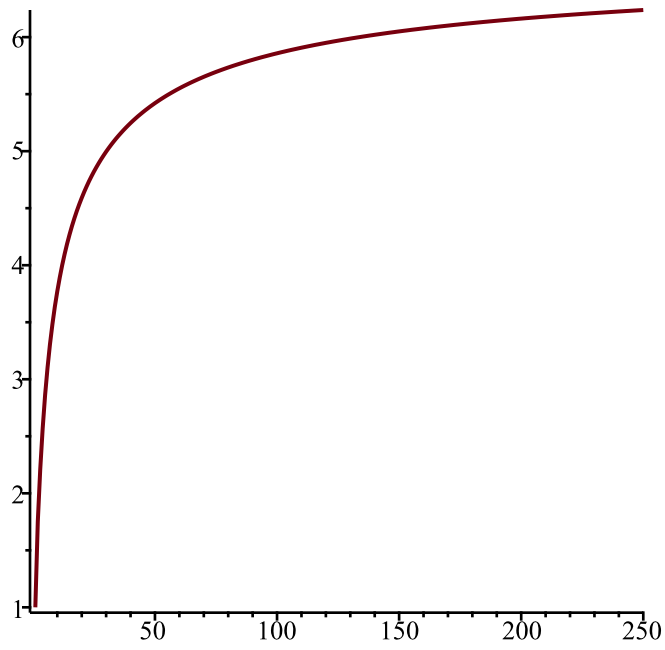
```
> n:='n':
factorial(n)^2*(3^3/2^2)^n*subs(K=3,mu)^(n^(1/3))*n^(-2);
plot([seq([i,(evalf(subs(n=i,%)))/rr3[2*i,i]],i=1..500)]);
```

$$\frac{n!^2 \left(\frac{27}{4} \right)^n \left(e^{\frac{3 \operatorname{AiryAiZeros}(1) 2^{1/3} 3^{1/3}}{2}} \right)^{n^{1/3}}}{n^2}$$



```
[> #N1 := floor(NN/2);
#plot([seq([i,evalf(rr3[2*N1+i,N1-i]/factorial(2*N1+i))],i=0..20)
]);
```

```
[> N2 := floor(NN/2);
                                     N2 := 250                                (1.2.3)
=
> for n from 0 to N2 do
    r3n[n] := rr3[2*n,n];
end:
=
> for n from 1 to N2 do
    q1[n] := r3n[n]/r3n[n-1]/n^2;
end:
=
> plot([seq([n,q1[n]],n=1..N2)]);
```



```
> evalf(q1[N2]);
expgrowth3 := 27/4:
expgrowth3,evalf(%);
```

6.237844002

$\frac{27}{4}, 6.750000000$

(1.2.4)

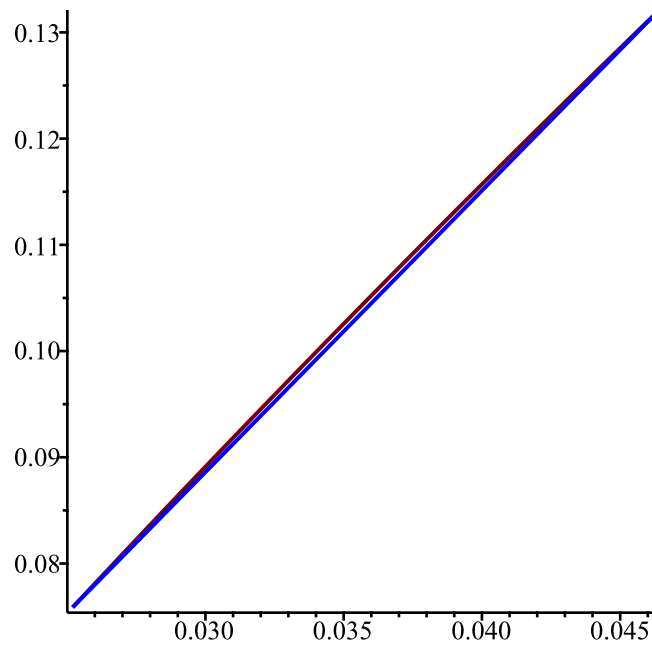
```
> #evalf(expgrowth3/4);
```

```
> for n from 1 to N2 do
  q2[n] := q1[n]/expgrowth3;
end:
```

we seem to have $\sigma=1/3$

```
> N0 := 100:
sigma := 1/3;
P1 := plot([seq([n^(sigma-1), 1-q2[n]], n=N0..N2)]):
P2 := plot([[N0^(sigma-1), 1-q2[N0]], [N2^(sigma-1), 1-q2[N2]]],
color=blue):
display(P1, P2);
```

$\sigma := \frac{1}{3}$



```
> q1[2];
```

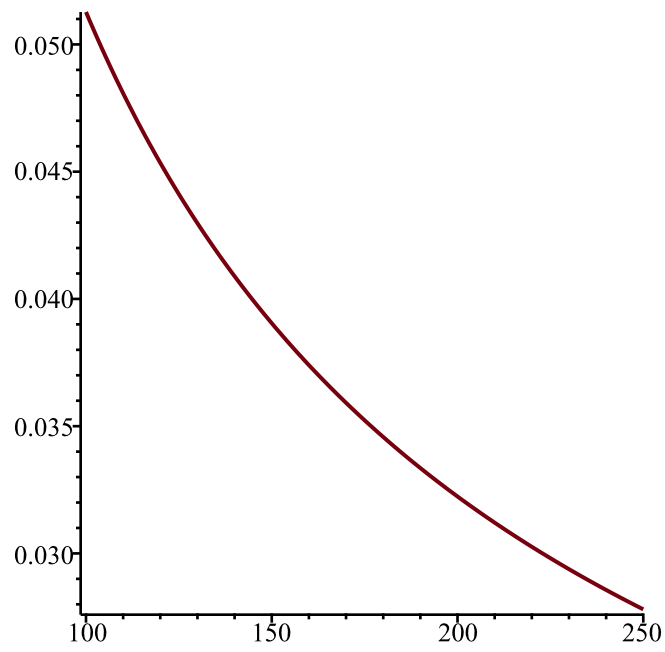
$$\frac{7}{4}$$

(1.2.5)

```
> for n from 3 to N2 do
  #q2[n] := (q1[n]-1)/(q1[n-1]-1);
  q2[n] := log(q1[n])/log(q1[n-1]);
  #q2[n] := log(1+10*n^(sigma-1))/log(1+10*(n-1)^(sigma-1));
end:
```

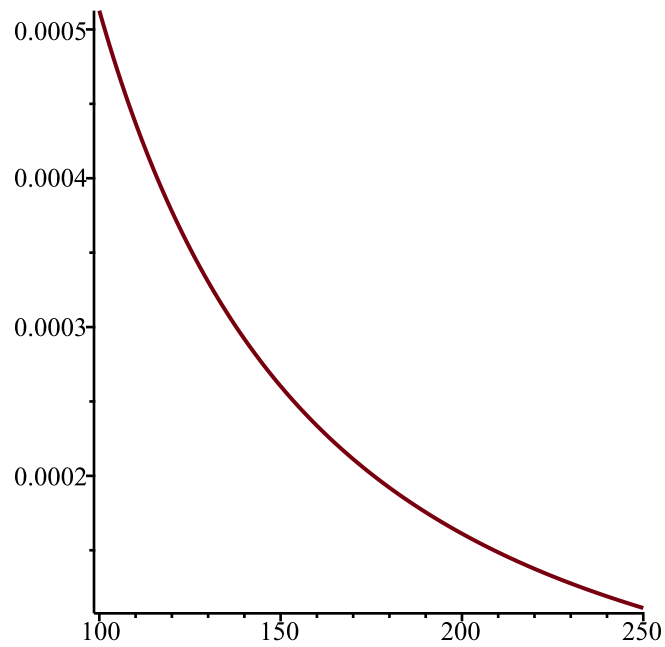
```
> N0 := 100;
sigma := 1/3;
plot([seq([n, n*(q2[n]-1)], n=N0..N2)]);
```

$$\sigma := \frac{1}{3}$$



```
> N0 := 100:
  sigma := 1/3;
  plot([seq([n, log(q2[n])], n=N0..N2)]);
```

$$\sigma := \frac{1}{3}$$



Relaxed 4-ary trees

```
Compute the number up to size NN
> for n from 0 to NN do
  for m from 0 to NN do
```

```

    rr4[n,m] := 0:
end:
end:

#initial conditions
for n from 0 to NN do
    rr4[n,0] := 1:
end:

for n from 1 to NN do
    for m from 1 to floor(n/3) do
        rr4[n,m] := rr4[n,m-1]+(m+1)*rr4[n-1,m];
    end do:
end do:

```

print the array

```

> for m from 7 to 0 by -1 do
  for n from 0 to 10 do
    printf("%10.0f ", rr4[n,m]);
  end;
  printf("\n");
end;

```

| | | | | | | |
|---|----|----|-----|------|------|------|
| | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1000 | 7255 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 15 | 76 | 291 | 1000 | 3255 | |
| 0 | 0 | 0 | 0 | 1 | 3 | |
| 7 | 15 | 31 | 63 | 127 | 255 | |
| | 1 | 1 | 1 | 1 | 1 | |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

<https://oeis.org/A102102>

deterministic completely defined initially connected acyclic automata with 4 inputs and n transient unlabeled states (and a unique absorbing state) with $a(0)=1$

```

> seq(rr4[3*n,n], n=0..min(floor(NN/3), 12));

```

1, 1, 15, 1000, 189035, 79278446, 63263422646, 86493299281972, 187766975052827491, (1.3.1)
 611024291011881918991, 2849262494779035461688236,
 18362167739517547774072439880, 158759599858376078627687256207242

Relaxed 5-ary trees

Compute the number up to size NN

```

> for n from 0 to NN do
  for m from 0 to NN do

```

```

    rr5[n,m] := 0:
end:
end:

#initial conditions
for n from 0 to NN do
    rr5[n,0] := 1:
end:

for n from 1 to NN do
    for m from 1 to floor(n/4) do
        rr5[n,m] := rr5[n,m-1]+(m+1)*rr5[n-1,m];
    end do:
end do:

```

print the array

```

> for m from 7 to 0 by -1 do
    for n from 0 to 10 do
        printf("%10.0f ", rr5[n,m]);
    end;
    printf("\n");
end;

```

```

0          0          0          0          0          0
0          0          0          0          0          0
0          0          0          0          0          0
0          0          0          0          0          0
0          0          0          0          0          0
0          0          0          0          0          0
0          0          0          0          0          0
0          0          0          0          0          0
0          0          0          31         156         595
3          7          15         31         63         127
1          1          1          1          1          1

```

not in the OEIS

```

> seq(rr5[4*n,n], n=0..min(floor(NN/4), 12));

```

```

1, 1, 31, 6631, 5470431, 12703473581, 68149976969707, 737746252883320473,
14607511868932281551079, 491034897540036851729430160,
26433393473562746491056524678041, 2174200324312090709906465225146948928,
262905488055517484872519169314942799083971

```

(1.4.1)

Weighted Dyck meanders, i.e., rescaled recurrence

(code from the Maple session accompanying [Elvey Price, Fang, Wallner 2021]; see <https://dmg.tuwien.ac.at/mwallner/>)

```
k-ary
> K := 3;
                                     K := 3
(1.5.1)
```

```
> NN;
                                     1000
(1.5.2)
```

compute the weighted Dyck meanders of size up to NN

```
> for n from 0 to NN do
  for m from -1 to NN do
    dd[n,m] := 0:
  end: end:

  dd[0,0] := 1:
  for n from 1 to NN do
    for m from 0 to n do
      dd[n,m] := (1+(n-m)/K) / ((m+(K-1)*n)/K) * dd[n-1,m-1] + dd[n-1,
m+K-1];
    end do:
  end do:
```

print the array

```
> for m from 5 to 0 by -1 do
  for n from 0 to 10 do
    printf("%10.0f ", dd[n,m]*factorial((m+(K-1)*n)/K));
  end;
  printf("\n");
end;
```

| | | | | | | |
|---|----|----|-----|-----|------|---|
| | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 63 | 0 | 0 | 0 |
| 0 | 0 | 31 | 0 | 0 | 1567 | |
| 0 | 15 | 0 | 0 | 480 | 0 | |
| 7 | 0 | 0 | 139 | 0 | 0 | |
| 0 | 0 | 36 | 0 | 0 | 1036 | |
| 0 | 1 | 0 | 0 | 1 | 0 | |
| 0 | 7 | 0 | 0 | 139 | 0 | |

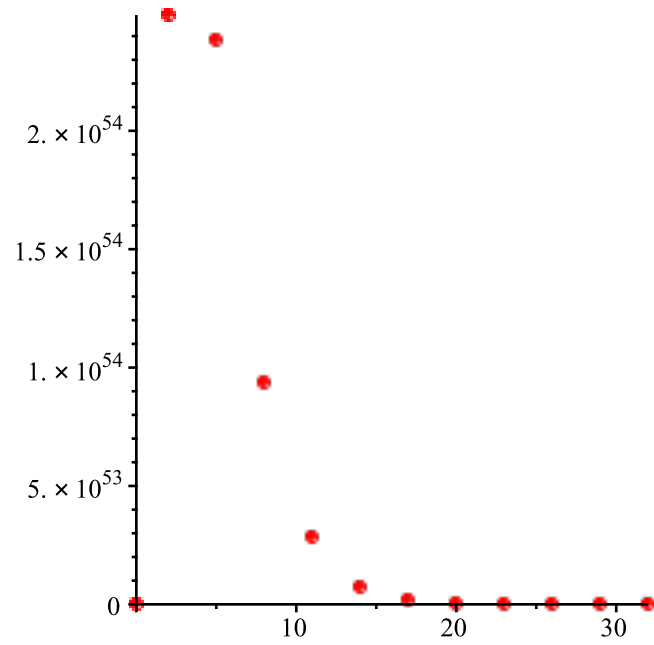
check

K=3: <http://oeis.org/A082162>

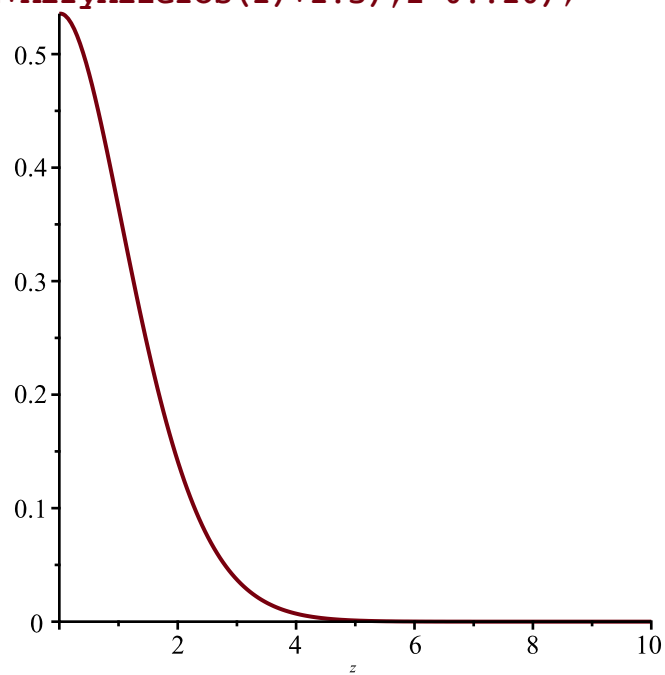
```
> seq(factorial((K-1)*n)/K * dd[n,0], n=0..min(NN, 10*K), K);
1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275,
183049273155939442
(1.5.3)
```

```
> ((K-1)*N)/K;
                                     2 N
                                     3
(1.5.4)
```

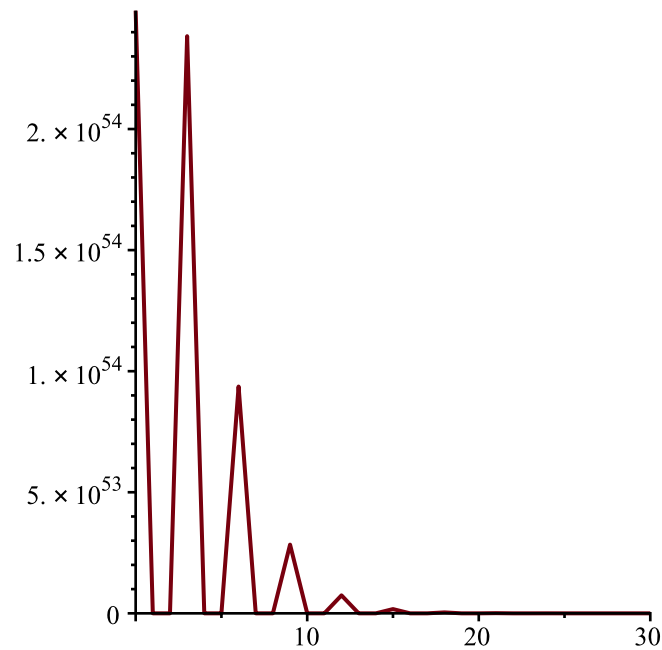
```
> N1 := K*floor(NN/K);
pointplot([[0,0], seq([i+K-1, dd[N1,i]], i=0..30, K)], color=red,
symbol=solidcircle, symbolsize=15);
NI := 999
```



```
> plot(AiryAi(z+AiryAiZeros(1)+1.3),z=0..10);
```



```
> plot([seq([i,dd[N1,i]],i=0..30)]);
```



```

> for n from 0 to NN do
  for m from -1 to NN do
    dd[n,m] := 0:
  end:end:

  dd[0,0] := 1:
  for n from 1 to NN do
    for m from 0 to n do
      #dd[n,m] := (1+(n-m)/K) / ((m+(K-1)*n)/K) / (1/(K-1)) * dd[n-1,
m-1]+dd[n-1,m+K-1];
      dd[n,m] := (1-K*(m-K+1) / ((K-1)*n+m)) * dd[n-1,m-1]+dd[n-1,m+
K-1];
    end do:
  end do:

```

print the array

```

> for m from 5 to 0 by -1 do
  for n from 0 to 10 do
    printf("%10.0f ", dd[n,m]*factorial((m+(K-1)*n)/K) * (1/(K-1))^(m+
(K-1)*n/K));
  end;
  printf("\n");
end;

```

| | | | | | | |
|---|----|----|-----|---|-----|------|
| | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 63 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 31 | 0 | 0 | 0 | 1567 |
| | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 15 | 0 | 0 | 0 | 480 | 0 |
| | 0 | 0 | 1 | 0 | 0 | 0 |
| 7 | 0 | 0 | 139 | 0 | 0 | 0 |
| | 0 | 1 | 0 | 0 | 0 | 3 |
| 0 | 0 | 36 | 0 | 0 | 0 | 1036 |
| | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 7 | 0 | 0 | 0 | 139 | 0 |

```

> seq(factorial((K-1)*n)/K*(1/(K-1))^( (0+(K-1)*n)/K)*dd[n,0],n=0.
.min(NN,10*K),K);
1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275,
183049273155939442

```

(1.5.5)

Weighted Dyck meanders, i.e., rescaled recurrence a bit different rescaling

(code from the Maple session accompanying [Elvey Price, Fang, Wallner 2021]; see <https://dmg.tuwien.ac.at/mwallner/>)

```

k-ary
> K := 3;

```

$K := 3$ (1.6.1)

```

> NN:=1000;

```

$NN := 1000$ (1.6.2)

compute the weighted Dyck meanders of size up to NN

```

> for n from 0 to NN do
  for m from -1 to NN do
    dd[n,m] := 0:
  end do:

  dd[0,0] := 1:
  for n from 1 to NN do
    for m from 0 to n do
      dd[n,m] := (1-K*(m-K+1)/((m+(K-1)*n)))*dd[n-1,m-1]+dd[n-1,
m+K-1];
    end do:
  end do:

```

print the array

```

> for m from 5 to 0 by -1 do
  for n from 0 to 10 do
    printf("%10.0f ",dd[n,m]*factorial((m+(K-1)*n)/K)*(1/(K-1))^(
(K-1)*n/K));
  end;
  printf("\n");
end;

```

| | | | | | | |
|----|---|----|-----|---|------|------|
| 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 200 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 3 | 0 |
| | 0 | 78 | 0 | 0 | 3949 | 0 |
| | 0 | 0 | 0 | 2 | 0 | 0 |
| | 0 | 30 | 0 | 0 | 960 | 0 |
| | 0 | 0 | 2 | 0 | 0 | 0 |
| 11 | 0 | 0 | 221 | 0 | 0 | 0 |
| | 0 | 1 | 0 | 0 | 4 | 0 |
| | 0 | 0 | 45 | 0 | 0 | 1305 |
| | 1 | 0 | 0 | 0 | 1 | 0 |
| | 0 | 7 | 0 | 0 | 139 | 0 |

check

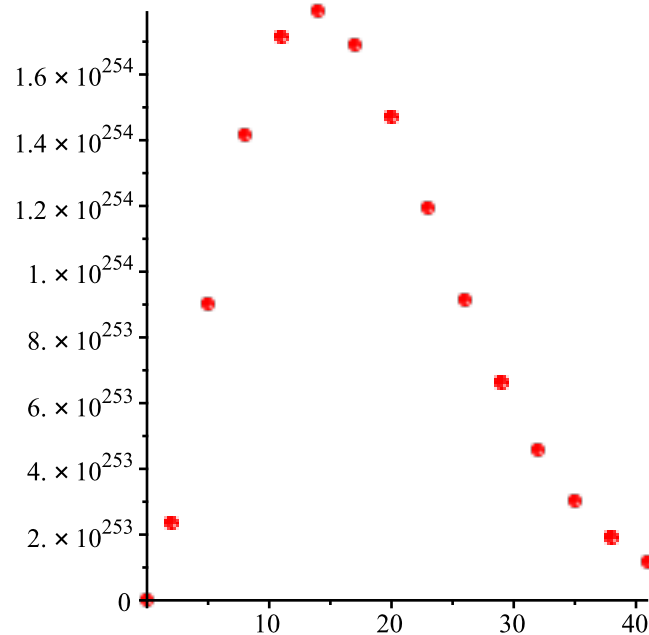
K=3: <http://oeis.org/A082162>

```
> seq(factorial((K-1)*n/K) * (1/(K-1))^(K-1)*n/K * dd[n,0], n=0..min
      (NN, 10*K), K);
1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275,
      183049273155939442
```

