De Jongh's Theorem for Intuitionistic Zermelo-Fraenkel Set Theory

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───── **Abstract** ─────

We prove that the propositional logic of intuitionistic set theory IZF is intuitionistic propositional logic **IPC**. More generally, we show that IZF has the de Jongh property with respect to every intermediate logic that is complete with respect to a class of finite trees. The same results follow for constructive set theory CZF.

1   Introduction

Originally motivated by philosophical concerns about the meaning of logical symbols such as $\lor$ and $\exists$, *intuitionistic logic* has been increasingly influential in computer science due to its constructive nature: in contexts of implementation, the abstract existence of a solution (roughly corresponding to the classical interpretation of $\exists$) is often less useful than the ability to construct such a solution (roughly corresponding to the intuitionistic interpretation of $\exists$).

As a consequence, we can see that computational systems used as automated theorem provers or proof assistants use constructive logic as their underlying logic (see, e.g., Wiedijk's discussion in [28, 127–129]).

However, you cannot just add mathematical axioms to a constructive logic and expect that the resulting system remains constructive: the most famous example of this is the fact that even in the context of intuitionistic logic, the Axiom of Choice proves the law of excluded middle, thus giving full classical logic [6, 159–160].

It is therefore important to determine which axiomatic frameworks for mathematics preserve which constructive logical systems. More formally, if T is any mathematical theory, we let $\mathbf{L}(T)$ be the propositional logic consisting of all propositional formulas φ such that all substitution instances of φ with sentences of the appropriate language are theorems of T (i.e., $T \vdash \varphi^\sigma$ for all substitutions σ of propositional letters for T-sentences). We are interested in determining for well-known constructive foundational systems T whether $\mathbf{L}(T)$ is intuitionistic propositional logic **IPC** or not. This property is known as *de Jongh's Theorem* for T.

The main result of this paper is that both intuitionistic Zermelo-Fraenkel set theory IZF and constructive Zermelo-Fraenkel set theory CZF satisfy de Jongh's Theorem.

Background: de Jongh's theorem and the de Jongh property.   The questions and basic concepts of this work originated in arithmetic; we give a brief historical overview. Heyting arithmetic HA, the intuitionistic counterpart to Peano arithmetic, is constructed on the basis of intuitionistic logic by adding certain arithmetical axioms and axiom schemes. Having defined the theory in this way, it follows immediately that the propositional validities of HA contain all of intuitionistic propositional logic **IPC**, i.e., $\mathbf{IPC} \subseteq \mathbf{L}(\mathsf{HA})$. De Jongh [7] proved that $\mathbf{L}(\mathsf{HA}) = \mathbf{IPC}$, i.e., that adding the axioms of HA does not entail any logical principle that goes beyond **IPC**.

That results like this are not obvious can be illustrated with an example of an arithmetical theory that does not satisfy de Jongh's theorem: Consider the theory $\mathsf{HA} + \mathsf{MP} + \mathsf{ECT}_0$, i.e., Heyting arithmetic extended with *Markov's Principle* (MP) and *Extended Church's Thesis* (ECT_0). Even though these principles are generally considered constructive, one can show that the propositional logic of this theory contains principles that are not provable in **IPC** but it can also not prove all of classical logic **CPC**, i.e., $\mathbf{IPC} \subsetneq \mathbf{L}(\mathsf{HA} + \mathsf{MP} + \mathsf{ECT}_0) \subsetneq \mathbf{CPC}$ (this follows from results of Rose [22] and McCarty [17]; for details see the discussion at the end of [9, Section 2]). The essence of this example is that – even though we construct the theory $\mathsf{HA} + \mathsf{MP} + \mathsf{ECT}_0$ on the basis of intuitionistic logic – its propositional logic contains principles that are not intuitionistically valid. Hence, the theory $\mathsf{HA} + \mathsf{MP} + \mathsf{ECT}_0$ does not satisfy *de Jongh's theorem.*

These arithmetical examples illustrate that de Jongh-style theorems are important as they guarantee that the logics of constructive systems are not strengthened by the mathematical axioms of the system. An in-depth history of *de Jongh's theorem* can be found in the paper [9] of de Jongh, Verbrugge and Visser.

Related Work.   Starting with de Jongh's classical result [7] that the propositional logic of Heyting Arithmetic HA is intuitionistic logic **IPC**, there has been an intensive examination of this phenomenon in arithmetic. Many authors (see, e.g., [5, 8, 23, 26, 27]) have refined and generalised de Jongh's original work for more logics or stronger arithmetical theories.

The de Jongh property was introduced and analysed for Heyting arithmetic by de Jongh, Verbrugge and Visser [9]: It is an interesting generalisation of de Jongh's theorem. Given an intuitionistic theory T and a propositional logic J, we can obtain a strengthened system $T(J)$ by adding all substitution instances of the rules in J to T. We then say that T has the *de Jongh property* with respect to the intermediate logic J whenever $\mathbf{L}(T(J)) = J$. We can think of the theory $T(J)$ as being constructed on the basis of intuitionistic logic enriched with the propositional principles from J.

In this article, we shall investigate the propositional logics of constructive set theory CZF and intuitionistic set theory IZF. In particular Aczel's constructive set theory CZF [1, 2, 3] has the status of a standard theory for constructive mathematics, also due to its type-theoretic interpretation (see [4]). The metamathematical properties of CZF have also been investigated: Rathjen [21] proved that CZF possesses the disjunction property, the numerical existence property and other common metamathematical properties, however Swan [24] showed that CZF does not have the existence property (the definitions can be found in the respective papers).

The *blended Kripke models* that we construct for the purpose of this article are inspired by the constructions of Iemhoff [10] and Lubarsky [13, 14, 15, 16], and combine Kripke semantics with classical models of set theory.

Bounded constructive set theory BCZF is obtained from CZF by restricting the collection schemes to bounded formulas. The present author used Iemhoff's construction to prove that BCZF has the de Jongh property with respect to every Kripke-complete logic (see [20]).

With the techniques used there, it was (provably) not possible to extend the result to CZF. However, in the present work we will be able to derive a result for CZF as a corollary of the result for IZF.

Moreover, the author [19, Chapter 4] proved that the propositional logic of those of Lubarsky's models that are based on a Kripke frame with leaves contains the intermediate logic **KC** (axiomatised by $\neg\varphi \vee \neg\neg\varphi$). Consequently, the Lubarsky models based on such a frame cannot be used to prove de Jongh properties with respect to logics weaker than **KC**, such as **IPC**.

The blended models have more flexibility than Lubarsky's models and model a stronger set theory than Iemhoff's models, and can therefore be used to prove de Jongh's theorem for IZF and CZF. We will discuss the relation of Lubarsky's models and the blended models at the end of Section 3.1.

