Completion and Reduction Orders

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Abstract

We present three techniques for improving the Knuth–Bendix completion procedure and its variants: An order extension by semantic labeling, a new confluence criterion for terminating term rewrite systems, and inter-reduction for maximal completion.

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1 Introduction

Completion [11] is a procedure that takes an equational system and a reduction order to construct a conversion-equivalent complete (terminating and confluent) term rewrite system.

Consider the equational system for commuting group endomorphisms (CGE_2):

$$
\begin{aligned}
\mathsf{e} + x &\approx x & \mathsf{f}(x+y) &\approx \mathsf{f}(x) + \mathsf{f}(y) \\
\mathsf{i}(x) + x &\approx \mathsf{e} & \mathsf{g}(x+y) &\approx \mathsf{g}(x) + \mathsf{g}(y) \\
(x+y)+z &\approx x + (y+z) & \mathsf{f}(x) + \mathsf{g}(y) &\approx \mathsf{g}(y) + \mathsf{f}(x)
\end{aligned}
$$

This system is known as a challenging completion problem. Stump and Löchner [16] showed that it admits the following complete TRS consisting of 20 rewrite rules:

$$
\begin{aligned}
\mathsf{e} + x &\to x & \mathsf{f}(\mathsf{e}) &\to \mathsf{e} & \mathsf{i}(x+y) &\to \mathsf{i}(y) + \mathsf{i}(x) \\
x + \mathsf{e} &\to x & \mathsf{g}(\mathsf{e}) &\to \mathsf{e} & \mathsf{f}(x+y) &\to \mathsf{f}(x) + \mathsf{f}(y) \\
\mathsf{i}(x) + x &\to \mathsf{e} & \mathsf{i}(\mathsf{e}) &\to \mathsf{e} & \mathsf{g}(x+y) &\to \mathsf{g}(x) + \mathsf{g}(y) \\
x + \mathsf{i}(x) &\to \mathsf{e} & \mathsf{i}(\mathsf{i}(x)) &\to x & \mathsf{f}(x) + \mathsf{g}(y) &\to \mathsf{g}(y) + \mathsf{f}(x) \\
x + (\mathsf{i}(x) + y) &\to y & \mathsf{i}(\mathsf{f}(x)) &\to \mathsf{f}(\mathsf{i}(x)) & \mathsf{f}(x) + (\mathsf{f}(y) + z) &\to \mathsf{f}(x+y) + z \\
\mathsf{i}(x) + (x + y) &\to y & \mathsf{i}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{i}(x)) & \mathsf{g}(x) + (\mathsf{g}(y) + z) &\to \mathsf{g}(x+y) + z \\
(x+y)+z &\to x+(y+z) & & & \mathsf{g}(x) + (\mathsf{f}(y) + z)) &\to \mathsf{f}(y) + (\mathsf{g}(x) + z)
\end{aligned}
$$

The main difficulty is that termination of the complete TRS cannot be shown by standard reduction orders such as the Knuth–Bendix order (KBO) [11] and the lexicographic path order (LPO) [8]. Therefore, existing completion tools capable of handling such a system either employ termination tools or adopts the dependency pair method [1, 5], giving up direct termination proofs by reduction orders. Instances of the former are [18, 15, 21], and an instance of the latter is [14].

In this note we present another approach to the problem. The idea is easy. We simply develop powerful reduction orders to use them for (maximal) completion. To this end, we reformulate Zantema's semantic labeling [22] as an order extension method for reduction orders (in Section 3). In order to perform completion with powerful orders effectively,

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we introduce a new variant of maximal completion [10, 14] that integrates the feature of rule simplification [7, 3], known as inter-reduction (in Section 5). In addition to them, we show that confluence of terminating systems can be characterized by rewrite strategies (in Section 4). This results in a new critical pair criterion.

2 Preliminaries

We assume familiarity with the basic notions of term rewriting and completion [2, 17]. Here we shortly recapitulate terminology and notation that we use in this note.

An *abstract rewrite system* ARS $\mathcal{A}$ is a pair of a set A and a binary relation $\to_{\mathcal{A}}$ on the set A. An ARS $\mathcal{A} = (A, \to_{\mathcal{A}})$ is *terminating* if there exists no infinite rewrite sequence $a_1 \to_{\mathcal{A}} a_2 \to_{\mathcal{A}} t_3 \to_{\mathcal{A}} \cdots$. An ARS $\mathcal{A}$ is *confluent* if ${}_{\mathcal{A}}^{*}\!\!\leftarrow \cdot \to_{\mathcal{A}}^{*} \subseteq \downarrow_{\mathcal{A}}$ holds. Here $\downarrow_{\mathcal{A}}$ stands for the *joinability* relation $\to_{\mathcal{A}}^{*} \cdot {}_{\mathcal{A}}^{*}\!\!\leftarrow$. An element a is a *normal form* of $\mathcal{A}$ if there is no element b with $a \to_{\mathcal{A}} b$. The set of all normal forms is denoted by $\mathsf{NF}(\mathcal{A})$. When an ARS $\mathcal{A}$ is terminating, an arbitrary element a admits a normal form b such that $a \to_{\mathcal{A}}^{*} b$. By $a{\downarrow_{\mathcal{A}}}$ we denote some fixed normal form of a.

Terms are built from a signature $\mathcal{F}$ and a countable set $\mathcal{V}$ of variables. An *equational system* over $\mathcal{F}$ is a set of equations. Here we assume that *equations* are ordered pairs of terms over $\mathcal{F}$. We write $s \approx t$ for the equation (s, t). An equation $s \approx t$ is called a *rewrite rule*, denoted by $s \to t$, if s is a non-variable term and $\mathcal{V}\mathsf{ar}(t) \subseteq \mathcal{V}\mathsf{ar}(s)$ holds. A *term rewrite system* (TRS) over $\mathcal{F}$ is an equational system consisting of rewrite rules over $\mathcal{F}$. The rewrite step $\to_{\mathcal{R}}$ of a TRS $\mathcal{R}$ is defined as follows: $s \to_{\mathcal{R}} t$ if there exist a rule $\ell \to r \in \mathcal{R}$, a position p of s, and a substitution σ such that $s|_p = \ell\sigma$ and $t = s[r\sigma]_p$. Any TRS $\mathcal{R}$ can be regarded as the ARS comprising the set of terms and the rewrite relation $\to_{\mathcal{R}}$.

