Reachability in Dynamical Systems with Rounding

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Abstract

We consider reachability in dynamical systems with discrete linear updates, but with fixed digital precision, i.e., such that values of the system are rounded at each step. Given a matrix $M \in \mathbb{Q}^{d \times d}$, an initial vector $x \in \mathbb{Q}^d$, a granularity $g \in \mathbb{Q}_+$ and a rounding operation $[\cdot]$ projecting a vector of $\mathbb{Q}^d$ onto another vector whose every entry is a multiple of g, we are interested in the behaviour of the orbit $\mathcal{O} = \langle [x], [M[x]], [M[M[x]]], \ldots \rangle$, i.e., the trajectory of a linear dynamical system in which the state is rounded after each step. For arbitrary rounding functions with bounded effect, we show that the complexity of deciding point-to-point reachability – whether a given target $y \in \mathbb{Q}^d$ belongs to $\mathcal{O}$ – is **PSPACE**-complete for hyperbolic systems (when no eigenvalue of M has modulus one). We also establish decidability without any restrictions on eigenvalues for several natural classes of rounding functions.

1   Introduction

A *discrete-time linear dynamical system* in ambient space $\mathbb{Q}^d$ is specified via a linear transformation together with a starting point. The state of the system is then updated at each step by applying the linear transformation, giving rise to an *orbit* (or infinite trajectory) in $\mathbb{Q}^d$.

One of the most well-known questions for such systems is the *Skolem Problem*, which asks whether the orbit ever hits a given $(d-1)$-dimensional hyperplane.[1] This problem has long eluded decidability, although instances of dimension $d \leq 4$ are known to be solvable (see, e.g., the survey [30]). Another natural problem is *point-to-point reachability*[2], known to be decidable in polynomial time [22]. In both cases, however, one assumes *arbitrary* precision, which arguably is unrealistic for simulations carried out on digital computers. In this paper, we therefore turn our attention to instances of these problems in which the numerical state of the system is rounded to finite precision at each time step. This leads us to the following definition:

▶ **Problem** (Rounded Point-to-Point Reachability (Rounded P2P)). *Given a matrix $M \in \mathbb{Q}^{d \times d}$, an initial vector $x \in \mathbb{Q}^d$, a target vector $y \in \mathbb{Q}^d$, a granularity $g \in \mathbb{Q}_+$, and a rounding operation $[\cdot]$ projecting a vector of $\mathbb{Q}^d$ onto another vector whose every entry is a multiple of g, let the orbit $\mathcal{O}$ of this system be the infinite sequence $\langle [x], [M[x]], [M[M[x]]], \dots \rangle$, i.e., $x^{(0)} = [x]$ and $x^{(i+1)} = [Mx^{(i)}]$. The **Rounded Point-to-Point Reachability (Rounded P2P) Problem** asks whether $[y] \in \mathcal{O}$.*

Main contributions.   We make the following contributions, summarised in Figure 1:
1. We introduce a family of natural problems, Rounded P2P (parameterised by the rounding function), which to the best of our knowledge has not previously been studied.
2. We show that for hyperbolic systems (i.e., those whose associated linear transformation has no eigenvalue of modulus 1) the Rounded P2P Problem is solvable – and is in fact **PSPACE**-complete – for any "reasonable" (i.e., bounded-effect) rounding function. It is interesting to note, in contrast, that exact P2P reachability is known to be solvable in polynomial time. Our approach to solving the Rounded P2P Problem relies on the observation that, outside a ball of exponential size, the change in magnitude of the system state at each step dwarfs any effect due to rounding. It thus suffices to exhaustively examine the effect of the dynamics inside an exponentially bounded state space.
3. In the general case (without any restriction on the magnitude of eigenvalues), the effect of rounding may forever remain non-negligible, requiring a careful analysis. We have not been able to solve the problem in full generality, but we do provide a complete solution for certain natural classes of rounding functions. More precisely, assume that the linear transformation has been converted to Jordan normal form (now requiring us to work with complex algebraic numbers). We can then solve the Rounded P2P Problem under two natural classes of rounding functions:
   (a) *Polar rounding functions:* given a complex number of the form $Ae^{i\theta}$, such functions round A and θ independently. In such instances we can handle in **EXPSPACE** all reasonable rounding functions on A, and what we view as the only natural rounding function on θ.

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[1]   The Skolem Problem is usually formulated in terms of linear recurrence sequences, but is equivalent to the description given here.
[2]   Historically this problem has been known as the *orbit problem*, however there are now multiple "orbit problems" (polytope reachability, hyperplane reachability, (semi-)algebraic set reachability,... etc.) and so we specify point-to-point reachability.

Rounding type	Hyperbolic Systems	No restrictions on eigenvalues	
		Jordan normal form (Note: no hardness)	General
Polar $Ae^{i\theta}$	**PSPACE**-complete, Section 3	**EXPSPACE**, Section 4.1	Open but **PSPACE**-hard
Argand truncation or expansion		**EXPSPACE**, Section 4.2	
Argand minimal error		Open (difficulties highlighted in Section 5)	
Arbitrary bounded-effect		Open (Open Problem 21)	

■ **Figure 1** Decidability and complexity table for the Rounded P2P Problem.

   (b) *Argand rounding:* given a complex number of the form $a + bi$, the *Argand truncation* will round a and b independently downwards (in magnitude), ensuring that the modulus never increases. Similarly, the *Argand expansion* (which rounds a and b independently upwards) guarantees that the modulus can only increase. Under such rounding functions, we show decidability in **EXPSPACE**.

4. We highlight some limitations of our methods, identifying a simple but technically challenging open problem, which points to some of the key difficulties in solving the Rounded P2P Problem in full generality. More precisely, we consider minimal error rounding for a simple rotation in two-dimensional space, for which Rounded P2P is presently open.

▶ Remark. It is worth noting that the rounded versions of the Skolem Problem (does the rounded orbit ever hit a $(d-1)$-dimensional hyperplane?) and the Positivity Problem (does the rounded orbit ever hit a d-dimensional half-space?) remain at least as hard as their exact integer counterparts, since over the integers rounding has no effect; the decidability of these problems therefore remains open. However, the rounded versions of reaching a bounded polytope or a bounded semialgebraic set (problems not known to be decidable in the exact setting [14, 4]) reduce to a finite number of Rounded P2P reachability queries (since a bounded set can contain only finitely many rounded points). These observations together motivate our focus, in the present paper, on the Rounded P2P Problem.

It is interesting to consider rounded reachability problems in the stochastic setting, i.e., Markov chains. One observes that the state space $[\mathcal{S}] = \{[x] \in [0,1]^d \mid x \text{ sub-stochastic}\}$ is finite, which entails decidability of virtually any reachability problem, including Skolem and Positivity. This is somewhat arresting, since without rounding reachability problems are known to be exactly as hard for stochastic systems as for general systems [3]. In any event, one should note that ensuring that for all $x \in [\mathcal{S}]$, $[Mx] \in [\mathcal{S}]$ requires some care, as arbitrary rounding does not necessarily preserve (sub-)stochasticity.