Organisation of the Article. The main result of this article are de Jongh theorems for the set theories CZF and IZF. That is, $\mathbf{L}(\mathsf{IZF}) = \mathbf{IPC}$ and $\mathbf{L}(\mathsf{CZF}) = \mathbf{IPC}$. To prove these results, we introduce a new semantics for IZF, the so-called *blended Kripke models*, or *blended models* for short, that allow for controlling the logic of the set-theoretic Kripke model in a very precise way. To prove our results, it will be enough to refute one substitution instance of every propositional formula that is not intuitionistically valid. We will do so by imitating valuations on Kripke frames for propositional logic through set-theoretic sentences in a corresponding blended model.

Using our blended Kripke models, we show that intuitionistic set theory IZF has the de Jongh property with respect to every intermediate logic J that is characterised by a class of finite trees (see Definition 10 and Theorem 31). Examples of such logics are intuitionistic propositional logic **IPC**, Dummett's logic **LC**, the Gabbay-de Jongh logics $\mathbf{T}_n$ and the logics $\mathbf{BD}_n$ of bounded depth n (see Example 7 for the definitions of these logics). As constructive set theory CZF is a subtheory of IZF, all of these results also apply for CZF (see Corollary 35).

Section 2 discusses the preliminaries for our work. We introduce blended Kripke models in Section 3 and prove that they satisfy intuitionistic set theory IZF. In Section 4, we consider the propositional logic of blended Kripke models and prove de Jongh's theorem for IZF. We draw some conclusions and state a few questions for further research in Section 5.

2 Preliminaries

In this section, we will discuss the preliminaries for the later sections. After briefly discussing notation and intermediate logics in Section 2.1 and Section 2.2, respectively, we will introduce Kripke semantics for intuitionistic propositional logic in Section 2.3. We will then discuss the de Jongh property in Section 2.4.

2.1 Notation and Meta-Theory

We adopt the following notational policy: The symbol $\Vdash$ will be used for the forcing relation of Kripke models. As usual, we will use $\vDash$ for the classical modelling relation, and $\vdash$ for the provability relation.

The meta-theory of this article is ZFC + "there is a countable transitive model of ZFC", a theory that is strictly in strength between ZFC + Cons(ZFC) and ZFC + "there is an inaccessible cardinal".

Note that a *countable transitive model of* ZFC is a countable set $M \vDash \mathsf{ZFC}$ (where $\in$ is interpreted as usual set-membership) such that whenever $y \in x \in M$, then $y \in M$. The class of ordinals Ord^M of such a countable transitive model M of ZFC is a countable ordinal in the meta-universe, i.e., $\mathrm{Ord}^M \in \mathrm{Ord}$. We also refer to Ord^M as the *ordinal height* of M.

2.2   Intuitionistic and Intermediate Logics

We fix a countable set Prop of propositional variables for the scope of this article, and identify propositional logics J with the set of formulas they prove (i.e., $J \vdash \varphi$ if and only if $\varphi \in J$). As usual, we denote intuitionistic propositional logic by **IPC**, and classical propositional logic by **CPC**. We say that a logic J is an intermediate logic if $\mathbf{IPC} \subseteq J \subseteq \mathbf{CPC}$ (in particular, **IPC** and **CPC** are considered intermediate logics here). Intuitionistic predicate logic is called **IQC**.

2.3   Kripke Frames and Kripke Models

We will now introduce Kripke frames for intuitionistic logic. In particular, we will focus on Kripke frames that are trees.

▶ **Definition 1.** *A Kripke frame $(K, \leq)$ is a partial order. We call a Kripke frame $(K, \leq)$ a* tree *if for every $v \in K$, the set $K^{\leq v} = \{w \in K \mid w \leq v\}$ is well-ordered by $\leq$, and moreover, if there is a node $r \in K$ such that $r \leq v$ for all $v \in K$ (i.e., K is* rooted*, and r is its* root*). A Kripke frame is called* finite *whenever K is finite.*

All finite trees can be constructed recursively according to the following rules: First, every reflexive partial order with only one point is a finite tree. Second, given finitely many finite trees T_i with roots r_i, the partial order T obtained as the disjoint union of the T_i with an additional element r such that $r \leq x$ for all $x \in T$, is a tree. This recursive definition allows us to prove facts about trees by induction on construction complexity.

▶ **Definition 2.** *Given a Kripke frame $(K, \leq)$, we say that a node e is a* leaf *if e is maximal with respect to $\leq$. We denote the set of leaves of $(K, \leq)$ by E_K. A Kripke frame $(K, \leq)$ with leaves is a Kripke frame such that for every $v \in K$ there is some $e \in E_K$ with $v \leq e$. Given a node $v \in K$, let E_v denote the set of all leaves $e \in K$ such that $v \leq e$.*

The following combinatorial proposition will be useful later when we will determine the propositional logic of certain Kripke models. An *up-set X* in a Kripke frame $(K, \leq)$ is a set $X \subseteq K$ such that $v \in X$ and $v \leq w$ implies $w \in X$. Given a finite tree $(K, \leq)$ and a node $v \in K$, let U_v be the number of up-sets $X \subseteq K^{\geq v}$, where $K^{\geq v} = \{w \in K \mid w \geq v\}$.

▶ **Proposition 3.** *In a finite tree $(K, \leq)$, every node v is uniquely determined by U_v and E_v.*

Proof. This is an easy induction on the construction complexity of finite trees.   ◀

A *valuation* on a Kripke frame $(K, \leq)$ is a function $V : \mathsf{Prop} \to \mathcal{P}(K)$ that is *persistent*, i.e., if $w \in V(p)$ and $w \leq v$, then $v \in V(p)$. A *Kripke model for* **IPC** is a triple $(K, \leq, V)$ such that $(K, \leq)$ is a Kripke frame. We can now define, by induction on propositional formulas, the forcing relation $\Vdash$ for propositional logic at a node $v \in K$ in the following way:

1. $K, V, v \Vdash p$ if and only if $v \in V(p)$,
2. $K, V, v \Vdash \varphi \wedge \psi$ if and only if $K, V, v \Vdash \varphi$ and $K, V, v \Vdash \psi$,
3. $K, V, v \Vdash \varphi \vee \psi$ if and only if $K, V, v \Vdash \varphi$ or $K, V, v \Vdash \psi$,
4. $K, V, v \Vdash \varphi \to \psi$ if and only if for all $w \geq v$, $K, V, w \Vdash \varphi$ implies $K, V, w \Vdash \psi$,
5. $K, V, v \Vdash \bot$ never holds.

Sometimes we will write $v \Vdash \varphi$ instead of $K, V, v \Vdash \varphi$, and $K, V \Vdash \varphi$ if $K, V, v \Vdash \varphi$ holds for all $v \in K$. A formula φ is *valid in K* if $K, V, v \Vdash \varphi$ holds for all valuations V on K and $v \in K$, and φ is *valid* if it is valid in every Kripke frame K.

▶ **Proposition 4** (Persistence). *Let $(K, \leq, V)$ be a Kripke model for* **IPC**, *$v \in K$, and φ be a propositional formula such that $K, v \Vdash \varphi$ holds. Then $K, w \Vdash \varphi$ holds for all $w \geq v$.*

Proof. By induction on formulas. The base case follows from the definition of a valuation, and the other cases follow easily. ◀

We can now define the logic of a Kripke frame and of a class of Kripke frames.

