Hennessy-Milner Theorems via Galois Connections

Harsh Beohar
University of Sheffield, UK

Sebastian Gurke
Universität Duisburg-Essen, Germany

Barbara König
Universität Duisburg-Essen, Germany

Karla Messing
Universität Duisburg-Essen, Germany

Abstract

We introduce a general and compositional, yet simple, framework that allows to derive soundness and expressiveness results for modal logics characterizing behavioural equivalences or metrics (also known as Hennessy-Milner theorems). It is based on Galois connections between sets of (real-valued) predicates on the one hand and equivalence relations/metrics on the other hand and covers a part of the linear-time-branching-time spectrum, both for the qualitative case (behavioural equivalences) and the quantitative case (behavioural metrics). We derive behaviour functions from a given logic and give a condition, called compatibility, that characterizes under which conditions a logically induced equivalence/metric is induced by a fixpoint equation. In particular, this framework allows to derive a new fixpoint characterization of directed trace metrics.

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1 Introduction

In the verification of state-based transition systems, modal logics play a central role: they can be used to specify the properties that a system must satisfy and model-checking techniques allow to verify whether this is in fact the case. Modal logics also play a fundamental role in characterizing behavioural equivalences: van Glabbeek in his seminal paper [25] showed how a whole spectrum of behavioural equivalences and preorders can be characterized via modal logics. This characterization is also known as the Hennessy-Milner theorem [9], which says that two states x, y are equivalent (wrt. to some notion of behavioural equivalence) iff they satisfy the same formulas ϕ (of a given modal logic). Formally, $x \sim y \iff \forall \phi\colon (x \models \phi \iff y \models \phi)$.

For quantitative systems, the notion of behavioural equivalence is often too strict and small deviations in quantitative information, such as probabilities, can cause two states that intuitively behave very much alike to be inequivalent in a formal sense. Hence it is natural to consider various metrics for determining at what behavioural distance two states lie [7, 24]. This yields an extension of classical notions of behavioural equivalence which knows only distance 0 (two states behave the same) and distance 1 (two states behave differently). Such metrics have often been studied in probabilistic settings [7], but they can be studied in other quantitative contexts, for instance metric transition systems [6, 8].

In the quantitative case, equivalences are replaced by pseudo-metrics and evaluation of a formula ϕ results in a real-valued (as opposed to a boolean-valued) function $[\![\phi]\!]$, intuitively indicating to which degree a state satisfies a formula. Stated in this context the Hennessy-Milner property says that $d(x,y) = \bigvee_\phi |[\![\phi]\!](x) - [\![\phi]\!](y)|$, where d is the behavioural metric.

We present a general framework that allows to easily deduce the Hennessy-Milner property for a variety of equivalences, preorders and (directed) metrics in the qualitative and quantitative setting. We rely on a well-known property [2, 4, 5] for Galois connections that says under which conditions left adjoints preserve least fixpoints. Such Galois connections relate the logical with the behavioural universe and translate sets of (real-valued) predicates to equivalences (metrics) and vice versa. Our *first* contribution is the identification of adjunctions both in quantitative/qualitative settings, which are crucial in capturing bisimilarity and (decorated) trace versions of equivalences/preorders/metrics.

While most contributions to this area start with a behavioural equivalence (resp. metric) and define a corresponding characteristic logic, our approach goes in the other direction, with the slogan: *"Derive behaviour functions from a modal logic"*. The recipe, which is our *second* contribution, is as follows: we define a logic function living in the logical universe and check that it is compatible with the closure induced by the Galois connection. Compatibility ensures that the Hennessy-Milner property is satisfied when we transfer the logic function into a behaviour function living in the behavioural universe. More concretely, we can guarantee that the least fixpoint of the logic function (the set of all formulas) induces an equivalence (resp. metric) which is the least fixpoint of the behaviour function. Note that in the qualitative case, the Galois connection is contravariant, resulting in behavioural equivalence being the greatest fixpoint, as usual.

Related ideas have been considered in more categorical settings [13, 18], here we demonstrate that this can be done in a purely lattice-theoretical setup and in particular for behavioural metrics. To our knowledge, the adjunctions that we are considering here, have not yet been used to derive Hennessy-Milner theorems and behaviour functions. Our *third* contribution is the novel connection to up-to functions and compatibility and we show how closure properties for up-to functions can be employed to combine logics, leading to a modular framework. Furthermore, the behaviour function that we obtain for the trace metric case is, as far as we know, not yet known in the literature. Our *final* contribution is the characterisation of these behaviour functions in more concrete terms both in the qualitative (Theorem 4.12 and Corollary 4.14) and quantitative (Theorem 5.17 and Corollary 5.22) cases. In turn, these general results effortlessly instantiate into many of the equivalences in the van Glabbeek spectrum and immediately yield: logical characterizations, the hierarchy between them and also recursive characterizations, which are often hard to obtain (at least in the metric case).

The full version of this paper, including all proofs, is available from [3].