Related work

With the emerging use of numerical computations during the 80s, doubts were raised concerning the transferability of results about dynamical systems obtained by simulation in finite-state machines. In this direction, the sensitivity that a rounding function may have on the long-term behaviour of a dynamical system is studied in [5]. How rounded orbits can be simulated by actual orbits of the dynamical system is investigated in [20, 29].

The series of papers [6, 7, 8, 9] examines which statistical properties of a discrete dynamical system are preserved under the introduction of a rounding function, a good summary of which can be found in Blank's book [10, Chapter 5]. As the rounding is refined, some properties

of the discretized orbits follow probabilistic laws asymptotically, as shown in [16, 17]. The paper [18] studies how volatile statistical notions are in the presence of finite precision (such as the mean distance of two orbits of discrete dynamical systems).

Another line of research focuses on discretized rotations in $\mathbb{Z}^2$ and higher-dimensional lattices [24, 1]. A connection from roundoff problems in the 2-dimensional case to expanding maps on the p-adic integers is described in [11, 36]. Building on this, [35] conjectures periodicity of all orbits of these discretized rotations in $\mathbb{Z}^2$. It is shown in [2] that there are infinitely many periodic orbits, and [31] attempts to concisely describe points leading to periodic orbits.

In the context of model checking, continuous dynamical systems have been translated into discrete models, mainly timed automata that approximate the behaviour of the original system [26, 13, 32]. On a more general level, one can observe a growing interest in the systematic study of roundoff errors inherent in finite precision computations [19, 33, 21, 25, 27, 15].

2   Rounding functions

Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{A}$ be the naturals, integers, rationals, reals, and algebraic numbers respectively.

Rounding real numbers

Let $g \in \mathbb{R}_+$ be a granularity. We define our rounding functions taking values to integers, i.e., $g = 1$. For $g \neq 1$ we consider $[x] = g \cdot [x/g]$. Given a set S, we let $[S] = \{[x] \mid x \in S\}$.

The floor function $\lfloor x \rfloor$ and ceiling functions $\lceil x \rceil$ are well-known rounding functions in mathematics and computer science. We recall two further rounding functions:

- *Minimal error rounding* rounds to the *nearest* value: $[x] = \arg\min_{y \in \mathbb{Z}} |x - y|$. If $|x - y| = 0.5$ an arbitrary but deterministic choice must be made (e.g. to round up).
- *Truncation* ("towards zero rounding", to cut off the remaining bits): if $x > 0$ then $\lfloor x \rfloor$ else $\lceil x \rceil$, or *expansion*: if $x > 0$ then $\lceil x \rceil$ else $\lfloor x \rfloor$.

Whenever possible, we prefer to analyse the problems without choosing a specific rounding function, relying only upon the property of *bounded effect*:

▶ **Definition 1.** *A real rounding function* $[\cdot] \colon \mathbb{R} \to \mathbb{R}$ *has bounded effect if there exists* Δ *such that* $|x - [x]| \leq \Delta$ *for all* x.

Rounding complex numbers

Complex numbers have both a real and imaginary part. Thus one can consider rounding each of the components separately, which we call *Argand rounding*. Consider $x = a + bi$ with $a, b \in \mathbb{R}$, then let $[x] = [a] + [b]i$, where $[\cdot]$ can be any real rounding function (leading to *Argand truncation*, *Argand expansion* and *Argand minimal error rounding* functions).

However, complex numbers can also be readily represented using polar coordinates as follows: a number is represented as $x = Ae^{i\theta}$, where A is the modulus and θ is the angle between the 2-d coordinates $(1, 0)$ and (a, b) (when represented as $a + bi$). Then, a *polar rounding* function rounds A and θ independently, i.e. $[x] = [A]e^{i[\theta]}$. The rounding of $[A]$ can be any real rounding function. For the rounding of the angle we always assume minimal error rounding. That is, given granularity $\theta_g = \frac{\pi}{R}$ for some $R \in \mathbb{N}$, then $[\theta]$ is a multiple of θ_g with minimal error and arbitrary but deterministic tie breaking.

We generalise non-specific bounded-effect rounding to the complex numbers.

▶ **Definition 2.** *A complex rounding function* $[\cdot] \colon \mathbb{C} \to \mathbb{C}$ *has bounded effect **on the modulus** if there exists* Δ *such that* $||x| - |[x]|| \leq \Delta$ *for all* x.

Argand and polar roundings are both defined by applying bounded-effect real rounding functions to each component, and have bounded effect under Definition 2. However, note the distinction with Definition 1; polar rounding can exhibit arbitrary large effects (in the following sense: given any $\Delta > 0$, one can always find $x \in \mathbb{C}$ such that $|x - [x]| > \Delta$), but nevertheless has only bounded effect *on the modulus*.

▶ **Definition 3** ([K]-Ball). *Given a complex rounding function* $[\cdot]$ *and an integer* K *let a* [K]*-ball be the set of admissible points of modulus at most* K, *i.e.,* $\{[x] \mid x \in \mathbb{C}, \ |[x]| \leq K\}$.

Rounding vectors

In general, a rounding function on $\mathbb{K}$ induces a rounding function on vectors $\mathbb{K}^d$, where $[(x_1, \ldots, x_d)] = ([x_1], \ldots, [x_d])$, although not all rounding functions on vectors need take this form. We generalise non-specific bounded-effect rounding to vectors.

▶ **Definition 4.** *A rounding function* $[\cdot] \colon \mathbb{K}^d \to \mathbb{K}^d$ *has bounded effect on the modulus if there exists* Δ *such that* $||x|_k - |[x]|_k| \leq \Delta$ *for all* x *and every* $k \in \{1, 2, \ldots, d\}$.

Finally, we assume that all of our rounding functions can be computed in polynomial time and are fixed (rather than inputs) in our problems, and thus Δ is also a fixed parameter.

3 Hyperbolic systems

In this section we establish our first main result for hyperbolic systems, which we first define:

▶ **Definition 5** (Hyperbolic System [23, Section 1.2]). *A linear map represented by the matrix* $M \in \mathbb{R}^{d \times d}$ *is hyperbolic if all of its eigenvalues have modulus different from one.*

▶ **Theorem 6.** *The Rounded P2P Problem is* **PSPACE***-complete for hyperbolic linear maps represented by rational matrices and real rounding functions with bounded effect.*

We first demonstrate that the problem is in **PSPACE** for matrices in Jordan normal form, to which we will reduce the general case in a second step. As the passage to Jordan normal form inevitably introduces complex numbers, **PSPACE** membership will be shown for Jordan normal form matrices over the algebraic numbers and, accordingly, complex rounding functions with bounded effect on the modulus. To complete the picture we show hardness for hyperbolic systems (in fact, the hardness result applies even for non-hyperbolic systems, that is for matrices whose eigenvalues may include 1).