▶ **Definition 5.** *If $(K, \leq)$ is a Kripke frame for* **IPC**, *we define the* propositional logic $\mathbf{L}(K, \leq)$ *to be the set of all propositional formulas that are valid in K. For a class $\mathcal{K}$ of Kripke frames, we define the* propositional logic $\mathbf{L}(\mathcal{K})$ *to be the set of all propositional formulas that are valid in all Kripke frames $(K, \leq)$ in $\mathcal{K}$. Given an intermediate logic J, we say that $\mathcal{K}$ characterises J if $\mathbf{L}(\mathcal{K}) = J$.*

If $\leq$ is clear from the context, we shall write $\mathbf{L}(K)$ for $\mathbf{L}(K, \leq)$. Let us conclude this section with a few examples of intermediate logics and some classes of Kripke frames that characterise them. For proofs we refer to the literature. The following important proposition is well-known.

▶ **Proposition 6** (e.g., [25, Theorem 6.12]). *Intuitionistic propositional logic* **IPC** *is characterised by the class of all finite trees.*

▶ **Example 7.** We present some examples of logics from the paper of de Jongh, Verbrugge and Visser [9] that are characterised by classes of finite trees.

Dummett's logic The logic **LC** is obtained by extending **IPC** with the axiom

$$(p \rightarrow q) \vee (q \rightarrow p).$$

The logic **LC** is characterised by the class of finite linear orders.

Gabbay-de Jongh Logics The logics $\mathbf{T}_n$, for $n \in \mathbb{N}$, are characterised by the class of finite trees which have splittings of exactly n, i.e., every node is either a leaf or has exactly n successors. $\mathbf{T}_1$ coincides with **LC**, and the logics $\mathbf{T}_n$ are axiomatised by the following formulas:

$$\bigwedge_{k \leq n+1} \left(\left(\varphi_k \rightarrow \bigwedge_{j \neq k} \varphi_j \right) \rightarrow \bigwedge_{j \neq k} \varphi_j \right) \rightarrow \bigwedge_{k \leq n+1} \varphi_k.$$

Logics of Bounded Depth n The logics $\mathbf{BD}_n$, for $n \in \mathbb{N}$, are characterised by the finite trees of depth n. The logic of depth 1, $\mathbf{BD}_1$ is classical logic **CPC** axiomatised by Peirce's law,

$$\beta_1 = ((\varphi_1 \rightarrow \psi) \rightarrow \varphi_1) \rightarrow \varphi_1.$$

For every $n \in \mathbb{N}$, the logic $\mathbf{BD}_n$ is axiomatised by β_n as obtained recursively via:

$$\beta_{n+1} = ((\varphi_{n+1} \rightarrow \beta_n) \rightarrow \varphi_{n+1}) \rightarrow \varphi_{n+1}.$$

2.4 The de Jongh Property

Let φ be a propositional formula and let $\sigma : \mathsf{Prop} \rightarrow \mathcal{L}^{\mathsf{sent}}$ an assignment of propositional variables to sentences in a language $\mathcal{L}$. By φ^σ we denote the $\mathcal{L}$-sentence obtained from φ by replacing each propositional variable p with the sentence $\sigma(p)$.

▶ **Theorem 8** (de Jongh, [7])**.** *Let φ be a formula of propositional logic. Then* $\mathsf{HA} \vdash \varphi^\sigma$ *for all* $\sigma : \mathsf{Prop} \to \mathcal{L}_{\mathsf{HA}}^{\mathsf{sent}}$ *if and only if* $\mathbf{IPC} \vdash \varphi$.

Given a theory based on intuitionistic logic, we may consider its propositional logic, i.e., the set of propositional formulas that are derivable after substituting the propositional letters by arbitrary sentences in the language of the theory.

▶ **Definition 9.** *Let T be a theory in intuitionistic predicate logic, formulated in a language $\mathcal{L}$. A propositional formula φ will be called T-valid if and only if $T \vdash \varphi^\sigma$ for all $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$. The propositional logic $\mathbf{L}(T)$ is the set of all T-valid formulas.*

Given a theory T and an intermediate logic J, we denote by $T(J)$ the theory obtained by closing T under J.

▶ **Definition 10.** *We say that a theory T has the de Jongh property if $\mathbf{L}(T) = \mathbf{IPC}$. The theory T has the de Jongh property with respect to an intermediate logic J if $\mathbf{L}(T(J)) = J$.*

De Jongh's theorem is equivalent to the assertion that Heyting arithmetic has the de Jongh property. As explained in the introduction, the theory $\mathsf{HA} + \mathsf{MP} + \mathsf{ECT}_0$ does not have the de Jongh property.

3   Blended Models

This section introduces the new model construction for intuitionistic set theory IZF: the blended models. We will now construct blended Kripke models in Section 3.1, observe some of their basic properties in Section 3.2 and show that they satisfy intuitionistic Zermelo-Fraenkel set theory IZF in Section 3.3. Finally, Section 3.4 contains a simple example of a blended model.

3.1   Constructing Blended Models

For the sake of this construction, we fix a Kripke frame $(K, \leq)$ with leaves. Transitive models of ZFC have an ordinal height Ω; in our construction all models assigned will have the same ordinal height. To each leaf $e \in K$, we assign a transitive model $M_e \vDash \mathsf{ZFC}$ of height Ω. Note that Ω denotes the same ordinal in the meta-universe for all $e \in E_K$; we can therefore refer to this ordinal by Ω without specifying a particular $e \in E_K$.

Before giving the technical details of the construction, let us spark the readers intuition. We need to define a collection $\mathcal{D}_v$ of v-sets at every node $v \in K$ of the Kripke model. A v-set x will be a function that assigns to every node $w \geq v$ a collection of previously defined w-sets; $x(w)$ is the extension of x at the node w. Note that these assignments shall not be random but must happen in a coherent way: at every leaf e, the extension $x(e)$ must be a set of the transitive model M_e associated to the leaf e. Moreover, the extensions of x should be monotone along the $\leq$-relation of the Kripke frame to account for the persistence required in Kripke models for intuitionistic theories – once a member of x, always a member of x. More formally, we shall require for any $y \in x(v)$ that $y \restriction K^{\geq w} \in x(w)$. The truncation of y to $y \restriction K^{\geq w}$ is necessary to obtain the w-set $y \restriction K^{\geq w}$ from the v-set y.

The formal construction of blended models is conducted in three steps. We begin by constructing the collection of domains $\langle \mathcal{D}_v \,|\, v \in K \rangle$: first the domains for the leaves and, secondly, for all remaining nodes of the Kripke frame. The third step is to define the semantics.

Step 1. Domains for leaves. Let $e \in E_K$ be a leaf, and M_e be the transitive model associated to it. Instead of directly assigning the transitive model M_e as the domain at the node e, we will transform this model into a domain $\mathcal{D}_e$ of functions that is isomorphic to the original model. We define a function $f_e : M_e \to \mathrm{ran}(f)$ by $\in$-recursion via $f_e(x) = (e, f_e[x])$.