A TRS is *complete* if it is terminating and confluent. A complete TRS $\mathcal{R}$ is called *canonical* if for every rule $\ell \to r \in \mathcal{R}$ we have $r \in \mathsf{NF}(\mathcal{R})$ and $\ell \in \mathsf{NF}(\mathcal{R}')$, where $\mathcal{R}'$ consists of $\mathcal{R}$-rules that are not a variant of $\ell \to r$. We say that $\mathcal{R}$ is a TRS *for* an equational system $\mathcal{E}$ if they are conversion-equivalent, namely, $\leftrightarrow_{\mathcal{R}}^{*} = \leftrightarrow_{\mathcal{E}}^{*}$. The aim of completion procedures is to find a complete (or canonical) TRS for a given equational system. Let $\mathcal{R}$ be a terminating TRS and $\mathcal{E}$ a set of equations. Notation $\mathcal{E}{\downarrow_{\mathcal{R}}}$ stands for the set $\{s{\downarrow_{\mathcal{R}}} \approx t{\downarrow_{\mathcal{R}}} \mid s \approx t \in \mathcal{E} \text{ and } s{\downarrow_{\mathcal{R}}} \neq t{\downarrow_{\mathcal{R}}}\}$.

Reduction orders are well-founded orders on terms that are closed under contexts and substitutions. LPO and KBO are instances of reduction orders. We denote them by $\succ_{\mathsf{lpo}}$ and $\succ_{\mathsf{kbo}}$, respectively.

▶ **Theorem 1.** *A TRS $\mathcal{R}$ is terminating if $\mathcal{R} \subseteq \succ$ holds for some reduction order $\succ$.*

Confluence of terminating TRSs is characterized by the notion of critical pair.

▶ **Definition 2** ([6]). *Let $\mathcal{R}$ be a TRS. A tuple $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\sigma$ is an* overlap *of $\mathcal{R}$ if*

- *$\ell_1 \to r_1$ and $\ell_2 \to r_2$ are variants of rules in $\mathcal{R}$ with $\mathcal{V}\mathsf{ar}(\ell_1) \cap \mathcal{V}\mathsf{ar}(\ell_2) = \varnothing$,*
- *p is a function position of ℓ_2,*
- *σ is a most general unifier of ℓ_1 and $\ell_2|_p$, and*
- *$p \neq \epsilon$ or $\ell_1 \to r_1$ is not a variant of $\ell_2 \to r_2$.*

Such an overlap induces the critical peak *$(\ell_2\sigma)[r_1\sigma]_p \;{}_{\mathcal{R}}\!\!\leftarrow (\ell_2\sigma)[\ell_1\sigma]_p = \ell_2\sigma \xrightarrow{\epsilon}_{\mathcal{R}} r_2\sigma$, and the equation $(\ell_2\sigma)[r_1\sigma]_p \approx r_2\sigma$ is called a* critical pair *of $\mathcal{R}$. We write $t \;{}_{\mathcal{R}}\!\!\leftarrow\!\rtimes\!\to_{\mathcal{R}} u$ for critical pair (t, u).*

▶ **Theorem 3** ([11]). *A terminating TRS $\mathcal{R}$ is confluent if and only if ${}_{\mathcal{R}}\!\!\leftarrow\!\rtimes\!\to_{\mathcal{R}} \subseteq \downarrow_{\mathcal{R}}$ holds.*

Finally, we define terminologies for algebras. An $\mathcal{F}$-*algebra* $\mathcal{M}$ is a pair of a set A and the set of interpretations $f_{\mathcal{M}} : A^n \to A$ for each $f \in \mathcal{F}$, where n is the arity of f. Mappings from $\mathcal{V}$ to A are called *assignments*. Let $\mathcal{M} = (A, \{f_{\mathcal{M}}\}_{f \in \mathcal{F}})$ be an $\mathcal{F}$-algebra and α an assignment from $\mathcal{V}$ to A. The valuation $[\alpha]_{\mathcal{M}}(t)$ of a term t under α is inductively defined as follows:

$$[\alpha]_{\mathcal{M}}(t) = \begin{cases} \alpha(t) & \text{if } t \text{ is a variable} \\ f_{\mathcal{M}}([\alpha]_{\mathcal{M}}(t_1), \ldots, [\alpha]_{\mathcal{M}}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Suppose that A is non-empty and equipped with a well-founded order $>$. If every interpretation $f_{\mathcal{A}}$ is weakly monotone, $\mathcal{M}$ is said to be a *weakly monotone well-founded* algebra.

3 Reduction Orders Extended by Semantic Labeling

Semantic labeling introduced by Zantema [22] is a powerful transformation technique for proving termination of term rewrite systems. In this section we reformulate it as an order extension for reduction orders. This is technically trivial but it is useful for completion.

Semantic labeling employs a labeling function for terms. Let $\mathcal{F}$ be a signature. To each n-ary function symbol $f \in \mathcal{F}$ we assign a fresh n-ary function symbol $f^\sharp$. The union of $\mathcal{F}$ and $\{f^\sharp \mid f \in \mathcal{F}\}$ is denoted by $\mathcal{F}^\sharp$.

