More Models of Walks Avoiding a Quadrant

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—— Abstract ——————————————————

We continue the enumeration of plane lattice paths avoiding the negative quadrant initiated by the first author in [1]. We solve in detail a new case, the king walks, where all 8 nearest neighbour steps are allowed. As in the two cases solved in [1], the associated generating function is proved to differ from a simple, explicit D-finite series (related to the enumeration of walks confined to the first quadrant) by an algebraic one. The principle of the approach is the same as in [1], but challenging theoretical and computational difficulties arise as we now handle algebraic series of larger degree.

We also explain why we expect the observed algebraicity phenomenon to persist for 4 more models, for which the quadrant problem is solvable using the reflection principle.

1 Introduction

In this paper we continue the enumeration of plane lattice paths confined to non-convex cones initiated by the first author in [1]. Therein the two most natural models of walks confined to the three-quadrant cone $\mathcal{C} := \{(i, j) : i \geq 0 \text{ or } j \geq 0\}$ were studied: walks with steps $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$, and those with steps $\{\nearrow, \nwarrow, \swarrow, \searrow\}$. In both cases, the generating function that counts walks starting at the origin was proved to differ (additively) from a simple explicit D-finite series by an algebraic one. The tools essentially involved power series manipulations, coefficient extractions, and polynomial elimination.

Later, Raschel and Trotignon gave in [13] sophisticated integral expressions for 8 models, which imply that 3 additional models ($\{\nearrow, \leftarrow, \downarrow\}$, $\{\rightarrow, \uparrow, \swarrow\}$, and $\{\rightarrow, \nearrow, \uparrow, \leftarrow, \swarrow, \downarrow\}$) are D-finite. Their results use an analytic approach inspired by earlier work on probabilistic and enumerative aspects of quadrant walks [5, 12].

In this paper we first extend the results of [1] to the so-called *king walks*, which take their steps from $\{\rightarrow, \nearrow, \uparrow, \nwarrow, \leftarrow, \swarrow, \downarrow, \searrow\}$. We show that the *algebraicity phenomenon* of [1] persists: if $Q(x, y; t)$ (resp. $C(x, y; t)$) counts walks starting from the origin that are confined

to the non-negative quadrant $\mathcal{Q} := \{(i,j) : i \geq 0 \text{ and } j \geq 0\}$ (resp. to the cone $\mathcal{C}$) by the length (variable t) and the coordinates of the endpoint (variables x, y), then $C(x, y; t)$ differs from the series

$$\frac{1}{3}\left(Q(x,y;t) - Q(1/x,y;t)/x^2 - Q(x,1/y;t)/y^2\right)$$

by an algebraic series, as detailed in our main theorem below. Moreover, we expect a similar property to hold (with variations on the above linear combination of the series Q) for the 7 step sets of Figure 1, related to reflection groups, and for which the quadrant problem can be solved using the reflection principle [7]. However, we also expect the effective solution of these models to be extremely challenging in computational terms, mostly, because the relevant algebraic series have very large degree. This is illustrated by our main theorem below. There, and in the sequel, we use the shorthand $\bar{x} = 1/x$, $\bar{y} = 1/y$, and omit in the notation the dependencies on t, writing for instance $Q(x, y)$ instead of $Q(x, y; t)$.

▶ **Theorem 1.** *Take the step set $\{-1, 0, 1\}^2 \setminus \{(0,0)\}$ and let $Q(x, y)$ be the generating function of lattice walks starting from $(0,0)$ that are confined to the first quadrant $\mathcal{Q}$ (this series is D-finite and given in [3]). Then, the generating function of walks starting from $(0,0)$, confined to $\mathcal{C}$, and ending in the first quadrant (resp. at a negative abscissa) is*

$$\frac{1}{3}Q(x,y) + P(x,y), \qquad (resp. -\frac{\bar{x}^2}{3}Q(\bar{x},y) + \bar{x}M(\bar{x},y)), \tag{1}$$

where $P(x, y)$ and $M(x, y)$ are algebraic of degree 216 over $\mathbb{Q}(x, y, t)$. Of course, the generating function of walks ending at a negative ordinate follows, using the x/y-symmetry.

The series P is expressed in terms of M by:

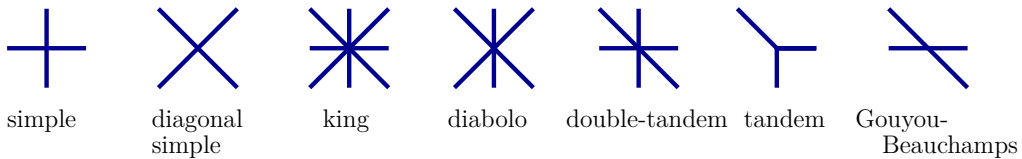$$P(x,y) = \bar{x}\big(M(x,y) - M(0,y)\big) + \bar{y}\big(M(y,x) - M(0,x)\big), \tag{2}$$

and M is defined by the following equation:

$$K(x,y)\,(2M(x,y) - M(0,y)) = \frac{2x}{3} - 2t\bar{y}(x + 1 + \bar{x})M(x,0) + t\bar{y}(y + 1 + \bar{y})M(y,0) \tag{3}$$
$$+ t(x - \bar{x})(y + 1 + \bar{y})M(0,y) - t\left(1 + \bar{y}^2 - 2\bar{x}\bar{y}\right)M(0,0) - t\bar{y}M_x(0,0),$$

where $K(x, y) = 1 - t(x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y})$. The specializations $M(x, 0)$ and $M(0, y)$ are algebraic each of degree 72 over $\mathbb{Q}(x, t)$ and $\mathbb{Q}(y, t)$, respectively, and $M(0, 0)$ and $M_x(0, 0)$ have degree 24 over $\mathbb{Q}(t)$.