Then define $\mathcal{D}_e = f_e[M_e]$. Hence, each $\mathcal{D}_e$ is a set of functions $x : K^{\geq e} \to \mathrm{ran}(x)$ (where $K^{\geq e} = \{e\}$). Moreover, for $\alpha \in \mathrm{Ord}^M$, let $\mathcal{D}_e^\alpha = f_e[(V_\alpha)^{M_e}]$. Then $\mathcal{D}_e^0 = \emptyset$ and it holds that

$$\bigcup_{\alpha \in \mathrm{Ord}^M} \mathcal{D}_e^\alpha = \mathcal{D}_e.$$

In Proposition 14 below, we will see that the domains of the leaves of a blended model are isomorphic to the classical model of set theory associated to the node (with respect to the equality and membership relations).

Step 2. Domains for all nodes. Now we are ready to define the domains at the remaining nodes. We do this simultaneously for all $v \in K \setminus E_K$ by induction on $\alpha \in \Omega$. Let $\mathcal{D}_v^\alpha$ consist of the functions $x : K^{\geq v} \to \mathrm{ran}(x)$ such that the following properties hold:

(i) for all leaves $e \geq v$, we have $x \upharpoonright \{e\} \in \mathcal{D}_e^\alpha$,
(ii) for all non-leaves $w \geq v$, we have $x(w) \subseteq \bigcup_{\beta < \alpha} \mathcal{D}_w^\beta$, and
(iii) for all nodes $u \geq w \geq v$ we have that $\{y \upharpoonright K^{\geq u} \mid y \in x(w)\} \subseteq x(u)$.

We define the domain $\mathcal{D}_v$ at the node v to be the set

$$\mathcal{D}_v = \bigcup_{\alpha \in \mathrm{Ord}^M} \mathcal{D}_v^\alpha.$$

For completing the definition of the domains of our model, we still require *transition functions* $f_{vw} : \mathcal{D}_v \to \mathcal{D}_w$ such that $f_{wu} \circ f_{vw} = f_{vu}$. The transition functions explain how the elements at a node v should be interpreted at a later node $w \geq v$ (see also step 3). For this purpose, we use the restriction maps f_{vw} with $x \mapsto x \upharpoonright K^{\geq w}$ as transition functions. Note that these maps are well-defined by the definition of the domains.

Moreover, by the definition of the maps f_{vw}, it is clear that condition (i) is just a special case of condition (iii). We state it separately as it requires special attention when working with blended models.

Step 3. Defining the semantics. We define, by induction on $\mathcal{L}_\in$-formulas, the forcing relation at every node of the Kripke model in the following way, where φ and ψ are formulas with all free variables shown, and, moreover, $\bar{y} = y_0, \ldots, y_{n-1}$ are elements of $\mathcal{D}_v$ for the node v considered on the left side:

1. $(K, \leq, \mathcal{D}), v \Vdash \bot$ never holds,
2. $(K, \leq, \mathcal{D}), v \Vdash \varphi(\bar{y}) \wedge \psi(\bar{y})$ if and only if $(K, \leq, \mathcal{D}), v \Vdash \varphi(\bar{y})$ and $(K, \leq, \mathcal{D}), v \Vdash \psi(\bar{y})$,
3. $(K, \leq, \mathcal{D}), v \Vdash \varphi(\bar{y}) \vee \psi(\bar{y})$ if and only if $(K, \leq, \mathcal{D}), v \Vdash \varphi(\bar{y})$ or $(K, \leq, \mathcal{D}), v \Vdash \psi(\bar{y})$,
4. $(K, \leq, \mathcal{D}), v \Vdash \varphi(\bar{y}) \to \psi(\bar{y})$ if and only if for all $w \geq v$, $(K, \leq, \mathcal{D}), w \Vdash \varphi(f_{vw}(\bar{y}))$ implies $(K, \leq, \mathcal{D}), w \Vdash \psi(f_{vw}(\bar{y}))$,
5. $(K, \leq, \mathcal{D}), v \Vdash x \in y$ if and only if $x \in y(v)$,
6. $(K, \leq, \mathcal{D}), v \Vdash a = b$ if and only if $a = b$,
7. $(K, \leq, \mathcal{D}), v \Vdash \exists x\, \varphi(x, \bar{y})$ if and only if there is some $a \in D_v$ with $(K, \leq, \mathcal{D}), v \Vdash \varphi(a, \bar{y})$,
8. $(K, \leq, \mathcal{D}), v \Vdash \forall x\, \varphi(x, \bar{y})$ if and only if for all $w \geq v$ and $a \in D_w$, $w \Vdash \varphi(a, f_{vw}(\bar{y}))$.

The negation $\neg\varphi$ is an abbreviation for $\varphi \to \bot$.

▶ **Definition 11.** *We call $(K, \leq, \mathcal{D})$ the* blended Kripke model *obtained from $\langle M_e \mid e \in E_K \rangle$.*

This finishes the definition of the blended models. If the collection $\langle M_e \mid e \in E_K \rangle$ is either clear from the context, or if it does not matter, we will also say that $(K, \leq, \mathcal{D})$ is a *blended Kripke model*. We will usually say *blended model* instead of blended Kripke model. Moreover, we might refer to an element $x \in \mathcal{D}_v$ as a *v-set*, and to $x(w)$ as the *extension of x at w*.

An $\mathcal{L}_\in$-formula φ is *valid in* $(K, \leq, \mathcal{D})$ if $v \Vdash \varphi$ holds for all $v \in K$, and φ is *valid* if it is valid in every Kripke frame K. We will call $(K, \leq)$ the *underlying Kripke frame* of $(K, \leq, \mathcal{D})$, or the frame that $(K, \leq, \mathcal{D})$ is based on. Moreover, let us call $[\![\varphi]\!]^{(K, \leq, D)} = \{v \in K \mid v \Vdash \varphi\}$ the *truth set* of a sentence φ in the language of set theory in a blended model $(K, \leq, D)$. When the model is clear from the context, we will also write $[\![\varphi]\!]^K$ or just $[\![\varphi]\!]$.

Before we continue with some basic properties of the blended models, let us briefly discuss this construction in comparison to Lubarsky's Kripke models [13, 14, 15, 16], which are constructed in a similar way. The crucial difference, however, is that our models are constructed in a top-down manner that allows to choose any (finite) collection of classical models of set theory of the same ordinal height at the leaves, whereas Lubarsky's bottom-up construction requires elementary equivalence of the models involved.

3.2 Basic Properties

We will now observe some basic properties of the blended models.

▶ **Proposition 12** (Persistence). *Let $(K, \leq, \mathcal{D})$ be a blended model and φ a formula in the language of set theory. If $v \Vdash \varphi(a_0, \ldots, a_{n-1})$ and $w \geq v$, then $w \Vdash \varphi(f_{vw}(a_0), \ldots, f_{vw}(a_{n-1}))$.*

Proof. This is proved by induction on $\mathcal{L}_\in$-formulas. ◀

▶ **Proposition 13.** *The blended models are sound with respect to* **IQC**.

