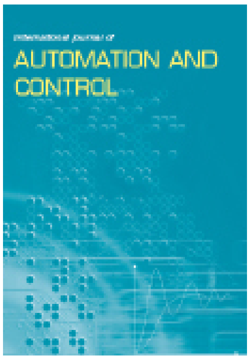

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Globally linearising control of linear time-fractional diffusion-advection-reaction systems

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Abstract: The globally linearising control (GLC) structure is adopted to solve both the step tracking and disturbance rejection problems for distributed parameter system described by a time-fractional partial differential equation. The actuation is assumed to be distributed in the spatial domain while the controlled output is defined as a spatial weighted average of the state. First, following a similar reasoning to geometric control and based on the late lumping approach, an infinite dimensional state feedback that yields a fractional finite dimensional system in closed loop is developed. Then, the input of this resulting closed-loop system is defined by means of a robust controller to cope with step disturbances. Assuming that the output shaping function is non-vanishing, on the spatial domain, it is demonstrated that the GLC strategy is stable. Two applications examples are presented to show, through simulation runs, the stabilisation, step tracking and disturbance rejection capabilities of the GLC scheme.

Keywords: distributed parameter system; time-fractional partial differential equation; distributed control; input-output linearisation; globally linearising control; GLC; diffusion-advection-reaction system.

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1 Introduction

The dynamical behaviour of distributed parameter systems (DPSs) can be described by two important classes of partial differential equations (PDEs): integer PDEs (IPDEs) and fractional or non-integer PDEs (FPDEs). The IPDEs and FPDEs are used to describe more accurately DPSs, which are characterised by normal and anomalous transport phenomena, respectively (Ray, 1989; Christofides, 2001b; Klages et al., 2008; Evangelista and Lenzi, 2018). The IPDEs involve derivative operators of integer order of the variable of interest, while FPDEs involve derivative operators of fractional (non-integer) order. Fractional derivative operators are non-local and they yield interesting information on the complex behaviour of some physical phenomena, which cannot be captured using derivatives of integer order (Terasov and Tarasova, 2020; Klages et al., 2008; Evangelista and Lenzi, 2018; Kesarkar and Selvaganesan, 2013; Khadhraoui and Jelassi, 2017; Nataraj and Kalla, 2010; Pourhashemi et al., 2019). Three types of FPDEs are distinguished: time FPDE (TFPDE) where the fractional derivative is only with respect to time, space FPDE (SFPDE) where the fractional derivative is only with respect to space and time-space FPDE (TSFPDE) where two fractional derivatives are present with respect to time and space. Nowadays, it has turned out that many physical, chemical, thermal, biological and other real-world systems can be modelled very successfully and more accurately by FPDEs (Deng et al., 2020; Evangelista and Lenzi, 2018; Klages et al., 2008; Liang et al., 2004; Liu, 2013; Băleanu and Lopes, 2019).

Due to the distributed nature of the characteristic variables of DPSs and the high complexity represented by the transport phenomena, the study of DPSs has attracted the attention of the control community (Christofides, 2001a; Meurer, 2013; Padhi and Faruque Ali, 2009; Si et al., 2018; Wang et al., 2011). Overall, for DPSs, two control design approaches can be used: early and late lumping approaches (Ray, 1989;

Christofides, 2001a; Meurer, 2013). The early lumping approach consists in deriving an approximate finite dimensional model of the DPS, and the controller is carried out based on the resulting approximate model (Morris, 2020; Liu, 2010). The late lumping approach constitutes an interesting and effective alternative, its principle consists in using the PDEs model in the process design of the controller, which yields superior performances in closed loop compared to the early lumping approach (Christofides and Daoutidis, 1996). Both approaches have been successfully applied to control DPSs described by IPDEs (see Christofides, 2001b, 2001a; Morris, 2020; Meurer, 2013; Liu, 2013 and references therein). However, the early lumping approach should be applied with care to avoid the spillover phenomenon, which means that the designed controller based on the approximate model may influence the states of the closed-loop system that are not used in this approximate model (Morris, 2020). Consequently, these neglected modes can be activated by the controller, which deteriorates the performance and destabilises the closed-loop system.