▶ **Definition 4.** *Let $\mathcal{F}$ and $\mathcal{G}$ be signatures with $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{F}^\sharp$, and let $\mathcal{M} = (A, \{f_{\mathcal{M}}\}_{f \in \mathcal{G}})$ be a $\mathcal{G}$-algebra. Given a term t over $\mathcal{F}$ and an assignment $\alpha : \mathcal{V} \to A$, the* labeled term $\mathsf{lab}_{\mathcal{M}}(t, \alpha)$ *is inductively defined as follows:*

$$\mathsf{lab}_{\mathcal{M}}(t, \alpha) = \begin{cases} t & \textit{if } t \textit{ is a variable} \\ f_a(\mathsf{lab}_{\mathcal{M}}(t_1, \alpha), \ldots, \mathsf{lab}_{\mathcal{M}}(t_n, \alpha)) & \textit{if } t = f(t_1, \ldots, t_n) \textit{ and } f^\sharp \in \mathcal{G} \\ f(\mathsf{lab}_{\mathcal{M}}(t_1, \alpha), \ldots, \mathsf{lab}_{\mathcal{M}}(t_n, \alpha)) & \textit{if } t = f(t_1, \ldots, t_n) \textit{ and } f^\sharp \notin \mathcal{G} \end{cases}$$

where, $a = [\alpha]_{\mathcal{M}}(f^\sharp(t_1, \ldots, t_n))$. Note that labeled terms are terms over the signature $\mathcal{F}_{\mathsf{lab}} := \mathcal{F} \cup \{f_a \mid f^\sharp \in \mathcal{G} \setminus \mathcal{F} \text{ and } a \in A\}$.

▶ **Example 5.** Consider the algebra $\mathcal{M} = (\mathbb{N}, \{\mathsf{g}_{\mathcal{M}}, \mathsf{f}_{\mathcal{M}}, \mathsf{f}^\sharp_{\mathcal{M}}\})$ with $\mathsf{g}_{\mathcal{M}}(x) = 0$, $\mathsf{f}_{\mathcal{M}}(x) = 1$, and $\mathsf{f}^\sharp_{\mathcal{M}}(x) = x$, and the assignment α defined by $\alpha(x) = 2$. Then, we have $\mathsf{lab}_{\mathcal{M}}(\mathsf{f}(\mathsf{g}(\mathsf{f}(x))), \alpha) = \mathsf{f}_0(\mathsf{g}(\mathsf{f}_2(x)))$. Here labels 0 and 2 are determined by $[\alpha]_{\mathcal{M}}(\mathsf{f}^\sharp(\mathsf{g}(\mathsf{f}(x)))) = 0$ and $[\alpha]_{\mathcal{M}}(\mathsf{f}^\sharp(x)) = 2$.

We now present an order extension by semantic labeling.

▶ **Definition 6.** *Suppose $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{F}^\sharp$. Let $(\mathcal{M}, >)$ be a weakly monotone well-founded $\mathcal{G}$-algebra, and $\succ$ a strict order on terms over $\mathcal{F}_{\mathsf{lab}}$. We define the binary relation $\succ^{\mathcal{M}}$ on terms over $\mathcal{F}$ as follows: $s \succ^{\mathcal{M}} t$ if for every assignment α the following inequalities hold:*

$$[\alpha]_{\mathcal{M}}(s) \geqslant [\alpha]_{\mathcal{M}}(t) \qquad\qquad \mathsf{lab}_{\mathcal{M}}(s, \alpha) \succ \mathsf{lab}_{\mathcal{M}}(t, \alpha)$$

Moreover, we define the TRS $\mathcal{D}\mathsf{ec}(\mathcal{M}, >)$ as follows:

$$\mathcal{D}\mathsf{ec}(\mathcal{M}, >) = \{f_a(x_1, \ldots, x_n) \to f_b(x_1, \ldots, x_n) \mid f^\sharp \in \mathcal{G} \setminus \mathcal{F} \textit{ and } a > b\}$$

where, $x_1, \ldots, x_n$ are pairwise distinct variables and n is the arity of f.

▶ **Theorem 7.** *Suppose $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{F}^\sharp$. Let $(\mathcal{M}, >)$ be a weakly monotone well-founded $\mathcal{G}$-algebra, and $\succ$ a reduction order on terms over $\mathcal{F}_{\mathsf{lab}}$. If $\mathcal{D}\mathsf{ec}(\mathcal{M}, >) \subseteq \succ$ holds, $\succ^{\mathcal{M}}$ is a reduction order on terms over $\mathcal{F}$.*

Proof. Immediate from [22, Theorem 8]. ◀

In the remaining part of the paper, the extended versions of KBO and LPO ($\succ_{\mathsf{kbo}}^{\mathcal{M}}$ and $\succ_{\mathsf{lpo}}^{\mathcal{M}}$) are referred to as EKBO and ELPO, respectively. We illustrate the use of EKBO by examples.

▶ **Example 8.** Consider the one-rule TRS $\mathcal{R}$:

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))$$

Let $\mathcal{M} = (\mathbb{N}, \{\mathsf{g}_{\mathcal{M}}, \mathsf{f}_{\mathcal{M}}, \mathsf{f}^{\sharp}_{\mathcal{M}}\})$ be the weakly monotone well-founded algebra given by the interpretations $\mathsf{g}_{\mathcal{M}}(x) = 0$, $\mathsf{f}_{\mathcal{M}}(x) = 1$, and $\mathsf{f}^{\sharp}_{\mathcal{M}}(x) = x$. The KBO $\succ_{\mathsf{kbo}}$ with the weight function given by

$$w(\mathsf{g}) = 0 \qquad w(\mathsf{f}_a) = 1 \quad \text{for all } a \in \mathbb{N} \qquad w(x) = 1 \quad \text{for all variables } x$$

and the well-founded precedence $\mathsf{g} \succ \cdots \succ \mathsf{f}_2 \succ \mathsf{f}_1 \succ \mathsf{f}_0$ satisfies the inclusion:

$$\mathcal{D}\mathsf{ec}(\mathcal{M}, >) = \{\mathsf{f}_a(x) \to \mathsf{f}_b(x) \mid a > b\} \subseteq \succ_{\mathsf{kbo}}$$