We have moreover a complete algebraic description of all the series needed to reconstruct $P(x, y)$ and $M(x, y)$ from (2) and (3), namely the univariate series $M(0, 0)$ and $M_x(0, 0)$, and the bivariate series $M(x, 0)$ and $M(0, y)$. In particular, both univariate series lie in the extension of $\mathbb{Q}(t)$ (the field of rational functions in t) generated in 3 steps as follows: first, $u = t + t^2 + \mathcal{O}(t^3)$ is the only series in t satisfying

$$(1 - 3u)^3(1 + u)t^2 + (1 + 18u^2 - 27u^4)t - u = 0, \tag{4}$$

simple   diagonal   king   diabolo   double-tandem tandem   Gouyou-
           simple   Beauchamps

■ **Figure 1** The seven step sets to which the strategy of this paper should apply. The first two are solved in [1], the third one in this paper.

then $v = t + 3t^2 + \mathcal{O}(t^3)$ is the only series with constant term zero satisfying

$$(1 + 3v - v^3)u - v(v^2 + v + 1) = 0, \tag{5}$$

and finally

$$w = \sqrt{1 + 4v - 4v^3 - 4v^4} = 1 + 2t + 4t^2 + \mathcal{O}(t^3). \tag{6}$$

Schematically, $\mathbb{Q}(t) \overset{4}{\hookrightarrow} \mathbb{Q}(t,u) \overset{3}{\hookrightarrow} \mathbb{Q}(t,v) \overset{2}{\hookrightarrow} \mathbb{Q}(t,w)$. Of particular interest is the series $M(0,0)$: by (1), this is also the series $C_{-1,0}$ that counts by the length walks in $\mathcal{C}$ ending at $(-1,0)$. It is algebraic, as conjectured in [13], and given by

$$M(0,0) = C_{-1,0} = \frac{1}{2t}\left(\frac{w(1+2v)}{1+4v-2v^3} - 1\right) = t + 2t^2 + 17t^3 + 80t^4 + 536t^5 + \mathcal{O}(t^6). \tag{7}$$

Due to the lack of space, the extensions of $\mathbb{Q}(x,t)$ generated by $M(x,0)$ and $M(0,x)$ will only be described in the long version of this paper.

Once the series $C(x,y)$ is determined, we can derive detailed asymptotic results, which refine general results of Denisov and Wachtel [4] and Mustapha [11] (who only obtain the following estimates up to a multiplicative factor).

▶ **Corollary 2.** *The number $c_{0,0}(n)$ of n-step king walks confined to $\mathcal{C}$ and ending at the origin, and the number $c(n)$ of walks of $\mathcal{C}$ ending anywhere satisfy for $n \to \infty$:*

$$c_{0,0}(n) \sim \left(\frac{2^{29}K}{3^7}\right)^{1/3} \frac{\Gamma(2/3)}{\pi} \frac{8^n}{n^{5/3}},$$

$$c(n) \sim \left(\frac{2^{32}K}{3^7}\right)^{1/6} \frac{1}{\Gamma(2/3)} \frac{8^n}{n^{1/3}},$$

where K is the unique real root of $101^6 K^3 - 601275603K^2 + 92811K - 1$.

Outline of the paper

We begin in Section 2 with a general discussion on models of walks with small steps confined to the cone $\mathcal{C}$, and on the related functional equations. The main part of the paper, Section 3, is devoted to the solution of the king model. We sketch in the final Section 4 what should be the starting point for the 4 rightmost models of Figure 1.

Some definitions and notation

Let $\mathbb{A}$ be a commutative ring and x an indeterminate. We denote by $\mathbb{A}[x]$ (resp. $\mathbb{A}[[x]]$) the ring of polynomials (resp. formal power series) in x with coefficients in $\mathbb{A}$. If $\mathbb{A}$ is a field, then $\mathbb{A}(x)$ denotes the field of rational functions in x, and $\mathbb{A}((x))$ the field of Laurent series in x, that is, series of the form $\sum_{n \geq n_0} a_n x^n$, with $n_0 \in \mathbb{Z}$ and $a_n \in \mathbb{A}$. The coefficient of x^n in a series $F(x)$ is denoted by $[x^n]F(x)$.

This notation is generalized to polynomials, fractions, and series in several indeterminates. If $F(x, x_1, \ldots, x_d)$ is a series in the x_i's whose coefficients are Laurent series in x, say

$$F(x, x_1, \ldots, x_d) = \sum_{i_1, \ldots, i_d} x_1^{i_1} \cdots x_d^{i_d} \sum_{n \geq n_0(i_1, \ldots, i_d)} a(n, i_1, \ldots, i_d) x^n,$$

then the *non-negative part of F in x* is the following formal power series in $x, x_1, \ldots, x_d$:

$$[x^{\geq 0}]F(x, x_1, \ldots, x_d) = \sum_{i_1, \ldots, i_d} x_1^{i_1} \cdots x_d^{i_d} \sum_{n \geq 0} a(n, i_1, \ldots, i_d) x^n.$$

We define similarly the negative part of F, its positive part, and so on. We denote with bars the reciprocals of variables: that is, $\bar{x} = 1/x$, so that $\mathbb{A}[x, \bar{x}]$ is the ring of Laurent polynomials in x with coefficients in $\mathbb{A}$.

If $\mathbb{A}$ is a field, a power series $F(x) \in \mathbb{A}[[x]]$ is *algebraic* (over $\mathbb{A}(x)$) if it satisfies a non-trivial polynomial equation $P(x, F(x)) = 0$ with coefficients in $\mathbb{A}$. It is *differentially finite* (or *D-finite*) if it satisfies a non-trivial linear differential equation with coefficients in $\mathbb{A}(x)$. For multivariate series, D-finiteness requires the existence of a differential equation *in each variable*. We refer to [8, 9] for general results on D-finite series.

As mentioned above, we usually omit the dependency in t of our series. For a series $F(x, y; t) \in \mathbb{Q}[x, \bar{x}, y, \bar{y}][[t]]$ and two integers i and j, we denote by $F_{i,j}$ the coefficient of $x^i y^j$ in $F(x, y; t)$. This is a series in $\mathbb{Q}[[t]]$.

2   Enumeration in the three-quarter plane

We fix a subset $\mathcal{S}$ of $\{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ and we want to count walks with steps in $\mathcal{S}$ that start from the origin $(0, 0)$ of $\mathbb{Z}^2$ and remain in the cone $\mathcal{C} := \{(x, y) : x \geq 0 \text{ or } y \geq 0\}$. By this, we mean that not only must every vertex of the walk lie in $\mathcal{C}$, but also every edge: a walk containing a step from $(-1, 0)$ to $(0, -1)$ (or vice versa) is not considered as lying in $\mathcal{C}$. We often say for short that our walks *avoid the negative quadrant*. The *step polynomial* of $\mathcal{S}$ is defined by

$$S(x, y) = \sum_{(i,j) \in \mathcal{S}} x^i y^j = \bar{y} H_-(x) + H_0(x) + y H_+(x) = \bar{x} V_-(y) + V_0(y) + x V_+(y),$$

for some Laurent polynomials H_-, H_0, H_+ and V_-, V_0, V_+ (of degree at most 1 and valuation at least -1) recording horizontal and vertical displacements, respectively. We denote by $C(x, y; t) \equiv C(x, y)$ the generating function of walks confined to $\mathcal{C}$, where the variable t records the length of the walk, and x and y the coordinates of its endpoints:

$$C(x, y) = \sum_{(i,j) \in \mathcal{C}} \sum_{n \geq 0} c_{i,j}(n) x^i y^j t^n = \sum_{(i,j) \in \mathcal{C}} x^i y^j C_{i,j}(t). \tag{8}$$

Here, $c_{i,j}(n)$ is the number of walks of length n that go from $(0, 0)$ to (i, j) and that are confined to $\mathcal{C}$.