3.1 Membership in PSPACE

We now prove the membership part of Theorem 6 under the additional assumption that the matrices are in Jordan normal form.

▶ **Lemma 7.** *The Rounded P2P Problem decidable in* **PSPACE** *for any complex rounding function with bounded effect on the modulus* Δ *and hyperbolic matrices* $M \in \mathbb{A}^{d \times d}$ *in Jordan normal form.*

Proof. We consider a single Jordan block of dimension d with eigenvalue λ. If the matrix M has multiple Jordan blocks, the algorithm can be run in lock step[3] for each block. Hence, without loss of generality we let

$$M = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

The idea will be to show that for $|\lambda| > 1$, for values large enough growth will outstrip the rounding, and the orbit will grow beyond the target, never to return. If $|\lambda| < 1$ and the orbit gets large enough, it will begin to contract again, so we choose a ball large enough to contain the whole orbit. We do not consider the case $|\lambda| = 1$ here.

Formally, in each dimension $k \in \{1, \ldots, d\}$ we compute a radius C_k, defining a $[C_k]$-ball of radius C_k about 0, containing x_k and y_k such that for all z in the orbit $\mathcal{O}$ if $z_k \notin [C_k]$-ball then $[Mz]_k \notin [C_k]$-ball. That is, if the orbit has left the ball, it will never come back. The algorithm proceeds by simulating the orbit from x until one of the following occurs.

- y is found, in which case RETURN YES, or
- a point repeats, in which case RETURN NO, or
- a point $x^{(i)}$ is found such that $\left|(x^{(i)})_k\right| \geq C_k$ for some k, in which case RETURN NO.

Since $B = [\{x \in \mathbb{R}^d \mid \text{for all } k \ |x_k| \leq C_k\}]$ is finite, one of the three must occur. Remembering all previous points would require too much space. Therefore we record a counter of the number of steps taken and once this exceeds the maximum number of points then we know some point must have been repeated (possibly many times by this point). Let $C = \max_i C_k$, then the bounding hyper-cube of B has $(2C/g)^d$ points, hence B has fewer points. We show this number has at most exponential size in the description length of the input, and hence can be represented in **PSPACE**.

▶ Case 1 (suppose $|\lambda| > 1$). For the dth component we have $(x^{(i+1)})_d = [\lambda(x^{(i)})_d]$. There is a bounded effect of the rounding Δ, ensuring $\left|(x^{(i+1)})_d\right| \geq |\lambda| \left|(x^{(i)})_d\right| - \Delta$. So when $|\lambda| \left|(x^{(i)})_d\right| - \Delta > \left|(x^{(i)})_d\right|$, this component must grow. Let $\ell = \max\{1, \Delta, |y_1|, \ldots, |y_d|\}$. We define the radius $C_d := \frac{\Delta}{|\lambda|-1} + \ell$, which satisfies the desired property described above.

Now suppose that the radius C_k is defined so that $C_k \leq \ell \sum_{j=0}^{d-k+1} (\frac{2}{|\lambda|-1})^j$ (holds for $k = d$) and assume that $\left|(x^{(i)})_j\right| \leq C_j$ for each $j \in \{k, \ldots, d\}$. For the $k-1$th dimension the update is of the form $(x^{(i+1)})_{k-1} = [\lambda(x^{(i)})_{k-1} + 1(x^{(i)})_k]$. Since $\left|(x^{(i)})_k\right| \leq C_k$, we have $\left|(x^{(i+1)})_{k-1}\right| \geq |\lambda| \left|(x^{(i)})_{k-1}\right| - \Delta - C_k$, and there is growth when $|\lambda| \left|(x^{(i)})_{k-1}\right| - \Delta - C_k > \left|(x^{(i)})_{k-1}\right|$, i.e., when $\left|(x^{(i)})_{k-1}\right| > \frac{\Delta + C_k}{|\lambda|-1}$. So, we may define $C_{k-1} := \frac{\Delta + C_k}{|\lambda|-1} + \ell$, which satisfies the property described above, and moreover, $C_{k-1} \leq \frac{2C_k}{|\lambda|-1} + \ell \leq \ell \sum_{j=0}^{d-(k-1)+1} (\frac{2}{|\lambda|-1})^j$ due to our choice of ℓ. Repeat for all remaining components $k-2, \ldots, 1$.

Now $C_k \leq \ell \sum_{j=0}^{d} (\frac{2}{|\lambda|-1})^j \leq \ell(d+1)(1 + (\frac{2}{|\lambda|-1})^d)$ for each k, and the claim follows.

▶ Case 2 (suppose $|\lambda| < 1$). We require the ball to have the property that if the orbit leaves, it will never come back. However for $|\lambda| < 1$, while initially there may be some growth (due to other components), once large enough $|\lambda|$ will dominate and the modulus will decrease. Therefore, we want to ensure we choose the ball large enough that the orbit will never leave the ball in the first place. The following definitions of the radii C_j can easily be altered to furnish this requirement.

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[3] By running processes in lock step, here and elsewhere, we mean running all of the processes simultaneously (interleaving instructions for each process) until either $x^{(i)} = y$ or one of the processes concludes non-reachability.

Consider the last component d: we have $\left|(x^{(i+1)})_d\right| \leq |\lambda| \left|(x^{(i)})_d\right| + \Delta$. Set again $\ell = \max\{1, \Delta, |y_1|, \ldots, |y_d|\}$ and define $C_d := \frac{\Delta}{1-|\lambda|} + \ell$; if $\left|(x^{(i)})_d\right| \leq C_d$, then $\left|(x^{(i+1)})_d\right| \leq C_d$.

Having fixed $C_{k'}$ for $k' \in \{k, \ldots, d\}$, consider component $k-1$: We have $(x^{(i+1)})_{k-1} = [\lambda(x^{(i)})_{k-1} + (x^{(i)})_k]$, and so $\left|(x^{(i+1)})_{k-1}\right| \leq |\lambda| \left|(x^{(i)})_{k-1}\right| + \left|(x^{(i)})_k\right| + \Delta$. Let us define $C_{k-1} := \frac{C_k + \Delta}{1-|\lambda|} + \ell$. Now if $\left|(x^{(i)})_{k-1}\right| \leq C_{k-1}$ then $\left|(x^{(i+1)})_{k-1}\right| \leq C_{k-1}$. Repeat for each remaining component. It can be shown, similar to the previous case, that $C_k \leq \ell(d+1)(1 + (\frac{2}{1-|\lambda|})^d)$ for each k, and this concludes the proof. ◄

Reducing the general form to Jordan normal form

In the previous section we assumed that the matrix is always in Jordan normal form, which is a significant restriction. In this section we will not assume Jordan normal form, which means we cannot make any assumption about the rounding, other than being of bounded effect, to prove Theorem 6. After a change of basis properties such as "rounding towards zero" may not be preserved.