Compared to the control of IPDEs, which has attained a level of maturity, control of FPDEs remains an explored research area and few contributions are reported in the literature. Through hybrid simulation (numerical and symbolic simulations), Liang et al. (2004) proved that two boundary controllers that achieve both stabilisation and disturbance rejection for integer wave equation can be applied to a time-fractional diffusion-wave equation. Both state and output feedback control problems of cascade connection of a linear time-fractional diffusion-reaction system, with spatially varying diffusivity, and a linear fractional ordinary differential equation are investigated by Chen et al. (2020) in the framework of backstepping. Ge et al. (2016) solved the boundary stabilisation problem of a time-fractional diffusion-reaction equation by converting it to a Mittag-Leffler stable linear system using an invertible coordinate transformation. Then, the controller design problem is reduced to solve a linear hyperbolic PDE. The same control problem but with Robin boundary conditions was solved by Chen et al. (2017) using the backstepping method. Both Dirichlet and Neumann boundary controllers that stabilise an unstable time-fractional reaction-diffusion equation are developed, in the framework of backstepping, by Zhou and Guo (2018). The stabilisation of a time-fractional diffusion-reaction equation with a non-constant diffusion coefficient using an observer-based output feedback was proposed by Chen et al. (2018) using the backstepping method. Zhou et al. (2019) used the active disturbance rejection control (ADRC) to achieve a Mittag-Leffler stabilisation by boundary control of an unstable time-fraction diffusion-reaction equation. Spatial FPDEs are less investigated by the control community and few results are available. A distributed controller of spatial fractional diffusion equation was developed by Maidi and Corriou (2019) in the framework of the geometric approach. The same control design technique was considered by Maidi and Corriou (2020) to design a Neumann boundary feedback controller for the same equation using the concept of extended operator.

The literature review revealed that most contributions deal with boundary control. The distributed (in-domain) control problem is rarely investigated by the community. Also, the case of a fraction diffusion-reaction equation is the most frequently studied. Motivated by these considerations, in the present paper, we consider the distributed control of the time-fractional diffusion-advection-reaction equation that captures the dynamical behaviour of many DPSs. This kind of FPDE represents a benchmark model for many real-world systems characterised by anomalous transport phenomena.

The input-output linearisation approach or geometric control (Isidori, 1995; Kravaris and Kantor, 1990a,b; Corriou, 2018) has been extended with success for DPSs (Christofides and Daoutidis, 1996; Maidi and Corriou, 2011b, 2014, 2016). It is worth noting that the geometric control technique allows dealing with controller design of DPSs following the late lumping approach, that is, using directly the PDEs model without any prior approximation or reduction of the original DPS, which yields an infinite dimension controller with enhanced performance (Christofides and Daoutidis, 1996). In addition, by adopting the globally linearising control (GLC) strategy, the full potential of an existing linear control theory can be exploited to cope with disturbance rejection and modelling errors (Kravaris and Kantor, 1990b; Isidori, 1995; Corriou, 2018). The GLC has been applied successfully to the boundary control of counter-current and parallel-flow heat exchangers (Maidi et al., 2009, 2010), to the boundary control of the linear and nonlinear Stefan problems (Maidi and Corriou, 2014, 2016), and to the distributed control of the spatial fraction diffusion equation (Maidi and Corriou, 2019).

Motivated by the fact that GLC strategy has not yet been applied for FPDEs, in this paper, the GLC strategy is extended to the linear time-fractional diffusion-advection-reaction equation, which constitutes the first-time application of GLC to this important FPDE that efficiently represents many dynamical systems (Klages et al., 2008). The control problem consists in controlling an output defined as the spatial weight average of the state by manipulating a distributed actuation. Thus, under some reasonable assumptions on both the shaping functions of the actuator and the sensor, an infinite dimensional state feedback that yields a fractional finite dimensional system in closed loop is designed following the same reasoning of geometric control (Isidori, 1995). Then, to cope with disturbances, an external controller is introduced to define the external reference input involved in the state feedback (the input of the resulting finite fractional dimensional system). It is demonstrated that the internal stability of the resulting finite dimension system guarantees the stability of the GLC strategy. The developed GLC scheme is then applied with success to stabilise an unstable time-fractional diffusion-reaction equation, and to achieve the step tracking with disturbance rejection of a time fractional diffusion-advection-reaction equation.

The paper is structured as follows: Section 2 present some mathematical tools from fractional calculus which are used in this study. Section 3 is devoted to the TFPDE control problem statement. In Section 4, the proposed design approach of the GLC is presented and its stability is investigated. Section 5 present two application examples that show the stabilisation, step tracking and disturbance rejection capabilities of the GLC. Section 6 concludes the paper.

2 Mathematical preliminary

Fractional calculus theory extended both the differentiation and the integration of a mathematical function to arbitrary order (Kilbas et al., 2006; Podlubny and Thimann, 1999). In this section, some tools from fractional calculus used along this study are presented.

Several definitions have been proposed for the fractional derivative in the literature (Capelas de Oliveira and Tenreiro Machado, 2014; Sales Teodoro et al., 2019). Among these definitions, the Caputo derivative is useful to tackle problems where initial conditions are set with respect to the variables of interest and the respective derivatives of integer-order, which are provided in practice and with clear physical interpretations. Note that most physical fractional systems that are encountered in practice fulfill this requirement dealing with the initial conditions. These important considerations have motivated the use of the Caputo derivative in this paper.

