Axiomatizing Propositional Dependence Logics*

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Abstract

We give sound and complete Hilbert-style axiomatizations for propositional dependence logic ($\mathcal{PD}$), modal dependence logic ($\mathcal{MDL}$), and extended modal dependence logic ($\mathcal{EMDL}$) by extending existing axiomatizations for propositional logic and modal logic. In addition, we give novel labeled tableau calculi for $\mathcal{PD}$, $\mathcal{MDL}$, and $\mathcal{EMDL}$. We prove soundness, completeness and termination for each of the labeled calculi.

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1 Introduction

Functional dependences occur everywhere in science, e.g., in descriptions of discrete systems, in database theory, social choice theory, mathematics, and physics. Modal logic is an important formalism utilized in the research of numerous disciplines including many of the fields mentioned above. With the aim to express functional dependences in the framework of logic Väänänen [9] introduced *dependence logic*. Dependence logic extends first-order logic with novel atomic formulae called *dependence atoms*. The intuitive meaning of the first-order dependence atom $=(t_1, \dots, t_n)$ is that the value of the term t_n is functionally determined by the values of the terms t_1, ..., t_{n-1}. With the aim to express functional dependences in the framework of modal logic, Väänänen [10] introduced *modal dependence logic* ($\mathcal{MDL}$). Modal dependence logic extends modal logic with *propositional dependence atoms*. A propositional dependence atom $\mathrm{dep}(p_1, \dots, p_n, q)$ intuitively states that the truth value of the proposition q is functionally determined by the truth values of the propositions $p_1, \dots, p_n$. It was soon realized that $\mathcal{MDL}$ lacks the ability to express temporal dependencies; there is no mechanism in $\mathcal{MDL}$ to express dependencies that occur between different points of the model. This is due to the restriction that only proposition symbols are allowed in the dependence atoms of modal dependence logic. To overcome this defect Ebbing et al. [1] introduced *extended modal dependence logic* ($\mathcal{EMDL}$) by extending the scope of dependence atoms to arbitrary modal formulae. Dependence atoms in extended modal dependence logic are of the form $\mathrm{dep}(\varphi_1, \dots \varphi_n, \psi)$, where $\varphi_1, \dots, \varphi_n, \psi$ are formulae of modal logic.

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In recent years the research around modal dependence logic has been active. The focus has been in the computational complexity and in the expressive powers of related formalisms. Sevenster [8] proved that the satisfiability problem for modal dependence logic is NEXPTIME-complete, whereas Ebbing and Lohmann [2] showed that the related model checking problem is NP-complete. Ebbing et al. [1] extended these results to handle also $\mathcal{EMDL}$. Subsequently Virtema [11] showed that the validity problems for $\mathcal{MDL}$ and $\mathcal{EMDL}$ are NEXPTIME-hard and contained in $\mathsf{NEXPTIME}^{\mathsf{NP}}$. Moreover he showed that the corresponding problem for the propositional fragment $\mathcal{PD}$ (see Section 2.1 for a definition) of $\mathcal{MDL}$ is NEXPTIME-complete.

Hella et al. [4] gave *a van Benthem–style characterization* of the expressive power of $\mathcal{EMDL}$ via the so-called *team k-bisimulation*. In the article it was also shown that the expressive powers of $\mathcal{EMDL}$ and $\mathcal{ML}(\varovee)$ (modal logic extended with intuitionistic disjunction) coincide. More recently Kontinen et al. (in the manuscript [6]) gave another van Benthem–style characterization for the expressive power of the so-called *modal team logic*. Moreover, in the manuscript [7], Sano and Virtema gave a Goldblatt–Thomason theorem for $\mathcal{MDL}$ and $\mathcal{EMDL}$. They also showed that with respect to frame definability $\mathcal{MDL}$ and $\mathcal{EMDL}$ coincide with a fragment of modal logic extended with the universal modality in which the universal modality occurs only positively. These characterization truly demonstrate the naturality of the related languages.

In this paper we give sound and complete axiomatizations for variants of propositional and modal dependence logics ($\mathcal{PD}$, $\mathcal{PL}(\varovee)$, $\mathcal{MDL}$, $\mathcal{EMDL}$, and $\mathcal{ML}(\varovee)$). We give Hilbert-style axiomatizations for these logics by extending existing axiomatizations for propositional logic and modal logic. In addition, we give novel labeled tableau calculi for these logics. This paper is one of the first articles on proof theory of propositional and modal dependence logics. The only other work known by the authors of this article is the PhD thesis of Fan Yang [12] and the subsequent manuscript [13]. Among other things, in her thesis, Yang presents axiomatizations of variants of propositional dependence logic based on natural deduction. Our Hilbert style axiomatization of $\mathcal{PD}$ coincides in essence with the natural deduction system given by Yang. However our axiomatization avoids the complexity of the system of Yang by concentrating on the proof-theoretic essence of the axiomatization. Provided that a Hilbert-style axiomatization for the negation normal form fragment of propositional logic is given, we specify *one* inference rule which gives us an axiomatization of $\mathcal{PD}$.

The article is structured as follows. In Section 2 we introduce the required notions and definitions. In Section 3 we give Hilbert-style axiomatizations for propositional and modal dependence logics. In Section 4 we present labeled tableau calculi for these logics.

2 Preliminaries

The syntax of propositional logic ($\mathcal{PL}$) and modal logic ($\mathcal{ML}$) could be defined in any standard way. However, when we consider extensions of $\mathcal{PL}$ and $\mathcal{ML}$ by dependence atoms, it is useful to assume that all formulae are in *negation normal form*, i.e., negations occur only in front of atomic propositions. Thus we will define the syntax of $\mathcal{PL}$ and $\mathcal{ML}$ in negation normal form. When φ is a formula of $\mathcal{PL}$ or $\mathcal{ML}$, we denote by $\varphi^{\perp}$ the equivalent formula that is obtained from $\neg\varphi$ by pushing all negations to the atomic level. Furthermore, we define $\varphi^{\top} := \varphi$. When $\vec{a}$ is a tuple of symbols of length k, we denote by a_j the jth element of $\vec{a}$, $j \leq k$. When φ is a formula, $|\varphi|$ denotes the number of symbols in φ excluding negations and brackets. When A is a set $|A|$ denotes the number of elements in A. When $f : A \to B$ is a function and $C \subseteq A$, we define $f[C] := \{f(a) \mid a \in C\}$.

2.1 Propositional logic with team semantics

Let PROP $= \{z_i \mid i \in \mathbb{N}\}$ denote the set of exactly all *propositional variables*, i.e., *proposition symbols*. We mainly use metavariables p, q, p_1, p_2, q_1, q_2, etc., in order to refer to the variable symbols in PROP. Let D be a finite, possibly empty, subset of PROP. A function $s : D \to \{0,1\}$ is called an *assignment*. A set X of assignments $s : D \to \{0,1\}$ is called a *propositional team*. The set D is the *domain* of X. Note that the empty team $\emptyset$ does not have a unique domain; any subset of PROP is a domain of the empty team. By $\{0,1\}^D$, we denote the set of all assignments $s : D \to \{0,1\}$.

Let Φ be a set of proposition symbols. The set of formulae for propostional logic $\mathcal{PL}(\Phi)$ is generated by the grammar: $\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi)$, where $p \in \Phi$.

By $\models_{\mathcal{PL}}$, we denote the ordinary satisfaction relation of propositional logic defined via assignments. Next we define the team semantics of propositional logic.

▶ **Definition 1.** Let Φ be a set of atomic propositions and let X be a propositional team. The satisfaction relation $X \models \varphi$ for $\mathcal{PL}(\Phi)$ is defined as follows. Note that, we always assume that the proposition symbols that occur in φ are also in the domain of X.

$$\begin{aligned}
X \models p &\quad\Leftrightarrow\quad \forall s \in X : s(p) = 1.\\
X \models \neg p &\quad\Leftrightarrow\quad \forall s \in X : s(p) = 0.\\
X \models (\varphi \wedge \psi) &\quad\Leftrightarrow\quad X \models \varphi \text{ and } X \models \psi.\\
X \models (\varphi \vee \psi) &\quad\Leftrightarrow\quad Y \models \varphi \text{ and } Z \models \psi, \text{ for some } Y, Z \text{ such that } Y \cup Z = X.
\end{aligned}$$

▶ **Proposition 2** ([8]). *Let φ be a formula of propositional logic and X a propositional team. Then $X \models \varphi \Leftrightarrow \forall s \in X : s \models_{\mathcal{PL}} \varphi$. In particular the equivalence $\{s\} \models \varphi \Leftrightarrow s \models_{\mathcal{PL}} \varphi$ holds for every assignment s.*

The syntax of *propositional logic with intuitionistic disjunction* $\mathcal{PL}(\varovee)(\Phi)$ is obtained by extending the syntax of $\mathcal{PL}(\Phi)$ by the grammar rule $\varphi ::= (\varphi \varovee \varphi)$. The syntax of *propositional dependence logic* $\mathcal{PD}(\Phi)$ is obtained by extending the syntax of $\mathcal{PL}(\Phi)$ by the grammar rules $\varphi ::= \mathrm{dep}(p_1, \ldots, p_n, q)$, where $p_1, \ldots, p_n, q \in \Phi$ and $n \in \mathbb{N}$. The intuitive meaning of the *propositional dependence atom* $\mathrm{dep}(p_1, \ldots, p_n, q)$ is that the truth value of the proposition symbol q is completely determined by the truth values of the proposition symbols $p_1, \ldots, p_n$. We define the semantics for the intuitionistic disjunction and the propositional dependence atoms as follows:

$$\begin{aligned}
X \models (\varphi \varovee \psi) &\quad\Leftrightarrow\quad X \models \varphi \text{ or } X \models \psi\\
X \models \mathrm{dep}(p_1, \ldots, p_n, q) &\quad\Leftrightarrow\quad \forall s, t \in X : s(p_1) = t(p_1), \ldots, s(p_n) = t(p_n)\\
&\qquad\quad \text{implies that } s(q) = t(q).
\end{aligned}$$

The next proposition is very useful. The proof is very easy and analogous to the corresponding proof for first-order dependence logic [9].