Proof (upper bound of Theorem 6). Let Δ be the fixed, bounded effect on the modulus of $[\cdot]$. Let us consider hyperbolic $M = PJP^{-1} \in \mathbb{Q}^{d \times d}$. We ask whether $x^{(i+1)} = y$ for some i. Observe that $x^{(i+1)} = [Mx^{(i)}] = Mx^{(i)} + e(Mx^{(i)})$ where $e(x) := [x] - x \in [-\Delta, \Delta]^d$ for any x since $[\cdot]$ has bounded effect. Now if we define $z^{(i)} := P^{-1}x^{(i)}$ we have that

$$z^{(i+1)} = P^{-1}x^{(i+1)} = P^{-1}(Mx^{(i)} + e(Mx^{(i)})) = Jz^{(i)} + P^{-1}e(PJz^{(i)}) = (\!|Jz^{(i)}|\!)$$

where $(\!|z|\!) := z + P^{-1}e(Pz)$ for any z. The question $x^{(i)} \stackrel{?}{=} y$ for some i now becomes equivalent to $z^{(i)} \stackrel{?}{=} P^{-1}y$. But note that the system for $z^{(i)}$ is in Jordan normal form and the rounding function $(\!|\cdot|\!)$ has bounded effect on the modulus, with bound $\Delta' \leq \max_{1 \leq k \leq d} \max_{e \in [-\Delta, \Delta]^d} (P^{-1}e)_k$. Since Δ is fixed and P^{-1} is computable in polynomial time [12], then Δ' is of polynomial size. Hence, we have produced in polynomial time an instance of the Rounded P2P problem with a matrix in Jordan normal form. As the proof of Lemma 7 shows that this problem is solvable in **PSPACE** even if Δ is given as input, we can conclude that the **PSPACE** upper bound holds also for the general case. ◄

3.2 PSPACE-hardness

We will prove **PSPACE**-hardness (i.e., the lower bound of Theorem 6) by reduction from quantified boolean formula (QBF), which is **PSPACE**-complete [34]. We do this by first encoding a simple programming language into the rounded P2P Problem. Then, we show that reachability in this language can solve QBF. Whilst a direct reduction is possible, we provide exposition via the language for two reasons; first, we will show that the language is robust to choice of rounding function (Remark 9), and secondly the reduction results in an instance where all eigenvalues have modulus 1, but by a small perturbation, we observe that the problem remains hard when all of the eigenvalues do not have modulus 1 (Remark 10).

The language will consist of m instructions, operating over d variables. Each instruction is a boolean map $f_i : [0,1]^d \to [0,1]^d$, where each dimension i is updated using a logical formula of the d inputs. Each of the m instructions is conducted in turn and updating the d variables is *simultaneous* in each step. Thus, references to variable in a function are the evaluation in the previous step. Once the m instructions are complete, the system returns to the first instruction and repeats ($x^{(i)} = (f_m \circ f_{m-1} \circ \cdots \circ f_2 \circ f_1)(x^{(i-1)})$, see also Algorithm 1).

An instruction is encoded into the rounded dynamical system using a map $f_i : \mathbb{N}^d \to \mathbb{N}^d$ for $0 \leq i \leq m-1$, where instructions are of the form $(f_i(x))_j = \lfloor (p_j \cdot x) \rceil$ where p_j in $\mathbb{Q}^d$. We demonstrate how to encode the required logical operations in a rounded dynamical

■ **Algorithm 1** System behaviour of the language.

Input: $x \in [0,1]^d$ initial vector, $y \in [0,1]^d$ target vector

while $x \neq y$ **do**

$\quad\Big|\quad x \leftarrow f_1(x)$

$$x \leftarrow f_2(x) \overset{e.g.}{=} \begin{cases} x_1 \leftarrow x_2 \vee (x_5 \wedge x_3) \\ x_2 \leftarrow \texttt{if } (x_1 \vee x_3) \texttt{ then } x_6 \texttt{ else } x_2 \\ x_3 \leftarrow \texttt{true} \\ \vdots \\ x_d \leftarrow x_4 \end{cases}$$

$\quad\quad \vdots$

$\quad\Big|\quad x \leftarrow f_m(x)$

end

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system: and $(x_i \leftarrow x_j \wedge x_k = \left\lfloor \frac{1+x_j+x_k}{3} \right\rfloor)$, or $(x_i \leftarrow x_j \vee x_k = \left\lfloor \frac{1+x_j+x_k}{2} \right\rfloor)$, negation $(x_i \leftarrow \neg x_j = \lfloor 1 - x_j \rfloor)$, resetting a variable to $\texttt{false}$ $(x_i \leftarrow \lfloor 0 \rfloor)$, copying a variable without change $(x_i \leftarrow \lfloor x_i \rfloor)$ or moving/duplicating a variable $(x_i \leftarrow \lfloor x_j \rfloor)$. To enable this, we will assume there is always access to the constant 1 (or $\texttt{true}$) by an implicit dimension, fixed to 1.

In multiple steps any logical formula can be evaluated. This can be done with auxiliary variables to store partial computations, where the instructions will in fact be multi-step instructions making use of a finite collection of auxiliary variables which will not be referenced explicitly. Meanwhile any unused variables can be copied without change. In particular the syntax $x_1 \leftarrow \texttt{if } (x_2) \texttt{ then } x_3 \texttt{ else } x_4$ can be encoded, by equivalence with the logical formula $x_1 \leftarrow ((x_2 \implies x_3) \wedge (\neg x_2 \implies x_4))$.

We ask, given some initial configuration $x^{(0)}$, and a target y: does there exist i such that $x^{(i)} = y$. If there was just one step function, the system dynamics would be a direct instance of the rounded orbit semantics. When there are m functions, we remark the sequence of functions can be encoded by taking m copies of each variable, and each function f_i, can transfer the function from one copy to the next, zeroing the previous set of variables. That is, let

$$M = \begin{bmatrix} 0 & & & & f_m \\ f_1 & 0 & & & \\ & f_2 & 0 & & \\ & & & \ddots & \\ & & & f_{m-1} & 0 \end{bmatrix}.$$

Then the initial configuration becomes $(x^{(0)}, 0, \ldots, 0)$, and the target becomes $(y, 0, \ldots, 0)$.

An abstraction of the language is depicted in Algorithm 1. It remains to show that QBF can be encoded in the language.

▶ **Lemma 8.** *Reachability in this language can solve QBF.*

Proof. Formally we write a program in our language to decide the truth of a formula of the form $\forall x_1 \exists x_2 \forall x_3 \ldots \exists x_n \psi(x_1, \ldots, x_n)$, where ψ is a quantifier free boolean formula. For convenience we assume it starts with $\forall$, ends with $\exists$ and alternates. Formulae not in this form can be padded if necessary with variables which do not occur in the formula ψ.