Thus, the EKBO $\succ_{\mathsf{kbo}}^{\mathcal{M}}$ is a reduction order. Let $\ell \to r$ denote the rule of the TRS. We have the inequalities $[\alpha]_{\mathcal{M}}(\ell) = 1 \geqslant 1 = [\alpha]_{\mathcal{M}}(r)$ and $\mathsf{lab}_{\mathcal{M}}(\ell, \alpha) = \mathsf{f}_1(\mathsf{f}_{\alpha(x)}(x)) \succ_{\mathsf{kbo}} \mathsf{f}_0(\mathsf{g}(\mathsf{f}_{\alpha(x)}(x)))) = \mathsf{lab}_{\mathcal{M}}(r, \alpha)$ for all assignments α. Therefore, $\ell \succ_{\mathsf{kbo}}^{\mathcal{M}} r$ holds. Hence, $\mathcal{R}$ is terminating.

▶ **Example 9.** We show termination of the complete TRS $\mathcal{R}$ for CGE_2 in the introduction. Let $\mathcal{M} = (\mathbb{N}, \{\mathsf{e}_{\mathcal{M}}, \mathsf{f}_{\mathcal{M}}, \mathsf{g}_{\mathcal{M}}, \mathsf{i}_{\mathcal{M}}, +_{\mathcal{M}}, +^{\sharp}_{\mathcal{M}}\})$ be the weakly monotone algebra with the interpretations:

$$\mathsf{e}_{\mathcal{M}} = 0 \qquad \mathsf{f}_{\mathcal{M}}(x) = 0 \qquad \mathsf{g}_{\mathcal{M}}(x) = 1 \qquad \mathsf{i}_{\mathcal{M}}(x) = x \qquad x +_{\mathcal{M}} y = x + y \qquad x +^{\sharp}_{\mathcal{M}} y = x$$

The KBO $\succ_{\mathsf{kbo}}$ comprising the weight function

$$\begin{aligned} w(\mathsf{i}) &= 0 & w(+_a) &= 0 \quad \text{for all } a \in \mathbb{N} \\ w(\mathsf{g}) = w(\mathsf{f}) = w(\mathsf{e}) &= 1 & w(x) &= 1 \quad \text{for all variables } x \end{aligned}$$

and the well-founded precedence $\mathsf{i} \succ \mathsf{g} \succ \cdots \succ +_2 \succ +_1 \succ +_0 \succ \mathsf{e} \succ \mathsf{f}$ satisfies the inclusion:

$$\mathcal{D}\mathsf{ec}(\mathcal{M}, >) = \{x +_a y \to x +_b y \mid a > b\} \subseteq \succ_{\mathsf{kbo}}$$

Thus, $\succ_{\mathsf{kbo}}^{\mathcal{M}}$ is a reduction order. It is easy to verify that $[\alpha]_{\mathcal{M}}(\ell) \geqslant [\alpha]_{\mathcal{M}}(r)$ holds for every rules $\ell \to r \in \mathcal{R}$ and assignment α. The inequality $\mathsf{lab}_{\mathcal{M}}(\ell, \alpha) \succ_{\mathsf{kbo}} \mathsf{lab}_{\mathcal{M}}(r, \alpha)$ holds too:

$$\begin{array}{rcc} \mathsf{e} +_0 x \succ_{\mathsf{kbo}} x & \mathsf{f}(\mathsf{e}) \succ_{\mathsf{kbo}} \mathsf{e} & \mathsf{i}(x +_a y) \succ_{\mathsf{kbo}} \mathsf{i}(y) +_b \mathsf{i}(x) \\ x +_a \mathsf{e} \succ_{\mathsf{kbo}} x & \mathsf{g}(\mathsf{e}) \succ_{\mathsf{kbo}} \mathsf{e} & \mathsf{f}(x +_a y) \succ_{\mathsf{kbo}} \mathsf{f}(x) +_0 \mathsf{f}(y) \\ \mathsf{i}(x) +_a x \succ_{\mathsf{kbo}} \mathsf{e} & \mathsf{i}(\mathsf{e}) \succ_{\mathsf{kbo}} \mathsf{e} & \mathsf{g}(x +_a y) \succ_{\mathsf{kbo}} \mathsf{g}(x) +_1 \mathsf{g}(y) \\ x +_a \mathsf{i}(x) \succ_{\mathsf{kbo}} \mathsf{e} & \mathsf{i}(\mathsf{i}(x)) \succ_{\mathsf{kbo}} x & \mathsf{f}(x) +_0 \mathsf{g}(y) \succ_{\mathsf{kbo}} \mathsf{g}(y) +_1 \mathsf{f}(x) \\ x +_a (\mathsf{i}(x) +_a y) \succ_{\mathsf{kbo}} y & \mathsf{i}(\mathsf{f}(x)) \succ_{\mathsf{kbo}} \mathsf{f}(\mathsf{i}(x)) & \mathsf{f}(x) +_0 (\mathsf{f}(y) +_0 z) \succ_{\mathsf{kbo}} \mathsf{f}(x +_a y) + z \\ \mathsf{i}(x) +_a (x +_a y) \succ_{\mathsf{kbo}} y & \mathsf{i}(\mathsf{g}(x)) \succ_{\mathsf{kbo}} \mathsf{g}(\mathsf{i}(x)) & \mathsf{g}(x) +_1 (\mathsf{g}(y) +_1 z) \succ_{\mathsf{kbo}} \mathsf{g}(x +_a y) + z \\ (x +_a y) +_{a+b} z \succ_{\mathsf{kbo}} x +_a (y +_b z) & & \mathsf{g}(x) +_1 (\mathsf{f}(y) +_0 z)) \succ_{\mathsf{kbo}} \mathsf{f}(y) +_0 (\mathsf{g}(x) +_1 z) \end{array}$$

where, $a = \alpha(x)$ and $b = \alpha(y)$. Therefore, $\mathcal{R} \subseteq \succ_{\mathsf{kbo}}^{\mathcal{M}}$ holds. Hence, $\mathcal{R}$ is terminating.