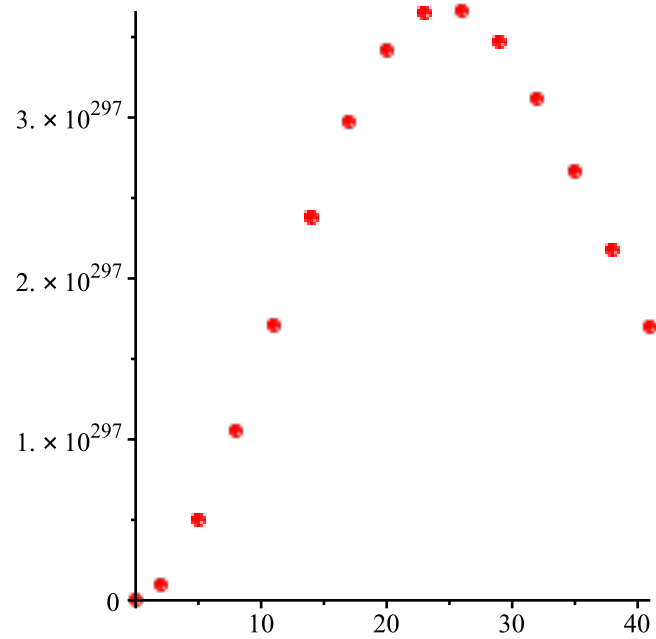
 (1.6.3)

new rescaling much better than old one

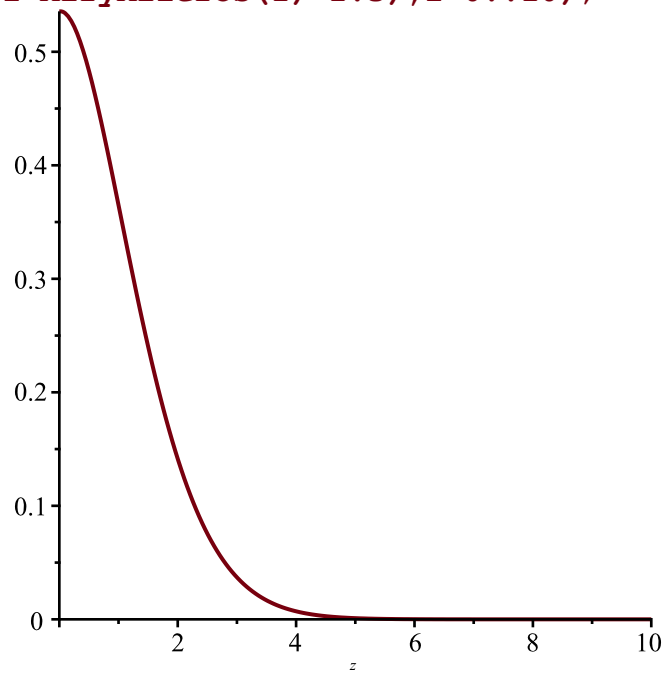
```
> N1 := K*floor(NN/K);
#N1 := 50*K;
pointplot([[0,0], seq([i+K-1, dd[N1,i] * (1/(K-1))^(K-1)*N1/K *
factorial((i+(K-1)*N1)/K) / factorial(ceil(((K-1)*N1+i)/2/K))
/factorial(ceil(((K-1)*N1+i)/(K-1)/2))], i=0..40, K)], color=red,
symbol=solidcircle, symbolsize=15);
NI := 999
```



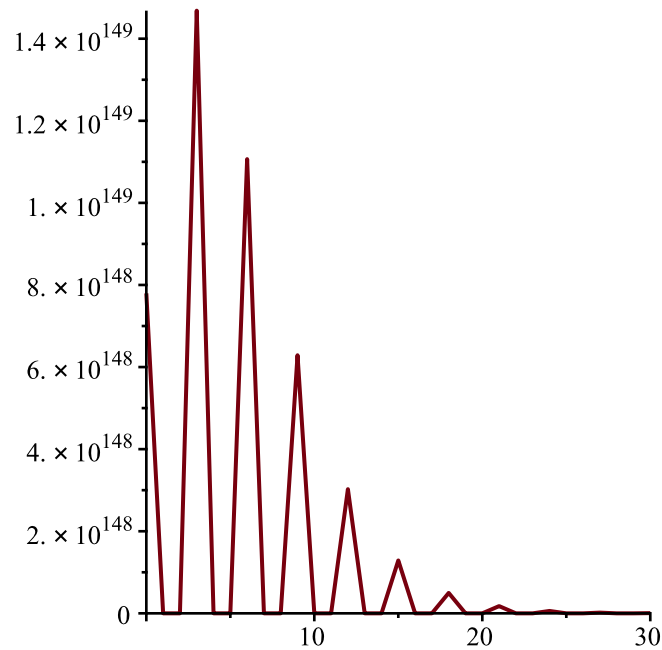
```
> N1 := K*floor(NN/K);
N1 := 290*K;
pointplot([[0,0], seq([i+K-1, dd[N1,i] * (1/(K-1))^(K-1)*N1/K *
factorial((i+(K-1)*N1)/K) / factorial(ceil(((K-1)*N1+i)/K))^2*ceil(
((K-1)*N1+i)/K)^(ceil(((K-1)*N1+i)/K))], i=0..40, K)], color=red,
symbol=solidcircle, symbolsize=15);
NI := 999
NI := 870
```

```
> pplot(AiryAi(z+AiryAiZeros(1)+1.3),z=0..10);
```



```
> plot([seq([i,dd[N1,i]],i=0..30)]);
```



```

> for n from 0 to NN do
  for m from -1 to NN do
    dd[n,m] := 0:
  end:end:

  dd[0,0] := 1:
  for n from 1 to NN do
    for m from 0 to n do
      #dd[n,m] := (1+(n-m)/K) / ((m+(K-1)*n)/K) / (1/(K-1)) * dd[n-1,
m-1]+dd[n-1,m+K-1];
      dd[n,m] := (1-K*(m-K+1) / ((K-1)*n+m)) * dd[n-1,m-1]+dd[n-1,m+
K-1];
    end do:
  end do:

```

```

print the array
> for m from 5 to 0 by -1 do
  for n from 0 to 10 do
    printf("%10.0f ", dd[n,m]*factorial((m+(K-1)*n)/K) * (1/(K-1))^(m+
(K-1)*n/K));
  end;
  printf("\n");
end;

```

| | | | | | | |
|---|----|----|-----|---|-----|------|
| | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 63 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 31 | 0 | 0 | 0 | 1567 |
| | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 15 | 0 | 0 | 0 | 480 | 0 |
| | 0 | 0 | 1 | 0 | 0 | 0 |
| 7 | 0 | 0 | 139 | 0 | 0 | 0 |
| | 0 | 1 | 0 | 0 | 0 | 3 |
| 0 | 0 | 36 | 0 | 0 | 0 | 1036 |
| | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 7 | 0 | 0 | 0 | 139 | 0 |

```

> seq(factorial((K-1)*n)/K * (1/(K-1)) ^ ((0+(K-1)*n)/K) * dd[n,0], n=0.
.min(NN,10*K),K);
1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275,
183049273155939442

```

(1.6.4)

Weighted Dyck meanders, i.e., rescaled recurrence a bit different rescaling again

(code from the Maple session accompanying [Elvey Price, Fang, Wallner 2021]; see <https://dmg.tuwien.ac.at/mwallner/>)

```

k-ary
> K := 3;

```

$K := 3$ (1.7.1)

```

> NN:=500;

```

$NN := 500$ (1.7.2)

```

compute the weighted Dyck meanders of size up to NN
> for i from 0 to NN do
  aa[0,i]:=0:
  for j from -1 to NN do
    aa[i,j] := 0:
  end:end:

  aa[0,0] := 1:

  for i from 1 to NN do
    for j from 0 to i do
      aa[i,j] := (i-j+K)/((K-1)*i+j)*aa[i-1,j-1]+aa[i-1,j+K-1];
    end do:
  end do:
> nij := (i,j) -> ((K-1)*i+j)/K;

```

$nij := (i,j) \mapsto \frac{(K-1) \cdot i + j}{K}$ (1.7.3)

```

print the array
> for j from 5 to 0 by -1 do
  for i from 0 to 10 do
    printf("%10.0f ",aa[i,j]*factorial(nij(i,j)));
  end;
  printf("\n");
end;

```

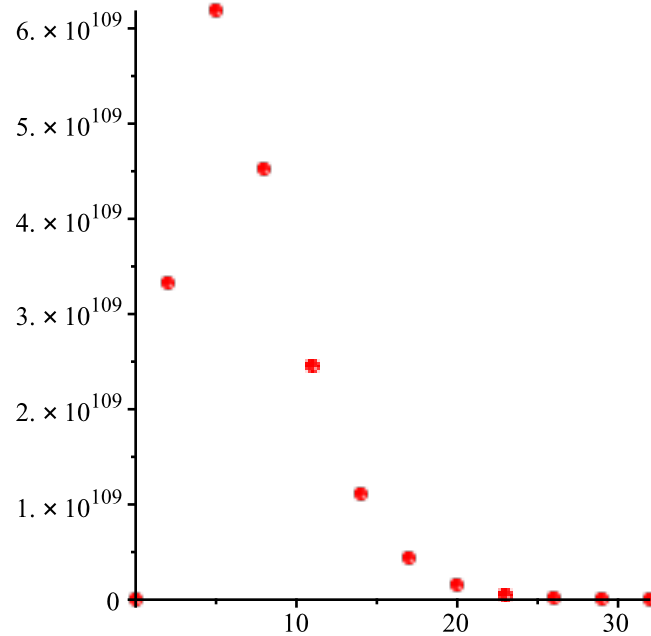
| | | | | | | |
|---|----|----|----|---|-----|------|
| | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 63 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 31 | 0 | 0 | 0 | 1567 |
| | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 15 | 0 | 0 | 0 | 480 | 0 |

$$\left[\frac{7}{24}, \frac{1}{8}, \frac{1}{720}, 0 \right]$$

(1.7.7)

new rescaling much better than old one

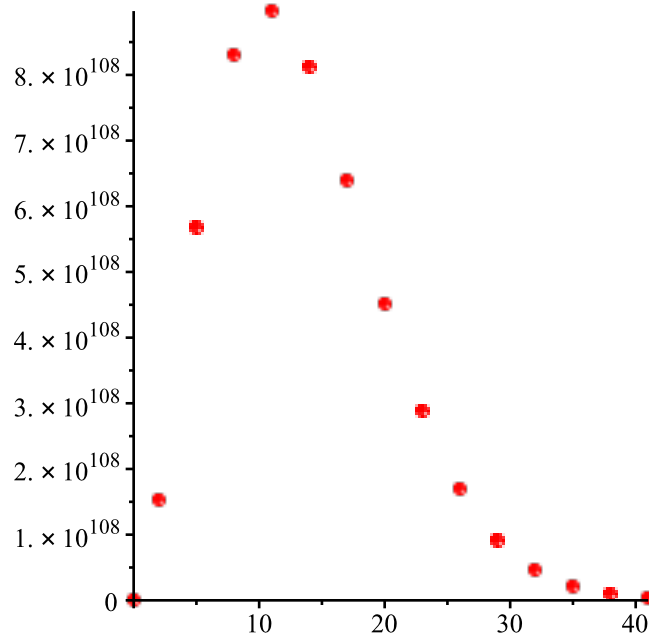
```
> #N1 := K*floor(NN/K);
N1 := 150*K;
pointplot([[0,0],seq([j+K-1,aa[N1,j]*factorial(nij(N1,j))/(
factorial(nij(N1,j))/2^nij(N1,j) ) ],j=0..30,K)],color=red,
symbol=solidcircle,symbolsize=15);
NI := 450
```



```
> N1 := K*floor(NN/K);
N1 := 150*K;
pointplot([[0,0],seq([i+K-1,dd[N1,i]*(1/(K-1))^(K-1)*N1/K)*
factorial(nij(N1,i))/factorial(ceil(nij(N1,i)/2))/factorial(ceil(
(nij(N1,i)-1)/2)) ],i=0..40,K)],color=red,symbol=solidcircle,
symbolsize=15);
```

NI := 999

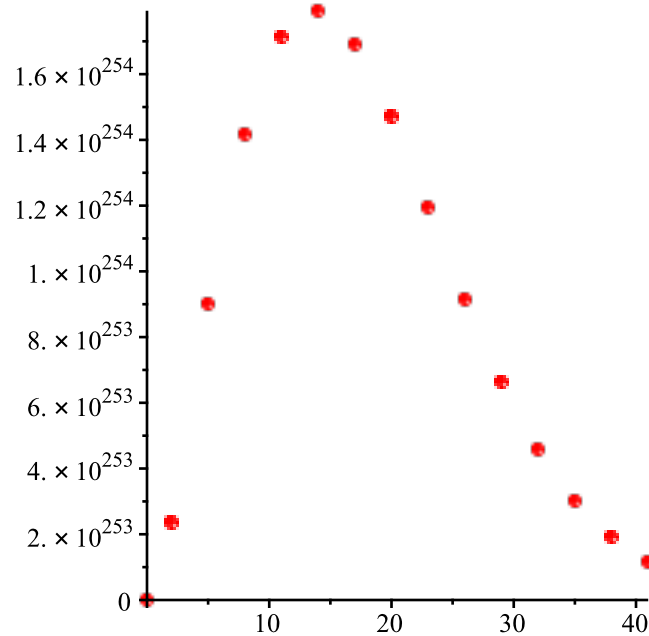
NI := 450



```

> N1 := K*floor(NN/K);
#N1 := 50*K;
pointplot([[0,0], seq([i+K-1, dd[N1,i] * (1/(K-1)) ^ ((K-1)*N1/K) *
factorial((i+(K-1)*N1)/K) / factorial(ceil(((K-1)*N1+i)/2/K))
/factorial(ceil(((K-1)*N1+i)/(K-1)/2))], i=0..40, K)], color=red,
symbol=solidcircle, symbolsize=15);
NI := 999

```



**Weighted Dyck meanders, i.e., rescaled recurrence a bit different
rescaling again**

(code from the Maple session accompanying [Elvey Price, Fang, Wallner 2021]; see <https://dmg.tuwien.ac.at/mwallner/>)

```
k-ary
> K := 3;
                                     K := 3
(1.8.1)
```

```
> NN:=1000;
                                     NN := 1000
(1.8.2)
```

```
compute the weighted Dyck meanders of size up to NN
> for i from 0 to NN do
  bb[0,i]:=0:
  for j from -1 to NN do
    bb[i,j] := 0:
  end:
end:

bb[0,0] := 1:

for i from 1 to NN do
  for j from 0 to i do
    #bb[i,j] := (K-1)*(i-j+K)/((K-1)*i+j)*bb[i-1,j-1] + bb[i-1,
j+K-1];
    bb[i,j] := 2*(i-j+3)/(2*i+j)*bb[i-1,j-1] + bb[i-1,j+2];
  end do:
end do:
> nij := (i,j) -> ((K-1)*i+j)/K;
                                     nij := (i,j) ↦  $\frac{(K-1) \cdot i + j}{K}$ 
(1.8.3)
```

```
print the array
> for j from 5 to 0 by -1 do
  for i from 0 to 10 do
    printf("%10.0f ",bb[i,j]*factorial(nij(i,j))/2^nij(i,j));
  end;
  printf("\n");
end;

1      0      0      0      0      0
  0      0      0      63     0      0
0      0      31     0      0      1567
  0      0      0      0      1      0
0      15     0      0      480     0
  0      0      0      1      0      0
7      0      0      139    0      0
  0      0      1      0      0      3
0      0      36     0      0      1036
  1      0      0      0      1      0
0      7      0      0      139     0
```

```
check
K=3: http://oeis.org/A082162
> seq(factorial(nij(i,0))/2^nij(i,0)*bb[i,0],i=0..min(NN,10*K),K);
1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275,
(1.8.4)
```

183049273155939442

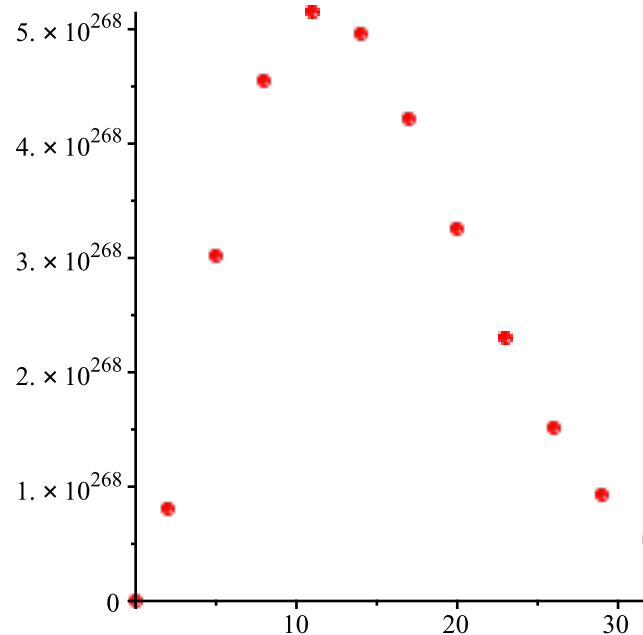
```
> nij(x,y);
```

$$\frac{2x}{3} + \frac{y}{3}$$

(1.8.5)

```
> #N1 := K*floor(NN/K);  
N1 := 200*K;  
pointplot([[0,0],seq([j+K-1,bb[N1,j]*factorial(nij(N1,j))  
/factorial(ceil(nij(N1,j)/2))/factorial(ceil((nij(N1,j)-1)/2))],  
j=0..30,K)],color=red,symbol=solidcircle,symbolsize=15);
```

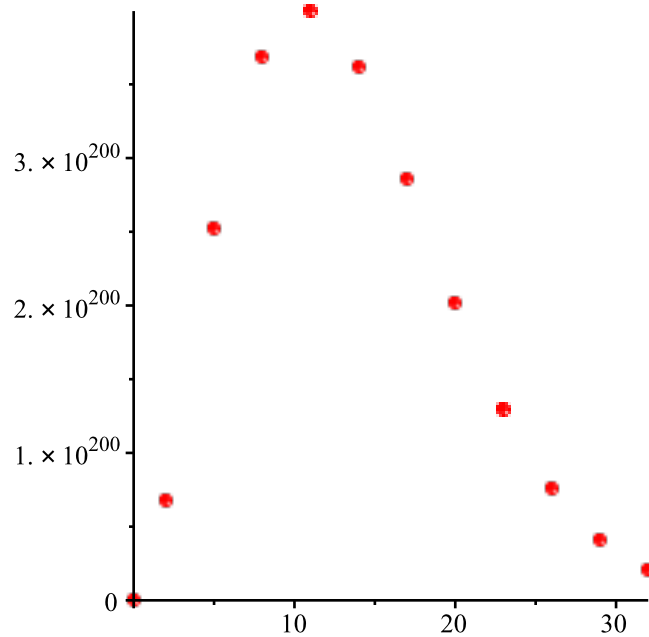
N1 := 600



new rescaling much better than old one

```
> #N1 := K*floor(NN/K);  
N1 := 150*K;  
pointplot([[0,0],seq([j+K-1,bb[N1,j]*2^nij(N1,j)],j=0..30,K)],  
color=red,symbol=solidcircle,symbolsize=15);
```

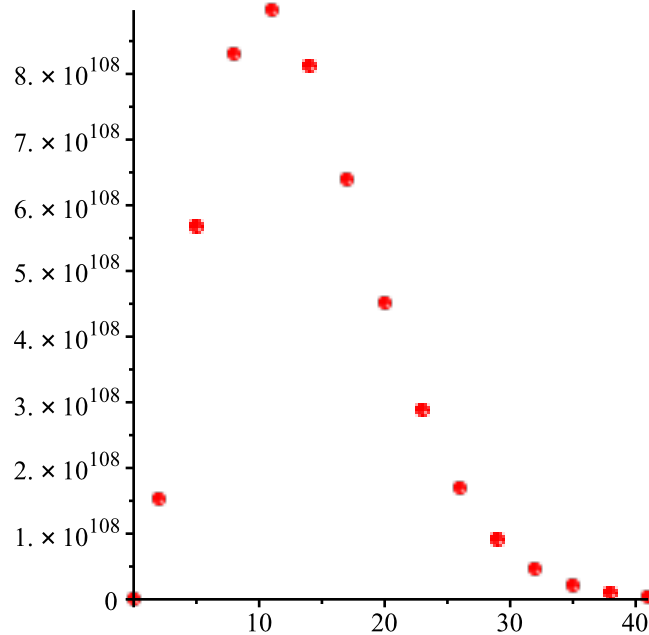
N1 := 450



```
> N1 := K*floor(NN/K);
N1 := 150*K;
pointplot([[0,0],seq([i+K-1,dd[N1,i]*(1/(K-1))^(K-1)*N1/K)*
factorial(nij(N1,i))/factorial(ceil(nij(N1,i)/2))/factorial(ceil(
(nij(N1,i)-1)/2))],i=0..40,K)],color=red,symbol=solidcircle,
symbolsize=15);
```

NI := 999

NI := 450

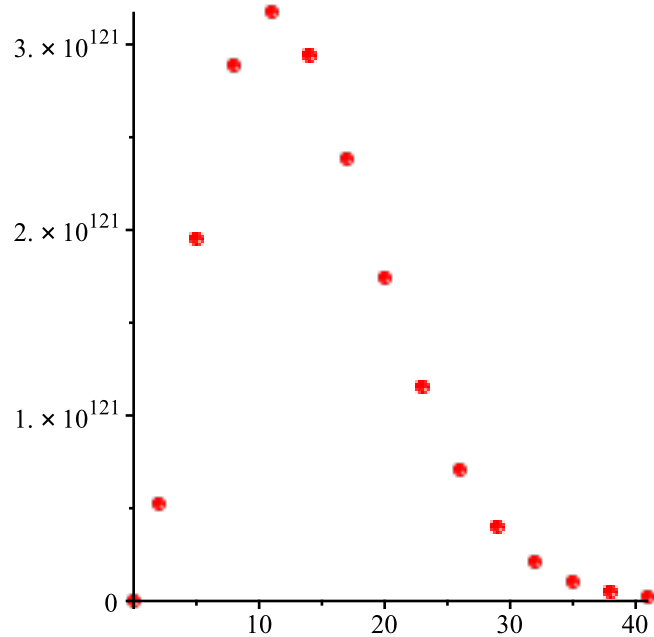


```
> N1 := K*floor(NN/K);
#N1 := 50*K;
pointplot([[0,0],seq([i+K-1,dd[N1,i]*(1/(K-1))^(K-1)*N1/K)*
factorial((i+(K-1)*N1)/K)/factorial(ceil((K-1)*N1+i)/2/K))
```

```

/factorial(ceil(((K-1)*N1+i)/(K-1)/2)),i=0..40,K],color=red,
symbol=solidcircle,symbolsize=15);
NI := 498

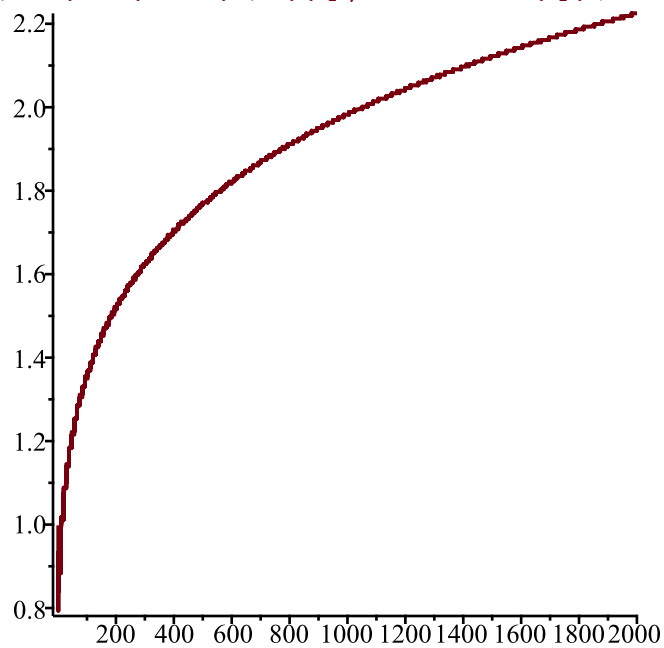
```



```

> plot([seq([n, factorial(ceil(n/2))*factorial(ceil((n-1)/2)) /
(factorial(n)/2^(n-1)*n^(1/3))],n=1..2000)]);