Proof. This follows from the more general soundness result for Kripke models for predicate logics with respect to **IQC**. See, for example, [25, Theorem 6.6]. ◀

We will now make the essential observation that the domains at the leaves are isomorphic to the models they were obtained from.

▶ **Proposition 14.** *Let $(K, \leq, \mathcal{D})$ be a blended model, and $e \in E_K$ a leaf. Then $(K, \leq, \mathcal{D}), e \Vdash \varphi(f_e(a_0), \ldots, f_e(a_{n-1}))$ if and only if $M_e \vDash \varphi(a_0, \ldots, a_{n-1})$ for all elements $a_0, \ldots, a_{n-1} \in M_e$.*

Proof. Let us first argue that the function $f_e : M_e \to \mathcal{D}_e$ as introduced in Step 1 is a bijection. Define g by $\in$-recursion with $(e, x) \mapsto g[x]$. It follows by induction that $g \circ f_e = \mathrm{id}_{M_e}$ and $f_e \circ g = \mathrm{id}_{\mathcal{D}_e}$. Hence, f_e is a bijection.

It suffices to prove the claim for the atomic cases: equality and set-membership. The case for equality follows from the definition of the semantics and the fact that f is bijective. For set-membership observe that if $M_e \vDash x \in y$, then $f_e(x) \in f_e(y)(e)$ and hence $e \Vdash f_e(x) \in f_e(y)$. Conversely, if $e \Vdash f_e(x) \in f_e(y)$, then $f_e(x) \in f_e(y)(e)$ and hence $x = g(f_e(x)) \in g(f_e(y)) = y$. The other cases follow trivially as the intuitionistic interpretation of the logical symbols in a leaf coincides with the classical interpretation in the model M_e. ◀

Extensionality $\forall a \, \forall b \, (\forall x \, (x \in a \leftrightarrow x \in b) \rightarrow a = b)$

Empty set $\exists a \, \forall x \in a \, \bot$

Pairing $\forall a \, \forall b \, \exists y \, \forall x \, (x \in y \leftrightarrow (x = a \lor x = b))$

Union $\forall a \, \exists y \, \forall x \, (x \in y \leftrightarrow \exists u(u \in a \land x \in u))$

Power set $\forall a \, \exists y \, \forall x \, (x \in y \leftrightarrow x \subseteq a)$

Infinity $\exists a \, (\exists x \, x \in a \land \forall x \in a \, \exists y \in a \ x \in y)$

Set Induction $\forall a \, (\forall x \in a \, \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \, \varphi(a)$, for all formulas $\varphi(x)$.

Separation $\forall a \, \exists y \, \forall x \, (x \in y \leftrightarrow (x \in a \land \varphi(x)))$, for all formulas $\varphi(x)$.

Collection $\forall a \, (\forall x \in a \, \exists y \, \varphi(x, y) \rightarrow \exists b \, \forall x \in a \, \exists y \in b \, \varphi(x, y))$, for all formulas $\varphi(x, y)$, where b is not free in $\varphi(x, y)$.

■ **Figure 1** The axioms of IZF. Note that the formulas $\varphi(x)$ appearing in the axiom schemes are allowed to have parameters.

3.3 Intuitionistic Set Theory Holds in Blended Models

In this section, we will show that the axioms of IZF (see Figure 1) hold in blended models. For the sake of this section, let $(K, \leq, \mathcal{D})$ be a blended model obtained from $\langle M_e \,|\, e \in E_K \rangle$.

Intuitionistic set theory IZF is classically equivalent to ZF set theory. With Proposition 14 we note that IZF holds true at every leaf because the models associated with the leaves are models of ZF set theory and classical logic holds in the leaves.

▷ **Claim 15.** The model $(K, \leq, \mathcal{D})$ satisfies the axiom of extensionality.

Proof. Let $v \in K$ and $a, b \in \mathcal{D}_v$. We have to show that

$$v \Vdash \forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b.$$

So assume that $w \Vdash \forall x(x \in a \leftrightarrow x \in b)$ for all $w \geq v$, i.e., $a(w) = b(w)$ for all $w \geq v$. Hence, a and b are equal as functions with domain $K^{\geq v}$, and so they are equal. ◁

▷ **Claim 16.** The model $(K, \leq, \mathcal{D})$ satisfies the axiom of pairing.

Proof. Let $v \in K$ and $a, b \in \mathcal{D}_v$. Let c be the function with $c(w) = \{f_{vw}(a), f_{vw}(b)\}$ for all $w \geq v$.

Let us first show that $c \in \mathcal{D}_v$. For condition (i), let $e \geq v$ be a leaf. As $a, b \in \mathcal{D}_v$ it follows from the definition that $f_{ve}(a), f_{ve}(b) \in \mathcal{D}_e$. Hence, by pairing in M_e, we have that $c \restriction \{e\} \in \mathcal{D}_e$, where $c(e) = \{f_{ve}(a), f_{ve}(b)\}$. Conditions (ii) and (iii) of the definition of $\mathcal{D}_v$ follow directly from the definition of c.

Now it is straightforward to check that c constitutes a witness for the axiom of pairing for a and b at the node v. ◁

▷ **Claim 17.** The model $(K, \leq, \mathcal{D})$ satisfies the axiom of union.

Proof. Let $v \in K$ and $a \in \mathcal{D}_v$. Define a function b with domain $K^{\geq v}$ with $b(w) = \bigcup_{c \in a(w)} c(w)$ for all $w \geq v$.

Again, we need to show that $b \in \mathcal{D}_v$. For condition (i), observe that $f_{ve}(a) \in \mathcal{D}_e$ for every leaf $e \geq v$. As the axiom of union holds in M_e, it follows that there is a witness $b' \in \mathcal{D}_e$. By transitivity of M_e, it must then hold that $b \restriction \{e\} = b' \in \mathcal{D}_e$. As in the previous proposition, conditions (ii) and (iii) follow directly from the definition of b. Then b witnesses the axiom of union for a. ◁

▷ **Claim 18.** The model $(K, \leq, \mathcal{D})$ satisfies the axiom of empty set.

Proof. For every $v \in K$ consider the function 0_v with domain $K^{\geq v}$ such that $0_v(w) = \emptyset$ for all $w \geq v$. This is an element of $\mathcal{D}_v$ and witnesses the axiom of empty set. ◁

▷ **Claim 19.** The model $(K, \leq, \mathcal{D})$ satisfies the axiom of infinity.

Proof. By recursion on natural numbers, we will define elements $n_v \in \mathcal{D}_v$ simultaneously for every $v \in K$. Let 0_v be the empty set as defined in the proof of Claim 18. Then, if m_v has been defined for all $m < n$, let n_v be the function with $n_v(w) = \{0_w, \ldots, (n-1)_w\}$ for all $w \geq v$. This finishes the recursive definition. It follows inductively that every $n_v \in \mathcal{D}_v$, again paying special attention at the leaves: the sets n_e correspond to the finite ordinal $n \in M_e$.