By using SAT/SMT solvers one can easily implement a program to find suitable parameters for EKBOs and ELPOs. See [12] for SAT/SMT encoding technique. As a final remark in the section, ELPO is almost same as the lexicographic version of the semantic path order (SPO) [8]; see [22] for discussions on the relation between semantic labeling and SPO.

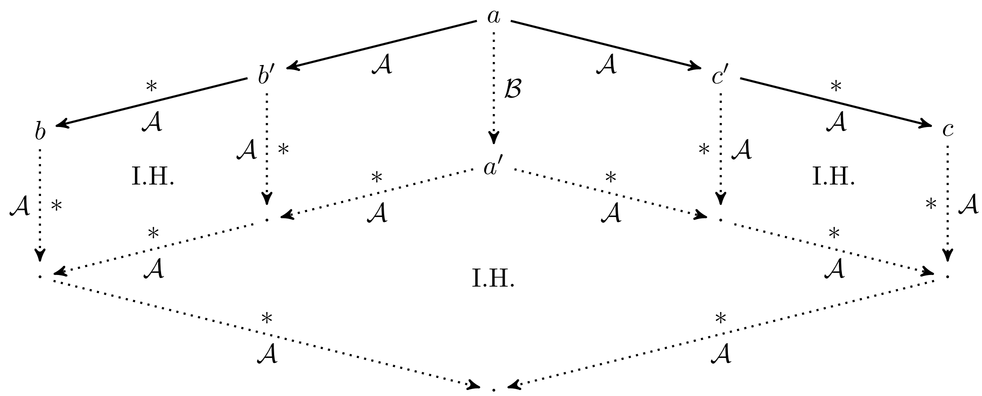

Figure 1 Proof of Theorem 11.

4 Confluence via Rewrite Strategies

In this section we present a new confluence criterion based on *rewrite strategies.*

▶ **Definition 10** ([17, Section 9.1]). *Let $\mathcal{A} = (A, \rightarrow_{\mathcal{A}})$ be an ARS. We say that an ARS $\mathcal{B} = (A, \rightarrow_{\mathcal{B}})$ is a rewrite strategy if $\rightarrow_{\mathcal{B}} \subseteq \rightarrow_{\mathcal{A}}^{+}$ and $\mathsf{NF}(\mathcal{A}) = \mathsf{NF}(\mathcal{B})$.*

▶ **Theorem 11.** *A terminating ARS $\mathcal{A}$ is confluent if and only if the inclusion ${}_{\mathcal{B}}\!\leftarrow \cdot \rightarrow_{\mathcal{A}} \subseteq \downarrow_{\mathcal{A}}$ holds for some rewrite strategy $\mathcal{B}$ of $\mathcal{A}$.*

Proof. The "only if"-direction is trivial as we can take $\mathcal{B} = \mathcal{A}$. We show the "if"-direction. Let $\mathcal{A}$ be a terminating ARS and $\mathcal{B}$ a rewrite strategy for $\mathcal{A}$ with ${}_{\mathcal{B}}\!\leftarrow \cdot \rightarrow_{\mathcal{A}} \subseteq \downarrow_{\mathcal{A}}$. Suppose $b \;{}_{\mathcal{A}}^{*}\!\leftarrow a \rightarrow_{\mathcal{A}}^{*} c$. As $\mathcal{A}$ is terminating, $\rightarrow_{\mathcal{A}}^{+}$ is a well-founded order. So we perform well-founded induction on a with respect to $\rightarrow_{\mathcal{A}}^{+}$ to show $b \downarrow_{\mathcal{A}} c$. If $b = a$ then $b \rightarrow_{\mathcal{A}}^{*} c$. Thus, $b \downarrow_{\mathcal{A}} c$ holds. Similarly, if $a = c$ then $b \;{}_{\mathcal{A}}^{*}\!\leftarrow c$. Thus, $b \downarrow_{\mathcal{A}} c$ holds. Otherwise, there exist b' and c' such that $b \;{}_{\mathcal{A}}^{*}\!\leftarrow b' \;{}_{\mathcal{A}}\!\leftarrow a \rightarrow_{\mathcal{A}} c' \rightarrow_{\mathcal{A}}^{*} c$ holds. Because $\mathcal{B}$ is a rewrite strategy, $a \notin \mathsf{NF}(\mathcal{A}) = \mathsf{NF}(\mathcal{B})$. Thus, there exists an element a' with $a \rightarrow_{\mathcal{B}} a'$. Since a', b', and c' are smaller than a with respect to $\rightarrow_{\mathcal{A}}^{+}$, the corresponding induction hypotheses and the assumption ${}_{\mathcal{B}}\!\leftarrow \cdot \rightarrow_{\mathcal{A}} \subseteq \downarrow_{\mathcal{A}}$ yield the diagram indicated in Figure 1. ◀

Using this characterization, we develop a new critical pair criterion. Let $\xrightarrow{\alpha}_{\mathcal{R}}$ be a rewrite strategy for a TRS $\mathcal{R}$. We say that a critical peak $t \;{}_{\mathcal{R}}\!\leftarrow s \xrightarrow{\epsilon}_{\mathcal{R}} u$ is an α-critical peak if $s \xrightarrow{\alpha}_{\mathcal{R}} t$. The corresponding critical pair (t, u) is denoted by $t \;{}_{\mathcal{R}}\!\xleftarrow{\alpha}\rtimes\!\rightarrow_{\mathcal{R}} u$. For instance, the innermost strategy $\xrightarrow{\mathsf{i}}_{\mathcal{R}}$ is a rewrite strategy. Innermost critical pairs ${}_{\mathcal{R}}\!\xleftarrow{\mathsf{i}}\rtimes\!\rightarrow_{\mathcal{R}}$ correspond to *prime* critical pairs.[1]