```



```

> 3

```

Drift considerations

for large u , this recurrence has nearly 0 drift! (at least for $w=o(u)$)

important to capture the Airy function

```
> newdrift := (K-1)^2*(K + u - w)/(K*u - u + w) - (K-1);
  subs(w = q*u,%);      # q=w/u
  map(simplify,series(%,u=infinity,3));
  series(simplify(subs(q=0,%)),u=infinity,2); # for q very small,
  at least q=o(1)
```

$$\begin{aligned} \text{newdrift} := & \frac{(K-1)^2 (K+u-w)}{Ku-u+w} - K + 1 \\ & \frac{(K-1)^2 (-qu+K+u)}{Ku+qu-u} - K + 1 \\ & - \frac{(K-1)^2 (q-1)}{K+q-1} - K + 1 + \frac{(K-1)^2 K}{(K+q-1)u} \\ & \frac{(K-1)K}{u} \end{aligned} \tag{1.9.1}$$

```
> olddrift := (K + u - w)/(K*u - u + w) - (K-1);
  subs(w = q*u,%);      # q=w/u
  map(simplify,series(%,u=infinity,3));
  series(simplify(subs(q=0,%)),u=infinity,2); # for q very small,
  at least q=o(1)
```

$$\begin{aligned} \text{olddrift} := & \frac{K+u-w}{Ku-u+w} - K + 1 \\ & \frac{-qu+K+u}{Ku+qu-u} - K + 1 \\ & \frac{-q+1}{K+q-1} - K + 1 + \frac{K}{(K+q-1)u} \\ & - \frac{K(K-2)}{K-1} + \frac{K}{(K-1)u} \end{aligned} \tag{1.9.2}$$

Weighted Dyck meanders now with zero drift

k-ary

```
> K := 3;
                                     K := 3 \tag{1.10.1}
```

```
> NN:=100;
                                     NN := 100 \tag{1.10.2}
```

compute the weighted Dyck meanders of size up to NN

```
> for i from 0 to NN do
  bb[0,i]:=0:
  for j from -1 to NN do
  bb[i,j] := 0:
  end:end:
```

```

bb[0,0] := 1:

for i from 1 to NN do
  for j from 0 to i do
    bb[i,j] := (K-1)^2*(i-j+K)/((K-1)*i+j)*bb[i-1,j-1] + bb
[i-1,j+K-1];
  end do:
end do:

```

number of righth steps when reaching (i,j)
> nij := (i,j) -> ((K-1)*i+j)/K

$$nij := (i,j) \mapsto \frac{(K-1) \cdot i + j}{K} \quad (1.10.3)$$

print the array

```

> for j from 5 to 0 by -1 do
  for i from 0 to 10 do
    printf("%10.0f ",bb[i,j]*factorial(nij(i,j))/(K-1)^(2*nij(i,j)));
  end;
  printf("\n");
end;

```

| | | | | | | |
|---|----|----|----|-----|-----|------|
| | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 63 | 0 | 0 | 0 |
| 0 | 0 | 31 | 0 | 0 | 1 | 1567 |
| 0 | 15 | 0 | 0 | 1 | 480 | 0 |
| 7 | 0 | 0 | 1 | 139 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 3 |
| 0 | 0 | 36 | 0 | 0 | 0 | 1036 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 7 | 0 | 0 | 0 | 139 | 0 |

check

K=3: <http://oeis.org/A082162>

```

> seq(factorial(nij(i,0))/(K-1)^(2*nij(i,0))*bb[i,0],i=0..min(NN,
10*K),K);

```

1, 1, 7, 139, 5711, 408354, 45605881, 7390305396, 1647470410551, 485292763088275, 183049273155939442 (1.10.4)

recurrence with balanced weights (zero drift)

```

> eerec := ee[_n,_m] = (K-1)^2*(K+_n-_m)/((K-1)*_n+_m)*ee[_n-1,
_m-1]+ee[_n-1,_m+K-1];

```

$$eerec := ee_{_n, _m} = \frac{(K-1)^2 (K + _n - _m) ee_{_n-1, _m-1}}{(K-1) _n + _m} + ee_{_n-1, _m+K-1} \quad (2)$$

```
> A1 := AiryAiZeros(1);
```

```
AI := AiryAiZeros(1) \quad (3)
```

Newton polygons

Programs from [Elvey Price, Fang, Wallner 2021]

Get the maximal power of n^b for each m^a for $a=0..M$

```
> getMaxNewt := proc(M::posint, newt)
  local i, el, mnmax;

  for i from 0 to M do mnmax[i]:=-infinity end:
  for el in newt do
    if el[1]<=M then
      if mnmax[el[1]]<el[2] then mnmax[el[1]]:=el[2] end:
    end;
  end;

  return mnmax;
end:
```

Compute the slopes and corners of the convex hull

```
> maxslope := proc(ll,M,i)
  local j,sl,tmp,sj;
  sl := ll[i+1]-ll[i]: #initial slope between first 2 points
  #is this the one of the hull? find max starting from 0
  sj:=M;
  for j from i+1 to M do
    tmp := (ll[j]-ll[i])/(j-i);
    if tmp >= sl then sl:=tmp;sj:=j; end;
  end:
  return sl,sj;
end:
```

Compute the slopes and corners of the convex hull

```
> getslopes := proc(ll,M)
  local sl,sj,li,ls;
  sj:=0;
  li:=sj;
  ls:=0;
  #go on and find other slopes of convex hull
  while sj<M do
    sl,sj := maxslope(ll,M,sj):
    #save it
    li:=li,sj;
    ls:=ls,sl;
  end:

  li:=[li];
  ls:=[seq(ls[i],i=2..nops(li))];
  return ls,li;
```

```
end:
```

List programs - Arrays

```
expansion to array (better performance than list)
> ex2Array := proc(ex,M,N,p)
  local tt,mdeg,ndeg,ar,mm,nn:
  ar := Array(0..M,0..p*N):

  for tt in expand(ex) do
    mdeg := m*diff(tt,m)/tt;
    ndeg := n*diff(tt,n)/tt;

    if mdeg>=0 and mdeg<=M and ndeg>=-N and ndeg<=0 then
      mm := mdeg;
      nn := -ndeg*p;
      ar[mm,nn] := ar[mm,nn] + subs(m=1,n=1,tt);
    end;
  end;

  return ar;
end:

> mynewtArray := proc(ar,p)
  local newt,ad,mm,nn;
  newt := {}:

  ad := [ArrayDims(posCF)];

  for mm in `$`(ad[1]) do
    for nn in `$`(ad[2]) do
      if ar[mm,nn]<>0 then
        newt := {op(newt), [mm,-nn/p]};
      end;
    end;
  end:

  return newt;
end:

> getArray := proc(ar,p,a,b)
  return ar[a,-b*p];
end:
```

Shorthands

```
The largest root of the Airy function Ai(z)
> A1 := AiryAiZeros(1);
  evalf(%);

      Ai := AiryAiZeros(1)
      -2.338107410
```

(2.3.1)

```
This is the constant c in
s(n) = 2 + c*n^(-2/3) + ...
```

and will be responsible for the base of the stretched exponential

```
> csubs := c=a1*(K - 1)^(1/3)*K/2^(1/3);
```

$$csubs := c = \frac{a1 (K - 1)^{1/3} K 2^{2/3}}{2} \quad (2.3.2)$$

Labeling options to produce nice plots.

```
> myoptionsLo := labels=["i", "j"], symbolsize=25, symbol=diamond,
axesfont = ["HELVETICA", "ROMAN", 15], labelfont = ["HELVETICA",
18]:
myoptionsUp := labels=["i", "j"], symbolsize=20, symbol=diamond,
axesfont = ["HELVETICA", "ROMAN", 15], labelfont = ["HELVETICA",
18]:
```

```
> (*
fchoice3:
wsubs:
normal(isolate(%, lambda)):
subs(%, %%%);
*)
```

We introduce the following shorthands for the Airy function and its derivative

Keep the factor L undetermined; we will see it has to be $2/(K-1)$; it actually rescales n

```
> kaplam := normal(AiryAi(AiryAiZeros(1)+(L)^(1/3)*m/n^(1/3)))=
kappa, normal(AiryAi(1, AiryAiZeros(1)+(L)^(1/3)*m/n^(1/3)))=
lambda;
```

$$kaplam := \text{AiryAi}\left(\frac{\text{AiryAiZeros}(1) n^{1/3} + L^{1/3} m}{n^{1/3}}\right) = \kappa, \text{AiryAi}\left(1, \frac{\text{AiryAiZeros}(1) n^{1/3} + L^{1/3} m}{n^{1/3}}\right) = \lambda \quad (2.3.3)$$

```
> eerec;
```

$$ee_{_n, _m} = \frac{(K-1)^2 (K + _n - _m) ee_{_n-1, _m-1}}{(K-1) _n + _m} + ee_{_n-1, _m+K-1} \quad (2.3.4)$$

The factor (coefficient) of ee[n-1,m-1]

```
> fac1 := (K - 1)^2*(K + n - m)/((K - 1)*n + m);
```

$$fac1 := \frac{(K-1)^2 (K+n-m)}{(K-1) n + m} \quad (2.3.5)$$

The factor (coefficient) of ee[n-1,m+K-1]

```
> fac2 := 1;
```

$$fac2 := 1 \quad (2.3.6)$$

Expansions

```
> n:='n':m:='m':
```

We expand the Airy function around $a1+(L)^{1/3}*m/n^{1/3}$ up to chosen order ordAi

```
> FFy := AiryAi(AiryAiZeros(1)+(L)^(1/3)*m/n^(1/3)+y);
```

$$FFy := \text{AiryAi}\left(\text{AiryAiZeros}(1) + \frac{L^{1/3} m}{n^{1/3}} + y\right) \quad (2.4.1)$$

We use two different expansion orders for the upper and lower bound (to speed up the computations, and to produce the best pictures)

Here, we start with the **lower bound**, which needs less terms

```
> ordAiLo := 19;
   FFyserLo := map(expand, series(FFy, y, ordAiLo)) :
               ordAiLo := 19
```

(2.4.2)

For $y = x - (a1 + L * m / n^{1/3})$ we have that $FF(x) = Ai(x)$,
i.e. an expansion of the Airy function

```
> FFxserLo := map(normal, subs(y=x - (AiryAiZeros(1) + (L)^(1/3) * m/n^(1/3)), FFyserLo)) :
> indets(FFxserLo) :
   map(x->subs(n=1, diff(x, n)), select(has, remove(has, %, L), n)) :
   NordLo := min(%);
```

$$NordLo := -\frac{17}{3} \quad (2.4.3)$$

Replace the appearing Airy functions by our shorthands kappa and lambda

```
> indets(FFxserLo) ;
   FFxserLoKL := subs(kaplam, AiryAiZeros(1)=a1, FFxserLo) :
   indets(%);
```

$$\left\{ L, m, n, x, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{5/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3}, \text{AiryAi}\left(\frac{\text{AiryAiZeros}(1) n^{1/3} + L^{1/3} m}{n^{1/3}}\right), \text{AiryAi}\left(1, \frac{\text{AiryAiZeros}(1) n^{1/3} + L^{1/3} m}{n^{1/3}}\right) \right\}$$

```
> indets(FFxserLos) ;
   #FFxserLosKL := subs(kaplam, AiryAiZeros(1)=a1, FFxserLos) :
   #indets(%);
```

$$\left\{ L, a1, \kappa, \lambda, m, n, x, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{5/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3} \right\} \quad (2.4.4)$$

Then, we use the generic ansatz for factor of FFy, i.e. the Airy function

```
> ## general ansatz to determine the constants ##
   ## in order to get better bounds, one would need to kill more
   ## terms in the Newton polygons ##
   # facAiryLo := 1 + (add(q[i]*m^i, i=0..2))/(n) + add(qq[i]*m^i, i=0..1)/(n^(2/3))
   # + add(ss[i]*m^i, i=0..2)/(n^(4/3)) + add(rr[i]*m^i,
```



```
i=0..4)/(n^(5/3)) + add(uu[i]*m^i,i=0..4)/(n^(6/3));
```

```
## we will only need this ##
```

```
## we only need ss[2] and rr[3] in the end ##
```

```
facAiryLo := 1+(add(q[i]*m^i,i=0..2))/(n) + add(qq[i]*m^i,i=0..1) / (n^(2/3))
```

```
+ add(ss[i]*m^i,i=2..2)/(n^(4/3)) + add(rr[i]*m^i, i=3..3)/(n^(5/3));
```

$$facAiryLo := 1 + \frac{m^2 q_2 + m q_1 + q_0}{n} + \frac{m q q_1 + q q_0}{n^2 | 3} + \frac{ss_2 m^2}{n^4 | 3} + \frac{rr_3 m^3}{n^5 | 3} \quad (2.4.5)$$

This is our ansatz

(only the substitution is influenced by the parameters, i.e. not the ms and ns that are already in FFxser; that is what we want, as all of them should be expanded at $a1+(L)^{(1/3)}*m/n(1/3)$)

Note that due to the replacement with kappa and lambda, the expansions are fixed already at this point and the ms and ns in the arguments of Ai and Ai' are not influenced.

Note that we substitute now $a1+(L)^{(1/3)}*(m+1)/n^{(1/3)}$, i.e. $m+1$ instead of m around which we expanded above.

```
> XFL := (n0,m0) -> subs(n=n0,m=m0,facAiryLo)*subs(x=a1+(L)^(1/3)*(m0+1)/n0^(1/3),FFxserLoKL);
```

$$XFL := (n0, m0) \mapsto subs(n = n0, m = m0, facAiryLo) \cdot subs\left(x = a1 + \frac{L^{1/3} \cdot (m0 + 1)}{n0^{1/3}}, \quad (2.4.6)$$

FFxserLoKL

Then, we do the same for the **upper bound**, with a few more terms

```
> ordAiUp := 30;
```

```
FFyserUp := map(expand, series(FFy, y, ordAiUp)) :
```

```
ordAiUp := 30
```

(2.4.7)

For $y = x - (a1+(L)^{(1/3)}*m/n^{(1/3)})$ we have that $FF(x)=Ai(x)$,

i.e. an expansion of the Airy function

```
> FFxserUp := normal(subs(y=x-(AiryAiZeros(1)+(L)^(1/3)*m/n^(1/3)), FFyserUp)) :
```

Compute the real expansion order in n

```
> indets(FFxserUp) :
```

```
map(x->subs(n=1,diff(x,n)),select(has,remove(has,%,m),n)) :
```

```
NordUp := min(%);
```

$$NordUp := -\frac{26}{3}$$

(2.4.8)

```
> #kaplam;
```

Replace the appearing Airy functions by our shorthands kappa and lambda

```
> indets(FFxserUp) :
```

```
FFxserUpKL := subs(kaplam,AiryAiZeros(1)=a1,FFxserUp) :
```

```
indets(%);
```

$$\left\{ L, m, n, x, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, L^{10/3}, L^{11/3}, L^{13/3}, L^{14/3}, \frac{1}{n^{26/3}} \right\}$$

$$\frac{1}{n^{25/3}}, \frac{1}{n^{23/3}}, \frac{1}{n^{20/3}}, \frac{1}{n^{19/3}}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^8/3}, \frac{1}{n^7/3},$$

$$\frac{1}{n^5/3}, \frac{1}{n^2/3}, \frac{1}{n^1/3}, n^1/3, n^2/3, n^4/3, n^5/3, n^7/3, n^8/3, n^{10/3}, n^{11/3}, n^{13/3},$$

$$n^{14/3}, n^{16/3}, n^{17/3}, n^{19/3}, n^{20/3}, n^{22/3}, n^{23/3}, n^{25/3}, n^{26/3},$$

$$\text{AiryAi}\left(\frac{\text{AiryAiZeros}(1) n^{1/3} + L^{1/3} m}{n^{1/3}}\right), \text{AiryAi}\left(1, \frac{\text{AiryAiZeros}(1) n^{1/3} + L^{1/3} m}{n^{1/3}}\right)\right\}$$

$$\left\{L, aI, \kappa, \lambda, m, n, x, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, L^{10/3}, L^{11/3}, L^{13/3}, L^{14/3}, \right. \quad (2.4.9)$$

$$\frac{1}{n^{26/3}}, \frac{1}{n^{25/3}}, \frac{1}{n^{23/3}}, \frac{1}{n^{20/3}}, \frac{1}{n^{19/3}}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^8/3},$$

$$\frac{1}{n^7/3}, \frac{1}{n^5/3}, \frac{1}{n^2/3}, \frac{1}{n^1/3}, n^1/3, n^2/3, n^4/3, n^5/3, n^7/3, n^8/3, n^{10/3}, n^{11/3},$$

$$n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3}, n^{19/3}, n^{20/3}, n^{22/3}, n^{23/3}, n^{25/3}, n^{26/3}\left\}$$

In the upper bound, we use a extend the lower bound ansatz by additional terms $p[i]$ which are divided by n^2

> facAiryUp := facAiryLo + (add(p[i]*m^i, i=0..4))/(n^2);

$$\text{facAiryUp} := 1 + \frac{m^2 q_2 + m q_1 + q_0}{n} + \frac{m q q_1 + q q_0}{n^{2/3}} + \frac{ss_2 m^2}{n^{4/3}} + \frac{rr_3 m^3}{n^{5/3}} \quad (2.4.10)$$

$$+ \frac{p_4 m^4 + p_3 m^3 + p_2 m^2 + p_1 m + p_0}{n^2}$$

This is our ansatz for the upper bound (comparable to XFL)

(only the substitution is influenced by the parameters, i.e. not the m s and n s that are already in FFxser ; that is what we want, as all of them should be expanded at $a1+(L)^{(1/3)}*m/n^{(1/3)}$)

> XFU := (n0, m0) -> subs(n=n0, m=m0, facAiryUp) * subs(x=a1+(L)^(1/3)*(m0+1)/n0^(1/3), FFxserUpKL);

$$\text{XFU} := (n0, m0) \mapsto \text{subs}(n=n0, m=m0, \text{facAiryUp}) \cdot \text{subs}\left(x=a1 + \frac{L^{1/3} \cdot (m0+1)}{n0^{1/3}}, \right) \quad (2.4.11)$$

$$\text{FFxserUpKL})$$

> A1;

$$\text{AiryAiZeros}(1) \quad (2.4.12)$$

> series(AiryAi(A1+x), x=0, 2);
evalf(%);

$$\text{AiryAi}(1, \text{AiryAiZeros}(1)) x + O(x^2)$$

$$0.7012108227 x + O(x^2) \quad (2.4.13)$$

Finally, the ansatz for the **quotient** of $h(n)/h(n-1)$.

Note that pterm is a mnemonic for "polynomial term", as this value influences the polynomial term $n^{\{\alpha\}}$;

The other values have similar interpretations:

a exponential growth, i.e. a^n

b b will be zero

c stretched exponential

pterm related to the critical exponent in polynomial term in asymptotics

dup technical choice, to simplify the proofs

> SF := n -> a+b/n^(1/3)+c/n^(2/3)+pterm/n+d/n^(7/6);

$$SF := n \mapsto a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}} \quad (2.4.14)$$

Gen TCs - Lower bound

lower bound:

We will prove that this is positive

> i:='i':

posansatz := -XX(n,m)*SS(n) + Fac1*XX(n-1,m-1) + Fac2*XX(n-1,m+K-1);

subs (Fac1=fac1, Fac2=fac2, %);

posansatz := -XX(n,m) SS(n) + Fac1 XX(n-1, m-1) + Fac2 XX(n-1, m+K-1)

$$-XX(n,m) SS(n) + \frac{(K-1)^2 (K+n-m) XX(n-1, m-1)}{(K-1) n + m} + XX(n-1, m+K-1) \quad (2.5.1)$$

Xtilde : specific values from proof below for experiments

From summary after proof below

> varsLowerBound :=

[q[0] = 0, # arbitrary; not needed

q[1] = (7*K - 11)/(6*(K - 1)),

q[2] = -(K+2)/6/(K-1),

qq[0] = 0, # arbitrary, not needed

qq[1] = -a1*(K - 2)*2^(2/3)/(6*(K - 1)^(2/3)),

L=2/(K-1),

a=K,

b=0,

csubs,

pterm = K*(7*K-6)/6,

d=-1

];

$$varsLowerBound := \left[q_0 = 0, q_1 = \frac{7K-11}{6K-6}, q_2 = -\frac{K+2}{6(K-1)}, qq_0 = 0, qq_1 = \right. \quad (2.5.1.1)$$

$$\left. \begin{aligned} & -\frac{a1(K-2)2^{2/3}}{6(K-1)^{2/3}}, L = \frac{2}{K-1}, a=K, b=0, c = \frac{a1(K-1)^{1/3}K2^{2/3}}{2}, pterm \\ & = \frac{K(7K-6)}{6}, d = -1 \end{aligned} \right\}$$

Fix K for experiments

> **KK := 3;**

$$KK := 3$$

(2.5.1.2)

This is X tilde (this is derived later)

> **subs(varsLowerBound, a1=A1, K=KK, (1 + (m^2*q[2] + m*q[1] + q[0])/n + (m*qq[1] + qq[0])/n^(2/3))*AiryAi(a1 + L^(1/3)*(m + 1)/n^(1/3))**

):
Xansatz := (n, m) -> subs(K=KK, ((1 + (-m^2*(K + 2)/(6*(K - 1)) + m*(7*K - 11)/(6*K - 6))/n - m*AiryAiZeros(1)*(K - 2)*2^(2/3)/(6*(K - 1)^(2/3)*n^(2/3)))*AiryAi(AiryAiZeros(1) + 2^(1/3)*(1/(K - 1))^(1/3)*(m + 1)/n^(1/3))));

$$Xansatz := (n, m) \mapsto \text{subs} \left(K=KK, \left(1 + \frac{-\frac{m^2 \cdot (K+2)}{6 \cdot K-6} + \frac{m \cdot (7 \cdot K-11)}{6 \cdot K-6}}{n} \right. \right. \quad (2.5.1.3)$$

$$\left. \left. - \frac{m \cdot \text{AiryAiZeros}(1) \cdot (K-2) \cdot 2^{2/3}}{6 \cdot (K-1)^{2/3} \cdot n^{2/3}} \right) \cdot \text{AiryAi} \left(\text{AiryAiZeros}(1) + \frac{2^{1/3} \cdot \left(\frac{1}{K-1} \right)^{1/3} \cdot (m+1)}{n^{1/3}} \right) \right)$$

(this is derived later)

> **subs(varsLowerBound, a1=A1, a + b/n^(1/3) + c/n^(2/3) + pterm/n + d/n^(7/6));**

Sansatz := n -> subs(K=KK, K + AiryAiZeros(1)*(K - 1)^(1/3)*K*2^(2/3)/(2*n^(2/3)) + K*(7*K - 6)/(6*n) - 1/n^(7/6));

$$K + \frac{\text{AiryAiZeros}(1) \cdot (K-1)^{1/3} \cdot K \cdot 2^{2/3}}{2 \cdot n^{2/3}} + \frac{K(7K-6)}{6n} - \frac{1}{n^{7/6}}$$

$$Sansatz := n \mapsto \text{subs} \left(K=KK, K + \frac{\text{AiryAiZeros}(1) \cdot (K-1)^{1/3} \cdot K \cdot 2^{2/3}}{2 \cdot n^{2/3}} \right. \quad (2.5.1.4)$$

$$\left. + \frac{K \cdot (7 \cdot K - 6)}{6 \cdot n} - \frac{1}{n^{7/6}} \right)$$

> **posXS := map(simplify, subs(XX=Xansatz, SS=Sansatz, Fac1=fac1, Fac2=fac2, K=KK, posansatz)):indets(%);**

$$\left\{ m, n, \frac{1}{n^{7/6}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}, \text{AiryAi} \left(\text{AiryAiZeros}(1) + \frac{m}{(n-1)^{1/3}} \right), \text{AiryAi} \left(\text{AiryAiZeros}(1) + \frac{m+1}{n^{1/3}} \right), \text{AiryAi} \left(\text{AiryAiZeros}(1) \right) \right\} \quad (2.5.1.5)$$

$$\left. + \frac{m+3}{(n-1)^{1/3}} \right\}$$

For a large n this function of m seems to be positive: looks good :)