Definition 1 (Podlubny and Thimann, 1999): The Caputo fractional derivative of order $\alpha > 0$ for an absolutely continuous function $f(t)$ is defined by

$$ {}_{0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\xi)}{(t-\xi)^{\alpha+1-n}} d\xi \tag{1} $$

where $n \in \mathbb{N}$ ($\mathbb{N}$ being the set of natural numbers) is the smallest positive number such that $n-1 \leq \alpha < n$ and $\Gamma(.)$ is the gamma function.

Special functions play a key role in dealing with engineering problems involving differential equations. The Mittag-Leffler function is a kind of special functions that is used in solving fractional differential equations (Kilbas et al., 2006; Podlubny and Thimann, 1999). Various Mittag-Leffler functions have been developed and successfully used for solving engineering problems (Haubold et al., 2011). The two-parameter Mittag-Leffler function occurs both in the analysis of dynamical systems and control theory.

Definition 2 (Podlubny and Thimann, 1999): The two-parameter ($\alpha > 0$ and $\beta > 0$) Mittag-Leffler function is defined by

$$ E_{\alpha,\beta}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}. \tag{2} $$

The study of the asymptotic property of the solutions of fractional differential equations is reduced to the study of the asymptotic behaviour of Mittag-Leffler functions using explicit estimation formulas of these functions. For the two-parameter Mittag-Leffler function (2), several useful explicit estimation formulas have been developed by Wang et al. (2018). The following lemma gives the explicit estimation formula, along the negative axis, when $\alpha \in (0, 1]$ and $\beta = \alpha + 1$.

Lemma 1 (Wang et al., 2018, Lemma 2.5): Let $\lambda > 0$ be arbitrary constant and $t > 0$. For any $\alpha \in (0, 1]$ and $\beta = \alpha + 1$, we have

$$ |E_{\alpha,\beta}(-\lambda\, t^{\alpha})| \leq \frac{m(\alpha, \lambda)}{t^{2\,\alpha}} + \frac{1}{\lambda\, t^{\alpha}} \tag{3} $$

in which

$$ m(\alpha, \lambda) = \frac{\int_0^{\infty} e^{-r^{1/\alpha}}\, dr}{|\sin(\pi\,\alpha)|\, \pi\, \alpha\, \lambda^2}. \tag{4} $$

Definition 3 (Podlubny and Thimann, 1999): The Laplace transform of the Caputo derivative is

$$\mathcal{L}\left[{}_0^C D_t^\alpha f(t)\right] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \tag{5}$$

where s is the Laplace variable.

In the following, $\mathbb{R}$ and $\mathbb{R}^+$ denote the sets of real and positive real numbers, respectively. $L^2(\bar{\Omega})$ is the space of measurable integrable real functions defined on the closed domain $\bar{\Omega} = [0, 1]$ with the canonical inner product (Atkinson and Han, 2009)

$$\langle f(z),\, g(z)\rangle = \int_0^1 f(z)\, g(z)\, dz \tag{6}$$

that induces the following norm

$$\|f(z)\| = \left(\int_0^1 f^2(z)\, dz\right)^{\frac{1}{2}}. \tag{7}$$

The first and the second partial derivatives of the function $f(z, t)$ with respect to the z argument is noted as follows

$$\partial_z f(z,\, t) = \frac{\partial f(z,\, t)}{\partial z}, \quad \partial_{zz} f(z,\, t) = \frac{\partial^2 f(z,\, t)}{\partial z^2}. \tag{8}$$

3 Control problem formulation

The dynamical behaviour of a class of nonequilibrium processes characterised by anomalous transport-reaction phenomena (Klages et al., 2008; Evangelista and Lenzi, 2018) is described by the following linear time-fractional diffusion-advection-reaction

$$_0^C D_t^\alpha x(z,\, t) = \mathcal{A}x(z,\, t) + \mathcal{B}u(t), \quad z \in \Omega \tag{9}$$

subject to the boundary Dirichlet conditions

$$x(0,\, t) = f(t), \tag{10}$$
$$x(1,\, t) = g(t) \tag{11}$$

and the initial condition

$$x(z,\, 0) = \phi(z), \quad z \in \bar{\Omega}. \tag{12}$$

where $z \in \bar{\Omega}$ and $t \in \mathbb{R}^+$ denote the spatial and the time variables, respectively. $\bar{\Omega} = [0, 1]$ is the whole spatial domain, that is, the closure of the open set $\Omega = (0, 1)$. $\partial\Omega = \{0, 1\}$ is the boundary of Ω. The variable of interest $x \in L^2(\bar{\Omega})$ is the state that can represent a species concentration (mass transfer process) or a temperature (heat transfer process). $u(t) \in \mathbb{R}$ is the manipulated variable. $f(t) \in \mathbb{R}$ and $g(t) \in \mathbb{R}$ are smooth bounded functions assumed to be the disturbances. The linear spatial operators $\mathcal{A}$ and $\mathcal{B}$ are defined as follows:

$$\mathcal{A}x = a_2\,\partial_{zz}x + a_1\,\partial_z x + a_0\,x, \tag{13}$$

$$\mathcal{B}u = b(z)\,u, \tag{14}$$

where $b(z) \in L^2(\bar{\Omega})$ and $c(z) \in L^2(\bar{\Omega})$ are smooth continuous functions that characterise the spatial distribution of the actuation and the structure of the sensor, respectively. The constants a_2, a_1 and a_0 represent the diffusion coefficient, the velocity of the moving quantity and the reaction rate, respectively. Note that the algebraic sign of a_1 provides the information on the convective flow direction. The time-fractional derivative in equation (9) is taken in the sense of Caputo (see Definition 1), which is more suitable for nonequilibrium processes with nonzero initial conditions.

The control objective consists in solving both the step tracking and the disturbance rejection problems in the case of an output defined as the spatial weighted average of the state given as follows

$$y(t) = \mathcal{C}x(z,\,t) \tag{15}$$

by manipulating the distributed control variable $u(t)$, with

$$\mathcal{C}x = \langle c(z),\,x\rangle \tag{16}$$

Assumption 1: $c(z)$ is a continuous function with $\min_{z\in\bar{\Omega}} |c(z)| = c^* \neq 0$.

Assumption 2: The functions $b(z)$ and $c(z)$ are not orthogonal, that is, $\langle c(z),\,b(z)\rangle \neq 0$.

Remark 1: As the function $c(z)$ is continuous and $\bar{\Omega}$ is a compact domain, consequently $|c(z)|$ ($|.|$ being the absolute value) is bounded and admits a minimum in $\bar{\Omega}$ denoted by c^*, that is, $c^* = \min_{z\in\bar{\Omega}} |c(z)|$.

To tackle the formulated control problem, it is proposed to use the GLC strategy (Kravaris and Kantor, 1990b; Corriou, 2018). In the following section, a design approach of GLC is proposed for the control problem (9)–(15).

Remark 2: The control problem (9)–(15) is given with Dirichlet boundary conditions, but it can be assumed that the following development remains valid for other kinds of boundary conditions.

Remark 3: It is worth noting that since a boundary control problem can be converted to a distributed control one using the concept of extended operator (Stafford and Dowrick, 1977), hence the following theoretical developments remain valid when designing a GLC for a TFPDE with a boundary actuation. The design of a boundary state feedback based on the conversion of a boundary control problem to a distributed one is discussed in length in Maidi et al. (2010) and Maidi and Corriou (2011b, 2011a).

4 Globally linearising control

In this section, the GLC structure based on the input-output linearisation approach (Kravaris and Kantor, 1990a; Isidori, 1995; Corriou, 2018) is adopted to solve the formulated control problem (9)–(15). The design of the GLC proceeds in two steps. The first step an output stabilising state-feedback that yields in closed loop a linear system relating the *controlled output* and an *external reference input*. The resulting state feedback does not involve an integral action, so it cannot achieve a zero steady-state error in the presence of constant disturbances or modelling errors. Hence, the second step consists in designing an external controller around the linear system called *external reference input-controlled output*.

In the following, a GLC design approach is proposed to solve both the step tracking and disturbance rejection problems for the time-fractional PDE (9) with the output (15).

4.1 State feedback design

The application of the input-output linearisation approach (Isidori, 1995; Kravaris and Kantor, 1990b) has been extended successfully to hyperbolic DPSs described by quasi-linear first-order PDEs by Christofides and Daoutidis (1996) by introducing the concept of the characteristic index which is a generalisation to DPSs of the notion of lumped parameter system relative degree (Isidori, 1995). Then, interesting applications of the input-output linearisation have been reported in the literature (Shang et al., 2005; Liu, 2003; Gundepudi and Friedly, 1998; Maidi et al., 2009, 2010; Maidi and Corriou, 2011b, 2014, 2016). For SFPDE systems, the characteristic index can be determined by calculating the successive time integer derivatives of the controlled output (Maidi and Corriou, 2019, 2020). For TFPDE systems, the situation is considerably more complex since the characteristic index cannot be determined following the same derivation process since the time derivative of the interest variable $x(z, t)$ is of fractional type. In this subsection, we propose a design approach of the infinite dimensional state feedback that achieves a set point tracking of the controlled output (15).