The program will have the following variables: $x_1, \ldots, x_n, \hat{\psi}, s_1^0, \ldots, s_n^0, s_1^1, \ldots, s_n^1$ and $c_1, \ldots, c_n$. The bits $x_1, \ldots, x_n$ represent the current allocation to the corresponding bit

variables of ψ, and $\hat{\psi}$ will store the current evaluation of $\psi(x_1, \ldots, x_n)$. To cycle through all allocations to $x_1, \ldots, x_n$, the variables will be treated as a binary number and incremented by one many times, for this purpose the bits $c_1, \ldots, c_n$ represent the carry bits when incrementing $x_1, \ldots, x_n$.

The intuition of s_i^z is the following: for fixed $x_1, \ldots, x_{i-1}$ it stores the evaluation of $Q x_{i+1} \, Q' x_{i+2} \ldots \exists x_n \psi(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$ where $Q, Q' \in \{\exists, \forall\}$ as required by the formula. Therefore the overall formula is true if and only if $s_1^0 \wedge s_1^1$ is eventually true.

We define $3 + n$ instructions, and each run through $f_1 \to f_{3+n}$ will cover exactly one allocation to $x_1, \ldots, x_n$, with the next run through covering the next allocation that one gets by incrementing the rightmost bit. Once x_{i+1} has been in both the 1 state and the 0 state for all values below, we have enough information to set $s_{i+1}^{x_i}$. This is set when the carry-bit c_{i+1} is one, which indicates that x_{i+1} has visited both 0 and 1 and is being returned back to 0 (thus setting $x_{i+1} = \cdots = x_n$ back to 0).

We let the initial configuration be $(0, 0 \ldots, 0)$. Note that this is hiding the implicit dimension that is always 1. Each of the following step functions should be interpreted as copying any variable that is not explicitly set.

Step 1.

Evaluate ψ

$$f_1(\cdot) = \left\{ \hat{\psi} \leftarrow \psi(x_1, \ldots, x_n) \right.$$

Step 2.

Update either s_n^0 or s_n^1

$$f_2(\cdot) = \begin{cases} s_n^0 \leftarrow \texttt{if } (x_n = 0) \texttt{ then } \hat{\psi} \\ \qquad\qquad\qquad \texttt{else } s_n^0 \\ s_n^1 \leftarrow \texttt{if } (x_n = 1) \texttt{ then } \hat{\psi} \\ \qquad\qquad\qquad \texttt{else } s_n^1 \end{cases}$$

Step 3.

Start incrementing x_n

$$f_3(\cdot) = \begin{cases} x_n \leftarrow \neg x_n \\ c_n \leftarrow x_n \end{cases}$$

Step $3 + n - i$, for $i = n - 1$ to 1.

If there is a carry, update s_i^z and continue incrementing

i even (x_i universally quantified):

$$f_{3+n-i}(\cdot) = \begin{cases} x_i \leftarrow \texttt{if } (c_{i+1}) \texttt{ then } \neg x_i \texttt{ else } x_i \\ c_i \leftarrow c_{i+1} \wedge x_i \\ c_{i+1} \leftarrow 0 \\ s_i^0 \leftarrow \texttt{if } (c_{i+1} \wedge \neg x_i) \texttt{ then } s_{i+1}^0 \wedge s_{i+1}^1 \\ \qquad\qquad\qquad \texttt{else } s_i^0 \\ s_i^1 \leftarrow \texttt{if } (c_{i+1} \wedge x_i) \texttt{ then } s_{i+1}^0 \wedge s_{i+1}^1 \\ \qquad\qquad\qquad \texttt{else } s_i^1 \end{cases}$$

i odd (x_i existentially quantified):

$$f_{3+n-i}(\cdot) = \begin{cases} x_i \leftarrow \texttt{if } (c_{i+1}) \texttt{ then } \neg x_i \texttt{ else } x_i \\ c_i \leftarrow c_{i+1} \wedge x_i \\ c_{i+1} \leftarrow 0 \\ s_i^0 \leftarrow \texttt{if } (c_{i+1} \wedge \neg x_i) \texttt{ then } s_{i+1}^0 \vee s_{i+1}^1 \\ \qquad\qquad\qquad \texttt{else } s_i^0 \\ s_i^1 \leftarrow \texttt{if } (c_{i+1} \wedge x_i) \texttt{ then } s_{i+1}^0 \vee s_{i+1}^1 \\ \qquad\qquad\qquad \texttt{else } s_i^1 \end{cases}$$

Step $3 + n$.

Set every variable to 1 if QBF satisfied. After this step, the program returns to f_1.

$$f_{3+n}(\cdot) = \left\{ v \leftarrow \texttt{if } (s_1^0 \wedge s_1^1) \texttt{ then } 1 \texttt{ else } v \quad \text{(for all variables } v) \right.$$

The (3+n)th step ensures that configuration $(1, \ldots, 1)$ will be reached if and only if the given QBF formula is satisfied. ◀

▶ **Remark 9** (Choice of rounding function). The presentation here relies on specific choices of rounding function, but we observe that the language can easily exchange several different natural rounding functions, so the reduction is robust. The rounding is only useful in the **and** and **or** instructions. The floor function can be replaced by essentially any other rounding. For example $x_j \vee x_k = \left\lceil \frac{x_j + x_k}{2} \right\rceil$ and $x_j \wedge x_k = \left\lceil \frac{-1 + x_j + x_k}{2} \right\rceil$. Similarly, when $[\cdot]$ is minimal error rounding then $x_j \vee x_k = \left[\frac{1 + x_j + x_k}{3} \right]$ and $x_j \wedge x_k = \left[\frac{x_j + x_k}{3} \right]$ (the break point is not used). Thus, the problem will also be hard for any of these roundings.

▶ **Remark 10** (Perturbation: ensuring the eigenvalues are not modulus 1). Observe that under the perturbation that multiplies each operation by 1.1 (before taking floor) we obtain the same resulting operation. For example $x_i \leftarrow x_j \vee x_k = \left\lfloor \frac{1+x_j+x_k}{2} \right\rfloor$ is equivalent to $x_i \leftarrow x_j \vee x_k = \left\lfloor \left(\frac{1+x_j+x_k}{2}\right) * 1.1 \right\rfloor$. Hence, if the resulting matrix M has eigenvalues 1, taking $1.1M$ (or similar value to 1.1) will result in a matrix that does not with the same orbit; which shows that hardness is retained for matrices in which no eigenvalue has modulus 1.

▶ **Remark 11** (Dimension). The hardness result needs reachability instances of unbounded dimension. For a QBF formula with n variables and ℓ logical operations, the resulting instance of rounded P2P has dimension $(3n + 1 + \ell)(4n + 15 + \ell)$.