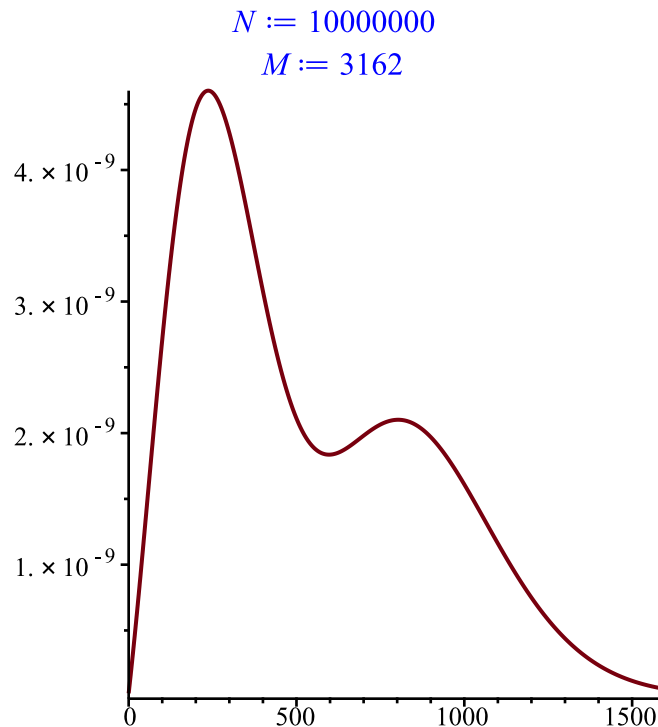
>

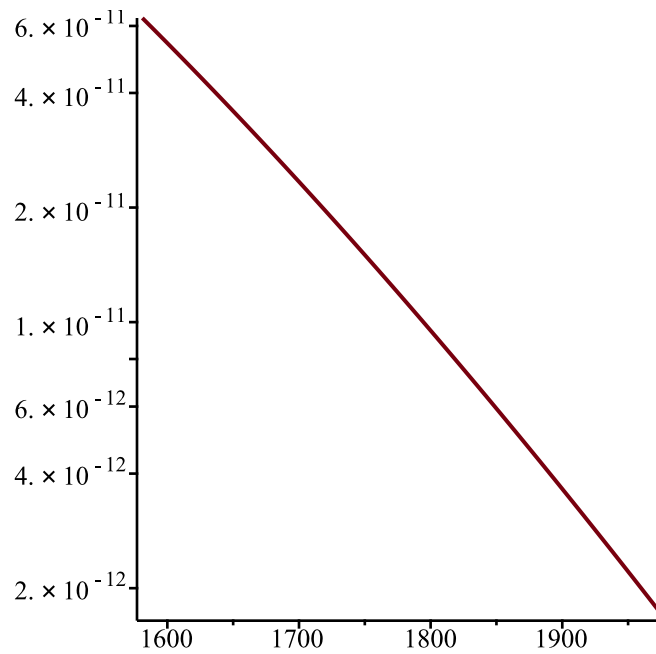
```

Digits:=20:
e1 := subs(csubs, a1=A1, pterm=1/6, posXS) : indets(%);
N := 10000000;
M := floor(N^(1/2));
P1 := plot([seq([mm, (subs(n=N, m=mm, e1))], mm=0..floor(M/2))])
:display(P1);
P2 := logplot([seq([mm, (subs(n=N, m=mm, e1))], mm=floor(M/2)..floor
(5*M/8))]) :display(P2);

```

$$\left\{ m, n, \frac{1}{n^{7/6}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}, \text{AiryAi}\left(\text{AiryAiZeros}(1)\right) + \frac{m}{(n-1)^{1/3}} \right\}, \text{AiryAi}\left(\text{AiryAiZeros}(1) + \frac{m+1}{n^{1/3}}\right), \text{AiryAi}\left(\text{AiryAiZeros}(1) + \frac{m+3}{(n-1)^{1/3}}\right) \left. \right\}$$





Prove it

Recall the general ansatz

```
> facAiryLo*AiryAi (a1+(L)^(1/3)*(m+1)/n^(1/3));
SF(n);
```

$$\left(1 + \frac{q_2 m^2 + q_1 m + q_0}{n} + \frac{m q q_1 + q q_0}{n^{2/3}} + \frac{ss_2 m^2}{n^{4/3}} + \frac{rr_3 m^3}{n^{5/3}} \right) \text{AiryAi} \left(a1 + \frac{L^{1/3} (m+1)}{n^{1/3}} \right)$$

$$a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}} \tag{2.5.2.1}$$

```
> subs (Fac1=fac1, Fac2=fac2, K=2, posansatz);
-XX(n, m) SS(n) + \frac{(2+n-m) XX(n-1, m-1)}{n+m} + XX(n-1, m+1) \tag{2.5.2.2}
```

Substitute ansatz into the sequence we want to be positive for large n and all m

```
> posF := map (expand, subs (XX=XFL, SS=SF, Fac1=fac1, Fac2=fac2, posansatz)) : indets (%), nops (expand (%));
{K, L, a, a1, b, c, d, κ, λ, m, n, pterm, q_0, q_1, q_2, qq_0, qq_1, rr_3, ss_2, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, L^{7/3}, L^{8/3}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, n^{17/3}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}}}, 11162 \tag{2.5.2.3}
```

```

> #subs(varsLowerBound, facAiryLo);
#Sansatz := n -> K + AiryAiZeros(1)*(K - 1)^(1/3)*K*2^(2/3)/(2*n^(2/3)) + K*(7*K - 6)/(6*n) - 1/n^(7/6);
> #XFLtest := (n0,m0) -> subs(n=n0,m=m0,1 + (-m^2*(K + 2)/(6*(K - 1)) + m*(7*K - 11)/(6*K - 6))/n - m*a1*(K - 2)*2^(2/3)/(6*(K - 1)^(2/3)*n^(2/3)))*subs(x=a1+(L)^(1/3)*(m0+1)/n0^(1/3),FFxserLoKL);
> #posFXtest := map(expand, subs(XX=XFLtest,SS=Sansatz,Fac1=fac1, Fac2=fac2,posansatz));
> #tmp := series(subs(K=K1+1,posFXtest),n=infinity,3) assuming K1>0,n::posint,m::posint;
> #tmp2 := collect(convert(tmp,polynomial),[n,m],simplify) assuming K1>0,n::posint,m::posint;
> #series(tmp2,n=infinity,2);

```

The error terms are (to check, expand posF)

UPDATE

```

> simplify(O((2^(1/3)*(m+1)/n^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo));
simplify(O((2^(1/3)*(m)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo));
simplify(O((2^(1/3)*(m+2)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo));

```

$$\begin{aligned}
& O\left(\frac{64 \cdot 2^{1/3}}{n^{19/3}}\right) \\
& O\left(\frac{64 \cdot 2^{1/3} \cdot m^{19} \cdot (n^{1/3} - (n-1)^{1/3})^{19}}{(n-1)^{19/3} \cdot n^{19/3}}\right) \\
& O\left(\frac{64 \cdot 2^{1/3} \cdot (-m \cdot (n-1)^{1/3} + n^{1/3} \cdot (m+2))^{19}}{n^{19/3} \cdot (n-1)^{19/3}}\right) \tag{2.5.2.4}
\end{aligned}$$

remove error terms

```

> posFd := convert(posF,polynomial):indets(%);

```

$$\left\{ K, L, a, a1, b, c, d, \kappa, \lambda, m, n, pterm, q_0, q_1, q_2, qq_0, qq_1, rr_3, ss_2, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, \tag{2.5.2.5}$$

$$\left. \begin{aligned}
& L^{7/3}, L^{8/3}, \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \\
& \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, \\
& n^{16/3}, n^{17/3}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}} \end{aligned} \right\}$$

Next we fit the display size of the plots in the n (Nord) and m (Mord) direction.

(Note that everything up to ordAi is computed, but possibly not shown)

```

> Nord := -ordAiLo/3;
Mord := floor(ordAiLo/3);
myview := view=[0..Mord,Nord..0];

```

$$Nord := -\frac{19}{3}$$

$$Mord := 6 \tag{2.5.2.6}$$

```

> Nord := -Mord;

```

(2.5.2.7)

$Nord := -6$

(2.5.2.7)

Expand again with respect to n,
these are then our unknowns

```
> posFe := series(posFd,n=infinity,ceil(-Nord)+1):indets(%);  
posFf := convert(%%,polynom):  
nops(expand(%));
```

$$\left\{ K, L, a, a1, b, c, d, \kappa, \lambda, m, n, pterm, q_0, q_1, q_2, qq_0, qq_1, rr_3, ss_2, L^{1/3}, L^{2/3}, L^{4/3}, L^{5/3}, \right.$$
$$L^{7/3}, L^{8/3}, L^{10/3}, L^{11/3}, L^{13/3}, L^{14/3}, L^{16/3}, L^{17/3}, \left(\frac{1}{n}\right)^{1/3}, \left(\frac{1}{n}\right)^{2/3},$$
$$\left(\frac{1}{n}\right)^{3/2}, \left(\frac{1}{n}\right)^{4/3}, \left(\frac{1}{n}\right)^{5/2}, \left(\frac{1}{n}\right)^{5/3}, \left(\frac{1}{n}\right)^{7/2}, \left(\frac{1}{n}\right)^{7/3}, \left(\frac{1}{n}\right)^{7/6},$$
$$\left(\frac{1}{n}\right)^{8/3}, \left(\frac{1}{n}\right)^{9/2}, \left(\frac{1}{n}\right)^{10/3}, \left(\frac{1}{n}\right)^{11/2}, \left(\frac{1}{n}\right)^{11/3}, \left(\frac{1}{n}\right)^{11/6}, \left(\frac{1}{n}\right)^{13/3},$$
$$\left(\frac{1}{n}\right)^{13/6}, \left(\frac{1}{n}\right)^{14/3}, \left(\frac{1}{n}\right)^{16/3}, \left(\frac{1}{n}\right)^{17/3}, \left(\frac{1}{n}\right)^{17/6}, \left(\frac{1}{n}\right)^{19/6}, \left(\frac{1}{n}\right)^{23/6},$$
$$\left.\left(\frac{1}{n}\right)^{25/6}, \left(\frac{1}{n}\right)^{29/6}, \left(\frac{1}{n}\right)^{31/6}, \left(\frac{1}{n}\right)^{35/6}, O\left(\frac{1}{n^6}\right)\right\}$$

12252

(2.5.2.8)

```
> #map(simplify,series(convert(coeff(subs(varsLowerBound,K=K1+1,  
posFf),rr[1]),polynom),n=infinity,3)) assuming K1>0;  
> #map(simplify,series(convert(coeff(subs(varsLowerBound,K=K1+1,  
posFf),ss[2]),polynom),n=infinity,3)) assuming K1>0;
```

lcm of denominators in exponents

```
> PP := 6;
```

$PP := 6$

(2.5.2.9)

convert the expansion into a list with respect to the degrees in m and n:

Format: [[mdeg,ndeg], coeff]

```
> posCF := ex2Array(posFf,Mord,-Nord,PP):ArrayNumElems(posCF);  
259
```

(2.5.2.10)

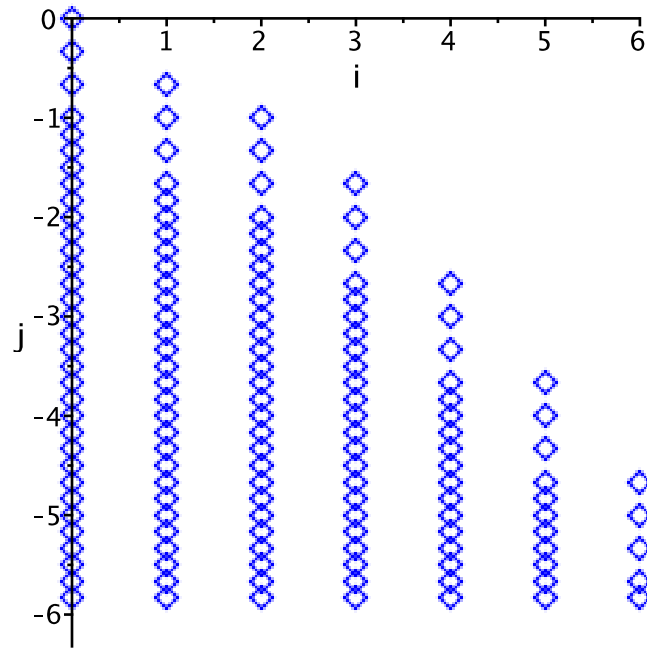
The mynewt function computes the Newton polygon of posFf

```
> newt1 := mynewtArray(posCF,PP):
```

First Newton polygon, where no unknowns have been fixed

Looks good, as usual!

```
> P1 := pointplot(newt1,myoptionsLo,color=blue):  
display(P1,myview);
```

Here, we want to kill the element (0,0)

```
> posCF[0,0];
simplify(%);
```

$$\kappa - \frac{2 \kappa K}{K-1} + \frac{\kappa K^2}{K-1} + \frac{\kappa}{K-1} - a \kappa$$

$$(K-a) \kappa$$

(2.5.2.11)

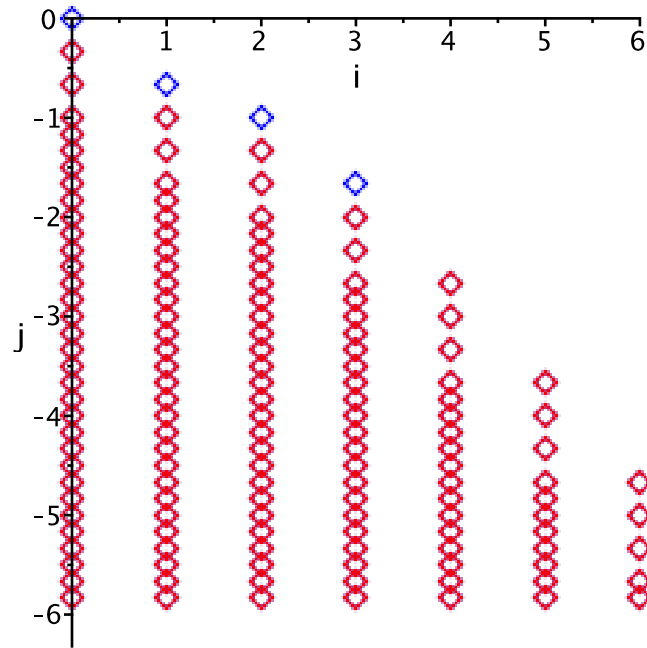
Set a=K

```
> posCFa := map(factor, subs(a=K, posCF)) :
```

```
> newta := mynewtArray(posCFa, PP) :
```

All blue points have been eliminated, and only the red ones remain:

```
> P1a := pointplot(newta, myoptionsLo, color=red) :
display(P1, P1a, myview) ;
```



These should all be zero:

we set $b=0$

```
> getArray(posCFa, PP, 0, -1/3) :
  collect(%, [kappa, lambda], factor) ;
                                      $-\kappa b$ 
```

(2.5.2.12)

$L = 2/(K-1)$

```
> getArray(posCFa, PP, 1, -1) :
  collect(subs(b=0, qq[2]=0, qqs[2]=0, %), [kappa, lambda],
  factor@simplify) assuming K::posint;
  solve(%, L) ;
```

$$\frac{K(LK - L - 2)\kappa}{2} \frac{2}{K-1}$$

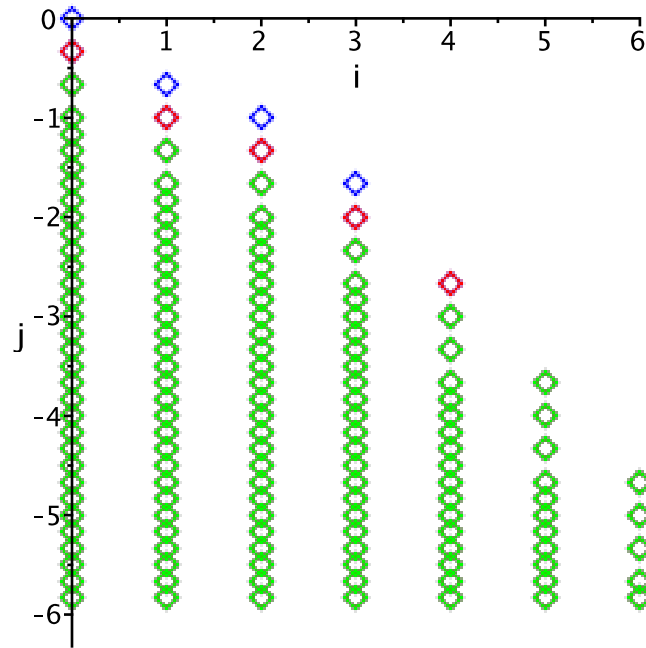
(2.5.2.13)

set $a=K$, $b=0$, $L=2/(K-1)$

```
> posCFab := map(factor, subs(b=0, L=2/(K-1),
  posCFa)) :
> newtab := mynewtArray(posCFab, PP) :
```

Now only the green points remain.

```
> Plab := pointplot(newtab, myoptionsLo, color=green) :
  display(P1, Pla, Plab, myview) ;
```



at this point we find our choice for c , which we heuristically computed already before in Section 3 also possible here to set $q[2]=-(K+2)/6/(K-1)$

```
> csubs ;
simplify(getArray(posCFab,PP,0,-2/3)) ;
collect(%,[lambda,kappa],factor@simplify) assuming K::posint;
simplify(subs(csubs,%)) assuming K::posint;
```

$$c = \frac{a l (K-1)^{1/3} K 2^{2/3}}{2}$$

$$\frac{\left(K a l 2^{2/3} (K-1) \left(\frac{1}{K-1} \right)^{2/3} - 2c \right) \kappa}{2}$$

$$\frac{\left(\frac{a l (K-1)^{1/3} K 2^{2/3}}{2} - c \right) \kappa}{0}$$

(2.5.2.14)

choose $q[2]$

```
> simplify(getArray(posCFab,PP,1,-4/3)) :
collect(subs(csubs,%),[lambda,kappa],factor@simplify) assuming
K::posint;
isolate(% ,q[2]);
```

$$\frac{2^{1/3} K (6 K q_2 + K - 6 q_2 + 2) \lambda}{3 (K-1)^{1/3}}$$

$$q_2 = \frac{-K-2}{6K-6}$$

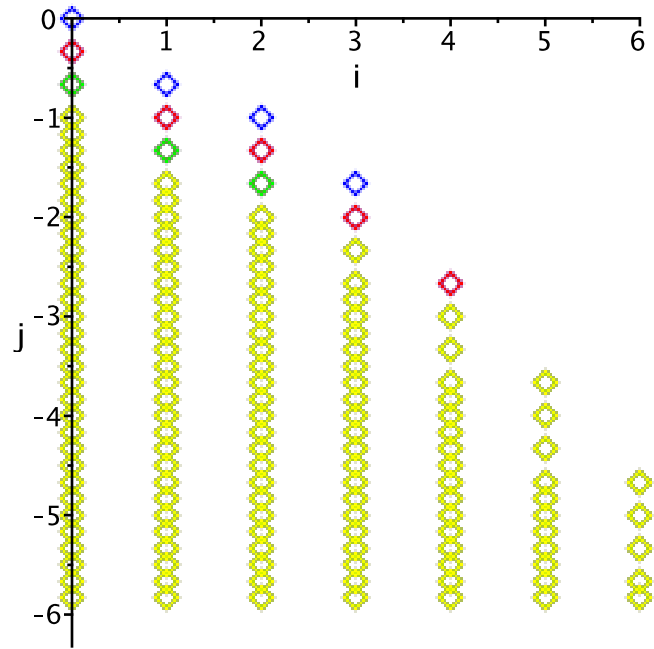
(2.5.2.15)

set $a=K, b=0, L=2/(K-1), c=a l * K * ((K-1)/2)^{(1/3)}, q[2]=-(K+2)/6/(K-1)$

```
> posCFabc := map(simplify@factor,subs(csubs,q[2]=-(K+2)/6/(K-1),
posCFab)) assuming K::posint:
```

```
> newtabc := mynewtArray(posCFabc,PP) :
```

```
> Plabc := pointplot(newtabc,myoptionsLo,color=yellow):
display(P1,Pla,Plab,Plabc,myview);
```



Here we get pterm

Interesting: the qq[1] term is fixed here and is not visible in the binary case! (K=2)

```
> factor(getArray(posCFabc,PP,0,-1)):
collect(%,[lambda,kappa],factor@simplify) assuming K::posint;
coeff(% ,kappa);
factor(isolate(% ,pterm));
coeff(%%%,lambda);
factor(isolate(% ,qq[1]));
```

$$\frac{K \left(3 K 2^{1/3} qq_1 + K (K-1)^{1/3} a l - 3 2^{1/3} qq_1 - 2 (K-1)^{1/3} a l \right) \lambda}{3 (K-1)^{1/3}} + \left(\frac{7}{6} K^2 \right.$$

$$\left. - K - pterm \right) \kappa$$

$$\frac{7}{6} K^2 - K - pterm$$

$$pterm = \frac{K(7K-6)}{6}$$

$$\frac{K \left(3 K 2^{1/3} qq_1 + K (K-1)^{1/3} a l - 3 2^{1/3} qq_1 - 2 (K-1)^{1/3} a l \right)}{3 (K-1)^{1/3}}$$

$$qq_1 = - \frac{a l (K-2) 2^{2/3}}{6 (K-1)^{2/3}}$$

(2.5.2.16)

For the polynomial term we need this correction by n^(-1/3)

```
> kaplam;
subs(m=1,n=2*n,AiryAi(AiryAiZeros(1) + (2/(K-1))^(1/3)*m/n^(1/3)))
);
```

```
map(simplify, series(% , n=infinity, 1));
```

$$\text{AiryAi}\left(\frac{\text{AiryAiZeros}(1) n^{1/3} + L^{1/3} m}{n^{1/3}}\right) = \kappa, \text{AiryAi}\left(1, \frac{\text{AiryAiZeros}(1) n^{1/3} + L^{1/3} m}{n^{1/3}}\right) = \lambda$$

$$\text{AiryAi}\left(\text{AiryAiZeros}(1) + \frac{\left(\frac{1}{K-1}\right)^{1/3}}{n^{1/3}}\right)$$

$$\text{AiryAi}(1, \text{AiryAiZeros}(1)) \left(\frac{1}{K-1}\right)^{1/3} \left(\frac{1}{n}\right)^{1/3} + O\left(\frac{1}{n}\right) \quad (2.5.2.17)$$

Conjectured polynomial term !!!