Evaluating the Caputo time derivative of the controlled output (15) for the TFPDE (9) yields

$$ {}_0^C D_t^\sigma y(t) = \mathcal{C} {}_0^C D_t^\sigma x(z, t) \tag{17} $$

and taking into account equation (9), it follows that

$$ {}_0^C D_t^\sigma y(t) = \mathcal{C}\mathcal{A}x(z, t) + \mathcal{C}\mathcal{B}u(t). \tag{18} $$

Now, according to the definitions of the operators $\mathcal{B}$ and $\mathcal{C}$ given by equations (14) and (16), respectively, it follows that

$$ \mathcal{C}\mathcal{B}u(t) = \langle c(z),\, b(z)\rangle\, u(t) \tag{19} $$

and by taking into account Assumption 2, that is,

$$ \langle c(z),\, b(z)\rangle \neq 0 \tag{20} $$

it can be seen from equation (18) that the manipulated input $u(t)$ appears explicitly in the Caputo time derivative of the controlled output $y(t)$. Thus, by assuming the following feedback control

$$v(t) = y(t) + \tau {}_0^C D_t^\sigma y(t) \tag{21}$$

and combining equations (15), (18) and (21), the external reference input $v(t)$ results

$$v(t) = \mathcal{C}x(z,\, t) + \tau\, \mathcal{C}\mathcal{A}x(z,\, t) + \tau\, \langle c(z),\, b(z)\rangle\, u(t) \tag{22}$$

where τ is the desired time constant.

Then, using equation (22), the following output stabilising control law results

$$u(t) = \frac{v(t) - y(t) - \tau\, \mathcal{C}\mathcal{A}x(z,\, t)}{\tau\, \langle c(z),\, b(z)\rangle} \tag{23}$$

which yields in closed loop the following fractional finite dimensional system (21). Then, by assuming zero initial conditions, equation (21) can be written in the Laplace domain (transfer function), according to the Definition 3, as follows

$$\frac{Y(s)}{V(s)} = \frac{1}{\tau\, s^\alpha + 1}. \tag{24}$$

Note that equation (24) is obtained by taking the Laplace transform of equation (21).

4.2 External controller design

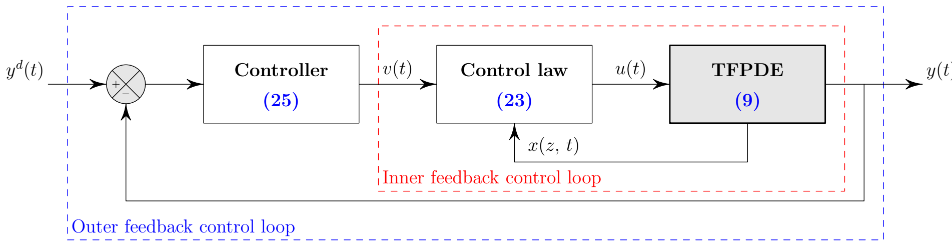
As stated before, the control law (23) does not guarantee a zero asymptotic error in the presence of step disturbances. From a practical viewpoint, since the state-feedback (23) yields a linear fractional finite dimensional system (21), a meaningful option to achieve a zero asymptotic error consists in defining an external reference input $v(t)$, around the linear fractional system $v(t) - y(t)$ given by equation (24), by means of a robust controller as follows (Kravaris and Kantor, 1990b; Corriou, 2018):

$$v(t) = \int_0^t G(t-\xi)\, e(t)\, d\xi \tag{25}$$

where $e(t) = y^d(t) - y(t)$ is the tracking error ($y^d(t)$ is the desired set-point reference), and the kernel $G(t-\xi)$ is an inverse of a transfer function determined so as to enforce a desirable behaviour of the controlled variable $y(t)$. Controller design approaches for fractional order systems can be found in Xue et al. (2007); Bijnan and Kamal (2015); Faieghi et al. (2012), Razminia and Torres (2013) and Petráš (2019).

The GLC structure is depicted in Figure 1.

Figure 1 GLC for TFPDE (9) (see online version for colours)



4.3 Stability of the control strategy

In this subsection, the stability of the GLC scheme in Figure 1 is investigated. Note that the stabilities of both the inner and the outer feedback control loops of the GLC structure imply its stability. Now, as the external controller designed around the fractional system $v(t) - y(t)$ meets the performance demands and stability margin in closed loop, hence the outer feedback control loop is stable. So it remains to investigate the stability of the inner feedback control loop, that is, the stability of the fractional finite dimensional system $v(t) - y(t)$.