4   Special cases on non-hyperbolic systems

In this section we consider certain cases when the eigenvalues can be of modulus one. In particular we work in the Jordan normal form and show that the problem can be solved for certain types of rounding. We fall short of arbitrary deterministic rounding, which would be required to show the problem in full generality through the Jordan normal form approach.

First, we show decidability for polar-rounding, along with an example with numbers requiring exponential space by the time the system becomes periodic – seeming to imply any "wait and see" approach would require **EXPSPACE**. We also show decidability for certain types of Argand rounding, in particular truncation and expansion, but minimal-error rounding remains open (which we discuss further in Section 5).

4.1   Polar rounding with updates in Jordan normal form

We restrict ourselves to a Jordan block M of dimension d, with eigenvalue λ of modulus 1. Since the polar rounding function has bounded effect *on the modulus*, the remaining blocks, which need not be of modulus 1 can be solved (Lemma 7) by running this algorithm in lock step with the algorithm for those blocks. All together, this gives us:

▶ **Theorem 12.** *The Rounded P2P Problem is decidable in* **EXPSPACE** *for the polar rounding function with* $\theta_g = \frac{\pi}{R}$, $R \geq 2$ *and matrices* $M \in \mathbb{A}^{d \times d}$ *in Jordan normal form.*

To prove Theorem 12 we show that each dimension $d, d-1, \ldots, 1$ will eventually be periodic on a fixed modulus, or permanently diverge beyond y_k (the target value in dimension k).

Let $\langle a, b \rangle$ be the smallest angle between vectors a and b – this is a value in $[0, \pi]$ and, in particular, it is always positive. It is used as a measure of alignment: the more a and b are aligned the smaller $\langle a, b \rangle$ is. We will assume that the system will round up if $[x] - x = 0.5$. The remaining case can be adapted by suitably adjusting the relevant inequalities. We say that a dimension $k \in \{1, \ldots, d\}$ is *just rotating* after position N, if for all $i \geq N$: $(x^{(i+1)})_k = [\lambda (x^{(i)})_k]$. Note that dimension d is just rotating after 0, by definition. Our goal is to show that every dimension k will eventually be just rotating (for which we would require it to have modulus $|y_k|$) or reach a point that lets us conclude it has permanently diverged past y_k. So we assume, henceforth, that dimension k is just rotating.

We let $\phi(i) = \langle \lambda(x^{(i)})_{k-1}, (x^{(i)})_k \rangle$. As $(x^{(i+1)})_{k-1} = [\lambda(x^{(i)})_{k-1} + (x^{(i)})_k]$, small values of $\phi(i)$ (between 0 and $\pi/2$) lead to an increase in modulus of $(x^{(i+1)})_{k-1}$, whereas large values (between $\pi/2$ and π) lead to a decrease when $\left|(x^{(i)})_{k-1}\right|$ is sufficiently large relative to $\left|(x^{(i)})_k\right|$. Our analysis relies on the fact that $\phi(i)$ can never increase:

▶ **Lemma 13.** *Suppose that dimension k is just rotating after step N. Then, for all $i \geq N+1$:* $\phi(i) \geq \phi(i+1)$.
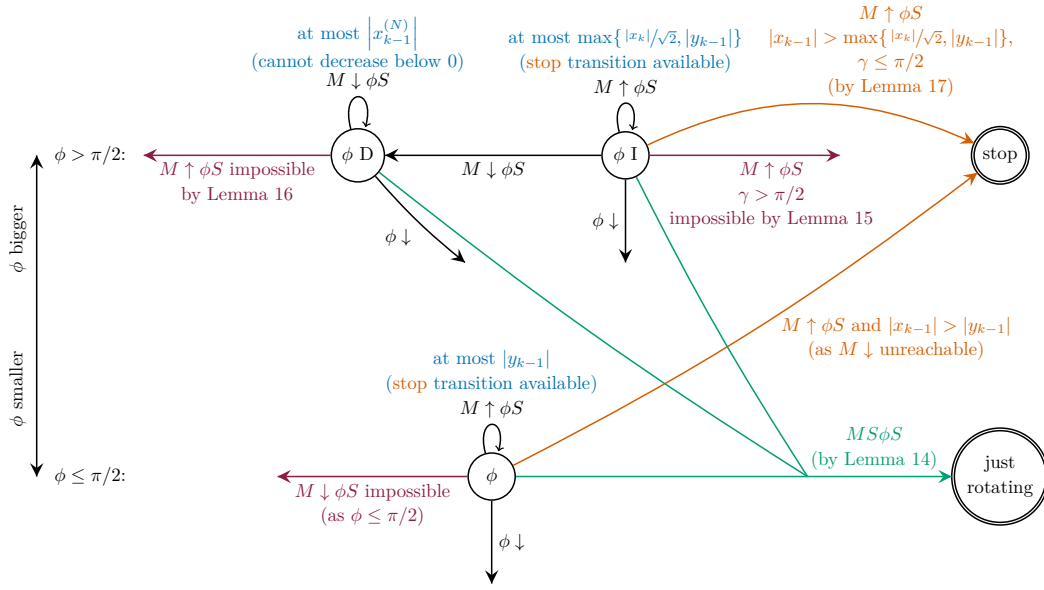
Figure 2 State diagram for ϕ whilst considering dimension $k-1$, assuming k is *just rotating*.

If dimension $k-1$ repeats its relative angle to k and its modulus in some step, we can conclude that $k-1$ is just rotating:

▶ **Lemma 14.** *Suppose that dimension k is just rotating after step N, that $\phi(N) = \phi(N+1)$ and $\left|(x^{(N)})_{k-1}\right| = \left|(x^{(N+1)})_{k-1}\right|$. Then, dimension $k-1$ is just rotating after N.*

If the precondition of Lemma 14 holds, we move to the next dimension $k-2$. Otherwise, we want to give a bound such that whenever $\left|(x^{(i)})_{k-1}\right|$ exceeds it, we can conclude that it never decreases back to $|y_{k-1}|$. We first introduce the angle $\gamma(i) = \left\langle \lambda(x^{(i)})_{k-1} + (x^{(i)})_k, \lambda(x^{(i)})_{k-1} \right\rangle$. The angle $\gamma(i)$ decreases with increasing $\left|(x^{(i)})_{k-1}\right|$, as dimension k is just rotating and hence does not change in modulus. We observe that $\gamma(i) \leq \phi(i)$ for all i. The following shows that an increase in modulus caused by crossing an "axis" (i.e. if $\gamma(i) > \pi/2$) can only happen once, as in the next step, the angle will have decreased.