2.1   Interesting step sets

As in the quadrant case [3], we can decrease the number of step sets that are worth being considered (*a priori*, there are 2^8 of them) thanks to a few simple observations:

- Since the cone $\mathcal{C}$ (as well as the quarter plane $\mathcal{Q}$) is x/y-symmetric, the models defined by $\mathcal{S}$ and by its mirror image $\overline{\mathcal{S}} := \{(j, i) : (i, j) \in \mathcal{S}\}$ are equivalent; the associated generating functions are related by $\overline{C}(x, y) = C(y, x)$.
- If all steps of $\mathcal{S}$ are contained in the right half-plane $\{(x, y) : x \geq 0\}$, then *all* walks with steps in $\mathcal{S}$ lie in $\mathcal{C}$, and the series $C(x, y) = 1/(1 - tS(x, y))$ is simply rational. The series $Q(x, y)$ is known to be algebraic in this case [6].
- If all steps of $\mathcal{S}$ are contained in the left half-plane $\{(x, y) : x \leq 0\}$, then confining a walk to $\mathcal{C}$ is equivalent to confining it to the upper half-plane: the associated generating function is then algebraic, and so is $Q(x, y)$.
- If all steps of $\mathcal{S}$ lie (weakly) above the first diagonal ($x = y$), then confining a walk to $\mathcal{C}$ is again equivalent to confining it to the upper half-plane: the associated generating function is then algebraic, and so is $Q(x, y)$.

- Finally, if all steps of $\mathcal{S}$ lie (weakly) above the second diagonal ($x + y = 0$), then all walks with steps in $\mathcal{S}$ lie in $\mathcal{C}$, and $C(x, y) = 1/(1 - tS(x, y))$ is simply rational. In this case however, the series $Q(x, y)$ is not at all trivial [3, 10]. Such step sets are sometimes called *singular* in the framework of quadrant walks.

Symmetric statements allow us to discard step sets that lie in the upper half-plane $\mathbb{Z} \times \mathbb{N}$, in the lower half-plane $\mathbb{Z} \times (-\mathbb{N})$, or weakly below the x/y diagonal.

In conclusion, one finds that there are exactly 51 essentially distinct models of walks avoiding the negative quadrant that are worth studying: the 56 models considered for quadrant walks (see Tables 1–4 in [3]) except the 5 singular models for which all steps of $\mathcal{S}$ lie weakly above the diagonal $x + y = 0$.

2.2 A functional equation

Constructing walks confined to $\mathcal{C}$ step by step gives the following functional equation:

$$C(x, y) = 1 + tS(x, y)C(x, y) - t\bar{y}H_-(x)C_{-,0}(\bar{x}) - t\bar{x}V_-(y)C_{0,-}(\bar{y}) - t\bar{x}\bar{y}C_{0,0}\mathbb{1}_{(-1,-1)\in\mathcal{S}},$$

where the series $C_{-,0}(\bar{x})$ and $C_{0,-}(\bar{y})$ count walks ending on the horizontal and vertical boundaries of $\mathcal{C}$ (but not at $(0,0)$):

$$C_{-,0}(\bar{x}) = \sum_{\substack{i<0 \\ n\geq0}} c_{i,0}(n)x^i t^n \in \bar{x}\mathbb{Q}[\bar{x}][[t]],$$

$$C_{0,-}(\bar{y}) = \sum_{\substack{j<0 \\ n\geq0}} c_{0,j}(n)y^j t^n \in \bar{y}\mathbb{Q}[\bar{y}][[t]].$$

On the right-hand side of the above functional equation, the term 1 accounts for the empty walk, the next term describes the extension of a walk in $\mathcal{C}$ by one step of $\mathcal{S}$, and each of the other three terms correspond to a "bad" move, either starting from the negative x-axis, or from the negative y-axis, or from $(0,0)$. Equivalently,

$$K(x, y)C(x, y) = 1 - t\bar{y}H_-(x)C_{-,0}(\bar{x}) - t\bar{x}V_-(y)C_{0,-}(\bar{y}) - t\bar{x}\bar{y}C_{0,0}\mathbb{1}_{(-1,-1)\in\mathcal{S}}, \qquad (9)$$

where $K(x, y) := 1 - tS(x, y)$ is the *kernel* of the equation.

The case of walks confined to the first (non-negative) quadrant $\mathcal{Q}$ has been much studied in the past 15 years. The associated generating function $Q(x, y) \equiv Q(x, y; t) \in \mathbb{Q}[x, y][[t]]$ is defined similarly to (8) and satisfies a similarly looking equation:

$$K(x, y)Q(x, y) = 1 - t\bar{y}H_-(x)Q_{-,0}(x) - t\bar{x}V_-(y)Q_{0,-}(y) + t\bar{x}\bar{y}Q_{0,0}\mathbb{1}_{(-1,-1)\in\mathcal{S}},$$

where now

$$Q_{-,0}(x) = \sum_{\substack{i\geq0 \\ n\geq0}} q_{i,0}(n)x^i t^n = Q(x, 0) \in \mathbb{Q}[x][[t]],$$

$$Q_{0,-}(y) = \sum_{\substack{j\geq0 \\ n\geq0}} q_{0,j}(n)y^j t^n = Q(0, y) \in \mathbb{Q}[y][[t]].$$

3 The king walks

In this section we focus on the case where the 8 steps of $\{-1, 0, 1\}^2 \setminus \{(0,0)\}$ are allowed. That is,

$$S(x, y) = (\bar{x} + 1 + x)(\bar{y} + 1 + y) - 1 = x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y}.$$

The functional equation (9) specializes to

$$K(x,y)C(x,y) = 1 - t\bar{y}(x+1+\bar{x})C_-(\bar{x}) - t\bar{x}(y+1+\bar{y})C_-(\bar{y}) - t\bar{x}\bar{y}C_{0,0}, \qquad (10)$$

where we have denoted $C_-(\bar{x}) = C_{-,0}(\bar{x}) = C_{0,-}(\bar{x})$ (by symmetry). Equivalently,

$$xyK(x,y)C(x,y) = xy - t(x^2+x+1)C_-(\bar{x}) - t(y^2+y+1)C_-(\bar{y}) - tC_{0,0}. \qquad (11)$$

The generating function $Q(x,y)$ of quadrant walks satisfies

$$xyK(x,y)Q(x,y) = xy - t(x^2+x+1)Q(x,0) - t(y^2+y+1)Q(0,y) + tQ_{0,0}. \qquad (12)$$