Divide by K and correct by -1/3

fits our conjecture :)

```
> factor((K*(7*K - 6)/6)/K-1/3);
subs(K=2,%); #binary case ok
```

$$\frac{7K}{6} - \frac{4}{3}$$

1

(2.5.2.18)

```
set set a=K, b=0, L=2/(K-1), c=a1*K*((K-1)/2)^(1/3), q[2]=-(K+2)/6/(K-1), qq[1]=-a1*(K-2)*2^(2/3)/6/(K-1)^(2/3), pterm=K*(7*K-6)/6
```

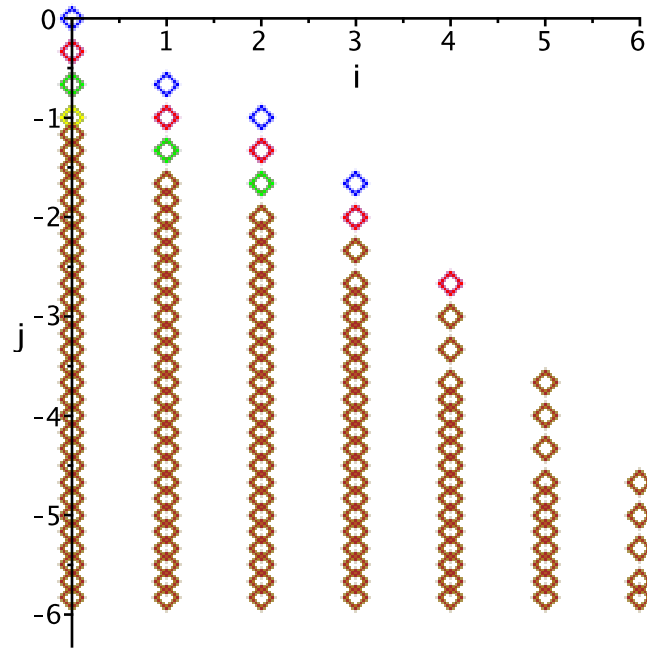
```
> posCFabcd := map(simplify, subs(pterm=K*(7*K-6)/6, qq[1]=-a1*(K-2)*2^(2/3)/6/(K-1)^(2/3), posCFabc)) assuming K::posint:
```

```
> newtabcd := mynewtArray(posCFabcd, PP):
```

only the brown points remain

Now all points are strictly below n^{-1}

```
> Plabcd := pointplot(newtabcd, myoptionsLo, color=brown):
display(P1, Pla, Plab, Plabc, Plabcd, myview);
```



Here are the dominating corners and we see that we have to choose $d=-1$ to have a positive term; note that we will see that the second term should be negative, as $\lambda=A_i$ is negative for large m

```
> getArray(posCFabcd, PP, 0, -7/6);
   getArray(posCFabcd, PP, 3, -14/6);
   (14/6-7/6)/3; #slope
```

$$-\frac{\kappa d}{18} \frac{(K+2)^2 \lambda K^{2^{1/3}}}{(K-1)^{4^{1/3}}}$$

(2.5.2.19)

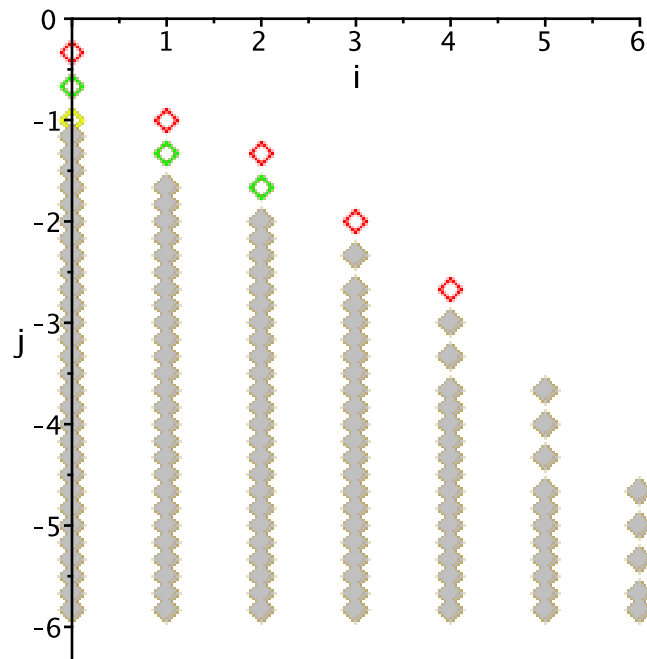
set $a=K$, $b=0$, c , p term=..., and $d=-1$

```
> posCFabcde := subs(d=-1, rr[4]=0, uu[4]=0, posCFabcd);
> newtabcde := mynewtArray(posCFabcde, PP):
```

This is the final result, where only the solid diamonds are non-zero

```
> Plabcde := pointplot(newtabcde, myoptionsLo, symbol=soliddiamond,
   color=gray);
   display(P1, Pla, Plab, Plabc, Plabcd, Plabcde, myview);
```

Warning, data could not be converted to float Matrix



Plot the boundary and the slopes of the Newton polygon;

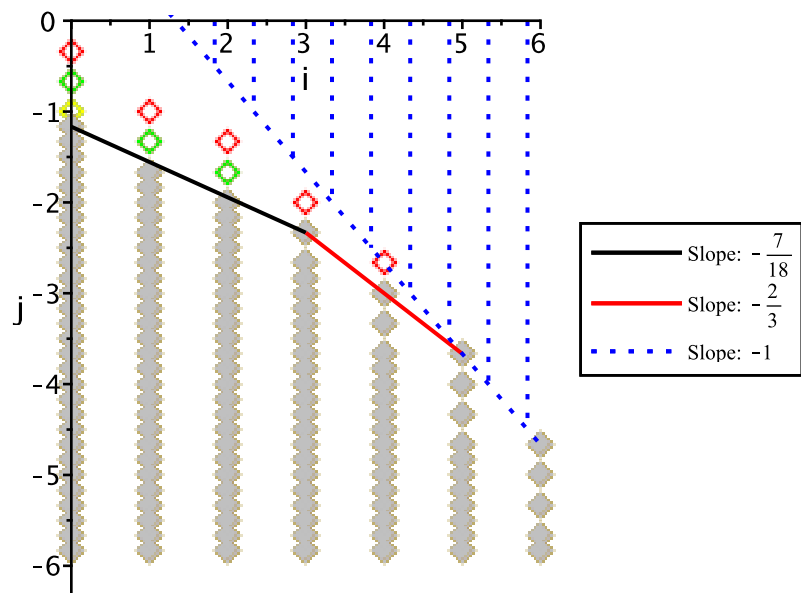
Note that we have already proved that there are now points above the blue dotted line

```
> LegendSize := size=[600,450]:
P1dom1 := plot(-7/6-(7/18)*m,m=0..3,color=black,legend=[typeset
("Slope: ", -7/18)],legendstyle=[location=right]):
P1dom2 := plot(-1/3-(2/3)*m,m=3..5,color=red,legend=[typeset
("Slope: ", -2/3)],legendstyle=[location=right]):
P1dom3a := plot(4/3-m,m=0..6,color=blue,linestyle=dot,legend=
[typeset("Slope: ", -1)],legendstyle=[location=right]):
P1all := display(P1,P1a,P1ab,P1abc,P1abcd,P1abcde,P1dom1,P1dom2,
P1dom3a,myview,LegendSize):

for i from 1 to 10 do
  P1dom3[i] := plot([[4/3+i/2,0],[4/3+i/2,-i/2]],color=blue,
linestyle=dot):
end:
display(P1all,seq(P1dom3[i],i=1..9));
```

[Warning, data could not be converted to float Matrix](#)

[Warning, data could not be converted to float Matrix](#)



This is the choice for SF

```
> SF(n);
subs(a=K,b=0,csubs,pterm=K*(7*K-6)/6,d=-1,%);
```

$$a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}}$$

$$K + \frac{a1(K-1)^{1/3}K2^{2/3}}{2n^{2/3}} + \frac{K(7K-6)}{6n} - \frac{1}{n^{7/6}} \quad (2.5.2.20)$$

recall

```
> subs(L=2/(K-1),[kplam]);
```

$$\left[\text{AiryAi} \left(\frac{\text{AiryAiZeros}(1) n^{1/3} + 2^{1/3} \left(\frac{1}{K-1} \right)^{1/3} m}{n^{1/3}} \right) = \kappa, \text{AiryAi} \left(1, \right. \right. \quad (2.5.2.21)$$

$$\left. \left. \frac{\text{AiryAiZeros}(1) n^{1/3} + 2^{1/3} \left(\frac{1}{K-1} \right)^{1/3} m}{n^{1/3}} \right) = \lambda \right]$$

Look at the corners

rr[3] will be fixed in lambda below (blue plot)

```
> getArray(posCFabcde,PP,0,-7/6);
factor(getArray(posCFabcde,PP,3,-14/6));
collect(getArray(posCFabcde,PP,4,-3),[lambda,kappa],factor)
assuming K>1;
collect(getArray(posCFabcde,PP,5,-11/3),[lambda,kappa],factor)
assuming K>1;
```

$$\frac{\kappa}{18(K-1)^{4/3}} \frac{(K+2)^2 \lambda K 2^{1/3}}{18(K-1)^{4/3}}$$

$$\frac{rr_3 2^{1/3} (K+2) K \lambda}{3 (K-1)^{1/3}} - \frac{K \kappa (K+2) (K^2 + K + 11)}{36 (K-1)^2}$$

$$\frac{\kappa rr_3 K (K^2 + K + 11)}{6 (K-1)} \quad (2.5.2.22)$$

Now split the solid diamonds into the contributions from Ai and Ai'

> **indets (posCFabcd) ;**

$$\left\{ K, a1, d, \kappa, \lambda, q_0, q_1, qq_0, rr_3, ss_2, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \right. \quad (2.5.2.23)$$

$$\frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}},$$

$$\left. \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}}, (K-1)^{1/3}, (K-1)^{2/3}, (K-1)^{4/3}, (K-1)^{5/3}, (K-1)^{7/3} \right\}$$

Sanity check that there are no other contributions

> **subs (kappa=0, lambda=0, posCFabcd) :**
ArrayTools [IsZero] (%) ;

true (2.5.2.24)

Extract the coefficients of kappa=Ai and lambda=Ai' and treat then seperately

> **posFk := subs (kappa=1, lambda=0, rr[4]=0, uu[4]=0, posCFabcd) : indets (%) ;**
posFl := subs (kappa=0, lambda=1, rr[4]=0, uu[4]=0, posCFabcd) : indets (%) ;

$$\left\{ K, a1, d, q_0, q_1, qq_0, rr_3, ss_2, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \right. \quad (2.5.2.25)$$

$$\frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}},$$

$$\left. \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}}, (K-1)^{1/3}, (K-1)^{2/3}, (K-1)^{4/3}, (K-1)^{5/3}, (K-1)^{7/3} \right\}$$

$$\left\{ K, a1, d, q_0, q_1, qq_0, rr_3, ss_2, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \right. \quad (2.5.2.25)$$

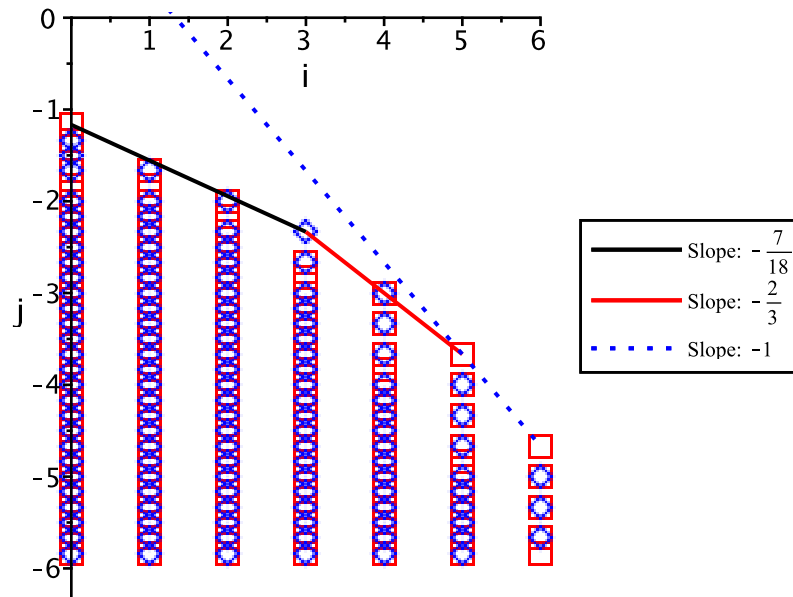
$$\frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}},$$

$$\left. \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}}, (K-1)^{1/3}, (K-1)^{2/3}, (K-1)^{4/3}, (K-1)^{5/3}, (K-1)^{7/3} \right\}$$

$$\{-1\}^{5/3}, (K-1)^{7/3}\}$$

We color the non-zero nodes of the lastNewton polygon into
 red squared coefficients of kappa=Ai
 blue diamonds coefficients of lambda=Ai'

```
> newt4a := mynewtArray(posFk, PP) :
newt4b := mynewtArray(posFl, PP) :
> P4a := pointplot(newt4a, labels=["m deg", "n deg"], symbolsize=25,
symbol=box, color=red) :
P4b := pointplot(newt4b, labels=["m deg", "n deg"], symbolsize=25,
symbol=diamond, color=blue) :
P1dom3s := plot(1-m, m=4..5, color=black) :
display(P4a, P4b, P1dom1, P1dom2, P1dom3a, myview, myoptionsLo,
LegendSize) ;
```



red extremes of Newton polygon

```
> mnmaxR := getMaxNewt(Mord, newt4a) :
seq([i, mnmaxR[i]], i=0..Mord) ;
```

$$\left[0, -\frac{7}{6}\right], \left[1, -\frac{5}{3}\right], [2, -2], \left[3, -\frac{8}{3}\right], [4, -3], \left[5, -\frac{11}{3}\right], \left[6, -\frac{14}{3}\right]$$

(2.5.2.26)

These are the specific values at these points;

we see that we still have some degree of freedom: d and (p[4] if used)

```
> for i from 0 to Mord do
i, factor(getArray(posFk, PP, i, mnmaxR[i])) ;
end;
```

$$1, -\frac{0, -d}{\left(-108 (K-1)^{5/3} rr_3 + 7 K^2 2^{2/3} aI + 4 K 2^{2/3} aI - 24 2^{2/3} aI\right) K}$$

$$2, -\frac{K (7 K^2 + 40 K - 26)}{36 (K-1)}$$

$$3, -\frac{1}{36 (K-1)^{5/3}} \left(K \left(-114 r r_3 (K-1)^{5/3} K + 2 K^3 2^{2/3} a l - 180 (K-1)^{5/3} r r_3 + 2 K^2 2^{2/3} a l + 11 K 2^{2/3} a l - 22 2^{2/3} a l \right) \right)$$

$$4, -\frac{K (K+2) (K^2 + K + 11)}{36 (K-1)^2}$$

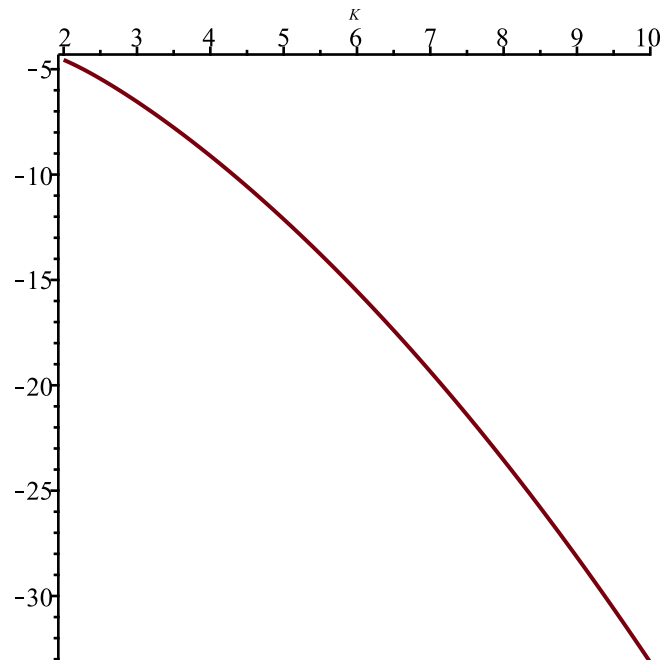
$$5, \frac{r r_3 K (K^2 + K + 11)}{6 (K-1)}$$

$$6, \frac{K r r_3 (K^4 + K^3 + 21 K^2 + 11 K - 99)}{90 (K-1)^2}$$

(2.5.2.27)

```
> -K*(7*K^2 + 40*K - 26)/(36*(K - 1));
plot(%,K=2..10);
```

$$-\frac{K (7 K^2 + 40 K - 26)}{36 K - 36}$$



These are the slopes of the convex hull where the corners are given by the second sequence; hence, in order to be positive when the slope > -1 , we need to choose $d > 0$, e.g. $d=1$; note that $p[4]$ is not important here, as the slope first slope $-5/12$ is less than $-7/18$; and in the later regimes it will be dominated by the blue points.

```
> ls, li := getslopes(mnmaxR, Mord);
```

$$ls, li := \left[-\frac{5}{12}, -\frac{1}{2}, -\frac{2}{3}, -1 \right], [0, 2, 4, 5, 6]$$

(2.5.2.28)

```
> colors := [green, black, brown, blue, olive, red];
styles := [spacedot, solid, dash, dot, dashdot, longdash,
spacedash];
```

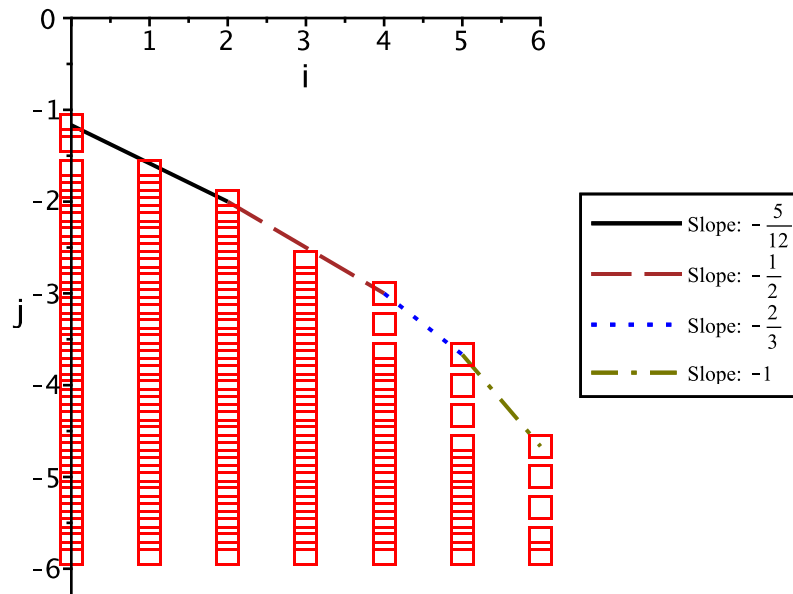
```
Draw the convex hull in red
```

```
> for i from 1 to nops(ls) do
```

```

ls[i];li[i];
tt[i] := plot((mnmaxR[li[i]]-ls[i]*li[i])+ls[i]*m,m=li[i]..li
[i+1],color=colors[i mod nops(colors)+1],linestyle=styles[i mod
nops(styles)+1],legend=[typeset("Slope: ", ls[i]),legendstyle=
[location=right]]):
end:
Pconvred := seq(tt[i],i=1..nops(ls)):
#display(%);
> display(Pconvred,P4a,myview,myoptionsLo,LegendSize);

```



```

> #getArray(posFk,PP,6,-5);

```

We continue with the blue diamonds, i.e., the coefficients of A_i'

```

> mnmaxB := getMaxNewt(Mord,newt4b):
seq([i,mnmaxB[i]],i=0..Mord);

```

$$\left[0, -\frac{4}{3}\right], \left[1, -\frac{5}{3}\right], [2, -2], \left[3, -\frac{7}{3}\right], [4, -3], [5, -4], [6, -5]$$

(2.5.2.29)

here we eliminate these corners: this is the reason to use $q[1]$, $ss[2]$, $rr[3]$
then the behavior will look up to $n^{(2/3-\epsilon)}$ like in the binary tree case
thus we can directly reuse the results/proofs from there

```

> for i from 0 to Mord-1 do
i, factor(getArray(posF1,PP,i,mnmaxB[i]));
end;

```

$$\begin{aligned}
& 0, \frac{(6Kq_1 - 7K - 6q_1 + 11)K2^{1/3}}{6(K-1)^{1/3}} \\
& 1, -\frac{2^{1/3}(K^2 2^{1/3} a l^2 - 36 s s_2 (K-1)^{4/3} - 4K 2^{1/3} a l^2 + 4 2^{1/3} a l^2)K}{18(K-1)^{2/3}} \\
& 2, -\frac{K(-18 2^{1/3} (K-1)^{5/3} r r_3 + K^2 a l - 4 a l)}{6(K-1)} \\
& 3, -\frac{(K+2)^2 K 2^{1/3}}{18(K-1)^{4/3}}
\end{aligned}$$

$$4, \frac{K r r_3 (K+2) 2^{1/3}}{3 (K-1)^{1/3}}$$

$$5, \frac{2^{1/3} r r_3 (K^3 + K^2 + K + 11) K}{30 (K-1)^{4/3}} \quad (2.5.2.30)$$

Differences to the binary case:

```
> factor (getArray (posF1, PP, 0, mnmaxB[0])) ;
factor (isolate (% , q[1])) ;
```

$$\frac{(6 K q_1 - 7 K - 6 q_1 + 11) K 2^{1/3}}{6 (K-1)^{1/3}}$$

$$q_1 = \frac{7 K - 11}{6 (K-1)} \quad (2.5.2.31)$$

this one is 0 in the binary case

```
> factor (getArray (posF1, PP, 1, mnmaxB[1])) ;
factor (isolate (% , ss[2])) ;
```

$$\frac{2^{1/3} (K^2 2^{1/3} a l^2 - 36 s s_2 (K-1)^{4/3} - 4 K 2^{1/3} a l^2 + 4 2^{1/3} a l^2) K}{18 (K-1)^{2/3}}$$

$$s s_2 = \frac{2^{1/3} a l^2 (K-2)^2}{36 (K-1)^{4/3}} \quad (2.5.2.32)$$

this one is also 0 in the binary case

```
> factor (getArray (posF1, PP, 2, mnmaxB[2])) ;
factor (isolate (% , rr[3])) ;
```

$$\frac{K (-18 2^{1/3} (K-1)^{5/3} r r_3 + K^2 a l - 4 a l)}{6 (K-1)}$$

$$r r_3 = \frac{a l (K-2) (K+2) 2^{2/3}}{36 (K-1)^{5/3}} \quad (2.5.2.33)$$

Again we derive the slopes of the convex hull and its corners;

note that if we choose $q[1]$ as shown above we eliminate the first term and decrease the slope.