▶ **Corollary 12** ([9]). *A terminating TRS $\mathcal{R}$ is confluent if and only if ${}_{\mathcal{R}}\!\xleftarrow{\mathsf{i}}\rtimes\!\rightarrow_{\mathcal{R}} \subseteq \downarrow_{\mathcal{R}}$ holds.*

This result can be refined by adopting the leftmost innermost strategy $\xrightarrow{\mathsf{li}}_{\mathcal{R}}$. Since $\xrightarrow{\mathsf{li}}_{\mathcal{R}}$ is a subrelation of $\xrightarrow{\mathsf{i}}_{\mathcal{R}}$, the inclusion ${}_{\mathcal{R}}\!\xleftarrow{\mathsf{li}}\rtimes\!\rightarrow_{\mathcal{R}} \subseteq {}_{\mathcal{R}}\!\xleftarrow{\mathsf{i}}\rtimes\!\rightarrow_{\mathcal{R}}$ holds in general.

▶ **Corollary 13.** *A terminating TRS $\mathcal{R}$ is confluent if and only if ${}_{\mathcal{R}}\!\xleftarrow{\mathsf{li}}\rtimes\!\rightarrow_{\mathcal{R}} \subseteq \downarrow_{\mathcal{R}}$ holds.*

Proof. Since ${}_{\mathcal{R}}\!\xleftarrow{\mathsf{li}}\rtimes\!\rightarrow_{\mathcal{R}} \subseteq \downarrow_{\mathcal{R}}$ and ${}_{\mathcal{R}}\!\xleftarrow{\mathsf{li}} \cdot \rightarrow_{\mathcal{R}} \subseteq \downarrow_{\mathcal{R}}$ are equivalent, Theorem 11 applies. ◀

[1] This was pointed out by Masahiko Sakai (personal communication).

delete	$(\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}) \vdash_{\succ} (\mathcal{E}, \mathcal{R})$	
orient_1	$(\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}) \vdash_{\succ} (\mathcal{E}, \mathcal{R} \cup \{s \to t\})$	if $s \succ t$
orient_2	$(\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}) \vdash_{\succ} (\mathcal{E}, \mathcal{R} \cup \{t \to s\})$	if $t \succ s$
$\mathsf{simplify}_1$	$(\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}) \vdash_{\succ} (\mathcal{E} \cup \{u \approx t\}, \mathcal{R})$	if $s \to_{\mathcal{R}} u$
$\mathsf{simplify}_2$	$(\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}) \vdash_{\succ} (\mathcal{E} \cup \{s \approx u\}, \mathcal{R})$	if $t \to_{\mathcal{R}} u$
$\mathsf{collapse}$	$(\mathcal{E}, \mathcal{R} \uplus \{t \to s\}) \vdash_{\succ} (\mathcal{E} \cup \{u \approx s\}, \mathcal{R})$	if $t \to_{\mathcal{R}} u$
$\mathsf{compose}$	$(\mathcal{E}, \mathcal{R} \uplus \{s \to t\}) \vdash_{\succ} (\mathcal{E}, \mathcal{R} \cup \{s \to u\})$	if $t \to_{\mathcal{R}} u$

Figure 2 Inference rules of abstract completion except deduce.

▶ **Example 14.** Consider the terminating TRS $\mathcal{R}$:

$$-\mathsf{0} \to \mathsf{0} \qquad x + \mathsf{0} \to x \qquad (-x) + x \to \mathsf{0} \qquad (-x) + (-x) \to \mathsf{0}$$

The TRS admits five overlaps and they form the five critical peaks (a–e):

$$\begin{tikzcd}[column sep=tiny] & \underline{\underline{\underline{(-\mathsf{0})} + \mathsf{0}}} \arrow[dl, "1"'] \arrow[dr, "\epsilon"] & \\ \mathsf{0} + \mathsf{0} & \approx & \mathsf{0} \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny] & \underline{\underline{(-\mathsf{0}) + \mathsf{0}}} \arrow[dl, "\epsilon"'] \arrow[dr, "\epsilon"] & \\ -\mathsf{0} & \approx & \mathsf{0} \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny] & \underline{\underline{(-\mathsf{0}) + \mathsf{0}}} \arrow[dl, "\epsilon"'] \arrow[dr, "\epsilon"] & \\ \mathsf{0} & \approx & -\mathsf{0} \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny] & \underline{\underline{\underline{(-\mathsf{0})} + (-\mathsf{0})}} \arrow[dl, "1"'] \arrow[dr, "\epsilon"] & \\ \mathsf{0} + (-\mathsf{0}) & \approx & \mathsf{0} \end{tikzcd} \quad \begin{tikzcd}[column sep=tiny] & \underline{\underline{(-\mathsf{0}) + \underline{(-\mathsf{0})}}} \arrow[dl, "2"'] \arrow[dr, "\epsilon"] & \\ (-\mathsf{0}) + \mathsf{0} & \approx & \mathsf{0} \end{tikzcd}$$

(a) (b) (c) (d) (e)

Out of the five, only (a) and (d) are leftmost innermost critical pairs (${}_{\mathcal{R}}\overset{\mathsf{li}}{\leftarrow}\!\rtimes\!\!\to_{\mathcal{R}}$), and they are joinable: $\mathsf{0} + \mathsf{0} \downarrow_{\mathcal{R}} \mathsf{0}$ and $\mathsf{0} + (-\mathsf{0}) \downarrow_{\mathcal{R}} \mathsf{0}$. Hence, confluence of the TRS $\mathcal{R}$ is concluded. Note that ${}_{\mathcal{R}}\overset{\mathsf{i}}{\leftarrow}\!\rtimes\!\!\to_{\mathcal{R}}$ contains one more critical pair (e).