Finally, let $\omega_v(w) = \{n_w \mid n < \omega\}$ for all $w \geq v$. To see that $\omega_v \in \mathcal{D}_v$ note that, for every leaf $e \geq v$, $f_{ve}(\omega_v) = \omega_e \in \mathcal{D}_e$ as M_e satisfies the axiom of infinity.

It follows that ω_v is a witness for the axiom of infinity at the node v. ◁

▷ **Claim 20.** The model $(K, \leq, \mathcal{D})$ satisfies the axiom scheme of separation.

Proof. Let $\varphi(x, y_0, \ldots, y_n)$ be a formula with all free variables shown. Let $v \in K$, $a \in \mathcal{D}_v$ and $b_0, \ldots, b_n \in \mathcal{D}_v$. Define c to be the function with domain $K^{\geq v}$ such that

$$c(w) = \{d \in a(w) \mid w \Vdash \varphi(d, b_0, \ldots, b_n)\}$$

holds for all $w \geq v$. We have that $c \in \mathcal{D}_v$ by the definition of the domains $\mathcal{D}_v$. Again, property (i) follows from the fact that separation holds in M_e for every leaf model M_e. Moreover, property (iii) follows by persistence. Finally, c witnesses separation from a by φ with parameters b_i. ◁

▷ **Claim 21.** If K is finite, then the model $(K, \leq, \mathcal{D})$ satisfies the axiom scheme of collection.

Proof. Let $v \in K$, $\varphi(x, y)$ be a formula (possibly with parameters), and $a \in \mathcal{D}_v$. We need to show that:

$$v \Vdash \forall x \in a \exists y\, \varphi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b\, \varphi(x, y).$$

Without loss of generality, assume that $v \Vdash \forall x \in a \exists y \varphi(x, y)$. In particular, by persistence, for every $w \geq v$ and every $x \in a(w)$ there exists some $y \in \mathcal{D}_w$ such that $w \Vdash \varphi(x, y)$. Let α be the minimal ordinal such that for every $w \geq v$ and $x \in a(w)$, there is some $y \in \mathcal{D}_w^\alpha$ with $w \Vdash \varphi(x, y)$. Note that $\alpha < \Omega$ as K is finite. Define b to be the function with domain $K^{\geq v}$ such that $b(w) = \mathcal{D}_w^\alpha$. It follows that $b \in \mathcal{D}_v$, where the case for leaves e follows from the fact that $(V_\alpha)^{M_e}$ is a set in M_e. Hence, b is a witness for the above instance of the collection scheme. ◁

▷ **Claim 22.** The model $(K, \leq, \mathcal{D})$ satisfies the powerset axiom.

Proof. Let $v \in K$ and $a \in \mathcal{D}_v$. Define a function b with domain $K^{\geq v}$ such that

$$b(w) = \{c \in \mathcal{D}_w \mid w \Vdash c \subseteq f_{vw}(a)\}$$

for all $w \geq v$. We have to show that $b \in \mathcal{D}_v$. Observe that for every leaf $e \geq v$, $f_{ve}(b)$ corresponds to $(\mathcal{P}(a))^{M_e}$, and hence condition (i) is satisfied. Conditions (ii) and (iii) follow easily. ◁

▷ **Claim 23.** The model $(K, \leq, \mathcal{D})$ satisfies the axiom scheme of set induction.

Proof. We shall show that the set-induction scheme holds for all $v \in K$, i.e., that

$$v \Vdash \forall a \, (\forall x \in a \, \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \, \varphi(a),$$

holds for all formulas $\varphi(x)$ and $v \in K$. So assume that $v \Vdash \forall a \, (\forall x \in a \, \varphi(x) \rightarrow \varphi(a))$. We have to show that $v \Vdash \forall a \, \varphi(a)$, i.e., for all $a \in \mathcal{D}_v$ we have that $v \Vdash \varphi(a)$. To do so, we will proceed by a simultaneous induction for all $v \in K$ on the rank of $a \in \mathcal{D}_v$, i.e., the minimal $\alpha < \Omega$ such that $a \in \mathcal{D}_v^{\alpha+1} \setminus \mathcal{D}_v^\alpha$.

The only v-set of rank 0 is the function x that assigns the empty set to every node $w \geq v$, so the assumption of set-induction applies and we have $v \Vdash \varphi(x)$. For the induction step, observe that the members of a v-set x of rank α are w-sets (for some $w \geq v$) of lower rank. Hence, the induction hypothesis applies and it follows by using the assumption of set-induction that $v \Vdash \varphi(x)$. This finishes the induction, and the proof of the claim. ◁

Let us summarise the results of this section in the following theorem.

▶ **Theorem 24.** *If K is finite, then the model $(K, \leq, \mathcal{D})$ satisfies* IZF. *For arbitrary K, the model $(K, \leq, \mathcal{D})$ satisfies* IZF − Collection.

We do not know whether there is an example of an infinite Kripke frame K and a model $(K, \leq, \mathcal{D})$ based on K that does not satisfy the collection scheme.

3.4 An Example

To illustrate our construction above, we will construct a Kripke model $(K, \leq, \mathcal{D})$ such that $(K, \leq, \mathcal{D}) \nVdash \mathsf{CH} \vee \neg\mathsf{CH}$, where CH is the continuum hypothesis. Take $(K, \leq)$ to be the three element Kripke frame $(K, \leq)$ with $K = \{v, e_0, e_1\}$ with $\leq$ being the reflexive closure of the relation defined by $v \leq e_0$ and $v \leq e_1$.

Now, let M be any countable transitive model of $\mathsf{ZFC} + \mathsf{CH}$, and take G to be generic for Cohen forcing over M. Then we associate the model M with node e_0, and $M[G]$ with e_1, i.e., $M_{e_0} = M$ and $M_{e_1} = M[G]$. By our construction above and Proposition 14, we know that $(K, \leq, \mathcal{D}), e_0 \Vdash \mathsf{CH}$ and $(K, \leq, \mathcal{D}), e_1 \Vdash \neg\mathsf{CH}$. Hence, persistence implies that $(K, \leq, \mathcal{D}), v \nVdash \mathsf{CH} \vee \neg\mathsf{CH}$.

In particular, observe that $[\![\mathsf{CH}]\!] = \{e_0\}$, $[\![\neg\mathsf{CH}]\!] = \{e_1\}$, $[\![\mathsf{CH} \vee \neg\mathsf{CH}]\!] = \{e_0, e_1\}$ and $[\![\top]\!] = K$. Hence, every up-set and therefore any valuation on K can be imitated with sentences in the language of set-theory evaluated in the blended model.

Moreover, this example also shows that $\mathsf{IZF} \nvdash \mathsf{CH} \vee \neg\mathsf{CH}$, i.e., the law of excluded middle does not hold for assertions regarding the continuum. One can easily generalise the above argument to obtain the following proposition.

▶ **Proposition 25.** *If φ is a sentence in the language of set theory such that there are models M and N of* ZFC *with the same ordinals such that $M \vDash \varphi$ and $N \vDash \neg\varphi$, then* $\mathsf{IZF} \nvdash \varphi \vee \neg\varphi$.