This will be useful in the proof of Lemma 5.3, in the same way as it was used in the proof of Lemma 4.4.

```
> ls, li := getslopes (mnmaxB, Mord) ;
```

$$ls, li := \left[-\frac{1}{3}, -\frac{2}{3}, -1 \right], [0, 3, 4, 6] \quad (2.5.2.34)$$

```
> colors := [green, olive, blue, black, brown, blue, red] ;
styles := [spacedot, solid, dot, dash, dashdot, longdash,
spacedash] ;
```

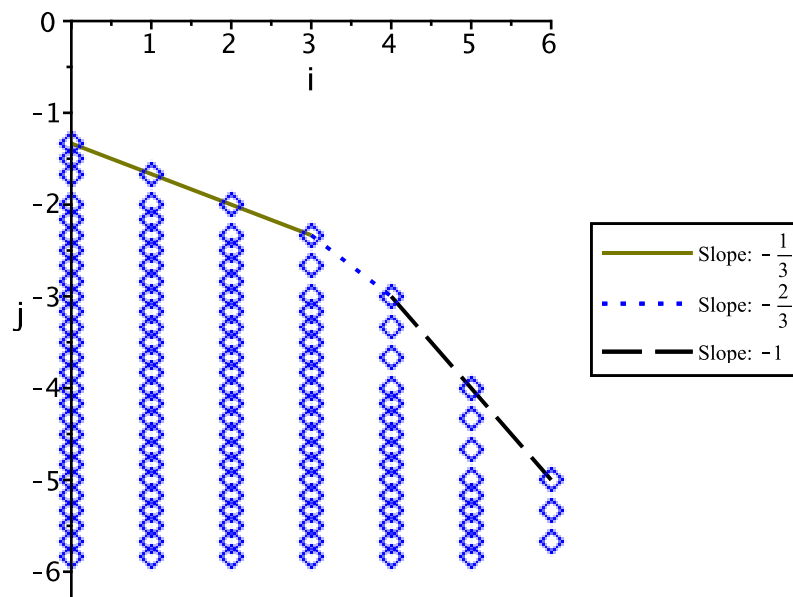
Draw the convex hull in blue which still includes the term of order $\Theta(n^{-4/3})$

```
> for i from 1 to nops (ls) do
  ls[i]; li[i];
  tt[i] := plot ((mnmaxB[li[i]] - ls[i] * li[i]) + ls[i] * m, m = li[i] .. li
[i+1], color = colors[i mod nops (colors) + 1], linestyle = styles[i mod
nops (styles) + 1], legend = [typeset ("Slope: ", ls[i])], legendstyle =
```

```

[location=right]):
end:
Pconvblue := seq(tt[i],i=1..nops(ls)):
#display(%);
> display(Pconvblue,P4b,myview,myoptionsLo,LegendSize);

```



recall and fix these choices

```

> redLsubs := [q[1] = (7*K - 11)/(6*(K - 1)),
ss[2] = 2^(1/3)*a1^2*(K - 2)^2/(36*(K - 1)^(4/3)),
rr[3] = a1*(K - 2)*(K + 2)*2^(2/3)/(36*(K - 1)^(5/3))];

```

$$\text{redLsubs} := \left[q_1 = \frac{7K - 11}{6K - 6}, ss_2 = \frac{2^{1/3} a_1^2 (K - 2)^2}{36 (K - 1)^{4/3}}, rr_3 \right. \\
\left. = \frac{a_1 (K - 2) (K + 2) 2^{2/3}}{36 (K - 1)^{5/3}} \right] \quad (2.5.2.35)$$

We kill this term by setting $q[1]$ as above and recompute the Newton polygons.

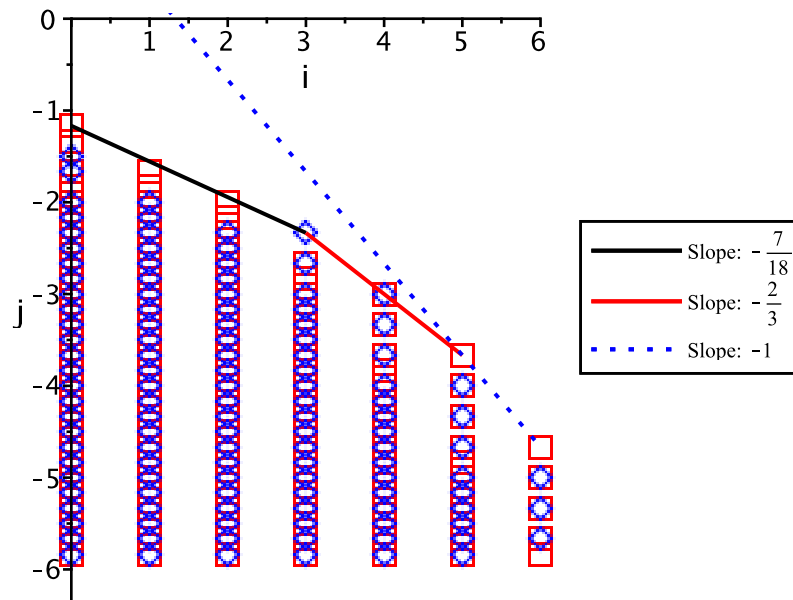
In the next picture the left-top blue point disappeared.

(note that the coefficients of lower order terms of posFk change as well, which is why we recompute them; however the picture will not change as the convex hull is not influenced by this change.)

```

> posFk2 := map(factor,subs(redLsubs,posFk)):
newt4a2 := mynewtArray(posFk2,PP):
P4a2 := pointplot(newt4a2,labels=["m deg", "n deg"], symbolsize=
25, symbol=box, color=red):
posF12 := map(factor,subs(redLsubs,posF1)):
newt4b2 := mynewtArray(posF12,PP):
P4b2 := pointplot(newt4b2,labels=["m deg", "n deg"], symbolsize=
25, symbol=diamond, color=blue):
display(P4a2,P4b2,P1dom1,P1dom2,P1dom3a,myview,myoptionsLo,
LegendSize);

```



Recompute blue with q[1], ss[2], and rr[3] set as above

```
> mnmaxB := getMaxNewt(Mord, newt4b2) :
  seq([i, mnmaxB[i]], i=0..Mord) ;
```

$$\left[0, -\frac{3}{2}\right], [1, -2], \left[2, -\frac{7}{3}\right], \left[3, -\frac{7}{3}\right], [4, -3], [5, -4], [6, -5] \quad (2.5.2.36)$$

here we see that the corners on the convex hull for $i=4,5,6$ are zero in the binary case but we do not need to kill them, as we will only need the lower bound up to $n^{(2/3-\epsilon)}$ maybe, we could kill them with right factors, but we don't do this here, and it is not clear, how many (infinitely many?) there are

```
> for i from 0 to Mord do
  i, factor(getArray(posF12, PP, i, mnmaxB[i])) ;
end;
```

$$0, -\frac{2^{1/3} d}{(K-1)^{1/3}}$$

$$1, -\frac{1}{45 (K-1)^2} \left(K \left(-15 K 2^{1/3} (K-1)^{5/3} q q_0 - 30 2^{1/3} (K-1)^{5/3} q q_0 + 2 K^4 a l - 35 K^3 a l + 170 K^2 a l - 275 K a l + 138 a l \right) \right)$$

$$2, \frac{(5 K^3 a l^3 - 30 K^2 a l^3 + 60 K a l^3 - 57 K^3 - 40 a l^3 + 168 K^2 + 63 K - 912) 2^{1/3} K}{540 (K-1)^{4/3}}$$

$$3, -\frac{(K+2)^2 K 2^{1/3}}{18 (K-1)^{4/3}}$$

$$4, \frac{K a l (K-2) (K+2)^2}{54 (K-1)^2}$$

$$5, \frac{a l (K-2) (K+2) (K^3 + K^2 + K + 11) K}{540 (K-1)^3}$$

$$6, \frac{a l (K-2) (K+2) (K^5 + K^4 + 36 K^3 + K^2 + 281 K - 279) K}{11340 (K-1)^4} \quad (2.5.2.37)$$

And we get new slopes, yet at the same m powers given in the second sequence; here we see that the term m^3 is negative, which will dominate in the regime when A_i is negative.

```
> ls, li := getslopes(mnmaxB, Mord-1);
```

$$ls, li := \left[-\frac{5}{18}, -\frac{2}{3}, -1 \right], [0, 3, 4, 5] \quad (2.5.2.38)$$

Draw the new convex hull in blue

```
> for i from 1 to nops(ls) do
  ls[i]; li[i];
  tt[i] := plot((mnmaxB[li[i]]-ls[i]*li[i])+ls[i]*m, m=li[i]..li
[i+1], color=colors[i mod nops(colors)+1], linestyle=styles[i mod
nops(styles)+1], legend=[typeset("Slope: ", ls[i])], legendstyle=
[location=right]):
end:
Pconvblue := seq(tt[i], i=1..nops(ls)):
#display(%);
```

Plot the difference to before:

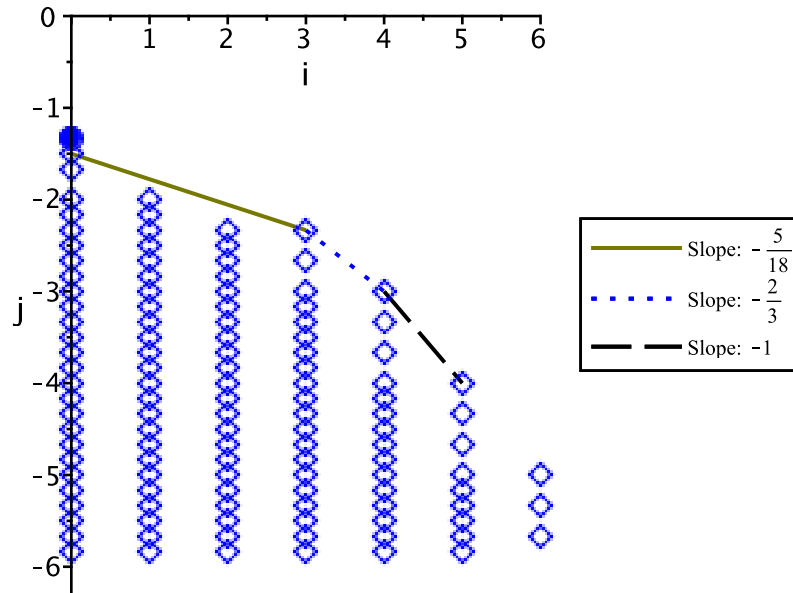
The solid circle on the top-left and the corners for $i=1,2$ disappeared

Like this, the convex hull looks up to $i=j^{2/3-\epsilon}$ as in the binary case

(note that in the binary case more terms vanish!!!)

We do not need these here!

```
> P4bdiffshort := pointplot([0, -4/3], labels=["m deg", "n deg"],
symbolsize=25, symbol=solidcircle, color=blue):
display(Pconvblue, P4bdiffshort, P4b2, myview, myoptionsLo,
LegendSize);
```



Summary

The general ansatz for the lower bound

```
> facAiryLo*Airy(a1+(L)^(1/3)*(m+1)/n^(1/3));
SF(n);
```


$$\left(1 + \frac{q_2 m^2 + q_1 m + q_0}{n} + \frac{m q q_1 + q q_0}{n^{2/3}} + \frac{ss_2 m^2}{n^{4/3}} + \frac{rr_3 m^3}{n^{5/3}} \right) \text{Airy} \left(a l + \frac{L^{1/3} (m+1)}{n^{1/3}} \right) + a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}} \quad (2.5.3.1)$$

Our choices

```
> varsLowerBound :=
[q[0] = 0, # arbitrary; not needed
q[1] = (7*K - 11)/(6*(K - 1)),
q[2] = -(K+2)/6/(K-1),
qq[0] = 0, # arbitrary; not needed
qq[1] = -a1*(K - 2)*2^(2/3)/(6*(K - 1)^(2/3)),
ss[2] = 2^(1/3)*a1^2*(K - 2)^2/(36*(K - 1)^(4/3)),
rr[3] = a1*(K - 2)*(K + 2)*2^(2/3)/(36*(K - 1)^(5/3)),
L=2/(K-1),
a=K,
b=0,
csubs,
pterm = K*(7*K-6)/6,
d=-1
];
```

$$\text{varsLowerBound} := \left[q_0 = 0, q_1 = \frac{7K - 11}{6K - 6}, q_2 = -\frac{K + 2}{6(K - 1)}, qq_0 = 0, qq_1 = -\frac{a l (K - 2) 2^{2/3}}{6(K - 1)^{2/3}}, ss_2 = \frac{2^{1/3} a l^2 (K - 2)^2}{36(K - 1)^{4/3}}, rr_3 = \frac{a l (K - 2) (K + 2) 2^{2/3}}{36(K - 1)^{5/3}}, L = \frac{2}{K - 1}, a = K, b = 0, c = \frac{a l (K - 1)^{1/3} K 2^{2/3}}{2}, pterm = \frac{K(7K - 6)}{6}, d = -1 \right] \quad (2.5.3.2)$$

```
> facL2 := subs(varsLowerBound, facAiryLo*Airy(a1+L^(1/3)*(m+1)/n^(1/3)));
```

$$\text{facL2} := \left(1 + \frac{-\frac{m^2 (K + 2)}{6(K - 1)} + \frac{m(7K - 11)}{6K - 6}}{n} - \frac{m a l (K - 2) 2^{2/3}}{6(K - 1)^{2/3} n^{2/3}} + \frac{2^{1/3} a l^2 (K - 2)^2 m^2}{36(K - 1)^{4/3} n^{4/3}} + \frac{a l (K - 2) (K + 2) 2^{2/3} m^3}{36(K - 1)^{5/3} n^{5/3}} \right) \text{Airy} \left(a l + \frac{2^{1/3} \left(\frac{1}{K - 1} \right)^{1/3} (m + 1)}{n^{1/3}} \right) \quad (2.5.3.3)$$

```
> i:='i':
facL1 := subs(varsLowerBound, SF(n));
product(subs(n=i,%), i=1..K*n);
```

$$facL1 := K + \frac{a1 (K-1)^{1/3} K 2^{2/3}}{2 n^{2/3}} + \frac{K (7K-6)}{6 n} - \frac{1}{n^{7/6}}$$

$$\prod_{i=1}^{Kn} \left(K + \frac{a1 (K-1)^{1/3} K 2^{2/3}}{2 i^{2/3}} + \frac{K (7K-6)}{6 i} - \frac{1}{i^{7/6}} \right) \quad (2.5.3.4)$$

simplification of the product:

First factor by $K^{(K*n)}$ and then perform an $\exp(\log(\dots))$ transform to get the stretched exponential
> $K^{(K*n)} * \text{product}(\text{subs}(n=i, \text{map}(x \rightarrow x/K, \text{facL1})), i=1..K*n) \text{ assuming } K>1, i>0;$

$$K^{Kn} \left(\prod_{i=1}^{Kn} \left(1 + \frac{a1 (K-1)^{1/3} 2^{2/3}}{2 i^{2/3}} + \frac{7K-6}{6 i} - \frac{1}{i^{7/6} K} \right) \right) \quad (2.5.3.5)$$

looks at the structure using only constants

**> $\text{product}(1-c1/i^{(2/3)}-c2/i, i=1..K*n);$
 $\text{sum}(\log(1-c1/i^{(2/3)}-c2/i), i=1..K*n);$
 $\#\text{series}(\%, n=\text{infinity}, 1) \text{ assuming } c1>0, c2>0, n>0, K>1;$
 $-\text{int}(c1/i^{(2/3)}+c2/i, i=1..K*n);$
 $\text{map}(\text{simplify}, \text{series}(\%, n=\text{infinity}, 1)) \text{ assuming } c1>0, c2>0, n>0, K>1;$**

$$\prod_{i=1}^{Kn} \left(1 - \frac{c1}{i^{2/3}} - \frac{c2}{i} \right)$$

$$\sum_{i=1}^{Kn} \ln \left(1 - \frac{c1}{i^{2/3}} - \frac{c2}{i} \right)$$

Warning, unable to determine if 0 is between 1 and $K*n$; try to use assumptions or use the AllSolutions option

$$- \left(\int_1^{Kn} \left(\frac{c1}{i^{2/3}} + \frac{c2}{i} \right) di \right)$$

$$-3 c1 K^{1/3} n^{1/3} - c2 \ln(K) - c2 \ln(n) + 3 c1 \quad (2.5.3.6)$$

the non-constant contributions

**> $c1val := -\text{subs}(\text{varsLowerBound}, c/K);$
 $c2val := -\text{subs}(\text{varsLowerBound}, \text{pterm}/K);$**

$$c1val := - \frac{a1 (K-1)^{1/3} 2^{2/3}}{2}$$

$$c2val := - \frac{7K}{6} + 1 \quad (2.5.3.7)$$

**> $\text{facL1b} := K^{(K*n)} * \exp(-3*c1*K^{(1/3)}*n^{(1/3)}) * n^{(-c2)};$
 $\text{subs}(c1=c1val, c2=c2val, \%);$**

$$facL1b := K^{Kn} e^{-3 c1 K^{1/3} n^{1/3}} n^{-c2}$$

$$K^{Kn} e^{\frac{3 a1 (K-1)^{1/3} 2^{2/3} K^{1/3} n^{1/3}}{2} - \frac{7K}{6} - 1} \quad (2.5.3.8)$$

polynomial growth part 2:

correct by $-1/3$, coming from the Airy function

and by $1-k/2$, coming from rewriting $((K-1)n)!$ into $n!^{(K-1)}$

```
> subs(m=0,n=K*n,a1=A1,Airy=AiryAi,facL2);
map(normal,series(% ,n=infinity,1));
simplify(-c2val-1/3+1-K/2);
```

$$\text{AiryAi}\left(\text{AiryAiZeros}(1) + \frac{2^{1/3} \left(\frac{1}{K-1}\right)^{1/3}}{(Kn)^{1/3}}\right)$$

$$\frac{\text{AiryAi}(1, \text{AiryAiZeros}(1)) 2^{1/3} \left(\frac{1}{K-1}\right)^{1/3} \left(\frac{1}{n}\right)^{1/3}}{K^{1/3}} + O\left(\frac{1}{n}\right)$$

$$\frac{2K}{3} - \frac{1}{3} \quad (2.5.3.9)$$

Gen TCs - Upper bound

upper bound: we want to prove positivity for large n and most m

```
> i:='i':
posansatz := XX(n,m)*SS(n) - Fac1*XX(n-1,m-1) - Fac2*XX(n-1,m+
K-1):
subs(Fac1=fac1,Fac2=fac2,%);
```

$$XX(n, m) SS(n) - \frac{(K-1)^2 (K+n-m) XX(n-1, m-1)}{(K-1)n+m} - XX(n-1, m+K-1) \quad (2.6.1)$$

Update Xhat: specific values from proof below for experiments

From summary after proof below

```
> varsUpperBound := [q[0] = 0, q[1] = (7*K - 11)/(6*K - 6), q[2] =
-(K + 2)/(6*(K - 1)), qq[0] = 0, qq[1] = -a1*(K - 2)*2^(2/3)/(6*
(K - 1)^(2/3)), ss[2] = 2^(1/3)*a1^2*(K - 2)^2/(36*(K - 1)^(4/3)
), rr[3] = a1*(K - 2)*(K + 2)*2^(2/3)/(36*(K - 1)^(5/3)), L = 2/
(K - 1), a = K, b = 0, c = a1*(K - 1)^(1/3)*K*2^(2/3)/2, pterm =
K*(7*K - 6)/6, d = 1, p[0] = 0, p[1] = 0, p[2] = 0, p[3] = 0, p
[4] = 1/3];
```

$$\text{varsUpperBound} := \left[q_0 = 0, q_1 = \frac{7K-11}{6K-6}, q_2 = -\frac{K+2}{6K-6}, qq_0 = 0, qq_1 = \right. \quad (2.6.1.1)$$

$$\left. -\frac{a1(K-2)2^{2/3}}{6(K-1)^{2/3}}, ss_2 = \frac{2^{1/3}a1^2(K-2)^2}{36(K-1)^{4/3}}, rr_3 = \frac{a1(K-2)(K+2)2^{2/3}}{36(K-1)^{5/3}}, L \right.$$

$$= \frac{2}{K-1}, a=K, b=0, c = \frac{a l (K-1)^{1/3} K 2^{2/3}}{2}, pterm = \frac{K(7K-6)}{6}, d=1, p_0$$

$$=0, p_1=0, p_2=0, p_3=0, p_4 = \frac{1}{3}$$

Fix K for experiments

> **KK := 3;**

KK := 3

(2.6.1.2)

This is X tilde (this is derived later)

> **subs(varsUpperBound, a1=A1, K=KK, (1 + (m^2*q[2] + m*q[1] + q[0])/n + (m*q[1] + q[0])/n^(2/3) + ss[2]*m^2/n^(4/3) + rr[3]*m^3/n^(5/3) + (m^4*p[4] + m^3*p[3] + m^2*p[2] + m*p[1] + p[0])/n^2)*AiryAi(a1 + L^(1/3)*(m + 1)/n^(1/3))):**
Xansatz := (n,m) -> subs(K=KK, ((1 + (-5/12*m^2 + 5/6*m)/n - m*AiryAiZeros(1)/(6*n^(2/3)) + AiryAiZeros(1)^2*m^2/(72*n^(4/3)) + (5*AiryAiZeros(1)*m^3)/(72*n^(5/3)) + m^4/(3*n^2))*AiryAi(AiryAiZeros(1) + (m + 1)/n^(1/3)))));

$$Xansatz := (n, m) \mapsto \text{subs} \left(K=KK, \left(1 + \frac{-\frac{5}{12} \cdot m^2 + \frac{5}{6} \cdot m}{n} - \frac{m \cdot \text{AiryAiZeros}(1)}{6 \cdot n^{2/3}} \right. \right. \quad (2.6.1.3)$$

$$\left. \left. + \frac{\text{AiryAiZeros}(1)^2 \cdot m^2}{72 \cdot n^{4/3}} + \frac{5 \cdot \text{AiryAiZeros}(1) \cdot m^3}{72 \cdot n^{5/3}} + \frac{m^4}{3 \cdot n^2} \right) \cdot \text{AiryAi} \left(\text{AiryAiZeros}(1) + \frac{m+1}{n^{1/3}} \right) \right)$$

(this is derived later)

> **subs(varsUpperBound, a1=A1, a + b/n^(1/3) + c/n^(2/3) + pterm/n + d/n^(7/6));**

Sansatz := n -> subs(K=KK, K + A1*(K - 1)^(1/3)*K*2^(2/3)/(2*n^(2/3)) + K*(7*K - 6)/(6*n) + 1/n^(7/6));

$$K + \frac{\text{AiryAiZeros}(1) (K-1)^{1/3} K 2^{2/3}}{2 n^{2/3}} + \frac{K(7K-6)}{6 n} + \frac{1}{n^{7/6}}$$

$$Sansatz := n \mapsto \text{subs} \left(K=KK, K + \frac{A1 \cdot (K-1)^{1/3} \cdot K \cdot 2^{2/3}}{2 \cdot n^{2/3}} + \frac{K \cdot (7 \cdot K - 6)}{6 \cdot n} + \frac{1}{n^{7/6}} \right) \quad (2.6.1.4)$$

> **posXS := map(simplify, subs(XX=Xansatz, SS=Sansatz, Fac1=fac1, Fac2=fac2, K=KK, posansatz)):indets(%);**

$$\left\{ m, n, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}}, \right. \quad (2.6.1.5)$$

$$\left. \frac{1}{(n-1)^{1/3}}, \text{AiryAi} \left(\text{AiryAiZeros}(1) + \frac{m}{(n-1)^{1/3}} \right), \text{AiryAi} \left(\text{AiryAiZeros}(1) \right) \right\}$$

$$\left. + \frac{m+1}{n^{1/3}} \right), \text{AiryAi}\left(\text{AiryAiZeros}(1) + \frac{m+3}{(n-1)^{1/3}}\right) \left. \right\}$$

For a large n this function of m seems to be positive: looks good :)