The time response of the fractional finite dimensional system (24), for a step reference $v(t)$ of an amplitude M, can be expressed with respect to the Mittag-Leffler function (2) as follows (Valério and da Costa, 2012):

$$y(t) = M\, t^{\alpha}\, E_{\alpha,\,\alpha+1}(-t^{\alpha}) \tag{26}$$

then

$$|y(t)| = |M|\, |t^{\alpha}|\, |E_{\alpha,\,\alpha+1}(-t^{\alpha})| \tag{27}$$

and using the result of Lemma 1, we have

$$|y(t)| \leq |M| \left(\frac{m(\alpha, 1)}{t^{\alpha}} + 1 \right) \tag{28}$$

hence

$$\lim_{t \to +\infty} |y(t)| \leq |M| \tag{29}$$

which means that the controlled output $y(t)$ is stable, that is, the fractional finite dimensional system (24) is externally stable (input-output stability). This is not sufficient to conclude about the stability of the inner feedback control loop, which is also linked to its internal stability (input-state stability).

Let us investigate the internal stability of the fractional finite dimensional system (24). According to Remark 1, its follows that

$$c^{*} \leq |c(z)|, \qquad \forall z \in \bar{\Omega}. \tag{30}$$

Multiplying both sides of formula (30) with $|x(z,\, t)|$ yields

$$c^* \, |x(z,\, t)| \leq |c(z)| \, |x(z,\, t)|, \qquad\qquad \forall z \in \bar{\Omega}. \tag{31}$$

and by integrating both sides of the equation with respect to z, we obtain

$$c^* \int_0^1 |x(z,\, t)| \, dz \leq \int_0^1 |c(z)| \, |x(z,\, t)| \, dz, \qquad \forall z \in \bar{\Omega}. \tag{32}$$

Now, using the definition of $y(t)$ given by equation (15), it follows that

$$|y(t)| \leq \int_0^1 |c(z)| \, |x(z,\, t)| \, dz \tag{33}$$

and by combining equations (32) and (33), we obtain

$$c^* \int_0^1 |x(z,\, t)| \, dz \leq |y(t)|, \qquad\qquad \forall z \in \bar{\Omega} \tag{34}$$

therefore

$$c^* \lim_{t \to \infty} \int_0^1 |x(z,\, t)| \, dz \leq \lim_{t \to \infty} |y(t)|, \qquad\qquad \forall z \in \bar{\Omega} \tag{35}$$

hence taking into account the result given by equation (29), we have

$$\lim_{t \to \infty} \int_0^1 |x(z,\, t)| \, dz \leq \frac{|M|}{c^*}, \qquad\qquad \forall z \in \bar{\Omega} \tag{36}$$

which implies that the state $x(z,\, t)$ is bounded, that is, the fractional system (24) is internally stable. Consequently, we conclude that the GLC structure based on the state feedback (23) is stable.

In the following section, the performance of the GLC structure is evaluated through numerical runs.

5 Application examples

In this section, the stabilisation, step tracking and disturbance rejection capabilities of the GLC are evaluated via numerical simulation.

The GLC is applied to control an unstable time-fractional diffusion-reaction and a time-fractional diffusion-advection-reaction systems expressed in terms of deviation variables. For both systems, the sensor shaping function and the actuation shaping function are respectively

$$c(z) = 1.5 - z, \quad b(z) = z\,(1 - z) \tag{37}$$

and the state feedback time constant is $\tau = 1$ s.

The external reference input $v(t)$ is defined by means of an optimal $\mathcal{H}_\infty$ controller. The design of the external controller is carried out using the mixed-sensitivity synthesis method (Gu et al., 2013; Amin and Aijun, 2017; Kaur and Ohri, 2014; Shao et al., 2017) by assuming an integer-order approximation (rational transfer function) of the fractional finite dimensional system (24). The reduced model is obtained using the Oustaloup approximation (Oustaloup, 1991) of the operator s^{α} by assuming six stable real poles and six stable real zeros within the frequency range $[10^{-3}, 10^{3}]$. The selected sensitivity and complementary sensitivity weights are given as follows

$$W_1(s) = \frac{1}{s + 0.01}, \quad W_3(s) = \frac{100}{0.01\,s + 100}. \tag{38}$$

The method of lines (Vande Wouwer and Schiesser, 2004) is used to approximate the TFPDE (9). Both the first and the second spatial derivatives are approximated using the second-order (three-points) centred finite-difference scheme. The number of discretisation points is $N = 150$. The integral term involved in the control law is evaluated using Simpson's quadrature (Burden et al., 2016).

Hence, the overall closed-loop system (GLC) is approximated by a multi-order system of differential equations, i.e., the integer ordinary differential equations of the external controller and the fractional differential equations that approximate the TFPDE. Then, the resulting multi-order system is integrated using the implicit product-integration of rectangular type solver (Garrappa, 2018).