▶ **Example 15.** The complete TRS for CGE_2 in the introduction admits 115 overlaps. Out of them, 18 overlaps are discarded by the condition of leftmost innermost critical pairs ($\overset{\mathsf{li}}{\leftarrow}\!\rtimes\!\!\to$). For this rewrite system $\overset{\mathsf{i}}{\leftarrow}\!\rtimes\!\!\to$ and $\overset{\mathsf{li}}{\leftarrow}\!\rtimes\!\!\to$ coincide.

Unfortunately, the outermost strategy $\overset{\mathsf{o}}{\to}_{\mathcal{R}}$ cannot be used for discarding critical pairs. The culprit is that ${}_{\mathcal{R}}\overset{\mathsf{o}}{\leftarrow}\!\rtimes\!\!\to_{\mathcal{R}} \subseteq \downarrow_{\mathcal{R}}$ does not imply ${}_{\mathcal{R}}\overset{\mathsf{o}}{\leftarrow} \cdot \to_{\mathcal{R}} \subseteq \downarrow_{\mathcal{R}}$ in general.

▶ **Example 16.** Consider the terminating TRS $\mathcal{R} = \{\mathsf{f}(\mathsf{f}(x)) \to \mathsf{a}\}$. Since the only critical peak

$$\mathsf{f}(\mathsf{a}) \;{}_{\mathcal{R}}\!\leftarrow \mathsf{f}(\mathsf{f}(\mathsf{f}(x))) \overset{\epsilon}{\to}_{\mathcal{R}} \mathsf{a}$$

is not an outermost critical peak, the inclusion ${}_{\mathcal{R}}\overset{\mathsf{o}}{\leftarrow}\!\rtimes\!\!\to_{\mathcal{R}} = \varnothing \subseteq \downarrow_{\mathcal{R}}$ holds. However, $\mathcal{R}$ is not confluent, as $\mathsf{f}(\mathsf{a})$ and a are not joinable.

5 Maximal Completion with Inter-reduction

In this section we present a new variant of maximal completion [10, 14], which incorporates inter-reduction of standard completion [7]. Figure 2 shows a subset of the inference rules of abstract completion [3], where the deduce rule is excluded. Inter-reduction corresponds to $\mathsf{collapse}$ and $\mathsf{compose}$. Due to absence of deduce, the derivation relation $\vdash_{\succ}$ fulfils the termination property. So for any finite equational system $\mathcal{E}$ the pair $(\mathcal{E}, \varnothing)$ has a normal form with respect to $\vdash_{\succ}$. We denote its arbitrary but fixed normal form by $\psi(\mathcal{E}, \succ)$.

We now formalize our procedure. Let $\mathcal{O}$ be a mapping from an equational system to a finite set of reduction order, and S a mapping from an equational system $\mathcal{E}$ to a set of equations $s \approx t$ satisfying $s \leftrightarrow^*_{\mathcal{E}} t$.

▶ **Definition 17.** *For an equational system $\mathcal{E}$ the partial function $\varphi(\mathcal{E})$ is defined as follows:*

$$\varphi(\mathcal{E}) = \begin{cases} \mathcal{R} & \text{if } \mathcal{E}' = \varnothing \text{ and } {}_{\mathcal{R}}\xleftarrow{\mathsf{li}}\rtimes\!\!\to_{\mathcal{R}} \subseteq {\downarrow_{\mathcal{R}}} \text{ for some } \succ \in \mathcal{O}(\mathcal{E}) \\ \varphi(\mathcal{E} \cup \mathsf{S}(\mathcal{E})) & \text{otherwise} \end{cases}$$

where $(\mathcal{E}', \mathcal{R}) = \psi(\mathcal{E}, \succ)$.

▶ **Theorem 18.** *If $\varphi(\mathcal{E})$ is defined then it is a complete presentation of $\mathcal{E}$.*

Proof. Immediate from $\leftrightarrow^*_{\mathcal{E}} = \leftrightarrow^*_{\mathcal{E}' \cup \mathcal{R}}$ and $\leftrightarrow^*_{\mathcal{E}} = \leftrightarrow^*_{\mathcal{E} \cup \mathsf{S}(\mathcal{E})}$. ◀

The procedure $\varphi(\mathcal{E})$ runs as follows: (1) $\mathcal{O}(\mathcal{E})$ generates reduction orders; (2) for each of them $\psi(\mathcal{E}, \succ)$ runs standard completion without the deduce rule; (3) if one of them results in a confluent TRS $\mathcal{R}$, the procedure returns $\mathcal{R}$; (4) otherwise $\mathcal{E}$ is extended by $\mathsf{S}(\mathcal{E})$. The second step ψ is a new ingredient to maximal completion [10, 14, 19].

In order to evaluate effectiveness of the presented framework we implemented it on the top of the completion tool Maxcomp [10].[2] In the implementation $\mathsf{S}(\mathcal{E})$ selects 21 smallest equations from the set:

$$\bigcup_{\succ \in \mathcal{O}(\mathcal{E})} (\mathcal{E}_{\succ} \cup \mathcal{R}_{\succ} \cup \mathsf{CP}_{\mathsf{li}}(\mathcal{R}_{\succ}){\downarrow_{\mathcal{R}_{\succ}}}) \setminus \mathcal{E}$$

where, $(\mathcal{E}_{\succ}, \mathcal{R}_{\succ}) = \psi(\mathcal{E}, \succ)$ and $\mathsf{CP}_{\mathsf{li}}(\mathcal{R})$ stands for ${}_{\mathcal{R}}\xleftarrow{\mathsf{li}}\rtimes\!\!\to_{\mathcal{R}}$. The definition of $\mathcal{O}$ is based on Sato and Winkler's heuristic method [14, 19]. The method aims to find *canonical* TRSs $\mathcal{P}$ for $\mathcal{E}$ such that $\mathcal{P} \subseteq \mathcal{E} \cup \mathcal{E}^{-1}$. Assume that we want to find k orders from a designated class $\mathcal{RO}$ of reduction orders. We define $\mathcal{RO}(\mathcal{E}, k)$ as $\mathcal{RO}(\mathcal{E}, 0) = \varnothing$ and $\mathcal{RO}(\mathcal{E}, k+1) = \mathcal{RO}(\mathcal{E}, k) \cup \{(\mathcal{P}, \succ)\}$. Here $\mathcal{P}$ is a TRS and $\succ$ is a reduction order in $\mathcal{RO}$ that minimizes the cardinality of $\mathcal{P}$ subject to the three constraints: The inclusion