3.1   Reduction to an equation with orbit sum zero

A key object in the study of walks confined to the first quadrant is a certain group of birational transformations that depends on the step set. For king walks, it is generated by $(x,y) \mapsto (\bar{x},y)$ and $(x,y) \mapsto (x,\bar{y})$. As in [1], the similarities between the equations for C and Q, combined with the structure of this group, lead us to define a new series $A(x,y)$ by

$$C(x,y) = A(x,y) + \frac{1}{3}\left(Q(x,y) - \bar{x}^2 Q(\bar{x},y) - \bar{y}^2 Q(x,\bar{y})\right). \qquad (13)$$

Then the combination of (11) and (12) gives

$$xyK(x,y)A(x,y) = \frac{2xy + \bar{x}y + x\bar{y}}{3} - t(x^2+x+1)A_-(\bar{x}) - t(y^2+y+1)A_-(\bar{y}) - tA_{0,0},$$

and it follows from this equation that $xyA(x,y)$ has *orbit sum* zero. By this, we mean:

$$xyA(x,y) - \bar{x}yA(\bar{x},y) + \bar{x}\bar{y}A(\bar{x},\bar{y}) - x\bar{y}A(x,\bar{y}) = 0. \qquad (14)$$

Theorem 1 states that $A(x,y)$ is algebraic. In Section 4 we define an analogous series A for all models of Figure 1 which we expect to be systematically algebraic.

The proof of Theorem 1 starts as in the case of the simple and diagonal walks in [1]. The first objective, achieved in Section 3.5, is to derive an equation that involves a single bivariate series, essentially $A_-(x)$ (and no trivariate series). In principle, the "generalized quadratic method" of [2] then solves it routinely. But in practise, the king model turns out to be much more difficult to solve than the other two, and raises serious computational difficulties. In what follows, we focus on the points of the derivation that differ from [1]. We have performed all computations with the computer algebra system Maple. The corresponding sessions will be available on the authors' webpages with the long version of the paper.

3.2   Reduction to a quadrant-like problem

We separate in $A(x,y)$ the contributions of the three quadrants, again using the x/y-symmetry of the step set:

$$A(x,y) = P(x,y) + \bar{x}M(\bar{x},y) + \bar{y}M(\bar{y},x),$$

where $P(x,y)$ and $M(x,y)$ lie in $\mathbb{Q}[x,y][[t]]$. Note that this identity defines P and M uniquely in terms of A. Replacing A by this expression, and extracting the positive part in x and y from the orbit equation (14) relates the series P and M by

$$xyP(x,y) = y\left(M(x,y) - M(0,y)\right) + x\left(M(y,x) - M(0,x)\right),$$

which is exactly the same as [1, Eq. (22)], and as Eq. (2) in Theorem 1. We then follow the lines of proof of [1, Sec. 2.3] to obtain the functional equation (3) for M.

3.3 An equation between $M(0,x)$, $M(0,\bar{x})$, and $M(x,0)$

Next we will cancel the kernel K. As a polynomial in y, the kernel admits only one root that is a formal power series in t:

$$Y(x) = \frac{1 - t(x+\bar{x}) - \sqrt{(1 - t(x+\bar{x}))^2 - 4t^2(x+1+\bar{x})^2}}{2t(x+1+\bar{x})} = (x+1+\bar{x})t + \mathcal{O}(t^2).$$

Note that $Y(x) = Y(\bar{x})$. We specialize (3) to the pairs $(x, Y(x)), (\bar{x}, Y(x)), (Y(x), x)$, and $(Y(x), \bar{x})$ (the left-hand side vanishes for each specialization since $K(x,y) = K(y,x)$), and eliminate $M(0,Y)$, $M(Y,0)$, and $M(\bar{x}, 0)$ from the four resulting equations. We obtain:

$$(x+1+\bar{x})\left(Y(x) - \frac{1}{Y(x)}\right)(xM(0,x) - 2\bar{x}M(0,\bar{x})) + 3(x+1+\bar{x})M(x,0)$$
$$- \frac{2\bar{x}Y(x)}{t} + 3M_{1,0} + (2Y(x) - x - \bar{x})M_{0,0} = 0. \tag{15}$$

3.4 An equation between $M(0,x)$ and $M(0,\bar{x})$

Let us denote the discriminant occurring in $Y(x)$ by

$$\Delta(x) := (1 - t(x+\bar{x}))^2 - 4t^2(x+1+\bar{x})^2 = (1 - t(3(x+\bar{x}) + 2))(1 + t(x+\bar{x} + 2)) \tag{16}$$

and introduce the notation

$$R(x) := t^2 M(x,0) = \frac{xt^2}{3} + \left(1 + \frac{x^2}{3}\right)t^3 + \mathcal{O}(t^4),$$
$$S(x) := txM(0,x) = x(1+x)t^2 + 2x(1+x+x^2)t^3 + \mathcal{O}(t^4). \tag{17}$$

Then (15) reads

$$\sqrt{\Delta(x)}\left(S(x) - 2S(\bar{x}) + \frac{R(0) - t\bar{x}}{t(x+1+\bar{x})}\right) = 3(x+1+\bar{x})R(x) + 3R'(0)$$
$$+ \frac{1 - t(x+\bar{x})(x+2+\bar{x})}{t(x+1+\bar{x})}R(0) - \frac{1 - t(x+\bar{x})}{1+x+x^2}. \tag{18}$$

Next, we square this equation and extract the negative part in x. The series $R(x)$ (mostly) disappears as it involves only non-negative powers of x. This gives an expression for the negative part of $\Delta(x)S(x)S(\bar{x})$. Using the symmetry of $\Delta(x)$ in x and $\bar{x}$, we then reconstruct an expression of $\Delta(x)S(x)S(\bar{x})$ that does not involve $R(x)$, as in [1, Sec. 2.5].

During these calculations, we have to extract the negative and non-negative parts in series of the form $F(x)/(1+x+\bar{x})^m$, where $F(x)$ is a series in t with coefficients in $\mathbb{Q}[x, \bar{x}]$. Upon performing a partial fraction expansion, and separating in F the negative and non-negative parts, we see that the key question is how to extract and express the non-negative part in series of the form $F(\bar{x})/(1 - \zeta_i x)^m$, where $F(x) \in \mathbb{C}[x][[t]]$ and

$$\zeta_1 := -\frac{1}{2} + \frac{i\sqrt{3}}{2} \quad \text{and} \quad \zeta_2 := -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

are the primitive cubic roots of unity. A simple calculation establishes the following lemma.