Of course, this result also follows from the fact that IZF is a subtheory of ZFC having the disjunction property (see [18, Corollary 1]).

4 The Propositional Logic of Blended Models

In this section, we will analyse the propositional logic of blended models and prove the de Jongh property for IZF with respect to intermediate logics that are characterised by a class of finite trees.

4.1   Faithful Blended Models

The aim of this section is to show that we can find a blended model based on every finite tree Kripke frame $(K, \leq)$ that allows us to imitate every valuation on $(K, \leq)$. Let us begin with a definition and several useful observations.

▶ **Definition 26.** *A blended model* $(K, \leq, \mathcal{D})$ *is called* faithful *if for every valuation V on the Kripke frame $(K, \leq)$ and every propositional letter $p \in$ Prop, there is an $\mathcal{L}_\in$-formula φ_p such that* $[\![\varphi_p]\!]^{(K, \leq, \mathcal{D})} = V(p)$.

This notion was first introduced in [19]. For further discussion and connections to the de Jongh property, see also [20].

Given a natural number n, let Γ_n be the following sentence[1] in the language of set theory:

$$\forall x_0, \dots, x_{n-1} \left(\bigwedge_{i < n} (\forall y \in x_i \forall z \in y \perp) \to \bigvee_{i < j < n} x_i = x_j \right).$$

Informally, this sentence asserts that given n subsets of $1 = \{\emptyset\}$, at least 2 of them are equal. The power set of 1 is crucial for distinguishing models of non-classical set theories; it is consistent with CZF that the power set of 1 is a proper class (see [13]). Note that Γ_1 is inconsistent and Γ_2 is a theorem of ZF set theory. If Γ_2 is not a theorem, then classical logic does not hold.

Recall that we defined U_v in Section 2.3 to be the number of up-sets $X \subseteq K^{\geq v}$. The following proposition holds for all Kripke frames with leaves and not only for finite trees. We also do not need to assume that U_v is finite.

▶ **Proposition 27.** *Let $(K, \leq)$ be a Kripke frame with leaves, $(K, \leq, \mathcal{D})$ be a blended model, and $v \in K$. For every natural number n, we have that $v \Vdash \Gamma_{n+1}$ if and only if $n \geq U_v$.*

Proof. Given any up-set $X \subseteq K^{\geq v}$, we define the element 1_X^v to be the function

$$K^{\geq v} \to \bigcup_{w \geq v} \mathcal{D}_w, \quad w \mapsto \begin{cases} \{0_w\}, & \text{if } w \in X, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Observe that $1_X^v \in \mathcal{D}_v$ as it is monotone because X is an up-set. Further, we have $1_X^v \neq 1_Y^v$ for up-sets $X \neq Y$ and therefore, $v \not\Vdash 1_X^v = 1_Y^v$. It follows that $v \Vdash \forall y \in 1_X^v \forall z \in y \perp$ for all up-sets X because $1_X^v(w)$ is either empty or contains the empty set for $w \geq v$. We conclude that $v \not\Vdash \Gamma_{n+1}$ for $n < U_v$ taking the 1_X^v as witnesses.

Conversely, assume that $n \geq U_v$. We will first show that whenever $v \Vdash \forall y \in x \forall z \in y \perp$ for some $x \in \mathcal{D}_v$, then x is actually of the form 1_X^v for some up-set $X \subseteq K^{\geq v}$. For contradiction, assume that x was not of the form 1_X^v for some up-set X. Then there is a node $w \geq v$ such that $x(w)$ contains an element y different from 0_w. But then there must be a node $u \geq w$ such that $y(w)$ is non-empty. This is a contradiction to $v \Vdash \forall y \in x \forall z \in y \perp$, and hence, every element $x \in \mathcal{D}_v$ satisfying the above formula must be of the form 1_X^v. As there are only U_v-many elements 1_X^v, we know that the conclusion of Γ_{n+1} must be true at the node v. Hence, $v \Vdash \Gamma_{n+1}$. ◀

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[1] The sentences Γ_n were also used in yet unpublished joint work with Lorenzo Galeotti and Benedikt Löwe on the logics of algebra-valued models of set theory; see also the discussion after Theorem 13 of [12]. We adapt them here for the case of Kripke semantics.

The following proposition is a special case of a more general proposition for Kripke models of predicate logic.

▶ **Proposition 28.** *Let* $(K, \leq)$ *be a Kripke frame with leaves,* $(K, \leq, \mathcal{D})$ *be a blended model and* $v \in K$. *If* $e \nVdash \varphi$ *for all leaves* $e \geq v$, *then* $v \Vdash \neg\varphi$.

Proof. By the definition of our semantics, we know that $v \Vdash \neg\varphi$ if and only if $w \nVdash \varphi$ for all $w \geq v$. Assume that there was a node $w \geq v$ such that $w \Vdash \varphi$. By persistence we can conclude that $e \Vdash \varphi$ for every leaf $e \geq w$. Hence, $w \nVdash \varphi$ for all $w \geq v$, so $v \Vdash \neg\varphi$. ◀

▶ **Theorem 29.** *Let* $(K, \leq, \mathcal{D})$ *be a blended model based on a finite tree* $(K, \leq)$ *with leaves* $e_0, \ldots, e_{n-1}$. *If there is a collection of* $\in$-*sentences* φ_i *for* $i < n$ *such that* $e_j \Vdash \varphi_i$ *if and only if* $i = j$, *then* $(K, \leq, \mathcal{D})$ *is faithful.*

Proof. Let $(K, \leq, \mathcal{D})$ be a blended model based on a finite tree $(K, \leq)$ with leaves $e_0, \ldots, e_{n-1}$ such that there is a collection of $\in$-sentences φ_i for $i < n$ such that $e_j \Vdash \varphi_i$ if and only if $i = j$.

As $(K, \leq)$ is a finite tree, we know by Proposition 3 that every node $v \in K$ is uniquely determined by U_v and the set of leaves $e \geq v$.

Let V be a valuation on $(K, \leq)$. For every $p \in \mathsf{Prop}$, we need to find a sentence ρ_p in the language of set theory such that $[\![\rho_p]\!]^{(K, \leq, \mathcal{D})} = V(p)$. Due to the finiteness of K, it suffices to consider up-sets of the form $K^{\geq v}$ for some $v \in K$ because general up-sets can be constructed by finitely many disjunctions.

We will now prove for every $v \in K$ that there is a sentence χ_v in the language of set theory such that $(K, \leq, \mathcal{D}), w \Vdash \chi_v$ if and only if $w \geq v$ (i.e., $w \in K^{\geq v}$). Let χ_v be the following sentence, where $n = U_v + 1$:

$$\Gamma_n \wedge \bigwedge_{v \nleq e_i} \neg\varphi_i$$

By Proposition 27 and Proposition 28 it is clear that $w \Vdash \chi_v$ for all $w \geq v$. For the converse direction, let $w \in K$ such that $w \ngeq v$. There are two cases.