2 Preliminaries

Functions and Relations

Given a function $f\colon X \to Y$ and $Z \subseteq X$ we write $f[Z]$ for $\{f(z) \mid z \in Z\}$. Similarly, for a relation $R \subseteq X \times X$ and $X' \subseteq X$, we define $R[X'] = \{y \in X \mid \exists x \in X'\colon (x,y) \in R\}$. Furthermore, Y^X denotes the set of all functions from X to Y and, for a given set $\mathcal{F} \subseteq Y^X$ of functions, by $\langle \mathcal{F} \rangle$ we denote a function of type $X \to Y^{\mathcal{F}}$ defined as $\langle \mathcal{F} \rangle(x)(f) = f(x)$. For $S \subseteq X$, $\chi_S\colon X \to \{0,1\}$ stands for the characteristic function of S.

A *congruence* is an equivalence relation $R \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ satisfying: $\bigcup_{i\in I} X_i \, R \, \bigcup_{i\in I} Y_i$ whenever $X_i \, R \, Y_i$ for all $i \in I$. Given any relation $R \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$, by *cong*$(R)$ we denote its *congruence closure*, i.e., the smallest congruence such that $R \subseteq cong(R)$.

The directed relation lifting $R_{\vec{H}} \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ for a relation $R \subseteq X \times X$ is defined as $X_1 \, R_{\vec{H}} \, X_2 \iff \forall x_1 \in X_1 \exists x_2 \in X_2 \colon x_1 \, R \, x_2$. Furthermore, we write $R_H = R_{\vec{H}} \cap (R_{\vec{H}})^{-1}$, which can be seen as a special case of the Hausdorff distance (see below).

Pseudo-metrics

We use truncated addition and subtraction on the interval $[0,1]$, i.e., for $r, s \in [0,1]$ we have $r \oplus s = \min\{r+s, 1\}$, $r \ominus s = \max\{0, r-s\}$.

A *directed pseudo-metric* or *hemimetric* on a set X is a function $d \colon X \times X \to [0,1]$ such that for all $x, y, z \in X$ (i) $d(x,x) = 0$, (ii) $d(x,z) \le d(x,y) \oplus d(y,z)$. It is called a *pseudo-metric* if in addition (iii) $d(x,y) = d(y,x)$ for all $x, y \in X$. Whenever $d(x,y) = 0$ implies $x = y$ we drop the prefix "pseudo-" and call d a *metric*. Given a directed pseudo-metric d on X, $\overline{d}$ refers to the *symmetrization* of d, i.e., $\overline{d}(x,y) = \max\{d(x,y), d(y,x)\}$, for every $x, y \in X$. Some examples of metrics used in this paper are the following:

- The discrete metric d_{disc} on a set A is $d_{\text{disc}}(a,b) = 1$ if $a \ne b$ and 0 otherwise.
- The *Euclidean* distance d on the interval $[0,1]$ given by $d(r,r') = |r - r'|$.
- The *sup-metric* d on $[0,1]^I$ is given by $d(p,p') = \sup_{i\in I} |p(i) - p'(i)|$.
- The product of two (pseudo)metric spaces (X, d_X) and (Y, d_Y) is a (pseudo)metric space $(X \times Y, d_X \otimes d_Y)$, where $(d_X \otimes d_Y)((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\}$.
- The directed Hausdorff lifting $d_{\vec{H}}$ of a pseudo-metric space (X,d) is a directed pseudo-metric on the power set $\mathcal{P}(X)$ given by $d_{\vec{H}}(U,V) = \sup_{x\in U} \inf_{y\in V} d(x,y)$. Intuitively, the Hausdorff distance between two sets is the farthest that any element of one set has to "travel" to reach the other set.
  It can equivalently be characterized as the infimum $d_{\vec{H}}(U,V) = \bigwedge\{\varepsilon \in [0,1] \mid U \subseteq V_\varepsilon\}$, where $V_\varepsilon = \{x \in X \mid \bigwedge_{v\in V} d(x,v) \le \varepsilon\}$. This means that we are looking for the least ε such that U is included in the union of all ε-balls around elements of V.
  Moreover, the Hausdorff lifting d_H of a pseudo-metric d is the symmetrization of $d_{\vec{H}}$.

Given a directed pseudo-metric $d \colon X \times X \to [0,1]$, a function $f \colon X \to [0,1]$ is called *non-expansive wrt.* d whenever for all $x, y \in X$: $f(x) \ominus f(y) \le d(x,y)$.

Lattices, Fixpoints and Galois Connections

A *complete lattice* $(\mathbb{L}, \sqsubseteq)$ consists of a set $\mathbb{L}$ with a partial order $\sqsubseteq$ such that each $Y \subseteq \mathbb{L}$ has a least upper bound $\bigsqcup Y$ (also called supremum, join) and a greatest lower bound $\bigsqcap Y$ (also called infimum, meet). In particular, $\mathbb{L}$ has a bottom element $\bot = \bigsqcap \mathbb{L}$ and a top element $\top = \bigsqcup \mathbb{L}$. Whenever the order is clear from the context, we simply write $\mathbb{L}$ for a complete lattice. For example:

- $([0,1], \le)$ has a lattice structure with infimum $\bigwedge$ and supremum $\bigvee$.
- The set $Eq(X)$ ($Pre(X)$) of equivalences (preorders) on X is a lattice with $\bigsqcap = \bigcap$ and the join $\bigsqcup \mathcal{R}$ is the least equivalence (resp. preorder) generated by $\bigcup \mathcal{R}$.
- The set $PMet(X)$ ($DPMet(X)$) of (directed) pseudo-metrics is lattice-ordered by $\le$.