>

```

Digits:=20:
e1 := subs(csubs, a1=A1, pterm=1/6, posXS) : indets(%);
N := 100000000;
M := floor(N^(1/2));
P1 := plot([seq([mm, (subs(n=N, m=mm, e1))] , mm=0..floor(M/2))]
:display(P1);
P2 := logplot([seq([mm, (subs(n=N, m=mm, e1))] , mm=floor(M/2)..floor
(5*M/8))] ) :display(P2);

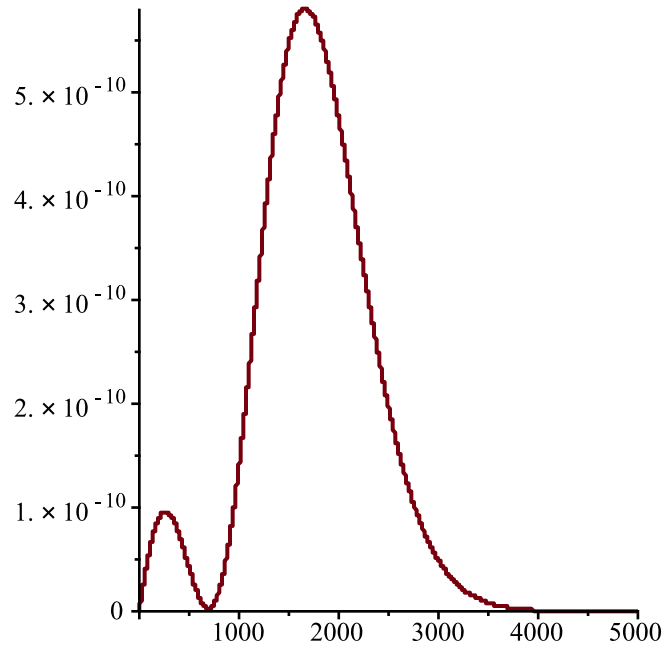
```

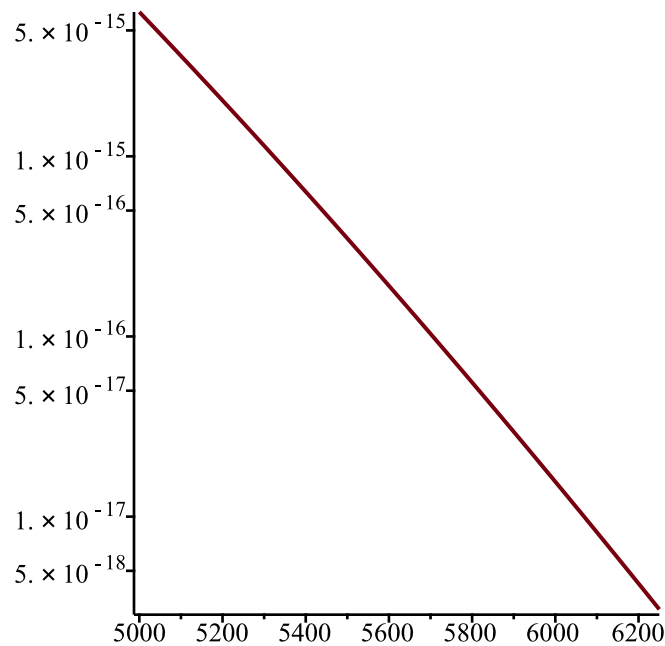
$$\left\{ m, n, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \frac{1}{n^{1/3}}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}}, \right.$$

$$\left. \frac{1}{(n-1)^{1/3}}, \text{AiryAi}\left(\text{AiryAiZeros}(1) + \frac{m}{(n-1)^{1/3}}\right), \text{AiryAi}\left(\text{AiryAiZeros}(1) + \frac{m+1}{n^{1/3}}\right), \text{AiryAi}\left(\text{AiryAiZeros}(1) + \frac{m+3}{(n-1)^{1/3}}\right) \right\}$$

$N := 100000000$

$M := 10000$





Prove it

Recall the general ansatz

> **facAiryUp*****AiryAi** (**a**l + (**L**) ^ (1/3) * (**m**+1) / **n**^(1/3)) ;
SF (**n**) ;

$$\left(1 + \frac{m^2 q_2 + m q_1 + q_0}{n} + \frac{m q q_1 + q q_0}{n^{2/3}} + \frac{ss_2 m^2}{n^{4/3}} + \frac{rr_3 m^3}{n^{5/3}} + \frac{p_4 m^4 + p_3 m^3 + m^2 p_2 + m p_1 + p_0}{n^2} \right) \text{AiryAi} \left(a l + \frac{L^{1/3} (m+1)}{n^{1/3}} \right)$$

$$a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{p_{term}}{n} + \frac{d}{n^{7/6}} \tag{2.6.2.1}$$

We will use the computed values and build on them;

note that only the term d=-1 will change to d=+1 and we will use additional terms of order 1/n^2

> **varsLowerBound** ;

$$\left[q_0 = 0, q_1 = \frac{7K-11}{6K-6}, q_2 = -\frac{K+2}{6(K-1)}, qq_0 = 0, qq_1 = -\frac{al(K-2)2^{2/3}}{6(K-1)^{2/3}}, ss_2 = \frac{2^{1/3} a l^2 (K-2)^2}{36(K-1)^{4/3}}, rr_3 = \frac{al(K-2)(K+2)2^{2/3}}{36(K-1)^{5/3}}, L = \frac{2}{K-1}, a = K, b = 0, c = \frac{al(K-1)^{1/3} K 2^{2/3}}{2}, p_{term} = \frac{K(7K-6)}{6}, d = -1 \right] \tag{2.6.2.2}$$

sub it into our ansatz

```
> subs (d=dup, varsLowerBound, facAiryUp*AiryAi (a1+(L)^(1/3)*(m+1)/n^(1/3)));
subs (d=dup, varsLowerBound, SF (n) );
```

$$\left(1 + \frac{-\frac{m^2(K+2)}{6(K-1)} + \frac{m(7K-11)}{6K-6}}{n} - \frac{ma1(K-2)2^{2/3}}{6(K-1)^{2/3}n^{2/3}} + \frac{2^{1/3}a1^2(K-2)^2m^2}{36(K-1)^{4/3}n^{4/3}} \right. \\ \left. + \frac{a1(K-2)(K+2)2^{2/3}m^3}{36(K-1)^{5/3}n^{5/3}} + \frac{p_4m^4 + p_3m^3 + m^2p_2 + mp_1 + p_0}{n^2} \right) \text{AiryAi} \left(a1 \right. \\ \left. + \frac{2^{1/3} \left(\frac{1}{K-1} \right)^{1/3} (m+1)}{n^{1/3}} \right) \\ K + \frac{a1(K-1)^{1/3}K2^{2/3}}{2n^{2/3}} + \frac{K(7K-6)}{6n} + \frac{dup}{n^{7/6}} \quad (2.6.2.3)$$

Substitute ansatz into the sequence we want to be positive for large n and all m

Note that we already fix the values we computed in the lower bound, as we want them to be the same. only dup will change, it will flip signs

```
> map (expand, subs (XX=XFU, SS=SF, Fac1=fac1, Fac2=fac2, posansatz) );
posF := subs (d=dup, varsLowerBound, %) : indets (%) , nops (expand (%)) ;
```

$$\left\{ K, a1, dup, \kappa, \lambda, m, n, p_0, p_1, p_2, p_3, p_4, \frac{1}{n^{26/3}}, \frac{1}{n^{25/3}}, \frac{1}{n^{23/3}}, \frac{1}{n^{20/3}}, \frac{1}{n^{19/3}}, \right. \\ \frac{1}{n^{17/3}}, \frac{1}{n^{14/3}}, \frac{1}{n^{13/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^{8/3}}, \frac{1}{n^{7/3}}, \frac{1}{n^{7/6}}, \frac{1}{n^{5/3}}, \frac{1}{n^{4/3}}, \frac{1}{n^{2/3}}, \\ \frac{1}{n^{1/3}}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3}, n^{14/3}, n^{16/3}, \\ n^{17/3}, n^{19/3}, n^{20/3}, n^{22/3}, n^{23/3}, n^{25/3}, n^{26/3}, \left(\frac{1}{K-1} \right)^{1/3}, \left(\frac{1}{K-1} \right)^{2/3}, \\ \left(\frac{1}{K-1} \right)^{4/3}, \left(\frac{1}{K-1} \right)^{5/3}, \left(\frac{1}{K-1} \right)^{7/3}, \left(\frac{1}{K-1} \right)^{8/3}, \left(\frac{1}{K-1} \right)^{10/3}, \\ \left(\frac{1}{K-1} \right)^{11/3}, \left(\frac{1}{K-1} \right)^{13/3}, \left(\frac{1}{K-1} \right)^{14/3}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \\ \left. \frac{1}{(K-1)^{2/3}}, (K-1)^{1/3}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}}, \frac{1}{(n-1)^{1/3}} \right\}, \\ 69956$$

The error terms are (to check, expand posF)

UPDATE

```
> simplify(O((2^(1/3)*(m+1)/n^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo));
simplify(O((2^(1/3)*(m)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo));
simplify(O((2^(1/3)*(m+2)/(n-1)^(1/3)-2^(1/3)*m/n^(1/3))^ordAiLo));
```

$$O\left(\frac{64 \cdot 2^{1/3}}{n^{19/3}}\right)$$

$$O\left(\frac{64 \cdot 2^{1/3} \cdot m^{19} \cdot (n^{1/3} - (n-1)^{1/3})^{19}}{(n-1)^{19/3} \cdot n^{19/3}}\right)$$

$$O\left(\frac{64 \cdot (-m \cdot (n-1)^{1/3} + n^{1/3} \cdot (m+2))^{19} \cdot 2^{1/3}}{n^{19/3} \cdot (n-1)^{19/3}}\right) \quad (2.6.2.5)$$

remove error terms

```
> posFd := convert(posF,polynomial):indets(%);
```

$$\left\{ K, a1, dup, \kappa, \lambda, m, n, p_0, p_1, p_2, p_3, p_4, \frac{1}{n^{55/3}}, \frac{1}{n^{52/3}}, \frac{1}{n^{47/3}}, \frac{1}{n^{43/3}}, \frac{1}{n^{40/3}}, \right. \quad (2.6.2.6)$$

$$\frac{1}{n^{35/3}}, \frac{1}{n^{31/3}}, \frac{1}{n^{28/3}}, \frac{1}{n^{23/3}}, \frac{1}{n^{19/3}}, \frac{1}{n^{16/3}}, \frac{1}{n^{11/3}}, \frac{1}{n^7}, \frac{1}{n^7}, \frac{1}{n^5},$$

$$\frac{1}{n^4}, \frac{1}{n^2}, \frac{1}{n}, n^{1/3}, n^{2/3}, n^{4/3}, n^{5/3}, n^{7/3}, n^{8/3}, n^{10/3}, n^{11/3}, n^{13/3},$$

$$n^{14/3}, n^{16/3}, n^{17/3}, n^{19/3}, n^{20/3}, n^{22/3}, n^{23/3}, n^{25/3}, n^{26/3}, \left(\frac{1}{K-1}\right)^{1/3},$$

$$\left(\frac{1}{K-1}\right)^{2/3}, \left(\frac{1}{K-1}\right)^{4/3}, \left(\frac{1}{K-1}\right)^{5/3}, \left(\frac{1}{K-1}\right)^{7/3}, \left(\frac{1}{K-1}\right)^{8/3},$$

$$\left(\frac{1}{K-1}\right)^{10/3}, \left(\frac{1}{K-1}\right)^{11/3}, \left(\frac{1}{K-1}\right)^{13/3}, \left(\frac{1}{K-1}\right)^{14/3}, \frac{1}{(K-1)^{5/3}},$$

$$\frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, (K-1)^{1/3}, \frac{1}{(n-1)^{5/3}}, \frac{1}{(n-1)^{4/3}}, \frac{1}{(n-1)^{2/3}},$$

$$\left. \frac{1}{(n-1)^{1/3}} \right\}$$

Next we fit the display size of the plots in the n (Nord) and m (Mord) direction.

(Note that everything up to ordAi is computed, but possibly not shown)

```
> Nord := -ordAiUp/3;
```

```
Mord := floor(ordAiUp/3);
```

```
myview := view=[0..Mord,Nord..0]:
```

```
Nord := -10
```

```
Mord := 10
```

(2.6.2.7)

```
> Nord := -Mord;
```

```
Nord := -10
```

(2.6.2.8)

Expand again with respect to n,
these are then our unknowns


```

> posFe := series(posFd,n=infinity,ceil(-Nord)+1):indets(%);
posFf := convert(%%,polynom):
nops(expand(%));

```

$$\left\{ K, a1, dup, \kappa, \lambda, m, n, p_0, p_1, p_2, p_3, p_4, \left(\frac{1}{n}\right)^{1/3}, \left(\frac{1}{n}\right)^{2/3}, \left(\frac{1}{n}\right)^{3/2}, \left(\frac{1}{n}\right)^{4/3}, \right. \\
\left(\frac{1}{n}\right)^{5/2}, \left(\frac{1}{n}\right)^{5/3}, \left(\frac{1}{n}\right)^{7/2}, \left(\frac{1}{n}\right)^{7/3}, \left(\frac{1}{n}\right)^{7/6}, \left(\frac{1}{n}\right)^{8/3}, \left(\frac{1}{n}\right)^{9/2}, \\
\left(\frac{1}{n}\right)^{10/3}, \left(\frac{1}{n}\right)^{11/2}, \left(\frac{1}{n}\right)^{11/3}, \left(\frac{1}{n}\right)^{11/6}, \left(\frac{1}{n}\right)^{13/3}, \left(\frac{1}{n}\right)^{13/6}, \left(\frac{1}{n}\right)^{14/3}, \\
\left(\frac{1}{n}\right)^{16/3}, \left(\frac{1}{n}\right)^{17/3}, \left(\frac{1}{n}\right)^{17/6}, \left(\frac{1}{n}\right)^{19/6}, \left(\frac{1}{n}\right)^{23/6}, \left(\frac{1}{n}\right)^{25/6}, \left(\frac{1}{n}\right)^{29/6}, \\
\left(\frac{1}{n}\right)^{31/6}, \left(\frac{1}{n}\right)^{35/6}, \left(\frac{1}{K-1}\right)^{1/3}, \left(\frac{1}{K-1}\right)^{2/3}, \left(\frac{1}{K-1}\right)^{4/3}, \left(\frac{1}{K-1}\right)^{5/3}, \\
\left(\frac{1}{K-1}\right)^{7/3}, \left(\frac{1}{K-1}\right)^{8/3}, \left(\frac{1}{K-1}\right)^{10/3}, \left(\frac{1}{K-1}\right)^{11/3}, \left(\frac{1}{K-1}\right)^{13/3}, \\
\left(\frac{1}{K-1}\right)^{14/3}, \left(\frac{1}{K-1}\right)^{16/3}, \left(\frac{1}{K-1}\right)^{17/3}, \frac{1}{(K-1)^{20/3}}, \frac{1}{(K-1)^{19/3}}, \\
\frac{1}{(K-1)^{17/3}}, \frac{1}{(K-1)^{16/3}}, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \\
\frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \\
\left. \frac{1}{(K-1)^{2/3}}, (K-1)^{1/3}, O\left(\frac{1}{n^6}\right) \right\}$$

18951 (2.6.2.9)

lcm of denominators in exponents

```

> PP := 6;

```

$PP := 6$

(2.6.2.10)

convert the expansion into a list with respect to the degrees in m and n:

Format: [[mdeg,ndeg], coeff]

```

> posCF := map(simplify,ex2Array(posFf,Mord+6,-Nord+6,PP)) assuming
K>1:

```

```

ArrayNumElems(posCF);

```

1649

(2.6.2.11)

```

> degree(posFf,m);

```

10

(2.6.2.12)

The mynewt function computes the Newton polygon of posFf

```

> newt1 := mynewtArray(posCF,PP):

```

Most terms are killed as expected from the choices of the lower bound

```

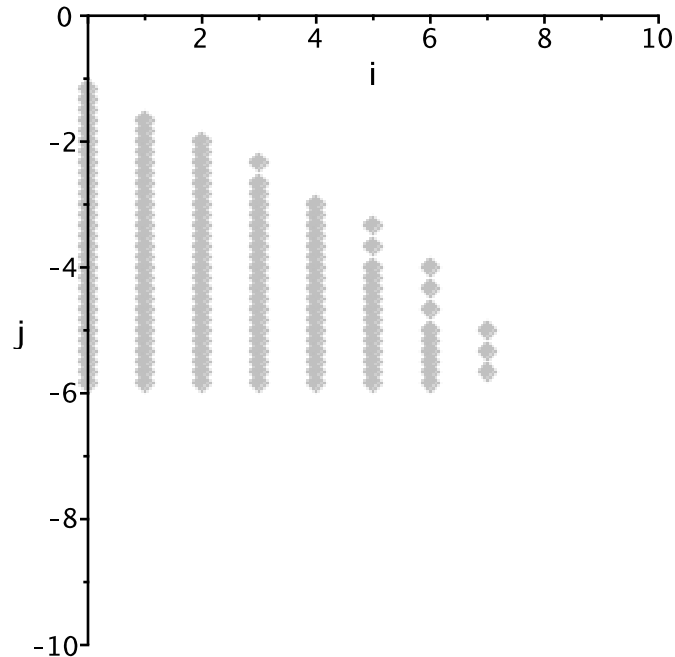
> P1 := pointplot(newt1,myoptionsUp,color=grey,symbol=soliddiamond)

```

```

:
display(P1,myview);

```



we want this term to be positive,
so we set dup=1

```

> getArray(posCF,PP,0,-7/6):
  collect(%, [kappa,lambda],factor);

```

κ dup

(2.6.2.13)

Plot the boundary and the slopes of the Newton polygon;

Note that we have already proved that there are now points above the blue dotted line

This picture looks exactly the same as in the binary case

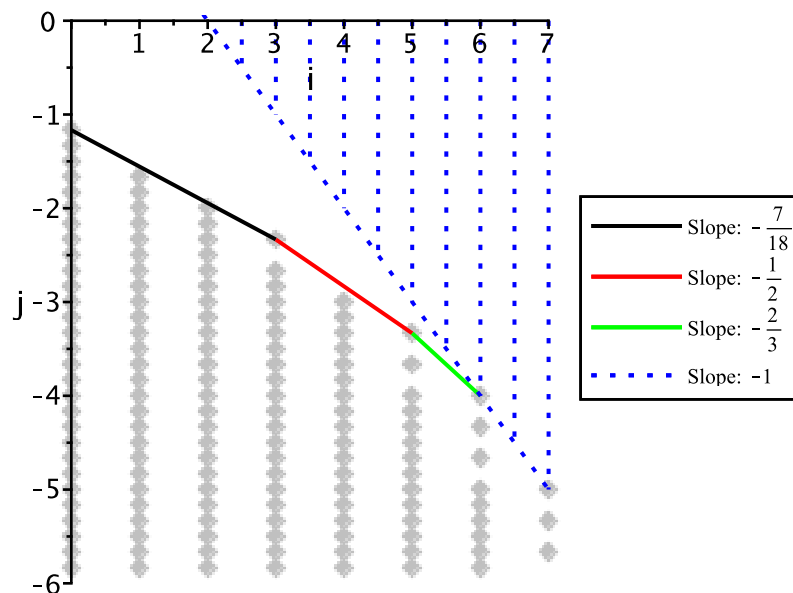
```

> LegendSize := size=[600,450]:
  P1dom1 := plot(-7/6-(7/18)*m,m=0..3,color=black,legend=[typeset
    ("Slope: ", -7/18)],legendstyle=[location=right]):
  P1dom2 := plot(-5/6-(1/2)*m,m=3..5,color=red,legend=[typeset
    ("Slope: ", -1/2)],legendstyle=[location=right]):
  P1dom2b := plot(0-(2/3)*m,m=5..6,color=green,legend=[typeset
    ("Slope: ", -2/3)],legendstyle=[location=right]):
  P1dom3a := plot(2-m,m=0..7,color=blue,linestyle=dot,legend=
    [typeset("Slope: ", -1)],legendstyle=[location=right]):
  P1all := display(P1,P1dom1,P1dom2,P1dom2b,P1dom3a,myview,
    LegendSize):

  for i from 1 to 10 do
    P1dom3[i] := plot([[2+i/2,0],[2+i/2,-i/2]],color=blue,
    linestyle=dot):
  end:

  myview := view=[0..7,0..-6]:
  display(P1all,seq(P1dom3[i],i=1..10),myview);

```



This is the choice for SF

```
> SF(n);
subs(a=K,b=0,csubs,pterm=K*(7*K-6)/6,d=-1,%);
```

$$a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{pterm}{n} + \frac{d}{n^{7/6}}$$

$$K + \frac{a1(K-1)^{1/3} K 2^{2/3}}{2 n^{2/3}} + \frac{K(7K-6)}{6n} - \frac{1}{n^{7/6}} \quad (2.6.2.14)$$

recall

```
> subs(L=2/(K-1), [kaplam]);
```

$$\left[\text{AiryAi} \left(\frac{\text{AiryAiZeros}(1) n^{1/3} + 2^{1/3} \left(\frac{1}{K-1} \right)^{1/3} m}{n^{1/3}} \right) = \kappa, \text{AiryAi} \left(1, \right. \right. \quad (2.6.2.15)$$

$$\left. \left. \frac{\text{AiryAiZeros}(1) n^{1/3} + 2^{1/3} \left(\frac{1}{K-1} \right)^{1/3} m}{n^{1/3}} \right) = \lambda \right]$$

Look at the corners

Here we see the necessary choice: $p[4] > (K+2)^2/72/(K-1)^2$ (note that for $m > n^{1/3}$ lambda is negative)

```
> factor(getArray(posCF,PP,0,-7/6));
factor(getArray(posCF,PP,3,-14/6));
factor(getArray(posCF,PP,5,-20/6));
factor(getArray(posCF,PP,6,-4));
factor(getArray(posCF,PP,7,-5));
```

$$\kappa \text{ dup}$$

$$- \frac{(72 K^2 p_4 - K^2 - 144 K p_4 - 4 K + 72 p_4 - 4) K \lambda 2^{1/3}}{18 (K-1)^{4/3}}$$

$$\begin{aligned}
& - \frac{(K+2) K \lambda 2^{1/3} p_4}{3 (K-1)^{1/3}} \\
& - \frac{\kappa K p_4 (K^2 + K + 11)}{6 (K-1)} \\
& - \frac{\kappa K p_4 (K^4 + K^3 + 21 K^2 + 11 K - 99)}{90 (K-1)^2}
\end{aligned} \tag{2.6.2.16}$$

```

> factor(getArray(posCF, PP, 3, -14/6));
collect(op(2, %), p[4], factor);
solve(%, p[4]);

```

$$\begin{aligned}
& - \frac{(72 K^2 p_4 - K^2 - 144 K p_4 - 4 K + 72 p_4 - 4) K \lambda 2^{1/3}}{18 (K-1)^{4/3}} \\
& 72 (K-1)^2 p_4 - (K+2)^2 \\
& \frac{(K+2)^2}{72 (K-1)^2}
\end{aligned} \tag{2.6.2.17}$$

This is important!!!

```

> p[4] = (K + 2)^2 / (72 * (K - 1)^2);
subs(K=2, %); # what we got in the binary case

```

$$\begin{aligned}
p_4 &= \frac{(K+2)^2}{72 (K-1)^2} \\
p_4 &= \frac{2}{9}
\end{aligned} \tag{2.6.2.18}$$

Now split the black dots into the contributions from Ai and Ai'

```

> indets(posCF);

```

$$\left\{ K, a1, dup, \kappa, \lambda, p_0, p_1, p_2, p_3, p_4, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \right. \tag{2.6.2.19}$$

$$\left. \frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \right.$$

$$\left. \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}} \right\}$$

Sanity check that there are no other contributions

```

> map(factor, subs(kappa=0, lambda=0, posCF)):
ArrayTools[IsZero](%);

```

true

(2.6.2.20)

Extract the coefficients of kappa=Ai and lambda=Ai'

and treat then separately

```

> posFk := subs(kappa=1, lambda=0, posCF):indets(%);
posFl := subs(kappa=0, lambda=1, posCF):indets(%);