▶ **Lemma 3** (Non-negative part at pole ρ). *Let $F(x) \in \mathbb{C}[x][[t]]$ and $\rho \in \mathbb{C}$. Then,*

$$[x^{\geq 0}]\frac{F(\bar{x})}{1 - \rho x} = \frac{F(\rho)}{1 - \rho x},$$
$$[x^{\geq 0}]\frac{F(\bar{x})}{(1 - \rho x)^2} = \frac{F(\rho)}{(1 - \rho x)^2} + \frac{\rho F'(\rho)}{1 - \rho x}.$$

One outcome of the extraction procedure is the following identity:

$$S(\zeta_1) = S(\zeta_2) = -\frac{R(0) + 3R'(0)}{1+t} = -t^2 - 11t^4 - 30t^5 + \mathcal{O}(t^6). \tag{19}$$

Using these results, we finally arrive at an equation relating $S(x)$ and $S(\bar{x})$:

$$\begin{aligned}
\Delta(x)\left(S(x)^2 + S(\bar{x})^2 - S(x)S(\bar{x}) + \frac{S(x)(xt - R(0)) + \bar{x}S(\bar{x})(\bar{x}t - R(0))}{t(x + 1 + \bar{x})}\right) = \\
(1+t)S(\zeta_1)\left(2(x + 1 + \bar{x})R(0) - \frac{(1 - t(x+\bar{x}))(t(x+\bar{x}) - 2R(0))}{t(x + 1 + \bar{x})}\right) \\
+ (1 + 4t)(x + \bar{x})R(0) - (t^2 + tR(0) + R(0)^2)(x^2 + \bar{x}^2) + \Delta_0,
\end{aligned} \tag{20}$$

where Δ_0 is the coefficient of x^0 in $\Delta(x)S(x)S(\bar{x})$.

3.5   An equation for $M(0, x)$ only

Equation (20) is almost ready for a positive part extraction, except for the mixed term $S(x)S(\bar{x})$. To eliminate it, we multiply (20) by $S(x) + S(\bar{x}) + \frac{x + \bar{x} - 2R(0)/t}{x + 1 + \bar{x}}$. Then we are able to extract the non-negative terms in x. Hereby we repeatedly apply Lemma 3. Additionally, we use $R(0) = tS'(0)$ and (19). Furthermore, we work with the real and imaginary parts of $\zeta_1 S'(\zeta_1)$ and $\zeta_2 S'(\zeta_2)$. More precisely, we define

$$(1+t)^2 \zeta_1 S'(\zeta_1) = B_1 + i\sqrt{3}B_2,$$
$$(1+t)^2 \zeta_2 S'(\zeta_2) = B_1 - i\sqrt{3}B_2.$$

(Note that B_1 and B_2 here are series in t.) In the end we get a cubic equation in $S(x)$:

$$\mathrm{Pol}(S(x), S'(0), S(\zeta_1), B_1, B_2, t, x) = 0, \tag{21}$$

where the polynomial $\mathrm{Pol}(x_0, x_1, x_2, x_3, x_4, t, x)$ is given in Appendix A.

3.6   The generalized quadratic method

We now use the results of [2] to obtain a system of four polynomial equations relating the series $S'(0), S(\zeta_1), B_1$, and B_2. Combined with a few initial terms, this system characterizes these four series. Unfortunately, it turned out to be too big for us to solve it completely, be it by bare hand elimination or using Gröbner bases: we did obtain a polynomial equation for $S'(0)$ and $S(\zeta_1)$, but not for the other two series. Instead, we have resorted to a guess-and-check approach, consisting in *guessing* such equations (of degree 12 or 24, depending on the series), and then *checking* that they satisfy the system. This guess-and-check approach is detailed in the next subsection. For the moment, let us explain how the system is obtained.

The approach of [2] instructs us to consider the fractional series X (in t), satisfying

$$\mathrm{Pol}_{x_0}(S(X), S'(0), S(\zeta_1), B_1, B_2, t, X) = 0, \tag{22}$$

where Pol_{x_0} stands for the derivative of Pol with respect to its first variable. The number and first terms of such series X depend only on the first terms of the series $S(x), S'(0), S(\zeta_1), B_1$, and B_2 (see [2, Thm. 2]). We find that 6 such series exist:

$$X_1(t) = i + 2t^2 + 4t^3 + (36 - 2i)t^4 + \mathcal{O}(t^5),$$
$$X_2(t) = -i + 2t^2 + 4t^3 + (36 + 2i)t^4 + \mathcal{O}(t^5),$$
$$X_3(t) = \sqrt{t} + t + \frac{3}{2}t^{3/2} + 3t^2 + \frac{51}{8}t^{5/2} + 14t^3 + \mathcal{O}(t^{7/2}),$$
$$X_4(t) = -\sqrt{t} + t - \frac{3}{2}t^{3/2} + 3t^2 - \frac{51}{8}t^{5/2} + 14t^3 + \mathcal{O}(t^{7/2}),$$
$$X_5(t) = i\sqrt{t} - it^{3/2} + 2it^{5/2} + t^3 - 4it^{7/2} + 2t^4 + \mathcal{O}(t^{9/2}),$$
$$X_6(t) = -i\sqrt{t} + it^{3/2} - 2it^{5/2} + t^3 + 4it^{7/2} + 2t^4 + \mathcal{O}(t^{9/2}).$$

Note that the coefficients of X_1 and X_2 (resp. X_5 and X_6) are conjugates of one another. As discussed in [2], each of these series X also satisfies

$$\mathrm{Pol}_x(S(X), S'(0), S(\zeta_1), B_1, B_2, t, X) = 0, \tag{23}$$

where Pol_x is the derivative with respect to the last variable of Pol, and (of course)

$$\mathrm{Pol}(S(X), S'(0), S(\zeta_1), B_1, B_2, t, X) = 0. \tag{24}$$

Using this, we can easily identify two of the series X_i: indeed, eliminating B_1 and B_2 between the three equations (22), (23), and (24) gives a polynomial equation between $S(X), S'(0), S(\zeta_1), t$, and X, which factors. Remarkably, its simplest non-trivial factor does not involve $S(X)$, nor $S'(0)$ nor $S(\zeta_1)$, and reads

$$X^2 - t(1 + X)^2(1 + X^2). \tag{25}$$

By looking at the first terms of the X_i's and the other factors, one concludes that the above equation holds for X_3 and X_4, which are thus explicit.

Let $D(x_1, \ldots, x_4, t, x)$ be the discriminant of $\mathrm{Pol}(x_0, \ldots, x_4, t, x)$ with respect to x_0. According to [2, Thm. 14], each X_i is a *double root* of $D(S'(0), S(\zeta_1), B_1, B_2, t, x)$, seen as a polynomial in x. Hence this polynomial, which involves 4 unknown series $S'(0), S(\zeta_1), B_1, B_2$, has (at least) 6 double roots. This seems more information than we need! In principle, 4 double roots should suffice to give 4 conditions relating the 4 unknown series. However, we shall see that there is some redundancy in the 6 series X_i, which comes from the special form of D.