Via the Knaster-Tarski theorem it is well-known that any monotone function $f \colon \mathbb{L} \to \mathbb{L}$ on a complete lattice $\mathbb{L}$ has a *least fixpoint* μf and a *greatest fixpoint* νf.

Let $\mathbb{L}$, $\mathbb{B}$ be two lattices. A *Galois connection* from $\mathbb{L}$ to $\mathbb{B}$ is a pair $\alpha \dashv \gamma$ of monotone functions $\alpha \colon \mathbb{L} \to \mathbb{B}$, $\gamma \colon \mathbb{B} \to \mathbb{L}$ such that for all $\ell \in L$: $\ell \sqsubseteq \gamma(\alpha(\ell))$ and for all $m \in \mathbb{B}$: $\alpha(\gamma(m)) \sqsubseteq m$. Equivalently, $\alpha(\ell) \sqsubseteq m \iff \ell \sqsubseteq \gamma(m)$, for all $\ell \in \mathbb{L}, m \in \mathbb{B}$. The function α (resp. γ) is also called the *left* (resp. *right*) *adjoint* and it preserves arbitrary joins (meets).

For an arbitrary function f, we define f^ω as $f^\omega(x) = \bigsqcup_{i\in\mathbb{N}} f^i(x)$. Given a function $f\colon X \to [0,1]$, the function $\tilde{f}\colon \mathcal{P}(X) \to [0,1]$ denotes the join-preserving function generated by f and is defined as $\tilde{f}(X') = \bigvee_{x\in X'} f(x)$ (for $X' \subseteq X$).

Closures

A *closure* c is a monotone, idempotent and extensive (i.e. $x \sqsubseteq c(x)$ for all x) function on a lattice. Given a Galois connection $\alpha \dashv \gamma$, the map $\gamma \circ \alpha$ is always a closure.

Given a set Z, a family $\mathcal{O}$ of operators on Z (of arbitrary, possibly infinite, arity) and a subset $Z' \subseteq Z$, we denote by $\mathrm{cl}^{\mathcal{O}}(Z')$ the least superset of Z' that is closed under all the operators from $\mathcal{O}$. The set $\mathcal{O}$ will sometimes be left implicit in favour of a more suggestive notation. For instance, given a set $\mathcal{S} \subseteq \mathcal{P}(X)$, $\mathrm{cl}^{\cup}(\mathcal{S})$ closes $\mathcal{S}$ under arbitrary unions and $\mathrm{cl}^{\cup,\cap}(\mathcal{S})$ under arbitrary unions and intersections. On the other hand $\mathrm{cl}^{\mathcal{O}}_f$ closes only under operators in $\mathcal{O}$ of finite arity (such as finite unions or intersections). Clearly, $\mathrm{cl}^{\mathcal{O}}$ and $\mathrm{cl}^{\mathcal{O}}_f$ are closures in the above sense.

A special case is the shift, where, given a set $\mathcal{F} \subseteq [0,1]^X$, $\mathrm{cl}^{\mathrm{sh}}(\mathcal{F})$ is the closure under constant shifts, i.e., operations $f \mapsto f \ominus c$, $f \mapsto f \oplus c$ for $c \in [0,1]$.

We end this subsection by a technical result which is needed to show that our "logic" function (cf. Section 3) is continuous.

▶ **Lemma 2.1.** *Let $(F_i \subseteq Z)_{i\in\mathbb{N}}$ be an increasing family of sets, i.e., $F_i \subseteq F_{i+1}$ for every $i \in \mathbb{N}$. If the set $\mathcal{O}$ (of operators on Z) contains operators of only finite arity, then $\mathrm{cl}^{\mathcal{O}}(\bigcup_{i\in\mathbb{N}} F_i) = \bigcup_{i\in\mathbb{N}} \mathrm{cl}^{\mathcal{O}}(F_i)$.*

Transition Systems

We will restrict to systems of the following kind in this paper.

▶ **Definition 2.2** ((Metric) Transition Systems)**.** *A* transition system *over an alphabet A is a pair $(X, \to)$ consisting of a state space X and a transition relation $\to \subseteq X \times A \times X$. We write $x \xrightarrow{a} x'$ for $(x, a, x') \in \to$. For $x \in X$, $\delta(x) = \{(a, x') \mid x \xrightarrow{a} x'\}$ and $\delta_a(x)$ denotes the a-successors of x. A transition system is* finitely branching *if $\delta(x)$ is finite for every x.*

For a set $\Delta \subseteq A \times X$ we denote by $\mathrm{lab}(\Delta)$ the set of labels of Δ, in other words the projection to the first argument, i.e. $\mathrm{lab}(\Delta) = \{a \mid \exists x \in X\colon (a, x) \in \Delta\}$. Similarly $\mathrm{tgt}(\Delta)$ is the set of targets and projects to the second argument.