First, if $w < v$, then $U_w > U_v = n$ and hence $w \nVdash \Gamma_n$ by Proposition 27. Hence, it follows that $w \nVdash \chi_v$.

Second, if $w \nless v$, then there must be a leaf $e_i \geq w$ such that $e_i \ngeq v$. By assumption $e_i \Vdash \varphi_i$ and hence, $w \nVdash \neg\varphi_i$. But this means that $w \nVdash \chi_v$.

This concludes the proof of the theorem. ◀

▶ **Theorem 30.** *Let* $(K, \leq)$ *be a finite tree. Then there is a faithful blended model* $(K, \leq, \mathcal{D})$ *based on* $(K, \leq)$.

Proof. Let $e_0, \ldots, e_{n-1}$ be the set of leaves of $(K, \leq)$. Let M be a countable transitive model of ZFC set theory. By set-theoretic forcing, we can obtain generic extensions $M[G_i]$ of M such that $M[G_i] \vDash 2^{\aleph_0} = \aleph_{i+1}$ for every $i < n$ (see, e.g., [11, Theorem 6.17] for details). Let $M_{e_i} = M[G_i]$, and $(K, \leq, \mathcal{D})$ be the blended model obtained from $\langle M_i \, | \, i < n \rangle$. Clearly, $M_{e_i} \vDash 2^{\aleph_0} = \aleph_{j+1}$ if and only if $i = j$. This implies, by Proposition 14, that $e_i \Vdash 2^{\aleph_0} = \aleph_{j+1}$ if and only if $i = j$. In this situation, we can apply Theorem 29 to conclude that $(K, \leq, \mathcal{D})$ is faithful. ◀

4.2   The de Jongh Property for IZF and CZF

In this section, we will draw conclusions regarding the de Jongh property for IZF and CZF from the main result of the previous section.

▶ **Theorem 31.** *Intuitionistic set theory* IZF *has the de Jongh property with respect to every intermediate logic J that is characterised by a class of finite trees.*

Proof. Let J be an intermediate logic with $\mathbf{L}(\mathcal{K}) = J$, where $\mathcal{K}$ is a class of finite trees. We have to show that $\mathbf{L}(\mathsf{IZF}(J)) = J$, i.e., for every propositional formula, we have that:

$J \vdash \varphi$ if and only if $\mathsf{IZF}(J) \vdash \varphi^{\sigma}$ for all substitutions $\sigma : \mathsf{Prop} \to \mathcal{L}_{\in}^{\mathsf{sent}}$.

The direction from left to right is immediate from the definition of $\mathsf{IZF}(J)$. We will prove the converse direction by contraposition.

Assume that there is φ such that $J \nvdash \varphi$. As J is characterised by $\mathcal{K}$, there is a frame $(K, \leq) \in \mathcal{K}$ and a valuation V such that $(K, \leq), V \nVdash \varphi$. By Theorem 30 and the assumption that $\mathcal{K}$ consists of finite trees, we can find a faithful blended model $(K, \leq, \mathcal{D})$ based on $(K, \leq)$. For every propositional letter $p \in \mathsf{Prop}$, let ψ_p be a sentence in the language of set theory such that $\llbracket \psi_p \rrbracket^{(K, \leq, \mathcal{D})} = V(p)$. Define an assignment $\sigma : \mathsf{Prop} \to \mathcal{L}_{\in}^{\mathsf{sent}}$ by $\sigma(p) = \psi_p$.

We prove by induction on propositional formulas χ, simultaneously for all $v \in K$ that:

$(K, \leq), v \Vdash \chi$ if and only if $(K, \leq, \mathcal{D}), v \Vdash \chi^{\sigma}$.

The base case for propositional letters follows directly from the definition of σ. Furthermore, the induction cases for the connectives $\to$, $\wedge$ and $\vee$ follow directly from the fact that their semantics coincide in Kripke models for **IPC** and in blended models. This finishes the induction.

Hence, it follows from the induction that $(K, \leq, \mathcal{D}) \nVdash \varphi^{\sigma}$, and therefore, $\varphi \notin \mathbf{L}(\mathsf{IZF}(J))$. This finishes the proof of the theorem. ◀

▶ **Corollary 32.** *Intuitionistic set theory* IZF *has the de Jongh property.*

Proof. By Proposition 6, we know that **IPC** is complete with respect to the class of all finite trees, i.e., this class characterises **IPC**. By the previous Theorem 31, this implies that IZF has the de Jongh property. ◀

More examples of logics that are characterised by classes of finite trees are Gödel-Dummett logic **LC**, the Gabbay-de Jongh logics $\mathbf{T}_n$, and the logics of bounded depth $\mathbf{BD}_n$ (see Example 7).

▶ **Corollary 33.** *Intuitionistic set theory* IZF *has the de Jongh property with respect to the logics* **LC**, $\mathbf{T}_n$ *and* $\mathbf{BD}_n$.

▶ **Lemma 34.** *If a theory T has the de Jongh property with respect to a logic J, then any theory $S \subseteq T$ has the de Jongh property with respect to J.*

Proof. We have to show that $J \vdash \varphi$ if and only if $S(J) \vdash \varphi^{\sigma}$ for all $\sigma : \mathsf{Prop} \to \mathcal{L}_{\in}^{\mathsf{sent}}$. The implication from left to right is trivial. We prove the other direction by contraposition. So assume that $J \nvdash \varphi$. By assumption, T has the de Jongh property with respect to J and hence there is some σ such that $T(J) \nvdash \varphi^{\sigma}$. As $S \subseteq T$, it follows that $S(J) \nvdash \varphi^{\sigma}$. ◀

▶ **Corollary 35.** *Constructive set theory* CZF *has the de Jongh property with respect to every intermediate logic J that is characterised by a class of finite trees. In particular,* CZF *has the de Jongh property with respect to the logics* **IPC**, **LC**, $\mathbf{T}_n$ *and* $\mathbf{BD}_n$.

In fact, Lemma 34 implies that Corollary 35 holds for any set theory $T \subseteq$ IZF based on intuitionistic logic. Indeed, any set theory T that is weaker than IZF has the de Jongh property with respect to every intermediate logic J that is characterised by a class of finite trees.

5   Open Questions

In this paper, we defined a class of Kripke models for intuitionistic set theory IZF, the blended Kripke models. We then used these models to prove a range of de Jongh properties for IZF and CZF. It would certainly be interesting to find a constructive proof of the results presented in this paper.

▶ **Question 36.** *Does intuitionistic set theory* IZF *have the de Jongh property with respect to every intermediate logic?*

Lubarsky used his Kripke models for independence results for CZF and IZF. For example, he proved that it is consistent with CZF that the power set of 1 is a proper class (see [13]; for more results in this area see, e.g., [14, 15, 16]). We wonder whether our blended models can be used for similar purposes.

▶ **Question 37.** *Can we obtain independence results for* IZF *with blended models?*

▶ **Question 38.** *Is it possible to vary the construction of blended models to provide proper models of* CZF (*i.e., models of* CZF *that are not also models of* IZF)*?*

───── **References** ─────

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