5.1 Unstable time-fractional diffusion-reaction system

In the first example, the stabilisation capability of the GLC structure is investigated. For this purpose, we consider the parameters α = 0.4, a_2 = 1, a_1 = 0 and a_0 = 10, and the initial condition $\phi(z) = \sin(\pi\, z)$. The boundary conditions are of Dirichlet type and are assumed to be homogeneous, that is, $f(t) = g(t) = 0$. The simulation results of the open-loop system with $u(t) = 0$ are given in Figure 2, from which it can be seen that the stationary point $x(z, t) = 0$ is open loop unstable. Thus, the closed loop stabilisation is studied around this point. Taking into account Assumption 1 and using the variational lemma (Zeidler, 1995), it follows from equation (15) that, to stabilise the time-fractional diffusion-reaction system, i.e., to have $\lim_{t \to +\infty} x(z, t) = 0$, one must force $y(t)$ to be zero when $t \to \infty$. Hence, this is can be achieved by taking $y^d(t) = 0$.

The application of GLC to the unstable time-fractional diffusion-reaction system yields the results depicted in Figure 3 that demonstrates the effectiveness of GLC in stabilising the system. It is noteworthy that GLC forces the closed-loop system state to gradually converge to the equilibrium profile $x(z, t) = 0$. Also, the control law works effectively with smooth moves to achieve an exponential stability of the closed-loop system.

5.2 Time-fractional diffusion-advection-reaction system

In the second example, the step tracking and disturbance rejection effectiveness of the GLC structure are both shown. The parameters of the system are α = 0.8, a_2 = 1, a_1 = –1.5 and a_0 = 2, and the initial condition $\phi(z) = 0$. The boundary conditions are again of Dirichlet type. The boundary condition at $z = 1$ is assumed to be homogeneous, that is, $g(t) = 0$, while the boundary condition at $z = 0$, i.e., the variable $f(t)$, is assumed to be a disturbance for the system.

Figure 2 Unstable time-fractional diffusion-reaction equation, (a) evolution of the output (b) the state in open loop (see online version for colours)

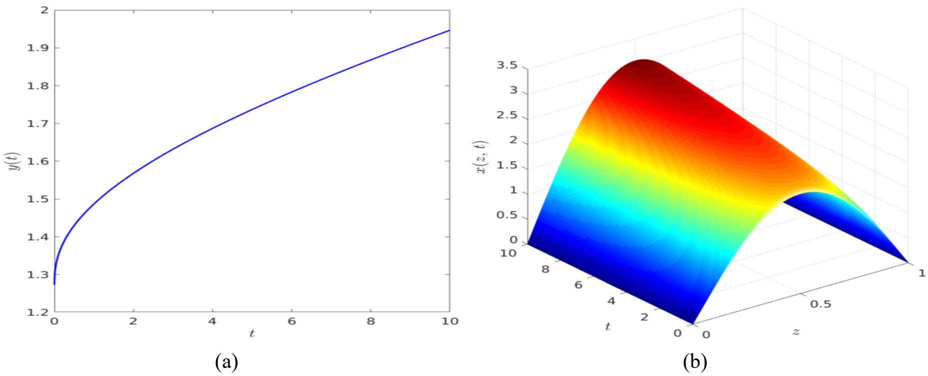
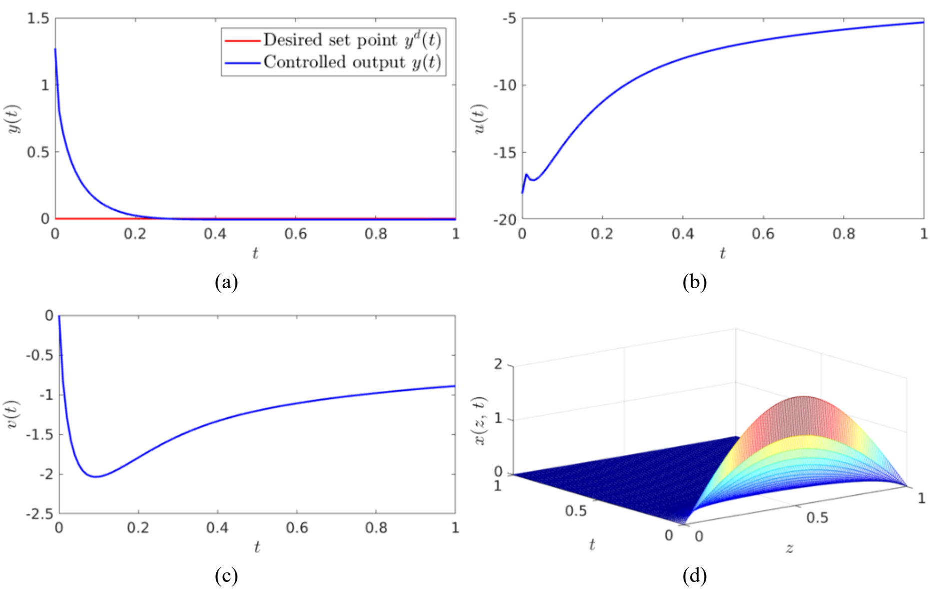