A metric transition system *over A is a triple $(X, \to, d_A)$ with a metric $d_A\colon A \times A \to [0,1]$.*

▶ **Definition 2.3** (Traces)**.** *For $x \in X, \sigma = a_1 \cdots a_n \in A^*$, we write $x \xrightarrow{\sigma} x'$ if $x \xrightarrow{a_1} \cdots \xrightarrow{a_n} x'$ and define $\mathrm{Tr}(x) = \{\sigma \mid \exists x'\colon x \xrightarrow{\sigma} x'\}$. We extend δ, δ_a to sequences $\hat{\delta}, \hat{\delta}_\sigma$ in the obvious way.*

Given a metric transition system, the distance of two traces is defined as $d_{\mathrm{Tr}}\colon A^* \times A^* \to [0,1]$ where $d_{\mathrm{Tr}}(\sigma_1, \sigma_2) = 1$ if $|\sigma_1| \neq |\sigma_2|$, $d_{\mathrm{Tr}}(\varepsilon, \varepsilon) = 0$ and $d_{\mathrm{Tr}}(a_1\sigma_1', a_2\sigma_2') = \max\{d_A(a_1, a_2), d_{\mathrm{Tr}}(\sigma_1', \sigma_2')\}$ (sup-metric).

3 General Framework

Our results are based on the following theorem that shows how fixpoints are preserved by Galois connections, a well-known property, see for instance [2, 4, 5].

We first introduce the notion of compatibility that has been studied in connection with up-to techniques, enhancing coinductive proofs [23].

▶ **Definition 3.1.** *Let* $\log, c\colon \mathbb{L} \to \mathbb{L}$ *be two monotone endo-functions on a lattice* $\mathbb{L}$*. We call* $\log$ *c-compatible whenever* $\log \circ c \sqsubseteq c \circ \log$*.*

▶ **Theorem 3.2.** *Let* $\mathbb{L}, \mathbb{B}$ *be two complete lattices with a Galois connection* $\alpha\colon \mathbb{L} \to \mathbb{B}$*,* $\gamma\colon \mathbb{B} \to \mathbb{L}$ *and two monotone endo-functions* $\log\colon \mathbb{L} \to \mathbb{L}$*,* $\mathrm{beh}\colon \mathbb{B} \to \mathbb{B}$*.*

1. *Then* $\alpha \circ \log = \mathrm{beh} \circ \alpha$ *implies* $\alpha(\mu \log) = \mu\, \mathrm{beh}$*.*
2. *Let* $c = \gamma \circ \alpha$ *be the closure operator corresponding to the Galois connection and assume that* $\mathrm{beh} = \alpha \circ \log \circ \gamma$*. Then c-compatibility of* $\log$ *implies* $\alpha(\mu \log) = \mu\, \mathrm{beh}$*.*
3. *Whenever* $\alpha \circ \log = \mathrm{beh} \circ \alpha$ *and* $\log$ *reaches its fixpoint in* ω *steps, i.e.,* $\mu \log = \log^\omega(\bot)$*, so does* beh*.*

Here $\mathbb{L}$ is the universe in which the logic lives and $\mathbb{B}$ is the universe in which equivalences respectively metrics live. Furthermore log is the "logic function", constructing modal logic formulas, and $\mu \log$ will be the set of all formulas. On the other hand, beh is the "behaviour function" whose least (respectively greatest) fixpoint is the behavioural metric (equivalence).

▶ Remark 3.3. Note that the above theorem is true even in more general situations, for example if $\mathbb{L}$ and $\mathbb{B}$ are only assumed to be complete partial orders. We however stick to complete lattices since they are more widely known. Also, on a complete lattice many notions of continuity, such as Scott-continuity or chain-continuity, coincide [19]. In the following we will therefore simply say that a monotone function on $\mathbb{L}$ or $\mathbb{B}$ is continuous if it preserves suprema of all (well-ordered) chains.

The recipe used in this paper is the following: first, define a logical universe $\mathbb{L}$ and a logic function $\log\colon \mathbb{L} \to \mathbb{L}$. Then choose a suitable Galois connection $\alpha \dashv \gamma$ to a behaviour universe $\mathbb{B}$ and show that log is c-compatible, where $c = \gamma \circ \alpha$ is the closure associated to the Galois connection. Then derive the behaviour function $\mathrm{beh} = \alpha \circ \log \circ \gamma\colon \mathbb{B} \to \mathbb{B}$ and from the results above, we automatically obtain the equality $\alpha(\mu \log) = \mu\, \mathrm{beh}$, which tells us that logical and behavioural equivalence respectively distance coincide (Hennessy-Milner theorem). This will be worked out in the following examples.

Combining logic functions results in the combination of the corresponding behaviour functions, which is essential in establishing Hennessy-Milner theorems compositionally.