▶ **Proposition 3** (Downwards closure). *Let φ be a formula of $\mathcal{PL}(\varovee)$ or $\mathcal{PD}$ and let $Y \subseteq X$ be propositional teams. Then $X \models \varphi$ implies $Y \models \varphi$.*

Note that, by downwards closure, $X \models (\varphi \vee \psi)$ iff $Y \models \varphi$ and $X \setminus Y \models \psi$ for some $Y \subseteq X$.

2.2 Modal logics

In order to keep the notation light, we restrict our attention to mono-modal logic, i.e., to modal logic with just the modal operators $\Diamond$ and $\Box$. However this is not really a restriction, since the definitions, results, and proofs of this article generalize, in a straightforward manner, to facilitate any number of modalities.

Let Φ be a set of atomic propositions. The set of formulae for *modal logic* $\mathcal{ML}(\Phi)$ is generated by the grammar: $\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \Box\varphi$, where $p \in \Phi$.

Note that, since negations are allowed only in front of proposition symbols, $\Box$ and $\Diamond$ are *not* interdefinable. The syntax of *modal logic with intuitionistic disjunction* $\mathcal{ML}(\varovee)(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the grammar rule $\varphi ::= (\varphi \varovee \varphi)$.

The *team semantics for modal logic* is defined via *Kripke models* and *teams*. In the context of modal logic, teams are subsets of the domain of the model.

▶ **Definition 4.** Let Φ be a set of proposition symbols. A *Kripke model* K over Φ is a tuple $\mathrm{K} = (W, R, V)$, where W is a nonempty set of *worlds*, $R \subseteq W \times W$ is a binary relation, and $V\colon \Phi \to \mathcal{P}(W)$ is a *valuation*. A subset T of W is called a *team* of K. Furthermore define

$$R[T] := \{w \in W \mid \exists v \in T \text{ s.t. } vRw\}, \quad R^{-1}[T] := \{w \in W \mid \exists v \in T \text{ s.t. } wRv\}.$$

For teams $T, S \subseteq W$, we write $T[R]S$ if $S \subseteq R[T]$ and $T \subseteq R^{-1}[S]$. Thus, $T[R]S$ holds if and only if for every $w \in T$ there exists some $v \in S$ such that wRv, and for every $v \in S$ there exists some $w \in T$ such that wRv.

We are now ready to define the team semantics for modal logic and modal logic with intuitionistic disjunction.

▶ **Definition 5.** Let Φ be a set of atomic propositions, K a Kripke model and T a team of K. The satisfaction relation $\mathrm{K}, T \models \varphi$ for $\mathcal{ML}(\Phi)$ is defined as follows.

$$\begin{aligned}
\mathrm{K}, T \models p \quad &\Leftrightarrow \quad w \in V(p) \text{ for every } w \in T.\\
\mathrm{K}, T \models \neg p \quad &\Leftrightarrow \quad w \notin V(p) \text{ for every } w \in T.\\
\mathrm{K}, T \models (\varphi \wedge \psi) \quad &\Leftrightarrow \quad \mathrm{K}, T \models \varphi \text{ and } K, T \models \psi.\\
\mathrm{K}, T \models (\varphi \vee \psi) \quad &\Leftrightarrow \quad \mathrm{K}, T_1 \models \varphi \text{ and } \mathrm{K}, T_2 \models \psi \text{ for some } T_1 \text{ and } T_2\\
&\qquad \text{such that } T_1 \cup T_2 = T.\\
\mathrm{K}, T \models \Diamond\varphi \quad &\Leftrightarrow \quad \mathrm{K}, T' \models \varphi \text{ for some } T' \text{ such that } T[R]T'.\\
\mathrm{K}, T \models \Box\varphi \quad &\Leftrightarrow \quad \mathrm{K}, T' \models \varphi, \text{ where } T' = R[T].
\end{aligned}$$

For $\mathcal{ML}(\varovee)$ we have the following additional clause:

$$\mathrm{K}, T \models (\varphi \varovee \psi) \quad \Leftrightarrow \quad \mathrm{K}, T \models \varphi \text{ or } \mathrm{K}, T \models \psi.$$

By $\models_{\mathcal{ML}}$, we denote the ordinary satisfaction relation of modal logic defined via pointed Kripke models.

▶ **Proposition 6** ([8]). *Let φ be an $\mathcal{ML}$-formula, K be a Kripke model, and T be a team of K. Then $\mathrm{K}, T \models \varphi \Leftrightarrow \forall w \in T : \mathrm{K}, w \models_{\mathcal{ML}} \varphi$ In particular, for every point w of K, the equivalence $\mathrm{K}, \{w\} \models \varphi \Leftrightarrow \mathrm{K}, w \models_{\mathcal{ML}} \varphi$ holds.*

The syntax for *modal dependence logic* $\mathcal{MDL}(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the rules $\varphi ::= \mathrm{dep}(p_1, \ldots, p_n, q)$, where $p_1, \ldots, p_n, q \in \Phi$ and $n \in \mathbb{N}$, for propositional dependence atoms. The syntax for *extended modal dependence logic* $\mathcal{EMDL}(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the rules $\varphi ::= \mathrm{dep}(\varphi_1, \ldots, \varphi_n, \psi)$, where $\varphi_1, \ldots, \varphi_n, \psi \in \mathcal{ML}(\Phi)$ and $n \in \mathbb{N}$, for *modal dependence atoms*. The intuitive meaning of the modal dependence atom $\mathrm{dep}(\varphi_1, \dots, \varphi_n, \psi)$ is that the truth value of the formula ψ is completely determined by the truth values of the formulae $\varphi_1, \dots, \varphi_n$. Formally:

$$\mathrm{K}, T \models \mathrm{dep}(\varphi_1, \dots, \varphi_n, \psi) \quad \Leftrightarrow \quad \forall w, v \in T : \bigwedge_{1 \le i \le n} (\mathrm{K}, \{w\} \models \varphi_i \Leftrightarrow \mathrm{K}, \{v\} \models \varphi_i)$$
$$\text{implies } (\mathrm{K}, \{w\} \models \psi \Leftrightarrow \mathrm{K}, \{v\} \models \psi).$$

The following result for $\mathcal{MDL}$ and $\mathcal{ML}(\varovee)$ is due to [10] and [3], respectively. For $\mathcal{EMDL}$ it follows via a translation from $\mathcal{EMDL}$ into $\mathcal{ML}(\varovee)$, see [1].

▶ **Proposition 7** (Downwards closure). *Let φ be a formula of $\mathcal{ML}(\varovee)$ or $\mathcal{EMDL}$, let* K *be a Kripke model and let $S \subseteq T$ be teams of* K*. Then* $\mathrm{K}, T \models \varphi$ *implies* $\mathrm{K}, S \models \varphi$.

2.3 Equivalence and validity in team semantics

We say that formulae φ and ψ of $\mathcal{PL}(\varovee)(\Phi)$ or $\mathcal{PD}(\Phi)$ are *equivalent* and write $\varphi \equiv \psi$, if the equivalence $X \models \varphi \Leftrightarrow X \models \psi$ holds for every propositional team X. Likewise, we say that formulae φ and ψ of $\mathcal{ML}(\varovee)(\Phi)$ or $\mathcal{EMDL}(\Phi)$ are *equivalent* and write $\varphi \equiv \psi$, if the equivalence $K, T \models \varphi \Leftrightarrow K, T \models \psi$ holds for every Kripke model K and team T of K.

A formula φ of $\mathcal{PL}(\varovee)(\Phi)$ or $\mathcal{PD}(\Phi)$ is said to be *valid*, if $X \models \varphi$ holds for every team X such that the proposition symbols that occur in φ are in the domain of X. Analogously, a formula ψ of $\mathcal{EMDL}(\Phi)$ or $\mathcal{ML}(\varovee)(\Phi)$ is said to be *valid*, if $\mathrm{K}, T \models \varphi$ holds for every Kripke model K (such that the proposition symbols in φ are mapped by the valuation of K) and every team T of K. When φ is a valid formula of $\mathcal{L}$, we write $\models_{\mathcal{L}} \varphi$.

The following proposition shown in [11, 12] will later prove to be very useful.

▶ **Proposition 8** ($\varovee$-disjunction property). *Let $\mathcal{L} \in \{\mathcal{PL}(\varovee), \mathcal{ML}(\varovee)\}$. For every φ, ψ in $\mathcal{L}$, $\models_{\mathcal{L}} (\varphi \varovee \psi)$ iff $\models_{\mathcal{L}} \varphi$ or $\models_{\mathcal{L}} \psi$.*

3 Extending axiomatizations of $\mathcal{PL}$ and $\mathcal{ML}$

In this section we show how to extend sound and complete axiomatizations for $\mathcal{PL}$ and $\mathcal{ML}$ into sound and complete axiomatizations for $\mathcal{PL}(\varovee)$ and $\mathcal{ML}(\varovee)$, respectively. We use the fact that both $\mathcal{PL}(\varovee)$ and $\mathcal{ML}(\varovee)$ have the $\varovee$-disjunction property. In addition, we obtain axiomatizations for $\mathcal{PD}$, $\mathcal{MDL}$, and $\mathcal{EMDL}$. The axiomatizations are based on compositional translations from $\mathcal{PD}$ into $\mathcal{PL}(\varovee)$, and from $\mathcal{MDL}$ and $\mathcal{EMDL}$ into $\mathcal{ML}(\varovee)$.