We first observe that D factors as

$$D(S'(0), S(\zeta_1), B_1, B_2, t, x) = 27x^2(1 + x + x^2)^2\Delta(x)D_1(S'(0), S(\zeta_1), B_1, B_2, t, x),$$

where $\Delta(x)$ is defined by (16), and D_1 has degree 24 in x. It is easily checked that none of the X_i's are roots of the prefactors, so they are double roots of D_1. But we observe that D_1 is symmetric in x and $\bar{x}$. More precisely,

$$D_1(S'(0), S(\zeta_1), B_1, B_2, t, x) = x^{12}D_2(S'(0), S(\zeta_1), B_1, B_2, t, x + 1 + \bar{x}),$$

for some polynomial $D_2(x_1, \ldots, x_4, t, s) \equiv D_2(s)$ of degree 12 in s. Since each X_i is a double root of D_1, each series $S_i := X_i + 1 + 1/X_i$, for $1 \leq i \leq 6$, is a double root of D_2. The series S_i, for $2 \leq i \leq 6$, are easily seen from their first terms to be distinct, but the first terms of S_1 and S_2 suspiciously agree: one suspects (and rightly so), that $X_2 = 1/X_1$, and carefully

concludes that D_2 has (at least) 5 double roots in s. Moreover, since X_3 and X_4 satisfy (25), the corresponding series S_3 and S_4 are the roots of $1 + t = tS_i^2$, that is, $S_{3,4} = \pm\sqrt{1 + 1/t}$. The other roots start as follows:

$$S_2 = 1 + 4t^2 + 8t^3 + \mathcal{O}(t^4), \qquad S_{5,6} = \mp\frac{i}{\sqrt{t}} + 1 + t^2 \pm it^{5/2} + \mathcal{O}(t^3).$$

But this is not the end of the story: indeed, D_2 appears to be almost symmetric in s and $1/s$. More precisely, we observe that

$$D_2(S'(0), S(\zeta_1), B_1, B_2, s) = s^6 D_3\left(S'(0), S(\zeta_1), B_1, B_2, ts + \frac{t+1}{s}\right),$$

for some polynomial $D_3(S'(0), S(\zeta_1), B_1, B_2, t, z) \equiv D_3(z)$ of degree 6 in z. It follows that each series $Z_i := tS_i + (1 + t)/S_i$, for $2 \leq i \leq 6$, is a root of $D_3(z)$, and even a double root, unless $tS_i^2 = 1 + t$, which precisely occurs for $i = 3, 4$. One finds $Z_{3,4} = \pm 2\sqrt{t(1 + t)}$,

$$Z_2 = 1 + 2t - 4t^2 + \mathcal{O}(t^3), \qquad Z_{5,6} = 2t + 2t^3 + \mathcal{O}(t^4).$$

Since Z_5 and Z_6 seem indistinguishable, we safely conclude that $D_3(z)$ has two double roots Z_2 and Z_5, and a factor $(z^2 - 4t(1 + t))$. Writing

$$D_3(z) = \sum_{i=0}^6 d_i z^i,$$

these properties imply, by matching the three monomials of highest degree, that

$$D_3(z) = \frac{\left(z^2 - 4\,t(1 + t)\right)\left(8\,z^2 d_6{}^2 + 4\,zd_5 d_6 + 16\,t^2 d_6{}^2 + 16\,td_6{}^2 + 4\,d_4 d_6 - d_5{}^2\right)^2}{64\,d_6{}^3}.$$

Extracting the coefficients of $z^0, \ldots, z^3$ gives 4 polynomial relations between the coefficients d_i, resulting in 4 polynomial relations between the 4 series $S'(0), S(\zeta_1), B_1, B_2$. One easily checks that this system, combined with the first terms of these series, defines them uniquely.

As explained at the beginning of this subsection, we have at the moment only been able to derive from this system polynomial equations (of degree 24) for $S'(0)$ and $S(\zeta_1)$. For the other two, we had to resort to a guess-and-check approach, which we now describe.

3.7   Guess-and-check

Guessing.   Returning to the functional equation (10) it is easy to extract a simple recurrence for the polynomials $c_n(x, y)$ that count walks of length n by the position of their endpoint. We implemented this recurrence in the programming language C using modular arithmetic and the Chinese remainder theorem to compute the explicit values of this sequence up to $n = 2000$. Then we were able to guess polynomial equations satisfied by $S'(0), S(\zeta_1), B_1$, and B_2 using the `gfun` package in MAPLE [14]. Of course, those obtained for $S'(0)$ and $S(\zeta_1)$ coincide with those that we derived from the system of the previous subsection. Details on the corresponding equations are shown below.

Generating function	Degree in GF	Degree in t	Number of terms
$S'(0)$	24	12	323
$S(\zeta_1)$	24	32	823
B_1	12	26	229
B_2	24	60	477

Checking that the guessed series satisfy the system turns out to be much easier once the algebraic structure of these series is elucidated, which we do below[1]. We have not tried a direct check.

The algebraic structure of $S'(0), S(\zeta_1), B_1,$ and B_2. We begin with the simplest series, B_1, of (conjectured) degree 12. Let $P(F, t)$ be its guessed monic minimal polynomial. Using the `Subfields` command of MAPLE for several fixed values of t, one conjectures that the extension $\mathbb{Q}(t, B_1)$ possesses a subfield $Q(t, u)$ of degree 4 over $\mathbb{Q}(t)$. MAPLE gives a possible generator u for fixed values of t, but how can we choose u for a generic t? Indeed, the value of u given by MAPLE for fixed t has no reason to be canonical. But the factorisation of $P(F, t)$ over $\mathbb{Q}(t, u)$, of the form $P_3(F)P_9(F)$ (with P_i of degree i), with coefficients in $\mathbb{Q}(t, u)$, *is* canonical. Hence we will compute this factorisation, first for fixed values of t. We proceed as follows: we factor $P(F, t)$ over $\mathbb{Q}(t, B_1)$, and find, for fixed $t = 3, \ldots, 50$, that

$$P(F, t) = (F - B_1)P_2(F, B_1)P_9(F, B_1),$$

where P_2 (resp. P_9) is a monic polynomial of degree 2 (resp. 9) in F. Hence the cubic factor $P_3(F) = F^3 + p_2F^2 + p_1F + p_0$ must be $(F - B_1)P_2(F, B_1)$, and we have just found its coefficients p_i in terms of B_1 (for t fixed). We now compute the minimal polynomial over $\mathbb{Q}$ of each p_i using a resultant or the `evala/Norm` command in MAPLE. If the above factorization persists for all t, as we expect, each p_i should have a minimal polynomial over $\mathbb{Q}(t)$ of degree (at most) 4. Having computed this polynomial for sufficiently many values of t, we reconstruct its generic form by rational reconstruction. We find that all p_i generate the same extension of degree 4 of $\mathbb{Q}(t)$, and we can take any of them as a first candidate for the generator u. We may simplify this generator further to end with the choice (4). Then we factor $P(F, t)$ over $\mathbb{Q}(t, u)$, and check that our guess was correct: the series B_1 is indeed cubic over $\mathbb{Q}(t, u)$. Moreover, it can be written rationally in terms of t and the series v given by (5).