▶ **Proposition 3.4.** *Let* $i \in \{1, 2\}$ *and* $\log_i, c\colon \mathbb{L} \to \mathbb{L}$ *be monotone functions on a complete lattice* $\mathbb{L}$ *such that* $\log_i$ *are c-compatible. Then* $\log_1 \sqcup \log_2$ *and* $\log_1 \circ \log_2$ *are also c-compatible.*

Let $\mathrm{beh}_i = \alpha \circ \log_i \circ \gamma$ *be the behaviour functions corresponding to* $\log_i$*. Then the behaviour functions of* $\log_1 \sqcup \log_2$ *and* $\log_1 \circ \log_2$ *are, respectively,* $\mathrm{beh}_1 \sqcup \mathrm{beh}_2$ *and* $\mathrm{beh}_1 \circ \mathrm{beh}_2$*.*

Furthermore every constant function k and the identity are c-compatible. Their corresponding behaviour functions are the constant function $b \mapsto \alpha(\ell)$ (where ℓ is the constant value of k) respectively the co-closure $\alpha \circ \gamma$.

We are using techniques for the construction of up-to functions studied in [23], but we are using them in a non-standard way. The point is subtle since the closure is usually supposed to be the up-to function, while in our notion of compatibility the logic function plays this role. Furthermore we are interested in least fixpoints, while the results of [23] consider post-fixpoints up-to in order to show that a lattice element is below the greatest fixpoint.

We end this section by characterising the compatibility property when the closure c is induced by an adjoint situation $\alpha \dashv \gamma$ (as in Theorem 3.2). This result is in turn used to relate with the notion of approximating family of predicates [14] in Section 6.

▶ **Lemma 3.5.** *Let* $\alpha \dashv \gamma$ *be a Galois connection between lattices* $\mathbb{L}$*,* $\mathbb{B}$ *(with* $c = \gamma \circ \alpha$*) and let* $\log\colon \mathbb{L} \to \mathbb{L}$ *be a monotone function. Furthermore let* $\ell \in \mathbb{L}$*. Then* $\log(c(\ell)) \sqsubseteq c(\log(\ell))$ *iff*

$$\forall \ell' \in \mathbb{L}\colon \big(\alpha(\ell') \sqsubseteq \alpha(\ell) \implies \alpha(\log(\ell')) \sqsubseteq \alpha(\log(\ell))\big).$$

4 Qualitative Case

We will start with the classical, qualitative case with behavioural equivalences on the one side and boolean-valued modal logics on the other side. In this way we will recreate parts of the theory of [25], incorporating it into the setting of adjunctions as described earlier. Throughout this section we fix a transition system $(X, \to)$ over A.

4.1 Bisimilarity

For bisimilarity we work with the lattices $\mathbb{L} = (\mathcal{P}(\mathcal{P}(X)), \subseteq)$ and $\mathbb{B} = (Eq(X), \supseteq)$. The Galois connection is given as follows, where $[x]_R$ is the equivalence class of x wrt. R:

$$\begin{aligned}\alpha_{\mathrm{b}}(\mathcal{S}) &= \{(x,x') \in X \times X \mid \forall S \in \mathcal{S}\colon (x \in S \iff x' \in S)\}\\ \gamma_{\mathrm{b}}(R) &= \{S \subseteq X \mid \forall (x,x') \in R\colon (x \in S \iff x' \in S)\} = \Big\{\bigcup\{[x]_R \mid x \in S\} \mid S \subseteq X\Big\}.\end{aligned}$$

Intuitively α_{b} generates an equivalence on X from a set of subsets of X and γ_{b} maps an equivalence to all subsets of X that are closed under this equivalence. Both functions are monotone and it is easy to see from the definition that it is indeed a Galois connection (see also Proposition 4.2 below). As logic function we consider $\log_{\mathrm{b}}\colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X))$ with $\log_{\mathrm{b}}(\mathcal{S}) = \bigcup_{a\in A} \Diamond_a[\mathrm{cl}_f^{\cup,\neg}(\mathcal{S})]$, where $\mathrm{cl}_f^{\cup,\neg}$ closes $\mathcal{S}$ under finite unions and complement (hence also finite intersections). Moreover, $\Diamond_a(S) = \{x \in X \mid \exists x' \in S:\ x \xrightarrow{a} x'\}$ for $a \in A$.

The set $\mu \log_{\mathrm{b}}$ of subsets of X is obtained by evaluating modal logic formulas consisting of constants *true*, *false* (empty conjunction/disjunction), binary conjunctions/disjunctions, negation and diamond modality, where the outermost operator is always the modality. Note that $\mu \log_{\mathrm{b}}$ is a strict subset of the usual modal logic formulas, but sufficient for expressivity.

▶ Remark 4.1. The continuity of $\log_{\mathrm{b}}$ deserves some attention. Note that the size of A (be it finite or infinite) has no effect on the continuity of $\log_{\mathrm{b}}$. Rather it follows from Lemma 2.1 and the fact the direct image of a function preserves arbitrary unions. As this argument remains unchanged in other contexts (e.g. simulation preorders and (bi)simulation metrics), we will henceforth tacitly state that our logic functions in the sequel are continuous.

We first study the closure associated to the Galois connection, which is important for showing compatibility later on, and the corresponding co-closure.