3.1 Axiomatizations for $\mathcal{PL}(\varovee)$ and $\mathcal{ML}(\varovee)$

In the definition below, we treat different occurrences of the same formulae as distinct entities.

▶ **Definition 9.** Let φ be a formula of $\mathcal{PL}(\varovee)$ or $\mathcal{ML}(\varovee)$. Let $\mathrm{SubOcc}(\varphi)$ denote the *set of exactly all occurrences of subformulas of* φ. Define

$$\mathrm{SubOcc}_{\varovee}(\varphi) := \{(\psi \varovee \theta) \mid (\psi \varovee \theta) \in \mathrm{SubOcc}(\varphi)\}.$$

We call a function $f : \mathrm{SubOcc}_{\varovee}(\varphi) \to \mathrm{SubOcc}(\varphi)$ a *$\varovee$-selection function for φ* if $f\big((\psi \varovee \theta)\big) \in \{\psi, \theta\}$, for every $(\psi \varovee \theta) \in \mathrm{SubOcc}_{\varovee}(\varphi)$. If f is a $\varovee$-selection function for φ, then φ^f denotes the formula that is obtained from φ by replacing simultaneously each $(\psi \varovee \theta) \in \mathrm{SubOcc}_{\varovee}(\varphi)$ by $f(\psi \varovee \theta)$.

Note that if $\varphi \in \mathcal{PL}(\varovee)(\Phi)$, $\psi \in \mathcal{ML}(\varovee)(\Psi)$, f is a $\varovee$-selection function for φ, and g is a $\varovee$-selection function for ψ, then $\varphi^f \in \mathcal{PL}(\Phi)$ and $\psi^g \in \mathcal{ML}(\Psi)$.

▶ **Proposition 10** ([11]). *Let φ be a formula of $\mathcal{PL}(\varovee)$ or $\mathcal{ML}(\varovee)$, and let F be the set of exactly all $\varovee$-selection functions for φ. Then, $\varphi \equiv \bigovee_{f\in F} \varphi^f$.*

Let $\mathbf{H}_{\mathcal{PL}}$ and $\mathbf{H}_{\mathcal{ML}}$ denote sound and complete axiomatizations of the negation normal form fragments of $\mathcal{PL}$ and $\mathcal{ML}$, respectively. For a logic $\mathcal{L}$, an $\mathcal{L}$-*context* is a formula of the logic $\mathcal{L}$ extended with the grammar rule $\varphi ::= *$. By $\varphi(\psi \,/\, *)$ we denote the formula that is obtained form φ by uniformly substituting each occurrence of $*$ in φ by ψ. We are now ready to define the axiomatizations for $\mathcal{PL}(\varovee)$ and $\mathcal{ML}(\varovee)$. We use $\mathcal{PL}(\varovee)$- and $\mathcal{ML}(\varovee)$-contexts in the following rules:

$$\frac{\varphi(\psi_i \,/\, *)}{\varphi\big((\psi_1 \varovee \psi_2) \,/\, *\,\big)}\;(I \varovee i) \qquad i \in \{1, 2\}.$$

Let $\mathbf{H}_{\mathcal{PL}(\varovee)}$ ($\mathbf{H}_{\mathcal{ML}(\varovee)}$, resp.) be the calculus $\mathbf{H}_{\mathcal{PL}}$ ($\mathbf{H}_{\mathcal{ML}}$, resp.) extended with the rules $(I \varovee 1)$ and $(I \varovee 2)$. In the calculi $\mathbf{H}_{\mathcal{PL}(\varovee)}$ and $\mathbf{H}_{\mathcal{ML}(\varovee)}$, the axioms and inference rules of $\mathbf{H}_{\mathcal{PL}}$ and $\mathbf{H}_{\mathcal{ML}}$ may only be applied to formulae of $\mathcal{PL}$ and $\mathcal{ML}$ (i.e, to formulae without $\varovee$), respectively.

▶ **Theorem 11.** *$\mathbf{H}_{\mathcal{PL}(\varovee)}$ and $\mathbf{H}_{\mathcal{ML}(\varovee)}$ are sound and complete.*

Proof. We will proof the soundness and completeness for $\mathbf{H}_{\mathcal{PL}(\varovee)}$. The case for $\mathbf{H}_{\mathcal{ML}(\varovee)}$ is completely analogous. Note first that from Proposition 2 it follows directly that $\mathbf{H}_{\mathcal{PL}}$ is complete for $\mathcal{PL}$ also in the context of team semantics.

For soundness, it suffices to show that the rule $(I \varovee 1)$ preserves validity. The case for $(I \varovee 2)$ is symmetric. Let φ be a $\mathcal{PL}(\varovee)$-context and let ψ_1 and ψ_2 be $\mathcal{PL}(\varovee)$-formulae. Assume that $\gamma_1 := \varphi(\psi_1 \,/\, *)$ is valid. We will show that then $\gamma_2 := \varphi\big((\psi_1 \varovee \psi_2) \,/\, *\,\big)$ is valid. Let F and G be the sets of exactly all $\varovee$-selection functions for γ_1 and γ_2, respectively. By Proposition 10, $\gamma_1 \equiv \bigovee_{f\in F}\gamma_1^f$ and $\gamma_2 \equiv \bigovee_{g\in G}\gamma_2^g$. Since γ_1 is valid, it follows by Proposition 8, that $\gamma_1^{f'}$ is valid for some $f' \in F$. Since clearly, for every $f \in F$, there exists some $g \in G$ such that $\gamma_1^f = \gamma_2^g$, it follows that there exists some $g' \in G$ such that $\gamma_2^{g'}$ is valid. Thus γ_2 is valid.

In order to prove completeness, assume that a $\mathcal{PL}(\varovee)$-formula φ is valid. Let F be the set of exactly all $\varovee$-selection functions for φ. By Propositions 10 and 8, there exists a function $f \in F$ such that the $\mathcal{PL}$-formula φ^f is valid. Since $\mathbf{H}_{\mathcal{PL}}$ is complete and $\mathbf{H}_{\mathcal{PL}(\varovee)}$ extends $\mathbf{H}_{\mathcal{PL}}$, φ^f is provable also in $\mathbf{H}_{\mathcal{PL}(\varovee)}$. Clearly by using the rules $(I \varovee 1)$ and $(I \varovee 2)$ repetitively to φ^f, we eventually obtain φ. Thus we conclude that $\mathbf{H}_{\mathcal{PL}(\varovee)}$ is complete. ◀

3.2 Axiomatizations for $\mathcal{PD}$, $\mathcal{MDL}$, and $\mathcal{EMDL}$

The following equivalence was observed by Väänänen in [10]:

$$\mathrm{dep}(p_1, \ldots, p_n, q) \equiv \bigvee_{a_1,\ldots,a_n \in \{\bot,\top\}} \bigwedge \{p_1^{a_1}, \ldots, p_n^{a_n}, (q \varovee q^\bot)\}. \tag{1}$$

Ebbing et al. ([1]) extended this observation of Väänänen into the following equivalence concerning $\mathcal{EMDL}$:

$$\mathrm{dep}(\varphi_1, \ldots, \varphi_n, \psi) \equiv \bigvee_{a_1,\ldots,a_n \in \{\bot,\top\}} \bigwedge \{\varphi_1^{a_1}, \ldots, \varphi_n^{a_n}, (\psi \varovee \psi^\bot)\}. \tag{2}$$

These equivalences demonstrate the existence of compositional translations from $\mathcal{PD}$ into $\mathcal{PL}(\varovee)$, and from $\mathcal{MDL}$ and $\mathcal{EMDL}$ into $\mathcal{ML}(\varovee)$, respectively.

We will use the insight that rises from combining the above equivalences with Propositions 8 and 10 in order to construct axiomatizations for $\mathcal{PD}$, $\mathcal{MDL}$, and $\mathcal{EMDL}$, respectively. Recall that when $\vec{a}$ is a finite tuple of symbols, we use a_j to denote the jth member of $\vec{a}$. For each natural number $n \in \mathbb{N}$ and function $f : \{\bot, \top\}^n \to \{\top, \bot\}$, we have the following rules:

$$\frac{\varphi\Big(\bigvee_{\vec{a}\in\{\bot,\top\}^n} \bigwedge\{p_1^{a_1}, \ldots, p_n^{a_n}, q^{f(\vec{a})}\} \;/\; *\Big)}{\varphi\big(\mathrm{dep}(p_1, \ldots, p_n, q) \;/\; *\big)} \;\big(\mathcal{PL}\,\mathrm{dep}(f)\,\big)$$

$$\frac{\varphi\Big(\bigvee_{\vec{a}\in\{\bot,\top\}^n} \bigwedge\{\varphi_1^{a_1}, \ldots, \varphi_n^{a_n}, \psi^{f(\vec{a})}\} \;/\; *\Big)}{\varphi\big(\mathrm{dep}(\varphi_1, \ldots, \varphi_n, \psi) \;/\; *\big)} \;\big(\mathcal{ML}\,\mathrm{dep}(f)\,\big)\dagger$$

where † means that $\varphi_1, \ldots, \varphi_n, \psi$ are required to be modal formulae.[1] Define $\mathcal{PL}\mathsf{dep} := \{(\mathcal{PL}\,\mathrm{dep}(f)) \mid f : \{\bot,\top\}^n \to \{\top,\bot\}$, where $n \in \mathbb{N}\}$ and $\mathcal{ML}\mathsf{dep} := \{(\mathcal{ML}\,\mathrm{dep}(f)) \mid f : \{\bot,\top\}^n \to \{\top,\bot\}$, where $n \in \mathbb{N}\}$. Let $\mathbf{H}_{\mathcal{PD}}$ and $\mathbf{H}_{\mathcal{MDL}}$ be the extensions of the calculi $\mathbf{H}_{\mathcal{PL}}$ and $\mathbf{H}_{\mathcal{ML}}$ by the rules of $\mathcal{PL}\mathsf{dep}$, respectively. Let $\mathbf{H}_{\mathcal{EMDL}}$ be the extension of $\mathbf{H}_{\mathcal{ML}}$ by the rules of $\mathcal{ML}\mathsf{dep}$.