Figure 3 Unstable time-fractional diffusion-reaction equation, (a) evolution of the output (b) the control (c) the external reference input (d) the state in closed loop (see online version for colours)



The simulation run consists in imposing the following desired output

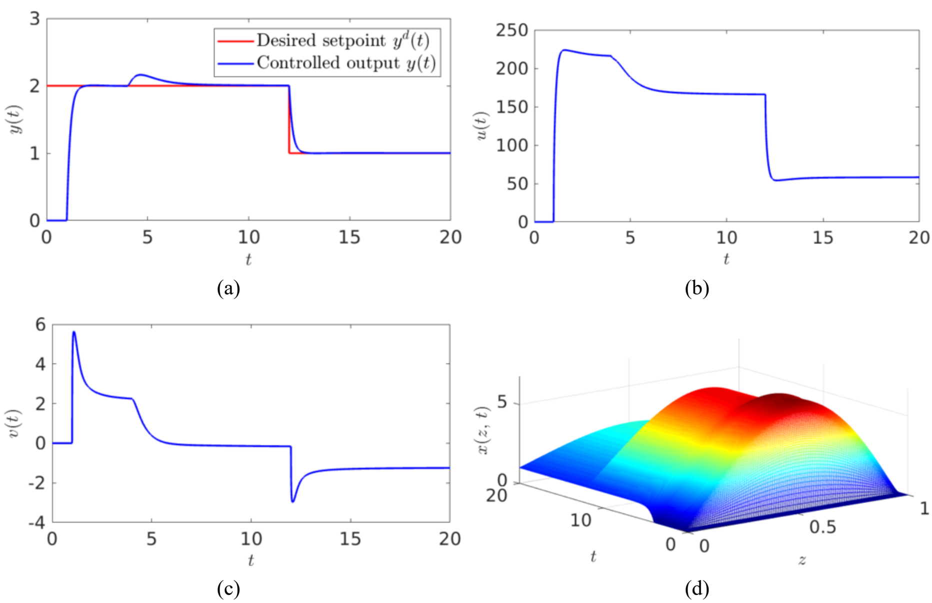
$$y^d(t) = \begin{cases} 0 & \text{for } t < 1 \\ 2 & \text{for } 1 \leq t < 12 \\ 1 & \text{for } t \geq 12 \end{cases} \tag{39}$$

with the following smooth disturbance

$$f(t) = \begin{cases} 0 & \text{for } t < 4 \\ 1 - e^{4-t} & \text{for } t \geq 4 \end{cases} \tag{40}$$

Figure 4 shows that the controlled output $y(t)$ tracks asymptotically the desired output despite the sudden disturbance. It is clearly observed that the influence of the disturbance is well attenuated with smooth moves of the manipulated variable $u(t)$. Consequently, the GLC achieves both the step tracking and disturbance rejection with satisfactory performance.

Figure 4 Time-fractional diffusion-advection-reaction, (a) evolution of the output (b) the control (c) the external reference input (d) the state in closed loop (see online version for colours)



Remark 4: The two benchmark models considered as application examples show that the obtained 3D state profiles are typical of anomalous diffusion-advection-reaction systems. The obtained profiles are shaped by the controller $u(t)$ through the assumed actuator structure characterised by the function $b(z)$. The controlled output $y(t)$ is related to the 3D state profile, through $c(z)$ that characterises the structure of the sensor. This is a typical response of the fractional lumped parameter system (21). Also, to achieve a step-point tracking or to compensate the effect of a disturbance, the control action works effectively to adapt the state profile so that the desired behaviour is achieved. The GLC strategy provides smooth moves of the actuation $u(t)$ that are physically reasonable. Overall, the GLC provides excellent performances in output tracking, disturbance rejection, control moves and stabilisation, which are in agreement with the phenomena arising in anomalous diffusion-advection-reaction systems.

6 Conclusions

In this paper, the GLC structure is adopted to control DPSs characterised by anomalous transport phenomena, which are described by TFPDEs. The objective is to control a spatial weighted average of the state by manipulating a distributed actuation. The proposed design approach consists in designing an infinite dimensional state feedback that yields in closed loop a fractional finite dimensional system given by a linear ordinary fractional differential equation. To cope with the step disturbance, it is proposed to define the input of the resulting fractional finite dimensional system by means of a robust controller. It is demonstrated that, under the assumption of a nonvanishing shaping output function, the resulting GLC scheme is stable. Finally, the GLC is applied with success to stabilise an unstable time-fractional diffusion-reaction system, and to enforce a step tracking despite the bounded disturbance in the case of a time-fractional diffusion-advection-reaction system.

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