▶ **Proposition 4.2.** *The closure $\mathrm{c}_{\mathrm{b}} = \gamma_{\mathrm{b}} \circ \alpha_{\mathrm{b}}$ closes a set $\mathcal{S} \subseteq \mathcal{P}(X)$ under arbitrary boolean operations (union, intersection, complement), while the co-closure $\alpha_{\mathrm{b}} \circ \gamma_{\mathrm{b}}$ is the identity.*

The next step is to show that the logic function is indeed c_{b}-compatible, so that we can invoke Theorem 3.2. Not being compatible basically means that the closure c_{b} introduces operators that clash with logical equivalence. For the proof of Proposition 4.4 we require the fact that the transition system is finitely branching. We first need the following lemma:

▶ **Lemma 4.3.** *Let $(X, \to)$ be a finitely branching transition system and $(X_i \subseteq X)_{i\in\mathcal{I}}$ be a sequence of sets of states. Then, for $a \in A$, we have $\Diamond_a\Big(\bigcap_{i\in\mathcal{I}} X_i\Big) = \bigcap_{\substack{\mathcal{I}_0\subseteq\mathcal{I}\\ \mathcal{I}_0 \text{ finite}}} \Diamond_a\Big(\bigcap_{i\in\mathcal{I}_0} X_i\Big)$.*

▶ **Proposition 4.4.** *For finitely branching transition systems, $\log_{\mathrm{b}}$ is c_{b}-compatible.*

This theorem would straightforwardly generalize to the case where the set of a-successors is finite for each a in the qualitative case, but not directly in the quantitative case which we treat later. Hence, in this paper, we require the transition system to be finitely branching for branching equivalences/metrics, a requirement that is unnecessary in the trace case.

As a result we can derive the behaviour function from the logic function via the Galois connection. Not surprisingly, this behaviour function is in fact the well-known function whose greatest fixpoint (remember the contravariance) is bisimilarity.

▶ **Proposition 4.5.** *The behaviour function* $\mathrm{beh_b}$ *can be characterized as:* $x_1\ \mathrm{beh_b}(R)\ x_2$ *iff*

$$\forall a \in A, y_1 \in \delta_a(x_1)\,\exists y_2 \in \delta_a(x_2)\colon y_1\ R\ y_2 \wedge \forall a \in A, y_2 \in \delta_a(x_2)\,\exists y_1 \in \delta_a(x_1)\colon y_1\ R\ y_2.$$

In particular this means that $(x_1, x_2) \in \alpha_b(\mu\,\mathrm{log_b}) = \mu\ \mathrm{beh_b}$ *iff* x_1, x_2 *are bisimilar.*

It is well known that the behaviour function $\mathrm{beh_b}$ for bisimilarity is continuous if the underlying transition system is finitely branching.

4.2 Simulation Preorders

In this section we show that not only equivalences, but also behavioural preorders can be integrated into our framework. Our logical and behavioural universes are given by the lattices $\mathbb{L} = (\mathcal{P}(\mathcal{P}(X)), \subseteq)$ and $\mathbb{B} = (\mathit{Pre}(X), \supseteq)$. The Galois connection is given as follows:

$$\alpha_s(\mathcal{S}) = \{(x_1, x_2) \mid \forall S \in \mathcal{S}\colon (x_1 \in S \Rightarrow x_2 \in S)\}$$
$$\gamma_s(R) = \{S \subseteq X \mid \forall s \in S\colon R[\{s\}] \subseteq S\}.$$

In other words, $\alpha_s(\mathcal{S})[x] = \bigcap\{S \in \mathcal{S} \mid x \in S\}$. As logic function we consider $\mathrm{log_s}\colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X))$ with $\mathrm{log_s}(\mathcal{S}) = \bigcup_{a\in A} \Diamond_a[\mathrm{cl}_f^{\cap}(\mathcal{S})]$, where $\mathrm{cl}_f^{\cap}$ closes a family of sets $\mathcal{S}$ under finite intersections. Hence the corresponding logic may use $\Diamond_a$ ($a \in A$), conjunction and *true* (the empty intersection), where we again consider only formulas where the outermost operator is a modality. The logic function $\mathrm{log_s}$ is continuous and $\mu\ \mathrm{log_s}$ contains all sets that are obtained from evaluating such formulas.

As desired, the closure induced by the Galois connection closes under union and intersection, but *not* under negation, an operation that should be disallowed in a logic characterizing simulation. The co-closure is instead the identity on preorders, as in Section 4.1.

▶ **Proposition 4.6.** *The closure* $\mathrm{c_s} = \gamma_s \circ \alpha_s$ *closes a family of subsets of* X *under arbitrary unions and intersections. Moreover, the co-closure* $\alpha_s \circ \gamma_s$ *is the identity on* $\mathit{Pre}(X)$*.*

We show that $\mathrm{log_s}$ is $\mathrm{c_s}$-compatible and subsequently state the main result of this section.