The proof of the following theorem is analogous to that of Theorem 11.

▶ **Theorem 12.** *Let $\mathcal{L} \in \{\mathcal{PD}, \mathcal{MDL}, \mathcal{EMDL}\}$, $\mathbf{H}_{\mathcal{L}}$ is sound and complete.*

4 Labeled tableaus for propositional dependence logics

The calculi presented in Section 3 have a few clear shortcomings. Foremost, the calculi miss the team semantic nature of these logics. Thus the calculi are in some parts quite complicated. Especially this is the case for the rules $\mathcal{PL}\,\mathsf{dep}$ and $\mathcal{ML}\,\mathsf{dep}$. This seems to be the case also for any concrete implementations of the axiomatizations $\mathbf{H}_{\mathcal{PL}}$ and $\mathbf{H}_{\mathcal{ML}}$ of the negation normal form fragments of $\mathcal{PL}$ and $\mathcal{ML}$, respectively.

In this section we give axiomatizations for $\mathcal{PD}$, $\mathcal{MDL}$, and $\mathcal{EMDL}$ that do not have the shortcomings of the calculi of Section 3. The proof rules of the labeled tableau calculi that are given in this section have a natural and simple correspondence with the truth definitions of connectives and modalities in team semantics.

4.1 Checking validity via small teams

The following result (observed, e.g., in [11]) follows directly from the fact that $\mathcal{PL}(\varovee)$ and $\mathcal{PD}$ are downwards closed, i.e., from Proposition 3.

▶ **Proposition 13.** *Let φ be a formula of $\mathcal{PL}(\varovee)$ or $\mathcal{PD}$ and let D be the set of exactly all proposition symbols that occur in φ. Then φ is valid iff $\{0,1\}^D \models \varphi$.*

Adapting a notion that was introduced by Jarmo Kontinen in [5] for first-order dependence logic, we say that an $\mathcal{ML}(\varovee)$- or $\mathcal{EMDL}$-formula φ is *n-coherent* if the condition

$$\mathrm{K}, T \models \varphi \quad \Leftrightarrow \quad \mathrm{K}, T' \models \varphi \text{ for all } T' \subseteq T \text{ such that } |T'| \leq n$$

holds for all Kripke models K and teams T of K.

[1] In the special case where φ is $*$ in the rule $(\mathcal{PL}\,\mathrm{dep}(f))$, the obtained rule coincides with the rule of Dependence Atom Introduction in [12, p.75].

The following result for $\mathcal{ML}(\varovee)$ was shown in [4]. The result for $\mathcal{EMDL}$ follows from the result for $\mathcal{ML}(\varovee)$ essentially via the following equivalence.

$$\mathrm{dep}(\varphi_1,\ldots,\varphi_n,\psi) \equiv \bigvee_{a_1,\ldots,a_n\in\{\bot,\top\}} \bigwedge \{\varphi_1^{a_1},\ldots,\varphi_n^{a_n},(\psi \varovee \psi^\bot)\}.$$

For $\varphi \in \mathcal{ML}(\varovee)$, we define $\mathrm{Rank}_\varovee(\varphi)$ to be the number of intuitionistic disjunctions in φ. For $\psi \in \mathcal{EMDL}$, we define $\mathrm{Rank}_\varovee(\psi)$ to be the number of intuitionistic disjunctions in the $\mathcal{ML}(\varovee)$ formula obtained by using the above equivalence. Note that $\mathrm{Rank}_\varovee(\varphi) \le |\varphi|$, whereas $\mathrm{Rank}_\varovee(\psi) \le 2^{|\psi|}$.

▶ **Theorem 14.** *Every formula φ of $\mathcal{ML}(\varovee)$ or $\mathcal{EMDL}$ is $2^{\mathrm{Rank}_\varovee(\varphi)}$-coherent.*

The following result follows directly from Theorem 14.

▶ **Corollary 15.** *Let φ be a formula of $\mathcal{ML}(\varovee)$ or $\mathcal{EMDL}$. The following holds:*

φ is valid iff $\mathrm{K}, T \models \varphi$ for every Kripke model K and every team T of K such that $|T| \le 2^{\mathrm{Rank}_\varovee(\varphi)}$.

4.2 Tableau Calculi for $\mathcal{PL}$, $\mathcal{PL}(\varovee)$, and $\mathcal{PD}$

We will now present labeled tableau calculi for $\mathcal{PL}$, $\mathcal{PL}(\varovee)$, and $\mathcal{PD}$. In Section 4.3 we will extend these calculi to deal with $\mathcal{ML}$, $\mathcal{MDL}$, and $\mathcal{EMDL}$.

Any finite, possibly empty, subset $\alpha \subseteq \mathbb{N}$ is called a *label*. We mainly use symbols $\alpha, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2$, etc, in order to refer to labels and symbols i, j, i_1, i_2, j_1, j_2, etc, in order to refer to natural numbers. Our tableau calculi are labeled, meaning that the formulae occurring in the tableau rules are *labeled formulae*, i.e., of the form $\alpha : \varphi$, where α a label and φ is a formula of some logic $\mathcal{L}$. Labels correspond to teams and the elements of labels, i.e., natural numbers, correspond to points in a model. The intended *top down* reading of the labeled formula $\alpha : \varphi$ is that α denotes some team that *falsifies* φ. A tableau in these calculi is just a well-founded, finitely branching tree in which each node is labeled by a labeled formula, and the edges represent applications of the tableau rules. The tableau rules needed for axiomatizing $\mathcal{PL}$, $\mathcal{PL}(\varovee)$, and $\mathcal{PD}$ are given in Figure 1.

In the construction of tableaus, we impose a rule that a labeled formula is never added to a tableau branch in which it already occurs. A *saturated branch* is a tableau branch in which no rules can be applied or the application of the rules have no effect on the branch. A *saturated tableau* is a tableau in which every branch is saturated. A branch of a tableau is called *closed* if it contains at least one of the following:

1. Both $\{i\} : p$ and $\{i\} : \neg p$, for some proposition symbol p and natural number $i \in \mathbb{N}$.
2. $\emptyset : \varphi$, for some formula φ.
3. $\{i\} : \mathrm{dep}(p_1,\ldots,p_n,q)$, for some proposition symbols $p_1,\ldots,p_n,q$ and $i,n \in \mathbb{N}$.

If a branch of a tableau is not closed it is called *open*. A tableau is called *closed* if every branch of the tableau is closed. A tableau is called *open* if at least one branch in the tableau is open.

Let $\mathbf{T}_{\mathcal{PL}}$ denote the calculi consisting of the rules (*Prop*), ($\neg$*Prop*), ($\wedge$), and ($\vee$) of Figure 1. Let $\mathbf{T}_{\mathcal{PL}(\varovee)}$ denote the extension of $\mathbf{T}_{\mathcal{PL}}$ by the rule ($\varovee$) of Figure 1, and $\mathbf{T}_{\mathcal{PD}}$ denote the extension of $\mathbf{T}_{\mathcal{PL}}$ by the rules (*Split*) and ($\mathcal{PL}\,$*dep*) of Figure 1.

Let φ be a formula of $\mathcal{L}(\Phi) \in \{\mathcal{PL}(\Phi), \mathcal{PL}(\varovee)(\Phi), \mathcal{PD}(\Phi)\}$ and $k := \min(|\Phi|, \mathrm{Rank}_\varovee(\varphi))$. We say that a tableau $\mathcal{T}$ is a *tableau for φ* if the root of $\mathcal{T}$ is $\{1,\ldots,2^k\} : \varphi$ and $\mathcal{T}$ is obtained

$$\frac{\{i_1,\ldots,i_k\} : p}{\{i_1\} : p \mid \ldots \mid \{i_k\} : p}\ (Prop) \qquad \frac{\{i_1,\ldots,i_k\} : \neg p}{\{i_1\} : \neg p \mid \ldots \mid \{i_k\} : \neg p}\ (\neg Prop) \qquad \frac{\alpha : (\varphi \wedge \psi)}{\alpha : \varphi \mid \alpha : \psi}\ (\wedge)$$

$$\frac{\alpha : (\varphi \vee \psi)}{\beta : \varphi \mid \alpha \setminus \beta : \psi}\ (\vee) \text{ where } \beta \subseteq \alpha \qquad \frac{\alpha : (\varphi \varovee \psi)}{\begin{array}{c}\alpha : \varphi\\ \alpha : \psi\end{array}}\ (\varovee)$$

$$\frac{\alpha : \mathrm{dep}(p_1,\ldots,p_n,q)}{\alpha_1 : \mathrm{dep}(p_1,\ldots,p_n,q) \mid \ldots \mid \alpha_k : \mathrm{dep}(p_1,\ldots,p_n,q)}\ (Split)\dagger$$

†: $\alpha_1,\ldots,\alpha_k$ are exactly all subsets of α of cardinality 2.

$$\frac{\{i_1,i_2\} : \mathrm{dep}(p_1,\ldots,p_n,q)}{\begin{array}{ccc}\{i_1\} : p_1^{g_1(1)} & \mid \ldots \mid & \{i_1\} : p_1^{g_k(1)}\\ \{i_2\} : p_1^{g_1(1)} & \mid \ldots \mid & \{i_2\} : p_1^{g_k(1)}\\ \vdots & \vdots & \vdots\\ \{i_1\} : p_n^{g_1(n)} & \mid \ldots \mid & \{i_1\} : p_n^{g_k(n)}\\ \{i_2\} : p_n^{g_1(n)} & \mid \ldots \mid & \{i_2\} : p_n^{g_k(n)}\\ \{i_1,i_2\} : q & \mid \ldots \mid & \{i_1,i_2\} : q\\ \{i_1,i_2\} : \neg q & \mid \ldots \mid & \{i_1,i_2\} : \neg q\end{array}}\ (\mathcal{PL}\,dep)\ddagger$$

‡: $g_1,\ldots g_k$ are exactly all functions with domain $\{1,\ldots,n\}$ and co-domain $\{\top,\bot\}$.