▶ **Lemma 15.** *Let $a = \lambda(x^{(i)})_{k-1}$ and $b = (x^{(i)})_k$. Suppose that $\theta_g \leq \pi/2$, $\gamma(i) > \pi/2$ and $|a + b| > |a|$. Then $\langle \lambda[a + b], [\lambda b] \rangle \leq \pi/2$, entailing $\gamma(i + 1) \leq \phi(i + 1) \leq \pi/2$.*

Furthermore, a decrease cannot be followed by an increase, unless the angle changes:

▶ **Lemma 16.** *Suppose dimension k is just rotating after N. It is not possible for $i - 1 \geq N$, to have $\phi(i - 1) = \phi(i) = \phi(i + 1) > \pi/2$ and $\left|(x^{(i+1)})_{k-1}\right| > \left|(x^{(i)})_{k-1}\right| < \left|(x^{(i-1)})_{k-1}\right|$.*

Finally, we place a limit on the number of consecutive increases until we can decide that dimension $k-1$ will not decrease below the current modulus in the future:

▶ **Lemma 17.** *Let $a = \lambda(x^{(N)})_{k-1}$ and $b = (x^{(N)})_k$ for some $N > 0$. Suppose that k is just rotating after N, $|a + b| > |a| + 0.5$ and $|a| \geq \frac{1}{\sqrt{2}}|b|$. Then, for all $i > N$: $\left|(x^{(i)})_{k-1}\right| > |a|$.*

With Lemmata 14-17 we are in a position to prove Theorem 12 (the proofs of the preceding lemmata, and the **EXPSPACE** analysis can be found in the full version).

Proof of Theorem 12. As described above, we consider each dimension separately, starting with $k = d$, and assume by induction that the previous dimension is just rotating. We describe an algorithm that tracks the value of ϕ and operates according to Figure 2. Each
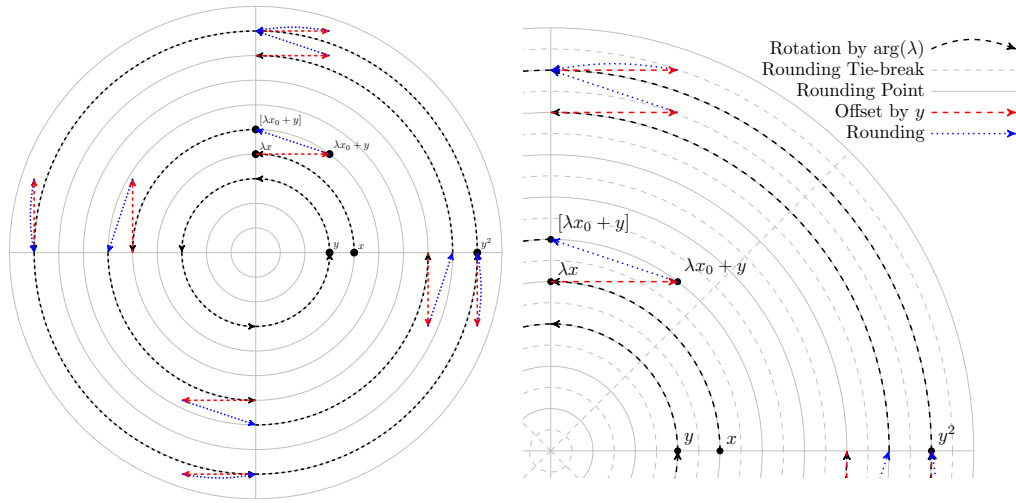
■ **Figure 3** Example where the system may become large before being periodic (see Example 18).

realisable value of ϕ relates to a copy of Figure 2 (we only draw one example of ϕ satisfying $\phi > \pi/2$ and $\phi \leq \pi/2$ respectively). For $\phi > \pi/2$ two states are used, one which encodes that the previous transition was decrementing the modulus (ϕ D), the other which indicates the previous was not decrementing (including first arrival) (ϕ I).

The algorithm moves on each update step according to the arrow, which denotes whether the update is modulus increasing $M \uparrow$, decreasing $M \downarrow$ or stationary MS. Similarly ϕ may decrease $\phi \downarrow$ or stay stationary ϕS, but never increase (Lemma 13). Whenever ϕ decreases we make progress through the DAG to a lower value of ϕ. All combinations $\{M \uparrow, M \downarrow, MS\} \times \{\phi \downarrow, \phi S\}$ are accounted for at each state.

Progress is made whenever we move through the DAG towards a stopping criterion. For self-loops a bound is provided (in blue) on the maximum time spent in this state. Since for each dimension we will ultimately end up in just rotating, or be able to stop early, the problem is decidable. ◀

▶ **Example 18** (System requiring **EXPSPACE** to be periodic). If $\phi \leq \pi/2$, the considered dimension will either diverge at some point, or become periodic. This depends, essentially, on whether $\left|(x^{(i)})_k\right| \cos(\phi) < 0.5$, in which case the rounding will not lead to an increase when $\left|(x^{(i)})_{k-1}\right|$ is sufficiently large relative to $\left|(x^{(i)})_k\right|$. We give an example where $\left|(x^{(i)})_{k-1}\right|$ grows to $\left|(x^{(i)})_k\right|^2$, and requires numbers of doubly exponential size (and exponential space) in d before becoming periodic. We assume that $\theta_g = \pi/2$ (so there are four possible angles) and integer modulus granularity. Let M be a single Jordan block of dimension d with eigenvalue $\lambda = e^{i\pi/2}$. The angle $\phi(i)$ remains constant, but the modulus grows while $\left|(x^{(i)})_k\right| < \left|(x^{(i)})_{k-1}\right| \leq \left|(x^{(i)})_k\right|^2$. We start at the point $x^{(0)} = ((3 + d, 0), \ldots, (6, 0), (5, 0), (4, 0))$, using the representation that $Ae^{i\theta}$ is written (A, θ). This system is periodic, with maximal component $x^{(N)} = ((4^{(2^{d-1})}, 0), \ldots, (4^{2 \cdot 2 \cdot 2}, 0), (4^{2 \cdot 2}, 0), (4^2, 0), (4, 0))$. Note that 4^{2^4} is larger than a 32-bit number. This idea is illustrated in Figure 3, where y represents $(x^{(i)})_k$ and is just rotating, and x represents $(x^{(i)})_{k-1}$, which grows to $|y|^2$.

Despite Example 18, which shows that waiting until becoming periodic may need exponential space, we conjecture the Rounded P2P can be solved in **PSPACE**. This is because if $(x^{(i)})_1$ exceeds a value representable in polynomial space we expect it will never return to the target y_1 (a value representable in polynomial space). However, we are unable to show at the moment that it never gets very large and subsequently returns to a small value.

4.2 Argand truncation or expansion in Jordan normal form

We now consider Argand truncation based rounding showing decidability in **EXPSPACE**. The rounding function is of the form $[a + bi] = [a] + [b]i$ where, for $x \in \mathbb{R}$, $[x] = \lfloor x \rfloor$ if $x \geq 0$ and $[x] = \lceil x \rceil$ if $x < 0$, which has a non-increasing effect on the modulus.