$$\mathcal{P} \subseteq \{s \to t \in \mathcal{E} \cup \mathcal{E}^{-1} \mid s \succ t\}$$

holds, all non-trivial equations in $\mathcal{E}$ are $\mathcal{P}$-reducible, and $\mathcal{P} \neq \mathcal{P}'$ for all $(\mathcal{P}', \succ') \in \mathcal{RO}(\mathcal{E}, k)$. Our tool employs $\mathcal{O}$ defined by $\mathcal{O}(\mathcal{E}) = \{ \succ \mid (\mathcal{P}, \succ) \in \mathcal{RO}(\mathcal{E}, 2) \}$.

▶ **Example 19.** Let $\mathcal{RO}$ be the class of EKBOs. Following our procedure, we complete the next equational system:

$$1\colon\quad \mathsf{s}(\mathsf{p}(x)) \approx x \qquad 2\colon\quad \mathsf{p}(\mathsf{s}(x)) \approx x \qquad 3\colon\quad \mathsf{s}(x) + y \approx \mathsf{s}(x + y)$$

The run of φ proceeds as follows: $\varphi(\{1,2,3\}) = \varphi(\{1,2,3,4,5\}) = \varphi(\{1,2,\ldots,8\})$, where:

$$\begin{array}{lll} 4\colon\ \mathsf{p}(\mathsf{s}(x) + y) \approx x + y & 6\colon\ \mathsf{s}((\mathsf{p}(x) + y) + z) \approx (x + y) + z & 8\colon\ \mathsf{p}(x) + y \approx \mathsf{p}(x + y) \\ 5\colon\ \mathsf{s}(\mathsf{p}(x) + y) \approx x + y & 7\colon\ \mathsf{p}((\mathsf{s}(x) + y) + z) \approx (x + y) + z & \end{array}$$

Let $\mathcal{E} = \{1, 2, \ldots, 8\}$. The function $\mathcal{RO}(\mathcal{E}, 2)$ yields $\{(\mathcal{P}_1, \succ_1), (\mathcal{P}_2, \succ_2)\}$, which pinpoints canonical TRSs for $\mathcal{E}$:

$$\mathcal{P}_1 = \{\ \mathsf{s}(\mathsf{p}(x)) \to x,\ \ \mathsf{p}(\mathsf{s}(x)) \to x,\ \ \mathsf{s}(x) + y \to \mathsf{s}(x + y),\ \ \mathsf{p}(x) + y \to \mathsf{p}(x + y)\ \}$$
$$\mathcal{P}_2 = \{\ \mathsf{s}(\mathsf{p}(x)) \to x,\ \ \mathsf{p}(\mathsf{s}(x)) \to x,\ \ \mathsf{s}(x + y) \to \mathsf{s}(x) + y,\ \ \mathsf{p}(x + y) \to \mathsf{p}(x) + y\ \}$$

Although they are ignored by $\mathcal{O}$, uniqueness of canonical TRSs [13] ensures that ψ reproduces the same TRSs: $\psi(\mathcal{E}, \succ_i) = (\varnothing, \mathcal{P}_i)$. Thus, $\varphi(\mathcal{E})$ returns one of them. Note that the EKBOs $\succ_1$ and $\succ_2$ employ algebras like $\mathsf{s}_{\mathcal{M}}(x) = \mathsf{p}_{\mathcal{M}}(x) = 0$ and $x +_{\mathcal{M}} y = 1$ to avoid unnecessary orientations for 4–7.

[2] https://www.jaist.ac.jp/project/maxcomp/

Table 1 Experimental results on 115 equational systems.

	LPO	ELPO	KBO	EKBO	ELPO + EKBO	KBCV	MaxcompDP
# of completed systems	81	89	82	85	96	86	97

▶ **Example 20.** Recall the equational system $\mathcal{E}$ of CGE_2. The procedure $\varphi(\mathcal{E})$ with the united class of ELPOs and EKBOs results in the same complete TRS in the introduction. At the last step φ maintains 120 equations. Sato and Winkler's method automatically constructs an EKBO like Example 9 to produce the 20-rule complete TRS $\mathcal{R}$ indicated in the introduction (or a symmetric variant that employs the right-associative rule $x + (y + z) \to (x + y) + z$).

Table 1 summaries experimental results on the standard set of completion problems.[3] The tests were single-threaded run on a system equipped with an Intel Core i7-1065G7 CPU with 1.3 GHz and 32 GB of RAM using a timeout of 600 seconds. We used SMT solver Z3[4] for computing $\mathcal{RO}(\mathcal{E}, k)$. See [10, 14] for the employed encoding techniques. Note that $k = 2$ is used in the implementation.

The first five columns indicate the results of our completion procedure with the classes of reduction orders LPO, ELPO, KBO, EKBO, and the union of ELPO and EKBO, respectively. Linear interpretations on natural numbers with $0, 1$-coefficients were employed for ELPO and EKBO. The union of ELPO and EKBO is the most powerful and subsumes all results of the other classes. The use of ordinary critical pairs did not change any number. For the comparison sake, we also included in the table the results of completion tools KBCV version 2.1.0.6 [15] and MaxcompDP [14].

6 Conclusion

We have presented an order extension by semantic labeling and maximal completion with inter-reduction as well as a confluence criterion based on rewrite strategies. Our primary future work is to evaluate these methods in the setting of (maximal) ordered completion [4, 20].

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[3] The problem set and detailed data are available from: `http://www.jaist.ac.jp/project/maxcomp/`
[4] `https://github.com/Z3Prover/`

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