Figure 1 Tableau Rules for $\mathbf{T}_{\mathcal{PL}}$, $\mathbf{T}_{\mathcal{PL}(\varovee)}$, and $\mathbf{T}_{\mathcal{PD}}$.

by applying the rules of $\mathbf{T}_{\mathcal{L}}$. We say that φ is *provable* in $\mathbf{T}_{\mathcal{L}}$ and write $\vdash_{\mathbf{T}_{\mathcal{L}}} \varphi$ if there exists a closed tableau for φ.

▶ **Example 16.** We show that the $\mathcal{PD}$-formula $\mathrm{dep}(p,p)$ is provable $\mathbf{T}_{\mathcal{PD}}$. Figure 2 is an illustration of a closed $\mathbf{T}_{\mathcal{PD}}$-tableau for $\mathrm{dep}(p,p)$.

Since the number of proposition symbols that occur in $\mathrm{dep}(p,p)$ is one, the root of the tableau is $\{1,2\} : \mathrm{dep}(p,p)$. We first apply the rule $(\mathcal{PL}dep)$ to $\{1,2\} : \mathrm{dep}(p,p)$ and branch into two branches as depicted in Figure 2. In the left (right) branch we apply the rule $(\neg Prop)$ to $\{1,2\} : \neg p$ ($(Prop)$ to $\{1,2\} : p$). Consequently, each branch of the tableau becomes closed due to the labeled formulae of the type $\{i\} : p$ and $\{i\} : \neg p$, $i \in \{1,2\}$. Therefore, $\mathrm{dep}(p,p)$ is a theorem of $\mathbf{T}_{\mathcal{PD}}$.

▶ **Theorem 17** (Termination of $\mathbf{T}_{\mathcal{PL}}$, $\mathbf{T}_{\mathcal{PL}(\varovee)}$, and $\mathbf{T}_{\mathcal{PD}}$)**.** *Let $\mathcal{L}$ be a logic in $\{\mathcal{PL}, \mathcal{PL}(\varovee), \mathcal{PD}\}$ and φ an $\mathcal{L}$-formula. Every tableau for φ in $\mathbf{T}_{\mathcal{L}}$ is finite.*

Proof. Let $\mathcal{T}$ be a tableau for φ. By definition, the root of $\mathcal{T}$ is $\alpha : \varphi$, for some finite α. Clearly every application of the tableau rules either decreases the size of the label or the length of the formula. Note also that the rule $(\vee)$ can be applied to any $\beta : \psi \in \mathcal{T}$ only finitely many times. Thus $\mathcal{T}$ must be finite. ◀

▶ **Lemma 18.** *If there exists a saturated open branch for φ then φ is not valid.*

Proof. Let $\mathcal{B}$ be a saturated open branch for φ and let Φ be the set of proposition symbols that occur in φ. Let $\alpha : \varphi$ denote the root of the branch $\mathcal{B}$. It is easy to check that if $\beta : \psi$ is a labeled formula in $\mathcal{B}$ then $\beta \subseteq \alpha$. For each $i \in \alpha$ we define an assignment $s_i : \Phi \to \{0,1\}$ such that

$$s_i(p) := \begin{cases} 1 & \text{if the labeled formula } \{i\} : \neg p \text{ occurs in the branch } \mathcal{B},\\ 0 & \text{otherwise.}\end{cases}$$

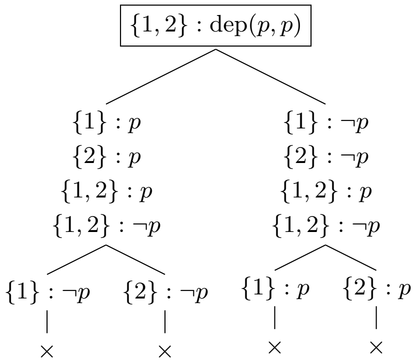

Figure 2 A tableau showing that the $\mathcal{PD}$-formula $\mathrm{dep}(p, p)$ is provable in $\mathbf{T}_{\mathcal{PD}}$.

It is easy to show by induction that if a labeled formula $\beta : \psi$ occurs in the branch $\mathcal{B}$ then $X_\beta \not\models \psi$, where $X_\beta = \{s_i \mid i \in \beta\}$. Thus φ is not valid. ◀

▶ **Theorem 19** (Completeness of $\mathbf{T}_{\mathcal{PL}}$, $\mathbf{T}_{\mathcal{PL}(\otimes)}$, and $\mathbf{T}_{\mathcal{PD}}$). *Let $\mathcal{L}$ be any of the logics in $\{\mathcal{PL}, \mathcal{PL}(\otimes), \mathcal{PD}\}$. The calculus $\mathbf{T}_\mathcal{L}$ is complete.*

Proof. Fix $\mathcal{L} \in \{\mathcal{PL}, \mathcal{PL}(\otimes), \mathcal{PD}\}$. Assume $\not\vdash_{\mathbf{T}_\mathcal{L}} \varphi$. Thus every tableau for φ is open. From Theorem 17 it follows that there exists a saturated open tableau for φ. Thus there exists a saturated open branch for φ. Thus, by Lemma 18, $\not\models_\mathcal{L} \varphi$. ◀

▶ **Definition 20.** Let $\mathcal{B}$ be a tableau branch and $\mathrm{Index}(\mathcal{B})$ the set of exactly all natural numbers that occur in $\mathcal{B}$. We say that $\mathcal{B}$ is *faithful* to a propositional team X by a mapping $f : \mathrm{Index}(\mathcal{B}) \to X$ if, for all $\alpha : \varphi \in \mathcal{B}$, $f[\alpha] \not\models \varphi$.

▶ **Lemma 21.** *Let $\mathcal{L}$ be a logic in $\{\mathcal{PL}, \mathcal{PL}(\otimes), \mathcal{PD}\}$. If $\varphi \in \mathcal{L}$ is not valid then there is an open saturated branch in every saturated tableau of φ in $\mathbf{T}_\mathcal{L}$.*

Proof. Assume $\not\models_\mathcal{L} \varphi$. Let Φ be the set of exactly all proposition symbols that occur in φ. By Proposition 13, $\{0,1\}^\Phi \not\models \varphi$. Put $\alpha := \{1, \ldots, 2^{|\Phi|}\}$ and fix a bijection $f : \alpha \to \{0,1\}^\Phi$. Let $\mathcal{T}$ be an arbitrary saturated tableau for φ. By Theorem 17, $\mathcal{T}$ is finite and, by definition, the root of $\mathcal{T}$ is $\alpha : \varphi$. Note that $\mathrm{Index}(\mathcal{B}) = \alpha$, for every branch $\mathcal{B}$ with the root $\alpha : \varphi$. We will show that there is an open saturated branch in $\mathcal{T}$.

First, we establish that $\mathcal{B}_0 := \{\alpha : \varphi\}$ is faithful to $\{0,1\}^\Phi$ by f. But, this is easy since $f[\alpha] = \{0,1\}^\Phi$. Second, assume that we have constructed a branch $\mathcal{B}_n$ such that $\mathcal{B}_n$ is faithful to $\{0,1\}^\Phi$ by f. We will show that at least one extension of $\mathcal{B}_n$ by rules of $\mathbf{T}_\mathcal{L}$ is faithful to $\{0,1\}^\Phi$ by f. Here we are concerned with the rule of $(\vee)$ alone. Assume that, from $\beta_1 : (\psi_1 \vee \psi_2) \in \mathcal{B}_n$ and the rule of $(\vee)$, we obtain two extensions $\{\beta_2 : \psi_1\} \cup \mathcal{B}_n$ and $\{\beta_1 \setminus \beta_2 : \psi_2\} \cup \mathcal{B}_n$ for $\beta_2 \subseteq \beta_1$. Our goal is to show that one of the extensions is faithful to $\{0,1\}^\Phi$ by f. By assumption, we obtain $f[\beta_1] \not\models (\psi_1 \vee \psi_2)$. By the semantic clause for $\vee$, $f[\beta_2] \not\models \psi_1$ or $f[\beta_1] \setminus f[\beta_2] \not\models \psi_2$. Since $f[\beta_1] \setminus f[\beta_2] \subseteq f[\beta_1 \setminus \beta_2]$, it follows from downwards closure that $f[\beta_2] \not\models \psi_1$ or $f[\beta_1 \setminus \beta_2] \not\models \psi_2$. This implies that at least one of the two extensions is faithful to $\{0,1\}^\Phi$ by f. We choose one of the faithful extensions as $\mathcal{B}_{n+1}$.