▶ **Theorem 19.** *The Rounded P2P Problem is decidable in* **EXPSPACE** *for deterministic Argand rounding function with a non-increasing effect on the modulus and matrices $M \in \mathbb{A}^{d \times d}$ in Jordan normal form.*

As a key ingredient of Theorem 19 we will make use of the following theorem:

▶ **Theorem 20** ([28, Corollary 3.12, p.41]). *Both x is a rational multiple of π and $\sin(x)$ is rational only at $\sin(x) = 0, \frac{1}{2}$, or 1. Both x is a rational multiple of π and $\cos(x)$ are rational only at $\cos(x) = 0, \frac{1}{2}$, or 1. Both x is a rational multiple of π and $\tan(x)$ are rational only at $\tan(x) = 0$, or ± 1.*

Proof sketch of Theorem 19. Without loss of generality we consider only a single Jordan block with $|\lambda| = 1$, as the remaining blocks can be handled in lock step (using the algorithm of Lemma 7 if the eigenvalue is not of modulus one). Consider the dth component. At each step, whenever rounding takes place, then there is some decrease in the modulus. Thus, either the coordinate hits zero (and stays forever), or it stabilises and becomes periodic (with no rounding ever occurring again). The dth coordinate can be simulated until this happens. At this point, if its modulus is not $|y_d|$, y will not be reached in the future and we return NO.

If dimension x_d reaches zero, then this dimension from some point on becomes irrelevant and the instance can be reduced to an instance of dimension $d - 1$. Note that this case must occur if $\arg(\lambda)$ is not a root of unity as an irrational point is found infinitely often.

In the case where x_d does not reach zero, then it is periodic at some modulus. This implies it never rounds again, and so surely hits integer points at every step. We show that this can only occur if $\arg(\lambda)$ is a multiple of $\pi/2$. Assume that $\arg(\lambda)$ is not a multiple of $\pi/2$: the rotation of a point with integer coordinate to integer coordinate leads to the conclusion of either rational tangent or rational sine and cosine. By Theorem 20 a rational tangent alongside a rational angle ($\arg(\lambda)$ is a root of unity) implies that the angle must be a multiple of $\pi/4$. It is not $\pi/4$, as there is no Pythagorean triangle with angle $\pi/4$. By Theorem 20 rational sine and cosine and rational angle concludes the angle must be a multiple of $\pi/2$. Finally, we show that when $\arg(\lambda)$ is a multiple of $\pi/2$ the system surely diverges at dimension $d-1$, and hence we can put a bound on how far we need to simulate. ◀

▶ Remark (Argand expansion in Jordan Normal Form). Instead of considering the rounding function to always decrease the modulus, we consider the rounding function to always increase the modulus. Then, by the same rationality argument either $\arg(\lambda)$ is a multiple of $\pi/2$ (so no rounding occurs and standard methods can be applied), or $\arg(\lambda)$ is not a multiple of $\pi/2$ and rounding is applied infinitely often. We observe that rounding infinitely often results in divergence. Suppose instead the modulus converges, in supremum, to C. However the $[C]$-ball is finite, thus rounding infinitely often must eventually exhaust the set, contradicting supremacy. Since divergence occurs in the dth component the system can be iterated until either $x^{(i)} = y$ or $(x^{(i)})_d$ exceeds y_d. (Unless $(x^{(0)})_d = y_d = 0$, in which case the dth component can be deleted.)

(a) $r = 10$, $\theta = \pi/42$ (b) $r = 15$, $\theta = \pi/91$ (c) $r = 10$, $\theta = \frac{2^{(0.4)}}{10}\pi$ (d) $r = 20$, $\theta = \pi/14$
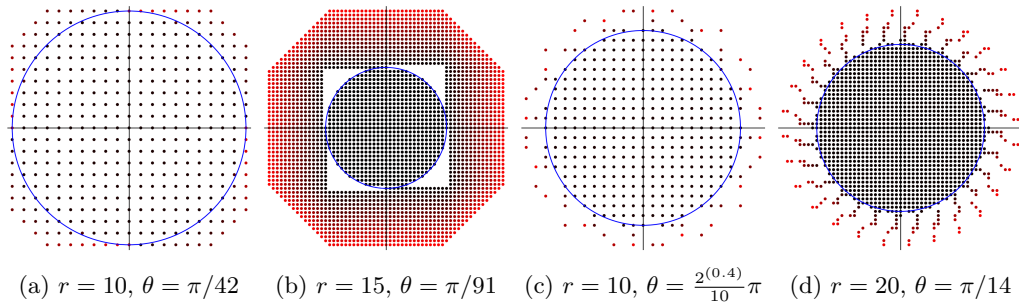
Figure 4 Rotational examples. We start with all points in the circle of radius r, and consider the effect of rotating every point by θ, followed by minimal error rounding. This can be seen as viewing the combined orbits, starting at several points. Redder points are added in later generations.

5 Discussion of open problems

In this section we consider the following open problem, which already exhibits a technical difficulty for a relatively simple instance.

▶ **Open Problem 21.** *Under which deterministic bounded-effect rounding functions does the Rounded P2P Problem become decidable (even when restricted to Jordan normal form)?*

In particular we emphasize that even decidability of the Rounded P2P Problem in the case of a 2D *rotation* matrix remains open. This should be compared to the papers [24, 11, 36, 35, 2, 31], which consider linear maps on $\mathbb{R}^2$ that are close to rotations, and the floor rounding $\lfloor \cdot \rfloor$ is used to induce discretized maps on $\mathbb{Z}^2$. The conjecture made in [35] that all orbits of these maps are eventually periodic (and thus finite) is, to the best of our knowledge, still open in general. This lack of understanding of the dynamics of rotations even on a 2-dimensional lattice is striking and hints at an intrinsic level of difficulty in dealing with eigenvalues of modulus 1.

We ran experiments on the behaviour of rounded orbits induced by rotations in the plane. Four prototypical results are depicted in Figure 4. We note that in every one of our examples the orbits eventually become periodic. Moreover, all experiments fall into the four categories of Figure 4, i.e., where the resulting set consists of (a) a square with cut-off corners, (b) this same square, but with a central square cut out, and (c) all points within the circle with some seemingly randomly added points outside (in the case of an irrational multiple of π), (d) the initial circle with added "tentacles" occuring in intervals corresponding to the rotational angle (in the case of a rational multiple of π). We have been unable construct a rotation with an infinite rounded orbit.

One could hope that other kinds of rounding functions simplify the analysis of the orbits. We have shown truncation based rounding, for example, either helps converge towards zero, or diverge towards infinity, and this can be exploited (particularly at the bottom dimension of a Jordan block). However, roundings which may either round up or down greatly complicate the analysis. Nevertheless, we conjecture that all rounded orbits obtained by rotation eventually become periodic.

Random rounding functions. Orbit problems for rounding functions which behave probabilistically are another line of open problems and are a natural candidate for future work.

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