▶ **Proposition 4.7.** *For finitely branching transition systems,* $\mathrm{log_s}$ *is* $\mathrm{c_s}$*-compatible.*

▶ **Theorem 4.8.** *The behaviour function* $\mathrm{beh_s}$ *can be characterized as follows:* $x_1\ \mathrm{beh_s}(R)\ x_2$ *iff* $\forall a \in A, y_1 \in \delta_a(x_1)\ \exists y_2 \in \delta_a(x_2)\colon y_1\ R\ y_2$*, i.e.,* $(x_1, x_2) \in \alpha_s(\mu\ \mathrm{log_s}) = \mu\ \mathrm{beh_s}$ *iff* x_2 *simulates* x_1*. Moreover, for finitely branching transition systems,* $\mathrm{beh_s}$ *is continuous.*

4.3 Trace Equivalence

We now follow the same storyline to set up a Galois connection and framework for trace equivalence, which will later be enriched to decorated traces like complete/failure/ready traces [25]. Note that we cannot use the Galois connections from the previous sections, since in particular *c*-compatibility would fail, due to the fact that negation and conjunction have to be disallowed in a logic using the diamond modality to characterize trace equivalence, while instead disjunction is permitted. On the logic side we use the same lattice $\mathbb{L} = (\mathcal{P}(\mathcal{P}(X)), \subseteq)$, however, the behaviour lattice $\mathbb{B} = (\mathit{Eq}(\mathcal{P}(X)), \supseteq)$ is the set of all equivalences over $\mathcal{P}(X)$ (instead of equivalences over X). Choosing powerset as a semantic domain seems natural due to determinization. The corresponding Galois connection is given as follows:

$$\alpha_t(\mathcal{S}) = \{(X_1, X_2) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid \forall S \in \mathcal{S}\colon (X_1 \cap S \neq \emptyset \iff X_2 \cap S \neq \emptyset)\}$$
$$\gamma_t(R) = \{S \subseteq X \mid \forall (X_1, X_2) \in R\colon (X_1 \cap S \neq \emptyset \iff X_2 \cap S \neq \emptyset)\}.$$

Now we consider $\log_t\colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X))$ with $\log_t(\mathcal{S}) = \bigcup_{a\in A} \Diamond_a[\mathcal{S}] \cup \{X\}$, which is again continuous. Then $\mu \log_t$ represents a set of subsets of X obtained by evaluating modal logic formulas consisting of the constant *true* (which evaluates to $\{X\}$) and iterated application of the diamond modalities.

▶ **Proposition 4.9.** *The closure* $c_t = \gamma_t \circ \alpha_t$ *closes a set of subsets of* X *under arbitrary unions, while the co-closure* $\alpha_t \circ \gamma_t$ *maps an equivalence on* $\mathcal{P}(X)$ *to its congruence closure.*

As indicated in the general "recipe", the next step is to show that the logic function is compatible with the closure. Intuitively this is true since diamond distributes over union.

▶ **Proposition 4.10.** *The logic function* $\log_t$ *is* c_t*-compatible.*

Finally the induced behaviour function is the one expected for trace equivalence: the bisimilarity function on the determinized transition system. This is true only for congruences, since beh_t automatically returns a congruence.

▶ **Proposition 4.11.** *On a congruence relation* $R \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$*, we have* $X_1 \mathrm{beh}_t(R)\, X_2$ *iff* $(X_1 = \emptyset \iff X_2 = \emptyset) \wedge \forall a \in A\colon \delta_a[X_1]\ R\ \delta_a[X_2]$*. The restriction of* beh_t *to congruences is continuous, independent of the branching type of the transition system.*

Since on congruences beh_t agrees with the usual fixpoint function for trace equivalence and beh_t preserves congruences, in the corresponding Kleene iteration we obtain only congruences and hence it agrees with the usual one, where one computes bisimilarity on the determinized transition system. Hence it is easy to see that $\mu\,\mathrm{beh}_t$ is indeed trace equivalence (cf. Theorem 4.12).

Decorated Trace Equivalences

We now consider completed trace/ready/failure/possible futures equivalence from the van Glabbeek spectrum [25] and explain how these equivalences can be obtained by adding fixed predicates. We parameterize over a family $\mathcal{S}$ of predicates over the state space (see Figure 1). We first characterize the fixpoint of the behaviour function, modified with an extra preorder as follows. The advantage of this characterisation is that it allows to state various decorated trace equivalences in terms of transfer properties as in the definition of bisimulation relations.

▶ **Theorem 4.12.** *Let* $R_0 \in Pre(X)$ *and consider the map* $\mathrm{beh}_{R_0} = \mathrm{beh}_t \cap (R_0)_H$*. Then* $\mu\,\mathrm{beh}_{R_0}$ *is equal to the set* $\Omega(R_0)$ *of those pairs* (X_1, X_2)*, such that if* $x_1 \in X_1$ *admits a trace* $x_1 \xrightarrow{\sigma} x_1'$*, then there exists* $x_2 \in X_2$*, such that* $x_2 \xrightarrow{\sigma} x_2'$ *and* $x_1'\ R_0\ x_2'$ *(and vice versa).*

In order to infer that $\mu\,\mathrm{beh}_t$ characterizes trace equivalence simply set $R_0 = X \times X$.