Finally, we factor the guessed minimal polynomials of $S'(0), S(\zeta_1),$ and B_2 over $\mathbb{Q}(t, v)$, and find that these three series all belong to the same quadratic extension of $\mathbb{Q}(t, v)$, generated by the series w given by (6). In particular,

$$S'(0) = \frac{1}{2}\left(\frac{w(1 + 2v)}{1 + 4v - 2v^3} - 1\right),$$

which coincides with (7), given the Definition (17) of $S(x)$.

Now that we have guessed rational expressions of $S'(0), S(\zeta_1), B_1,$ and B_2 in terms of $t, v,$ and w, the 4 equations obtained in Section 3.6 are readily checked to hold, using the minimal polynomials of v and w.

3.8 Back to $S(x)$ and $R(x)$

For $S(x)$ we start with Equation (21), with all one-variable series replaced by their expressions in terms of $t, v,$ and w. We eliminate w and v using resultants to arrive at an equation of degree 72 over $\mathbb{Q}(t, x)$ for $S(x) = txM(0, x)$.

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[1] For this section, we have greatly benefited from the help of Mark van Hoeij (`https://www.math.fsu.edu/~hoeij/`), who explained us how to find subextensions of $\mathbb{Q}(t, B_1)$, and "simple" series in these extensions.

We can simplify (21) by working with the depressed equation, i.e., removing the quadratic term by a suitable change of variable. Indeed, defining $T(x)$ by

$$S(x) = T(x) + \frac{3xS'(0) - 2x^2 - 1}{3(x^2 + x + 1)},$$

we find that $T(x)$ satisfies a cubic equation with no quadratic term, involving t and v but not w. That is, $T(x)$ has degree 36 over $\mathbb{Q}(t, x)$, instead of 72 for $S(x)$.

Introducing $T(x)$ also helps understanding the algebraic structure of $R(x)$. Returning to (18), we recall that $R(0) = tS'(0)$ and use (19) to express $R'(0)$ in terms of $t, v,$ and w. The left-hand side simply reads $\sqrt{\Delta(x)}(T(x) - 2T(\bar{x}))$, and is found to be an element of $w\mathbb{Q}(t, x, T(x))$. In the end, $R(x)$ has degree 72 and belongs to the same extension of $\mathbb{Q}(t, x)$ as $S(x)$. This ends the proof of our main result, Theorem 1.

4   More models

For each of the 7 step sets $\mathcal{S}$ of Figure 1, we are able to define a series $A(x, y)$ that
- satisfies the same equation as $C(x, y)$ (see (9)), but with a different constant term,
- satisfies an *orbit sum* identity similar to (14).

Explaining where this series comes from would require us to introduce the group associated to a step set. For the sake of conciseness, we simply define $A(x, y)$ without further justification.

For the first four step sets $\mathcal{S}$ of Figure 1, the series $A(x, y)$ is defined by (13) (with $Q(x, y)$ counting quadrant walks with steps in $\mathcal{S}$) as we have seen. For the next two step sets,

$$C(x, y) = A(x, y) + \frac{1}{5}\left(Q(x, y) - \bar{x}^2 y Q(\bar{x}y, y) + \bar{x}^3 Q(\bar{x}y, \bar{x}) + \bar{y}^3 Q(\bar{y}, x\bar{y}) - x\bar{y}^2 Q(x, x\bar{y})\right).$$

Finally, for the seventh one,

$$\begin{aligned}
C(x, y) = A(x, y) + \frac{1}{7}\Big(&Q(x, y) - \bar{x}^2 y Q(\bar{x}y, y) + \bar{x}^4 y Q(\bar{x}y, \bar{x}^2 y)\\
&-\bar{x}^4 Q(\bar{x}, \bar{x}^2 y) - \bar{y}^3 Q(x\bar{y}, \bar{y}) + x^2\bar{y}^3 Q(x\bar{y}, x^2\bar{y}) - x^2\bar{y}^2 Q(x, x^2\bar{y})\Big).
\end{aligned}$$

In all cases, the series $A(x, y)$ satisfies the following variant of (9):

$$K(x, y)A(x, y) = P_0(x, y) - t\bar{y}H_-(x)A_{-,0}(\bar{x}) - t\bar{x}V_-(y)A_{0,-}(\bar{y}) - t\bar{x}\bar{y}A_{0,0}\mathbb{1}_{(-1,-1)\in\mathcal{S}},$$

where $K(x, y) = 1 - tS(x, y)$ as before, and $P_0(x, y)$ is a Laurent polynomial. This equation is easily obtained by combining the equations for $C(x, y)$ and $Q(x, y)$.

Finally, the vanishing orbit sum, which is (14) for the first four models, reads

$$xyA(x, y) - \bar{x}y^2 A(\bar{x}y, y) + \bar{x}^2 y A(\bar{x}y, \bar{x}) - \bar{x}\bar{y}A(\bar{y}, \bar{x}) + x\bar{y}^2 A(\bar{y}, x\bar{y}) - x^2\bar{y}A(x, x\bar{y}) = 0$$

for the next two, and

$$\begin{aligned}
xyA(x, y) - \bar{x}y^2 A(\bar{x}y, y) &+ \bar{x}^3 y^2 A(\bar{x}y, \bar{x}^2 y) - \bar{x}^3 y A(\bar{x}, \bar{x}^2 y)\\
&+ \bar{x}\bar{y}A(\bar{x}, \bar{y}) - x\bar{y}^2 A(x\bar{y}, \bar{y}) + x^3\bar{y}^2 A(x\bar{y}, x^2\bar{y}) - x^3\bar{y}A(x, x^2\bar{y}) = 0
\end{aligned}$$

for the last one. We conjecture that the series $A(x, y)$ is systematically algebraic (this is now proved for the first three models). To support this conjecture, we have tried to guess (using the `gfun` package [14] in MAPLE), for the 4 models for which it is still open, a polynomial equation for the series $A_{-1,0}$, which, in all cases, coincides with the generating function $C_{-1,0}$

of walks ending at $(-1, 0)$ (for the second model we consider $A_{-2,0}$ instead, since $A_{-1,0} = 0$ due to the periodicity of the model). This series has degree 4 (resp. 8, 24) in the three solved cases. We could not guess anything for the 4th model (using the counting sequence for such walks up to length $n = 4000$), but we discovered equations of degree 24 for each of the next three.