Since $\mathcal{T}$ is finite and saturated, $\mathcal{B}_j$ is a saturated branch in $\mathcal{T}$ for some $j \in \mathbb{N}$. Moreover, since $\mathcal{B}_j$ is faithful to $\{0,1\}^\Phi$ by f, $\mathcal{B}_j$ is open. ◀

▶ **Theorem 22** (Soundness of $\mathbf{T}_{\mathcal{PL}}$, $\mathbf{T}_{\mathcal{PL}(\otimes)}$, and $\mathbf{T}_{\mathcal{PD}}$). *Let $\mathcal{L}$ be any of the logics in $\{\mathcal{PL}, \mathcal{PL}(\otimes), \mathcal{PD}\}$. The calculus $\mathbf{T}_\mathcal{L}$ is sound.*

$$\begin{array}{c} i_1 \mathsf{R} j_1 \\ \vdots \\ i_n \mathsf{R} j_n \\ \dfrac{\{i_1,\dots,i_n\} : \Diamond\varphi}{\{j_1,\dots,j_n\} : \varphi}\ (\Diamond) \end{array} \qquad \dfrac{\alpha : \Box\varphi}{\begin{array}{c} f_1(1)\mathsf{R}i_1 \mid \dots \mid f_k(1)\mathsf{R}i_1 \\ \vdots \qquad \vdots \qquad \vdots \\ f_1(t)\mathsf{R}i_t \mid \dots \mid f_k(t)\mathsf{R}i_t \\ \{i_1,\dots i_t\} : \varphi \mid \dots \mid \{i_1,\dots i_t\} : \varphi \end{array}}\ (\Box)\dagger$$

$$\dfrac{\{i_1,i_2\} : \mathrm{dep}(\varphi_1,\dots,\varphi_n,\psi)}{\begin{array}{c} \{i_1\} : \varphi_1^{h_1(1)} \mid \dots \mid \{i_1\} : \varphi_1^{h_k(1)} \\ \{i_2\} : \varphi_1^{h_1(1)} \mid \dots \mid \{i_2\} : \varphi_1^{h_k(1)} \\ \vdots \qquad \vdots \qquad \vdots \\ \{i_1\} : \varphi_n^{h_1(n)} \mid \dots \mid \{i_1\} : \varphi_n^{h_k(n)} \\ \{i_2\} : \varphi_n^{h_1(n)} \mid \dots \mid \{i_2\} : \varphi_n^{h_1(n)} \\ \{i_1,i_2\} : \psi \mid \dots \mid \{i_1,i_2\} : \psi \\ \{i_1,i_2\} : \psi^{\bot} \mid \dots \mid \{i_1,i_2\} : \psi^{\bot} \end{array}}\ (\mathcal{ML}\,dep)\ddagger$$

†: $t = 2^{\mathrm{Rank}_{\otimes}(\varphi)}$ and $f_1,\dots,f_k$ denote exactly all functions with domain $\{1,\dots,t\}$ and co-domain α, and $i_1,\dots,i_t$ are fresh and distinct.
‡: $h_1,\dots h_k$ denotes all the functions with domain $\{1,\dots,n\}$ and co-domain $\{\top,\bot\}$.

Figure 3 Additional Tableau Rules for $\mathbf{T}_{\mathcal{ML}}$, $\mathbf{T}_{\mathcal{ML}(\otimes)}$, $\mathbf{T}_{\mathcal{MDL}}$ and $\mathbf{T}_{\mathcal{EMDL}}$.

Proof. Fix $\mathcal{L} \in \{\mathcal{PL}, \mathcal{PL}(\otimes), \mathcal{PD}\}$. Assume that $\not\models_{\mathcal{L}} \varphi$. By Lemma 21, there is an open saturated branch in every saturated tableau of φ in $\mathbf{T}_{\mathcal{L}}$. Therefore, and since, by Theorem 17, every tableau of φ in $\mathbf{T}_{\mathcal{L}}$ is finite, there does not exists any closed tableau for φ in $\mathbf{T}_{\mathcal{L}}$. Thus $\not\vdash_{\mathbf{T}_{\mathcal{L}}} \varphi$. ◀

4.3 Tableau Calculi for $\mathcal{ML}$, $\mathcal{ML}(\otimes)$, $\mathcal{MDL}$, and $\mathcal{EMDL}$

In addition to labeled formulae, the tableau rules for modal logics contain *accessibility formulae* of the form $i\mathsf{R}j$, where $i, j \in \mathbb{N}$. The intended interpretation of $i\mathsf{R}j$ is that the point denoted by j is accessible by the relation R from the point denoted by i. The tableau rules for the calculi are given in Figures 1 and 3.

In the construction of tableaus, in addition to the rules given in Section 4.2, we impose that the tableau rule ($\Box$) is never applied twice to the same labeled formula in any branch. The definitions of open, closed and saturated tableau and branch are as in Section 4.2 with the following additional rule: A branch is called *closed* also if it contains a labeled formula $\{i\} : \mathrm{dep}(\varphi_1,\dots,\varphi_n,\psi)$, for some $i, n \in \mathbb{N}$ and $\varphi_1,\dots,\varphi_n,\psi \in \mathcal{ML}$.

Let $\mathbf{T}_{\mathcal{ML}}$, $\mathbf{T}_{\mathcal{ML}(\otimes)}$, and $\mathbf{T}_{\mathcal{MDL}}$ denote the extensions of $\mathbf{T}_{\mathcal{PL}}$, $\mathbf{T}_{\mathcal{PL}(\otimes)}$, and $\mathbf{T}_{\mathcal{PD}}$ by the rules ($\Diamond$) and ($\Box$) of Figure 3, respectively. Let $\mathbf{T}_{\mathcal{EMDL}}$ denote the extension of $\mathbf{T}_{\mathcal{ML}}$ by the rules (*Split*) of Figure 1 and ($\mathcal{ML}\,dep$) of Figure 3.

Let φ be a formula of $\mathcal{L} \in \{\mathcal{ML}, \mathcal{ML}(\otimes), \mathcal{MDL}, \mathcal{EMDL}\}$. We say that a tableau $\mathcal{T}$ is a *tableau for* φ if the root of $\mathcal{T}$ is $\{1,\dots,2^{\mathrm{Rank}_{\otimes}(\varphi)}\} : \varphi$ and $\mathcal{T}$ is obtained by applying the rules of $\mathbf{T}_{\mathcal{L}}$. We say that φ is *provable* in $\mathbf{T}_{\mathcal{L}}$ and write $\vdash_{\mathbf{T}_{\mathcal{L}}} \varphi$ if there exists a closed tableau for φ.

▶ **Example 23.** This example illustrates one difference between $\mathbf{T}_{\mathcal{PL}}$ and $\mathbf{T}_{\mathcal{MDL}}$ even for the same formula $\mathrm{dep}(p,p)$. Figure 4 is an illustration of a closed $\mathbf{T}_{\mathcal{MDL}}$-tableau for $\mathrm{dep}(p,p)$. When $\mathrm{dep}(p,p)$ is considered as a $\mathcal{PD}$-formula, the calculation starts with the label $\{1,2\}$

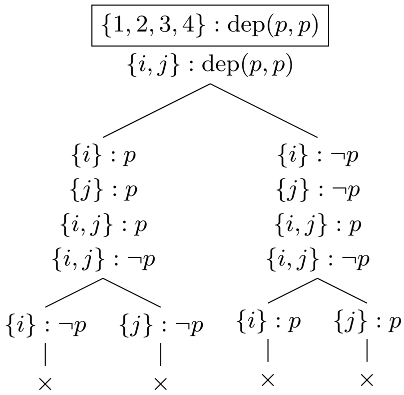

Figure 4 A tableau showing that the $\mathcal{MDL}$-formula dep(p, p) is provable in $\mathbf{T}_{\mathcal{MDL}}$.

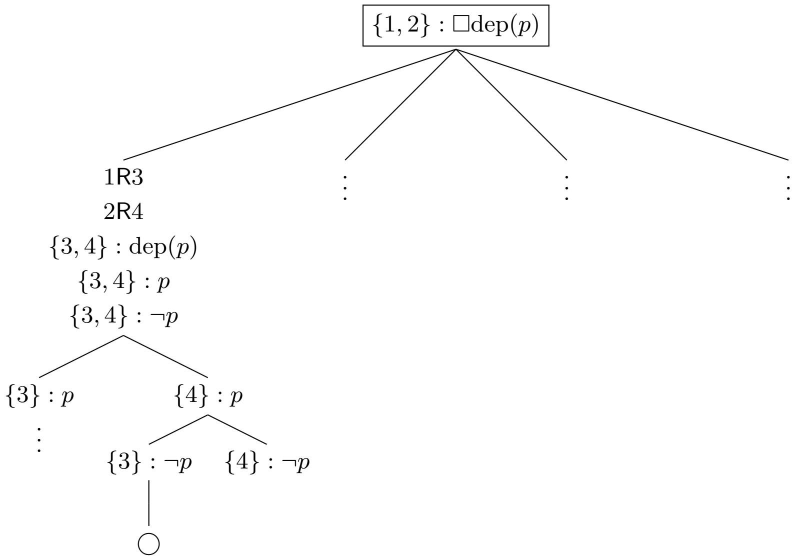

Figure 5 A tableau showing that the $\mathcal{MDL}$-formula $\Box$dep(p) is not valid.

(see Example 16 and Figure 2). However, when dep(p, p) is considered as an $\mathcal{MDL}$-formula, our definition leads us to start the calculation with the label $\{1, 2, 3, 4\}$.