```

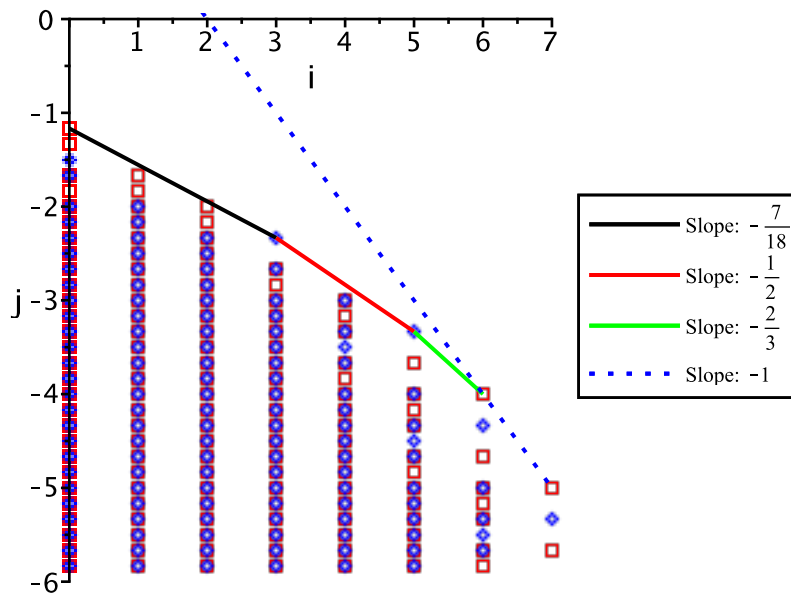
$$\left\{ K, a1, dup, p_0, p_1, p_2, p_3, p_4, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \right.$$

$$\left\{ \frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}} \right\}$$

$$\left\{ K, a1, dup, p_0, p_1, p_2, p_3, p_4, \frac{1}{(K-1)^{14/3}}, \frac{1}{(K-1)^{13/3}}, \frac{1}{(K-1)^{11/3}}, \frac{1}{(K-1)^{10/3}}, \frac{1}{(K-1)^{8/3}}, \frac{1}{(K-1)^{7/3}}, \frac{1}{(K-1)^{5/3}}, \frac{1}{(K-1)^{4/3}}, \frac{1}{(K-1)^{2/3}}, \frac{1}{(K-1)^{1/3}} \right\} \quad (2.6.2.21)$$

We color the non-zero nodes of the lastNewton polygon into red squares..... coefficients of kappa=Ai
blue diamonds coefficients of lambda=Ai'

```
> newt4a := mynewtArray(posFk, PP) :
  newt4b := mynewtArray(posF1, PP) :
> P4a := pointplot(newt4a, labels=["m deg", "n deg"], symbolsize=15,
  symbol=box, color=red) :
  P4b := pointplot(newt4b, labels=["m deg", "n deg"], symbolsize=15,
  symbol=diamond, color=blue) :
  P1dom3s := plot(1-m, m=4..5, color=black) :
  display(P4a, P4b, P1dom1, P1dom2, P1dom2b, P1dom3a, myview, myoptionsUp,
  LegendSize, myview) ;
```



red extremes of Newton polygon

```
> mnmaxR := getMaxNewt(Mord, newt4a) :
  seq([i, mnmaxR[i]], i=0..Mord) ;
```

$$\left[0, -\frac{7}{6} \right], \left[1, -\frac{5}{3} \right], [2, -2], \left[3, -\frac{8}{3} \right], [4, -3], \left[5, -\frac{11}{3} \right], [6, -4], [7, -5], [8, -\infty], \quad (2.6.2.22)$$

$$[9, -\infty], [10, -\infty]$$

These are the specific values at these points;
we see that we still have some degree in freedom: d and p[4]

```
> for i from 0 to Mord do
  i, factor(getArray(posFk, PP, i, mnmaxR[i]));
end;
```

$$\begin{aligned}
 & 0, \text{dup} \\
 & 1, \frac{a1 2^{2/3} (K^2 + K - 3) K}{9 (K - 1)^{2/3}} \\
 & 2, -\frac{K (216 K^2 p_4 - 7 K^2 - 432 K p_4 - 40 K + 216 p_4 + 26)}{36 (K - 1)} \\
 & 3, -\frac{(432 K^3 p_4 + 7 K^3 - 864 K^2 p_4 + 18 K^2 + 432 K p_4 - 142 K + 12) a1 K 2^{2/3}}{216 (K - 1)^{5/3}} \\
 & 4, -\frac{K (150 K^3 p_4 - K^3 - 72 K^2 p_4 - 3 K^2 - 306 K p_4 - 13 K + 228 p_4 - 22)}{36 (K - 1)^2} \\
 & 5, -\frac{(36 K^4 p_4 + K^4 - 36 K^3 p_4 + K^3 - 36 K^2 p_4 + 7 K^2 + 36 K p_4 - 4 K - 44) a1 2^{2/3} K}{216 (K - 1)^{8/3}} \\
 & 6, -\frac{K p_4 (K^2 + K + 11)}{6 (K - 1)} \\
 & 7, -\frac{K p_4 (K^4 + K^3 + 21 K^2 + 11 K - 99)}{90 (K - 1)^2}
 \end{aligned}$$

Error, (in getArray) bad index into Array

These are the slopes of the convex hull where the corners are given by the second sequence; hence, in order to be positive when the slope > -1, we need to choose d>0, e.g. d=1; note that p[4] is not important here, as the slope first slope -5/12 is less then -7/18; and in the later regimes it will be dominated by the blue points.

```
> ls, li := getslopes(mnmaxR, Mord);
```

$$ls, li := \left[-\frac{5}{12}, -\frac{1}{2}, -1, -\infty \right], [0, 2, 6, 7, 10] \quad (2.6.2.23)$$

```
> colors := [green, black, brown, blue, olive, red]:
  styles := [spacedot, solid, dash, dot, dashdot, longdash,
  spacedash]:
```

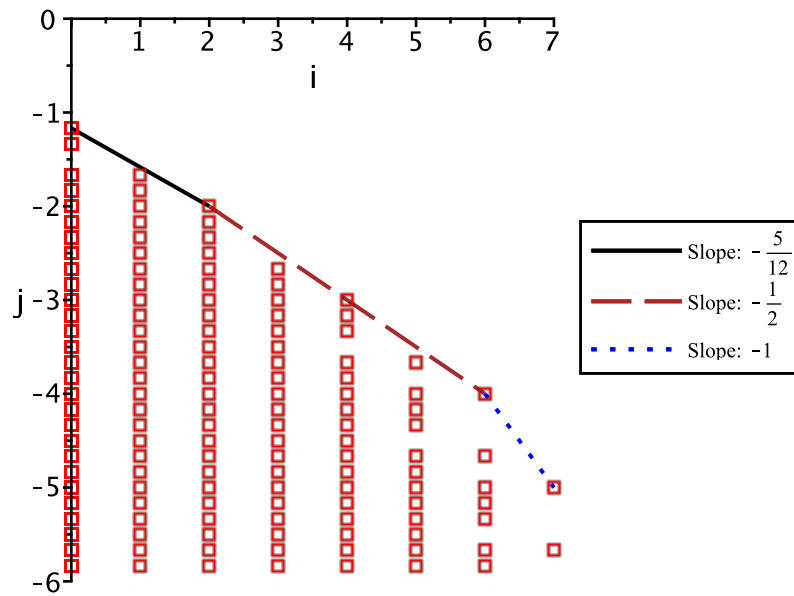
Draw the convex hull in red

picture is exactly the same as in the binary case!!!

```
> for i from 1 to nops(ls) do
  ls[i]; li[i];
  tt[i] := plot((mnmaxR[li[i]] - ls[i] * li[i]) + ls[i] * m, m = li[i] .. li
  [i+1], color = colors[i mod nops(colors) + 1], linestyle = styles[i mod
  nops(styles) + 1], legend = [typeset("Slope: ", ls[i])], legendstyle =
  [location = right]):
end:
Pconvred := seq(tt[i], i = 1 .. nops(ls)):
#display(%);
```

Warning, unable to evaluate the function to numeric values in the region; see the plotting command's help page to ensure the calling sequence is correct

```
> display(Pconvred, P4a, myview, myoptionsUp, LegendSize);
```



We continue with the blue diamonds, i.e., the coefficients of A_i

```
> mmaxB := getMaxNewt(Mord, newt4b) :
  seq([i, mmaxB[i]], i=0..Mord-1) ;
```

```
[0, -3/2], [1, -2], [2, -7/3], [3, -7/3], [4, -3], [5, -10/3], [6, -13/3], [7, -16/3], [8, -∞], [9, -∞]
```

(2.6.2.24)

```
> for i from 0 to Mord-1 do
  i, factor(getArray(posF1, PP, i, mmaxB[i])) ;
end;
```

$$0, \frac{\text{dup } 2^{1/3}}{(K-1)^{1/3}}$$

$$1, \frac{al(2K-3)(K^2-15K+46)K}{45(K-1)}$$

$$2, -\frac{1}{540(K-1)^{4/3}} \left(2^{1/3} (5K^3 al^3 - 30K^2 al^3 + 3240K^3 p_4 + 60Kal^3 - 57K^3 + 1620K^2 p_3 - 9720K^2 p_4 - 40al^3 + 168K^2 - 3240K p_3 + 9720K p_4 + 63K + 1620p_3 - 3240p_4 - 912) K \right)$$

$$3, -\frac{(72K^2 p_4 - K^2 - 144K p_4 - 4K + 72p_4 - 4) K 2^{1/3}}{18(K-1)^{4/3}}$$

$$4, -\frac{Kal(K-2)(18K^2 p_4 + K^2 - 36K p_4 + 4K + 18p_4 + 4)}{54(K-1)^2}$$

$$5, -\frac{(K+2)K 2^{1/3} p_4}{3(K-1)^{1/3}}$$

$$6, -\frac{(K^3 + K^2 + K + 11) K 2^{1/3} p_4}{30 (K - 1)^{4/3}}$$

$$7, -\frac{(K^5 + K^4 + 36 K^3 + K^2 + 281 K - 279) K 2^{1/3} p_4}{630 (K - 1)^{7/3}}$$

Error, (in getArray) bad index into Array

Again we derive the slopes of the convex hull and its corners;

```
> ls, li := getslopes(mnmaxB, Mord-1);
```

$$ls, li := \left[-\frac{5}{18}, -\frac{1}{2}, -1, -\infty \right], [0, 3, 5, 7, 9] \quad (2.6.2.25)$$

```
> colors := [brown, olive, purple, blue, olive, red, black]:
styles := [spacedot, solid, dash, dot, dashdot, longdash,
spacedash]:
```

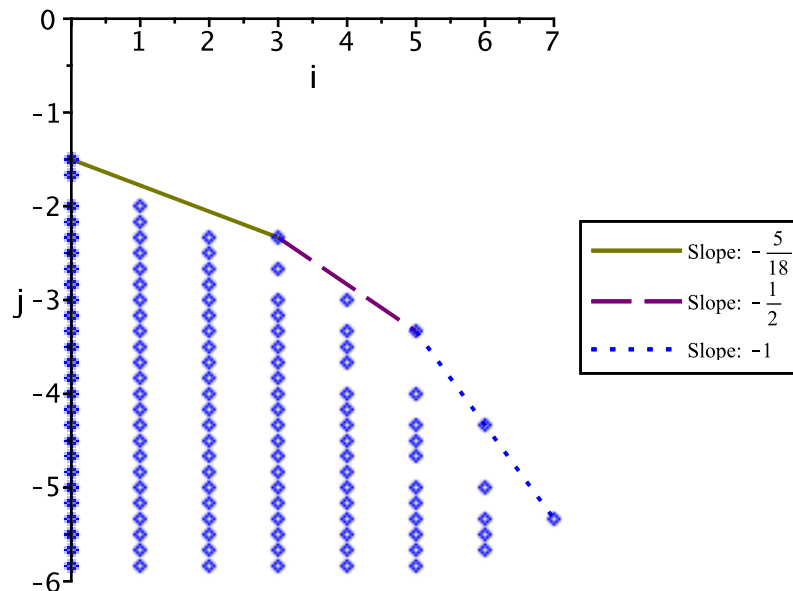
Draw the convex hull in blue which still includes the term of order Theta(n^{-4/3})

```
> for i from 1 to nops(ls) do
  ls[i]; li[i];
  tt[i] := plot((mnmaxB[li[i]]-ls[i]*li[i])+ls[i]*m, m=li[i]..li
[i+1], color=colors[i mod nops(colors)+1], linestyle=styles[i mod
nops(styles)+1], legend=[typeset("Slope: ", ls[i])], legendstyle=
[location=right]):
end:
Pconvblue := seq(tt[i], i=1..nops(ls)):
#display(%);
```

[Warning, unable to evaluate the function to numeric values in the region; see the plotting command's help page to ensure the calling sequence is correct](#)

picture is exactly the same as in the binary case!!!

```
> display(Pconvblue, P4b, myview, myoptionsUp, LegendSize);
```



```
> varsLowerBound;
```

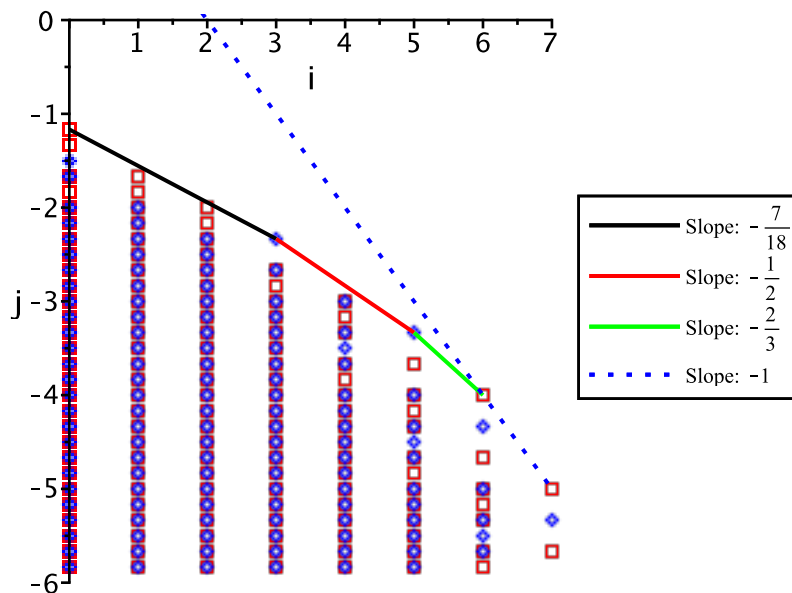
$$\left[q_0 = 0, q_1 = \frac{7K - 11}{6K - 6}, q_2 = -\frac{K + 2}{6(K - 1)}, qq_0 = 0, qq_1 = -\frac{al(K - 2) 2^{2/3}}{6(K - 1)^{2/3}}, ss_2 \right] \quad (2.6.2.26)$$

$$= \frac{2^{1/3} a l^2 (K-2)^2}{36 (K-1)^{4/3}}, rr_3 = \frac{a l (K-2) (K+2) 2^{2/3}}{36 (K-1)^{5/3}}, L = \frac{2}{K-1}, a=K, b=0, c$$

$$= \frac{a l (K-1)^{1/3} K 2^{2/3}}{2}, pterm = \frac{K (7K-6)}{6}, d = -1$$

We remove the other $p[0]$, $p[1]$, $p[2]$, and $p[3]$ contributions and recompute the Newton polygons. (note that the coefficients of lower order terms of $posFk$ change as well, which is why we recompute them; however the picture will not change as the convex hull is not influenced by this change.)

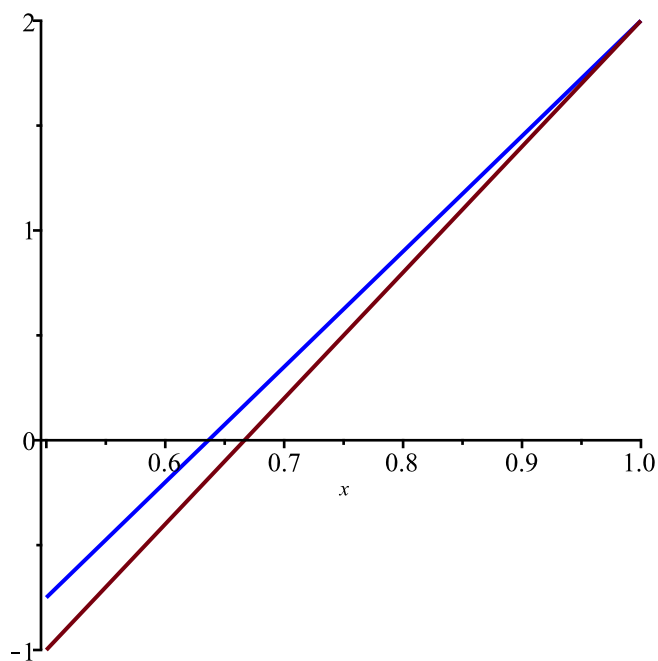
```
> posFk2 := map(factor, subs(p[3]=0, p[2]=0, p[1]=0, p[0]=0, q[0]=0,
posFk)):
newt4a2 := mynewtArray(posFk2, PP):
P4a2 := pointplot(newt4a2, labels=["m deg", "n deg"], symbolsize=
15, symbol=box, color=red):
posF12 := map(factor, subs(p[3]=0, p[2]=0, p[1]=0, p[0]=0, q[0]=0,
posF1)):
newt4b2 := mynewtArray(posF12, PP):
P4b2 := pointplot(newt4b2, labels=["m deg", "n deg"], symbolsize=
15, symbol=diamond, color=blue):
display(P4a2, P4b2, P1dom1, P1dom2, P1dom2b, P1dom3a, myview,
myoptionsUp, LegendSize);
```



Proving the fourth regime $1/2 < m \leq n^{(1-\epsilon)}$

blue domiantes red

```
> P1 := plot(6*x-4, x=1/2..1):
P2 := plot((5+1/2)*x-(10/3+1/6), x=1/2..1, color=blue):
display(%, %%);
```



Summary

The general ansatz for the lower bound

```
> facAiryUp*AiryAi (a1+(L)^(1/3)*(m+1)/n^(1/3));
SF(n);
```

$$\left(1 + \frac{m^2 q_2 + m q_1 + q_0}{n} + \frac{m q q_1 + q q_0}{n^{2/3}} + \frac{ss_2 m^2}{n^{4/3}} + \frac{rr_3 m^3}{n^{5/3}} + \frac{p_4 m^4 + m^3 p_3 + m^2 p_2 + m p_1 + p_0}{n^2} \right) \text{AiryAi} \left(a1 + \frac{L^{1/3} (m+1)}{n^{1/3}} \right)$$

$$a + \frac{b}{n^{1/3}} + \frac{c}{n^{2/3}} + \frac{p_{term}}{n} + \frac{d}{n^{7/6}}$$

(2.6.3.1)

Our choices additional to the lower case

Only d changes to d=+1 instead of d=-1 and p[4] is there additionally

```
> p[4]>(K+2)^2/(72*(K-1)^2);
p[0]=0,p[1]=0,p[2]=0,p[3]=0,
d=1;
```

$$\frac{(K+2)^2}{72(K-1)^2} < p_4$$

$$p_0=0, p_1=0, p_2=0, p_3=0, d=1$$

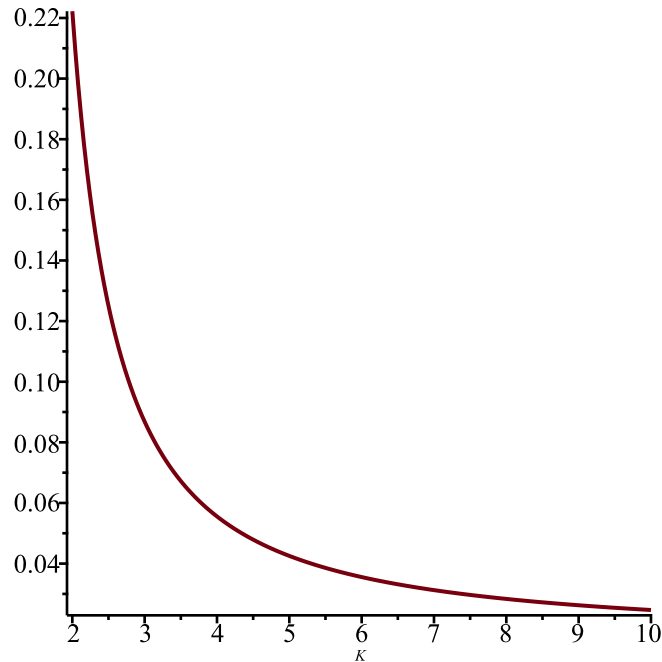
(2.6.3.2)

```
> p4lowbnd := (K+2)^2/(72*(K-1)^2);
subs(K=2,%);
```

$$p4lowbnd := \frac{(K+2)^2}{72(K-1)^2} \cdot \frac{2}{9} \quad (2.6.3.3)$$

this function lower bounding $p[4]$ is monotonically decreasing, so we can use the same as in the binary case: there we used $p[4]=1/3$

```
> plot(p4lowbnd,K=2..10);
simplify(diff(p4lowbnd,K));
```



$$\frac{-K-2}{12(K-1)^3} \quad (2.6.3.4)$$

a possible choice

```
> varsUpperBound := [op(varsLowerBound[1..-2]), d=1, p[0]=0, p[1]=0, p[2]=0, p[3]=0, p[4]=1/3];
```

$$\begin{aligned} \text{varsUpperBound} := & \left[q_0=0, q_1 = \frac{7K-11}{6K-6}, q_2 = -\frac{K+2}{6(K-1)}, qq_0=0, qq_1 = \right. \\ & \left. -\frac{a1(K-2)2^{2/3}}{6(K-1)^{2/3}}, ss_2 = \frac{2^{1/3}a1^2(K-2)^2}{36(K-1)^{4/3}}, rr_3 = \frac{a1(K-2)(K+2)2^{2/3}}{36(K-1)^{5/3}}, L \right. \\ & \left. = \frac{2}{K-1}, a=K, b=0, c = \frac{a1(K-1)^{1/3}K2^{2/3}}{2}, pterm = \frac{K(7K-6)}{6}, d=1, p_0 \right. \\ & \left. =0, p_1=0, p_2=0, p_3=0, p_4 = \frac{1}{3} \right] \end{aligned} \quad (2.6.3.5)$$

Set $p[4]=1/3$ as in the relaxed case works

```
> subs(varsUpperBound, facAiryUp*AiryAi(a1+(L)^(1/3)*(m+1)/n^(1/3)))
);
subs(varsUpperBound, SF(n));
```

$$\left[\left(1 + \frac{-\frac{m^2 (K+2)}{6 (K-1)} + \frac{m (7K-11)}{6K-6}}{n} - \frac{m a l (K-2) 2^{2/3}}{6 (K-1)^{2/3} n^{2/3}} + \frac{2^{1/3} a l^2 (K-2)^2 m^2}{36 (K-1)^{4/3} n^{4/3}} \right. \right. \\
\left. \left. + \frac{a l (K-2) (K+2) 2^{2/3} m^3}{36 (K-1)^{5/3} n^{5/3}} + \frac{m^4}{3 n^2} \right) \text{AiryAi} \left(a l \right. \right. \\
\left. \left. + \frac{2^{1/3} \left(\frac{1}{K-1} \right)^{1/3} (m+1)}{n^{1/3}} \right) \right. \\
\left. K + \frac{a l (K-1)^{1/3} K 2^{2/3}}{2 n^{2/3}} + \frac{K (7K-6)}{6 n} + \frac{1}{n^{7/6}} \right) \tag{2.6.3.6}$$