We believe that it would be worth exploring if the guiding principles of the present paper apply to these 4 other models. In all cases, we expect to face a *system* of quadrant-like equations rather than a single one. We plan to investigate at least some of these models.

To conclude, we recall that the 4 small step models that are algebraic for the quadrant problem are conjectured to be algebraic for the three-quadrant cone as well [1, Fig. 5]. In this case, the series $A(x, y)$ simply coincides with $C(x, y)$, as the orbit sum of $xyC(x, y)$ vanishes.

References

1 M. Bousquet-Mélou. Square lattice walks avoiding a quadrant. *J. Combin. Theory Ser. A*, 144:37–79, 2016. `doi:10.1016/j.jcta.2016.06.010`.

2 M. Bousquet-Mélou and A. Jehanne. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B*, 96:623–672, 2006. `doi:10.1016/j.jctb.2005.12.003`.

3 M. Bousquet-Mélou and M. Mishna. Walks with small steps in the quarter plane. In *Algorithmic probability and combinatorics*, volume 520 of *Contemp. Math.*, pages 1–39. Amer. Math. Soc., Providence, RI, 2010. `doi:10.1090/conm/520/10252`.

4 D. Denisov and V. Wachtel. Random walks in cones. *Ann. Probab.*, 43(3):992–1044, 2015. `doi:10.1214/13-AOP867`.

5 G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random walks in the quarter-plane: Algebraic methods, boundary value problems and applications*, volume 40 of *Applications of Mathematics*. Springer-Verlag, Berlin, 1999.

6 I. Gessel. A factorization for formal Laurent series and lattice path enumeration. *J. Combin. Theory Ser. A*, 28(3):321–337, 1980. `doi:10.1016/0097-3165(80)90074-6`.

7 I. M. Gessel and D. Zeilberger. Random walk in a Weyl chamber. *Proc. Amer. Math. Soc.*, 115(1):27–31, 1992. `doi:10.1090/S0002-9939-1992-1092920-8`.

8 L. Lipshitz. The diagonal of a D-finite power series is D-finite. *J. Algebra*, 113(2):373–378, 1988. `doi:10.1016/0021-8693(88)90166-4`.

9 L. Lipshitz. D-finite power series. *J. Algebra*, 122:353–373, 1989. `doi:10.1016/0021-8693(89)90222-6`.

10 M. Mishna and A. Rechnitzer. Two non-holonomic lattice walks in the quarter plane. *Theoret. Comput. Sci.*, 410(38-40):3616–3630, 2009. `doi:10.1016/j.tcs.2009.04.008`.

11 S. Mustapha. Non-D-finite walks in a three-quadrant cone. *Ann. Comb.*, 23(1):143–158, 2019. `doi:10.1007/s00026-019-00413-2`.

12 K. Raschel. Counting walks in a quadrant: a unified approach via boundary value problems. *J. Eur. Math. Soc. (JEMS)*, 14(3):749–777, 2012. `doi:10.4171/JEMS/317`.

13 K. Raschel and A. Trotignon. On walks avoiding a quadrant. *Electron. J. Combin.*, 26(3):Paper 3.31, 34, 2019. `doi:10.37236/8019`.

14 B. Salvy and P. Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994. `doi:10.1145/178365.178368`.

A   Final polynomial equation for $S(x)$ in the king model

The polynomial Pol involved in the cubic Equation (21) defining $S(x)$ is:

$$\mathrm{Pol}(x_0, x_1, x_2, x_3, x_4, t, x) =$$
$$- 3(x^2 + x + 1)^2(x^2 t + 2xt + x + t)(3x^2 t + 2xt - x + 3t)\,\boldsymbol{x_0^3}$$
$$+ 3(x^2 + x + 1)(x^2 t + 2xt + x + t)(3x^2 t + 2xt - x + 3t)(3x_1 x - 2x^2 - 1)\,\boldsymbol{x_0^2}$$
$$+ \big[3x^2(x^2 + x + 1)^2(2x_4 x_1 + x_4 - x_3) - 3x^2(t+1)^2(x^2 + x + 1)^2 x_2^2$$
$$+ 6x(t+1)(x^2 + x + 1)(x^4 t + 2x^2 t + x^2 + t)x_1 x_2$$
$$+ 3x(t+1)(x^2 + x + 1)(x^4 t - x^3 t - x^3 + x^2 t - xt - x + t)x_2 - 3\left(x^8 t^2 + 2x^7 t^2\right.$$
$$\left.+10x^6 t^2 + 20x^5 t^2 + 4x^5 t + 25x^4 t^2 + 20x^3 t^2 - 2x^4 + 4x^3 t + 10x^2 t^2 + 2xt^2 + t^2\right) x_1^2$$
$$- 3\left(x^8 t^2 - 11x^7 t^2 - x^7 t - 32x^6 t^2 - 9x^6 t - 53x^5 t^2 - 6x^5 t - 55x^4 t^2 + 3x^5 - 15x^4 t\right.$$
$$\left.-39x^3 t^2 - 6x^3 t - 16x^2 t^2 + x^3 - 5x^2 t^2 - 5xt^2 - xt + t^2\right) x_1 - 12x^8 t^2 - 30x^7 t^2 - 6x^7 t$$
$$- 51x^6 t^2 - 60x^5 t^2 + 3x^6 - 12x^5 t - 54x^4 t^2 - 36x^3 t^2 + 3x^4 - 6x^3 t - 21x^2 t^2 - 6xt^2$$
$$-3t^2\big]\,\boldsymbol{x_0} + x^2(x^2 + x + 1)\big[(2x_3 x^2 - 6x_4 x - 2x_3)x_1^2 - (x^2 + 2)x_3 + 3x_4 x^2$$
$$+(2x - 1)(3x_4 x + x_3(x+2))x_1\big] + 3x^3(t+1)^2(x^2 + x + 1)(x_1 - x)x_2^2$$
$$- 3x^2(t+1)x_2(x_1 - x)((2(x^2 + t(x^2 + 1)^2))x_1 + t(x^4 + x^2 + 1) - (t+1)x(x^2 + 1))$$
$$+ 3xt(x^2 + x + 1)^2(x_1 - x)(t(x^2 - x + 1)x_1^2 + (x^2 t - 5xt - x + t)x_1 + t(x^2 - x + 1)).$$