The equivalent $\mathcal{ML}(\varovee)$ formula that the $\mathcal{MDL}$-formula dep(p, p) translates into is

$$\bigvee_{a \in \{\top, \bot\}} \bigwedge \{p^a, p \varovee \neg p\}.$$

Therefore $\text{Rank}_{\varovee}(\text{dep}(p, p)) = 2$, and thus the root of any $\mathbf{T}_{\mathcal{MDL}}$-tableau for dep$(p, p)$ is $\{1, 2, 3, 4\} : \text{dep}(p, p)$. We first apply the rule $(Split)$ to $\{1, 2, 3, 4\} : \text{dep}(p, p)$ and obtain 6 branches. By symmetry, we may concentrate on one of the branches. We denote it by $\{i, j\}$ $(i \neq j)$. We then apply the rule $(\mathcal{PL}dep)$ to $\{i, j\} : \text{dep}(p, p)$ and branch into two branches as depicted in Figure 4. In the left (right) branch we apply the rule $(\neg Prop)$ to $\{i, j\} : \neg p$ $((Prop)$ to $\{i, j\} : p)$. Consequently, each branch of the tableau becomes closed due to the labeled formulae of the type $\{l\} : p$ and $\{l\} : \neg p$, $l \in \{i, j\}$. Therefore, dep(p, p) is a theorem of $\mathbf{T}_{\mathcal{MDL}}$.

▶ **Example 24.** We show that the $\mathcal{MDL}$ formula $\Box\mathrm{dep}(p)$ is not valid. Note that the equivalent $\mathcal{ML}(\otimes)$-formula that $\Box\mathrm{dep}(p)$ translates into is $\Box(p \otimes \neg p)$. Therefore $\mathrm{Rank}_{\otimes}(\Box\mathrm{dep}(p)) = 1$, and thus the root of any $\mathbf{T}_{\mathcal{MDL}}$-tableau for $\Box\mathrm{dep}(p)$ is $\{1,2\} : \Box\mathrm{dep}(p)$. We are going to find an open saturated branch for $\Box\mathrm{dep}(p)$.

First, we apply the rule $(\Box)$ for $\{1,2\} : \Box\mathrm{dep}(p)$. One of the branches that is obtained is depicted in Figure 5. We then apply the rule $(\mathcal{PL}dep)$ to $\{3,4\} : \mathrm{dep}(p)$. Then, by applying the rules $(Prop)$ and $(\neg Prop)$ to $\{3,4\} : p$ and $\{3,4\} : \neg p$, respectively, we obtain an open saturated branch as depicted in Figure 5. From the open saturated branch, we can construct the following Kripke model $\mathrm{K} = (W, R, V)$ that falsifies the $\mathcal{MDL}$-formula $\Box\mathrm{dep}(p)$. Define $W := \{w_1, w_2, w_3, w_4\}$, $R := \{(w_1, w_3), (w_2, w_4)\}$, $V(p) := \{w_3\}$. One can easily verify that $\mathrm{K}, \{w_1, w_2\} \not\models \Box\mathrm{dep}(p)$.

▶ **Definition 25.** Let $\mathcal{L} \in \{\mathcal{ML}, \mathcal{ML}(\otimes), \mathcal{MDL}, \mathcal{EMDL}\}$. Let $\mathcal{B}$ be a branch of a tableau in $\mathbf{T}_{\mathcal{L}}$ and let $\alpha : \varphi$ be the root of $\mathcal{B}$. Recall that $\mathrm{Index}(\mathcal{B})$ denotes the set of exactly all natural numbers that occur in $\mathcal{B}$. For $i, j \in \mathrm{Index}(\mathcal{B})$, we write $i \prec_{\mathcal{B}} j$ if iRj occurs in $\mathcal{B}$. By $\prec^*_{\mathcal{B}}$ and $\preceq^*_{\mathcal{B}}$, we mean the transitive closure and the reflexive and transitive closure of $\prec_{\mathcal{B}}$, respectively. Moreover, for $i \in \mathrm{Index}(\mathcal{B})$ and $n \in \mathbb{N}$, define

$$\mathrm{Level}_{\mathcal{B}}(i) := |\{j \in \mathrm{Index}(\mathcal{B}) \mid i_0 \prec^*_{\mathcal{B}} j \preceq^*_{\mathcal{B}} i, \text{ for some } i_0 \in \alpha\}|,$$
$$\mathrm{Layer}_{\mathcal{B}}(n) := \{j \in \mathrm{Index}(\mathcal{B}) \mid \mathrm{Level}_{\mathcal{B}}(j) = n\}.$$

It is easy to see that, for every branch $\mathcal{B}$, the graph $(\mathrm{Index}(\mathcal{B}), \prec_{\mathcal{B}})$ is a well-founded forest.

▶ **Theorem 26** (Termination of $\mathbf{T}_{\mathcal{ML}}$, $\mathbf{T}_{\mathcal{ML}(\otimes)}$, $\mathbf{T}_{\mathcal{MDL}}$, and $\mathbf{T}_{\mathcal{EMDL}}$). *Let φ be a formula of $\mathcal{ML}$, $\mathcal{ML}(\otimes)$, $\mathcal{MDL}$, or $\mathcal{EMDL}$. Every tableau for φ is finite.*

Proof. Let $\mathcal{T}$ be a tableau for φ and let $\alpha : \varphi$ denote the root of $\mathcal{T}$. By definition α is finite. Clearly, by the definitions of the tableau rules, if $\beta : \psi$ occurs in $\mathcal{T}$ then $|\beta| \le |\alpha|$. From this and from the definitions of the tableau rules, it is easy to see that $\mathcal{T}$ is a finitely branching tree. Thus from König's lemma it follows that $\mathcal{T}$ is infinite if and only if $\mathcal{T}$ has an infinite branch.

Let $\mathcal{B}$ be an arbitrary branch of $\mathcal{T}$. We will show that $\mathcal{B}$ is finite.

Claim 1. If $\alpha : \varphi$ occurs in $\mathcal{B}$ then, for every $i, j \in \alpha$, $\mathrm{Level}_{\mathcal{B}}(i) = \mathrm{Level}_{\mathcal{B}}(j)$.

Claim 2. For each $k \in \mathbb{N}$ the set $\mathrm{Layer}_{\mathcal{B}}(k)$ is finite.

Claim 3. There is a $k \in \mathbb{N}$ such that $\mathrm{Layer}_{\mathcal{B}}(k) = \emptyset$.

Note first that if $\mathrm{Layer}_{\mathcal{B}}(k) = \emptyset$ then $\mathrm{Layer}_{\mathcal{B}}(n) = \emptyset$, for every $n \ge k$. Thus from Claims 2 and 3 it follows that only finitely many labels may occur in $\mathcal{B}$. Note also that, for every labeled formula $\beta : \psi$ that occurs in $\mathcal{B}$, ψ is either a subformula of φ or a subformula of some $\theta^{\perp}$, where θ is an $\mathcal{ML}$ subformula of φ. Thus only finitely many labeled formulae may occur in $\mathcal{B}$. Thus $\mathcal{B}$ is finite.

Proof of Claim 1 is easy. We will sketch the proofs of Claims 2 and 3.

Proof sketch of Claim 2. Claim 2 follows from Claim 1 by induction: Clearly $\mathrm{Layer}_{\mathcal{B}}(0)$ is finite. $\mathrm{Layer}_{\mathcal{B}}(k+1)$ is generated via applications of the tableau rule $(\Box)$ to labeled formulae $\beta : \Box\psi$ of the branch $\mathcal{B}$, where $\beta \subseteq \mathrm{Layer}_{\mathcal{B}}(k)$ and $\Box\psi$ is either a subformula of φ or a subformula of some $\theta^{\perp}$, where θ is an $\mathcal{ML}$ subformula of φ. Since $\mathrm{Layer}_{\mathcal{B}}(k)$ is finite, $\mathrm{Layer}_{\mathcal{B}}(k+1)$ is as well.

Proof sketch of Claim 3. For finite labels β, define

$$m_{\mathcal{B}}(\beta) := \max\{|\varphi| \mid \beta_1 : \varphi \text{ occurs in } \mathcal{B} \text{ and } \beta_1 \cap \beta \neq \emptyset\}.$$

For finite labels β, define $M_{\mathcal{B}}(\beta : \psi) := (m_{\mathcal{B}}(\beta), |\psi|, |\beta|)$. The ordering between the tuples is defined as follows:

$$(i, j, k) < (k, l, m) \text{ iff } i < k \text{ or } (i = k \text{ and } j < l) \text{ or } (i = k \text{ and } j = l \text{ and } k < m).$$

Note that for every labeled formula $\beta : \psi$ that occurs in $\mathcal{B}$ it holds that $m_{\mathcal{B}}(\beta) < m_{\mathcal{B}}(\alpha)$, $|\psi| \leq |\varphi|$ and $|\beta| \leq |\alpha|$. Thus the ordering of the tuples is well-founded. Furthermore it is easy to check that an application of each tableau rule decreases the measure $M_{\mathcal{B}}$. For finite collections of labeled formulae Γ, define $\mathcal{M}_{\mathcal{B}}(\Gamma) := \max\{M_{\mathcal{B}}(\beta : \psi) \mid \beta : \psi \in \Gamma\}$. It is straightforward to show that, for every $k \in \mathbb{N}$, either $\mathcal{M}_{\mathcal{B}}\big(\mathrm{Layer}_{\mathcal{B}}(k+1)\big) < \mathcal{M}_{\mathcal{B}}\big(\mathrm{Layer}_{\mathcal{B}}(k)\big)$ or $\mathrm{Layer}_{\mathcal{B}}(k+1) = \emptyset$. From this the claim follows. ◀

▶ **Definition 27.** Let $\mathcal{B}$ be a tableau branch. We say that $\mathcal{B}$ is *faithful* to a Kripke model $\mathrm{K} = (W, R, V)$ if there exists a mapping $f : \mathrm{Index}(\mathcal{B}) \to W$ such that, $\mathrm{K}, f[\alpha] \not\models \varphi$ for all $\alpha : \varphi \in \mathcal{B}$, and $f(i)Rf(j)$ holds, for every $i\mathsf{R}j \in \mathcal{B}$.

▶ **Lemma 28.** *Let $\mathcal{L} \in \{\mathcal{ML}, \mathcal{ML}(\varovee), \mathcal{MDL}, \mathcal{EMDL}\}$. If $\varphi \in \mathcal{L}$ is not valid then there is an open saturated branch in every saturated tableau of φ in $\mathbf{T}_{\mathcal{L}}$.*

Proof. In this proof, we focus on $\mathcal{ML}(\varovee)$. Assume that $\varphi \in \mathcal{ML}(\varovee)$ is not valid. By Corollary 15, there is a Kripke model $\mathrm{K} = (W, R, V)$ and a team T of K such that $|T| \leq 2^{\mathrm{Rank}_{\varovee}(\varphi)}$ and $\mathrm{K}, T \not\models \varphi$. Put $\alpha_0 := \{1, \ldots, 2^{\mathrm{Rank}_{\varovee}(\varphi)}\}$. Let $\mathcal{T}$ be an arbitrary saturated tableau for φ. By Theorem 26, $\mathcal{T}$ is finite and, by definition, the root of $\mathcal{T}$ is $\alpha_0 : \varphi$. We will show that there is an open branch $\mathcal{B}$ in $\mathcal{T}$.

We first establish that $\mathcal{B}_0 := \{\alpha_0 : \varphi\}$ is faithful to K. Let $f : \alpha_0 \to W$ be any mapping (note: W is non-empty) such that $f[\alpha_0] = T$. Clearly $\mathrm{K}, f[\alpha_0] \not\models \varphi$, and thus $\mathcal{B}_0$ is faithful to K. Assume then that we have constructed a branch $\mathcal{B}_n$ such that $\mathcal{B}_n$ is faithful to K. Thus there is a mapping $g : \mathrm{Index}(\mathcal{B}_n) \to W$ such that, for all $\beta : \psi \in \mathcal{B}_n$, $\mathrm{K}, g[\beta] \not\models \psi$, and, for all $i\mathsf{R}j \in \mathcal{B}_n$, $g(i)Rg(j)$ holds. We will show that any rule-application to $\mathcal{B}_n$ generates at least one faithful extension $\mathcal{B}_{n+1}$ to K. Here we are concerned with the rules of ($\Diamond$) and ($\Box$) alone.

($\Diamond$) Assume that $\{i_1, \ldots, i_k\} : \Diamond\psi, i_1\mathsf{R}j_1, \ldots, i_k\mathsf{R}j_k \in \mathcal{B}_n$. Let $\alpha := \{i_1, \ldots, i_k\}$ and $\beta := \{j_1, \ldots, j_k\}$. We obtain from our assumption that $\mathrm{K}, g[\alpha] \not\models \Diamond\psi$ and $g[\alpha][R]g[\beta]$. From the semantics of $\Diamond$ it follows that $\mathrm{K}, g[\beta] \not\models \psi$. Thus $\mathcal{B}_{n+1} := \mathcal{B}_n \cup \{\beta : \psi\}$ is faithful to K. Clearly $\mathcal{B}_{n+1}$ is an extension of $\mathcal{B}$ by the rule ($\Diamond$).

($\Box$) Assume that $\alpha : \Box\psi \in \mathcal{B}_n$. We obtain from our assumption that $\mathrm{K}, g[\alpha] \not\models \Box\psi$. By the semantics of $\Box$, it follows that $\mathrm{K}, R[g[\alpha]] \not\models \psi$. Now, by Theorem 14, there exists a team $S \subseteq R[g[\alpha]]$ such that $0 < |S| \leq 2^{\mathrm{Rank}_{\varovee}(\psi)}$ and $\mathrm{K}, S \not\models \psi$. Fix such $S \subseteq R[g[\alpha]]$ and let $u_1, \ldots, u_m$ be the elements of S. Since $S \subseteq R[g[\alpha]]$ there exists a function $h : \{1, \ldots, m\} \to \alpha$ such that $g\big(h(l)\big)Ru_l$, for each $l \leq m$. Let $h' : \{1, \ldots, 2^{\mathrm{Rank}_{\varovee}(\psi)}\} \to \alpha$ denote the expansion of h defined such that $h'(l) := h(m)$ for $m < l \leq 2^{\mathrm{Rank}_{\varovee}(\psi)}$. We then extend our function g to a mapping g' to cover new fresh indexes $\beta := \{j_1, \ldots, j_{2^{\mathrm{Rank}_{\varovee}(\psi)}}\}$. We define that $g'(j_l) := u_l$, for $l \leq m$, and $g'(j_l) := u_m$ for $m < l \leq 2^{\mathrm{Rank}_{\varovee}(\psi)}$. By construction, we obtain that $\mathrm{K}, g'[\beta] \not\models \psi$ and $g'(h'(l))Rg'(j_l)$ for all $1 \leq l \leq 2^{\mathrm{Rank}_{\varovee}(\psi)}$. Therefore, together with our assumption, $\mathcal{B}_{n+1} := \mathcal{B}_n \cup \{h'(1)\mathsf{R}j_1, \ldots, h'(2^{\mathrm{Rank}_{\varovee}(\psi)})\mathsf{R}j_{2^{\mathrm{Rank}_{\varovee}(\psi)}}, \beta : \psi\}$ is faithful to K by g'. Clearly $\mathcal{B}_{n+1}$ is an extension of $\mathcal{B}$ by the rule ($\Box$).

Since $\mathcal{T}$ is finite and saturated, $\mathcal{B}_j$ is a saturated branch in $\mathcal{T}$ for some $j \in \mathbb{N}$. Moreover, since $\mathcal{B}_j$ is faithful to K, $\mathcal{B}_j$ is open. ◀

▶ **Theorem 29** (Soundness of $\mathbf{T}_{\mathcal{ML}}$, $\mathbf{T}_{\mathcal{ML}(\varovee)}$, $\mathbf{T}_{\mathcal{MDL}}$, and $\mathbf{T}_{\mathcal{EMDL}}$). *Let $\mathcal{L}$ be a logic in $\{\mathcal{ML}, \mathcal{ML}(\varovee), \mathcal{MDL}, \mathcal{EMDL}\}$. The calculus $\mathbf{T}_{\mathcal{L}}$ is sound.*

Proof. Fix $\mathcal{L} \in \{\mathcal{ML}, \mathcal{ML}(\varovee), \mathcal{MDL}, \mathcal{EMDL}\}$. Assume that $\not\models_{\mathcal{L}} \varphi$. By Lemma 28, there is an open saturated branch in every saturated tableau of φ in $\mathbf{T}_{\mathcal{L}}$. Therefore, and since, by Theorem 26, every tableau of φ in $\mathbf{T}_{\mathcal{L}}$ is finite, there does not exists any closed tableau for φ in $\mathbf{T}_{\mathcal{L}}$. Thus $\not\vdash_{\mathbf{T}_{\mathcal{L}}} \varphi$. ◀

▶ **Lemma 30.** *Let $\mathcal{L} \in \{\mathcal{ML}, \mathcal{ML}(\varovee), \mathcal{MDL}, \mathcal{EMDL}\}$. If there exists an open saturated branch for φ in $\mathbf{T}_{\mathcal{L}}$ then φ is not valid.*

Proof. Let $\mathcal{B}$ be an open saturated branch in a tableau $\mathcal{T}$ of $\mathbf{T}_{\mathcal{L}}$ starting with $\{1, \ldots, 2^{\mathrm{Rank}_{\varovee}(\varphi)}\} : \varphi$. Define the induced Kripke model $\mathrm{K}_{\mathcal{B}} = (W, R, V)$ from $\mathcal{B}$ as follows: $W := \mathrm{Index}(\mathcal{B})$; iRj iff $i\mathsf{R}j \in \mathcal{B}$; $V(p) := \{i \,|\, \{i\} : \neg p \in \mathcal{B}\}$ for any p occurring in $\mathcal{B}$, otherwise, $V(p) := \emptyset$. It is straightforward to prove by induction on χ that $\alpha : \chi \in \mathcal{B}$ implies $\mathrm{K}_{\mathcal{B}}, \alpha \not\models \chi$. Since $\{1, \ldots, 2^{\mathrm{Rank}_{\varovee}(\varphi)}\} : \varphi \in \mathcal{B}$, it follows that $\mathrm{K}_{\mathcal{B}}, \{1, \ldots, 2^{\mathrm{Rank}_{\varovee}(\varphi)}\} \not\models \varphi$. Thus φ is not valid. ◀

▶ **Theorem 31** (Completeness of $\mathbf{T}_{\mathcal{ML}}$, $\mathbf{T}_{\mathcal{ML}(\varovee)}$, $\mathbf{T}_{\mathcal{MDL}}$, and $\mathbf{T}_{\mathcal{EMDL}}$)**.** *Let $\mathcal{L}$ be a logic in $\{\mathcal{ML}, \mathcal{ML}(\varovee), \mathcal{MDL}, \mathcal{EMDL}\}$. The calculus $\mathbf{T}_{\mathcal{L}}$ is complete.*

Proof. Fix $\mathcal{L} \in \{\mathcal{ML}, \mathcal{ML}(\varovee), \mathcal{MDL}, \mathcal{EMDL}\}$. Assume that $\not\vdash_{T_{\mathcal{L}}} \varphi$. Thus every tableau for φ is open. From Theorem 26 it follows that there exists a saturated open tableau for φ. Thus there exists a saturated open branch for φ. Thus, by Lemma 30, $\not\models_{\mathcal{L}} \varphi$. ◀

5 Conclusion

We gave sound and complete Hilbert-style axiomatizations for $\mathcal{PL}$, $\mathcal{PL}(\varovee)$, $\mathcal{PD}$, $\mathcal{ML}(\varovee)$, $\mathcal{MDL}$, and $\mathcal{EMDL}$. In addition, we presented novel labeled tableau calculi for these logics. We proved soundness, completeness and termination for each of the calculi presented.

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