The idea is to fix a set $\mathcal{S}$ of predicates and add these to our trace logic, using $R_0 = \alpha_s(\mathcal{S})$ as the preorder required in the above theorem. In order to ensure that logical and behavioural equivalence coincide, we require that $\mathcal{S}$ has certain "good" properties.

▶ **Lemma 4.13.** *Let* $\mathcal{S} \subseteq \mathcal{P}(X)$ *such that* $\forall x \exists S \in \mathcal{S}\colon (x \in S \wedge \forall y\colon x\ \alpha_s(\mathcal{S})\ y \iff y \in S)$*. Then,* $\alpha_t(\mathcal{S})$ *coincides with the relation lifting* $(\alpha_s(\mathcal{S}))_H$*.*

$\mathcal{S}$	$x\ R_0\ y$	Behavioural equivalence
$\{T_X\}$	$\mathrm{lab}(\delta(x)) = \emptyset \implies \mathrm{lab}(\delta(y)) = \emptyset$	completed trace
$\{\mathrm{Ref}(B) \mid B \subseteq A\}$	$\mathrm{lab}(\delta(y)) \subseteq \mathrm{lab}(\delta(x))$	failure
$\{\mathrm{Ready}(B) \mid B \subseteq A\}$	$\mathrm{lab}(\delta(x)) = \mathrm{lab}(\delta(y))$	ready
$\mathrm{cl}^{\cap}(\mu\ \log_t \cup \neg(\mu\ \log_t))$	$\mathrm{Tr}(x) = \mathrm{Tr}(y)$	possible futures

Figure 1 Behavioural equivalences obtained from a logic of the form $\log_0(\mathcal{F}) = \log_t(\mathcal{F}) \cup \mathcal{S}$, respectively a behaviour function of the form $\mathrm{beh}_0 = \mathrm{beh}_t \cap (R_0)_H$.

The condition of Lemma 4.13 is for instance satisfied if $\mathcal{S}$ is closed under intersections. We obtain the following characterization of decorated trace logics.

▶ **Corollary 4.14.** *Assume that $\mathcal{S}$ satisfies the requirements of Lemma 4.13 and let $R_0 = \alpha_s(\mathcal{S})$. Consider the logic function $\log_{\mathcal{S}} = \log_t \cup \mathcal{S}$. Then $\alpha_t(\mu \log_{\mathcal{S}}) = \Omega(R_0) = \mu\ \mathrm{beh}_{R_0}$.*

Hence if we instantiate $\mathcal{S}$ as in Figure 1, where

$$T_X = \delta^{-1}(\emptyset) \qquad \mathrm{Ref}(B) = \{x \mid \mathrm{lab}(\delta(x)) \cap B = \emptyset\} \quad \mathrm{Ready}(B) = \{x \mid \mathrm{lab}(\delta(x)) = B\},$$

we obtain complete trace/failure/readiness equivalences as the least fixpoint of beh_{R_0}. In all these cases $\mathcal{S} \cup \{X\}$ satisfies the requirements of Lemma 4.13.

Note that $\{X\}$ is already generated by $\log_t$. The predicate T_X semantically corresponds to the predicate denoted 0 in [25] (satisfied by those states that have no outgoing transitions). Similarly, the predicate $\mathrm{Ref}(B)$ (resp. $\mathrm{Ready}(B)$) corresponds to the predicate $\tilde{B}$ (resp. B) in [25], which is satisfied by those states that refuse (resp. enable) all the actions from B.

5 Quantitative Case

After discussing the classical case of behavioural equivalences, we will now follow an analogous path to obtain behavioural distances in a quantitative setting. We will begin by first considering the bisimulation pseudo-metric, then directed simulation pseudo-metric, and lastly conclude with the directed (decorated) trace pseudo-metric, from which one can obtain the undirected version by symmetrization. In each case, we will again start out by defining the logics and derive the fixpoint equations for the corresponding behaviour function.

In addition, our decorated trace distance can be seen as the quantitative generalization of a decorated trace preorder, which when instantiated corresponds to (complete) trace/failure/ready inclusions. So, in this sense, our decorated trace distance is going to be parametric. Lastly, though the concrete trace distance is studied elsewhere (cf. [6, 8]), we are not aware of this fixpoint characterization of (decorated) trace distance in the literature. There is a recursive characterization in [8], but based on an auxiliary lattice that serves as memory.

In the rest of this section we fix a metric transition system $(X, \rightarrow, d_A)$ over A.

5.1 Bisimulation Pseudo-metrics

Recall the adjunction from Section 4.1, which we will enrich by replacing a predicate $S \subseteq X$ with a function $f\colon X \to [0,1]$, while pseudo-metrics now play the role of equivalences. In particular, our logical and behavioural universes are given by the lattices $\mathbb{L} = (\mathcal{P}([0,1]^X), \subseteq)$ and $\mathbb{B} = (PMet(X), \leq)$, respectively. Moreover, the Galois connection is given as follows:

$$\alpha_{\mathrm{B}}(\mathcal{F})(x_1,x_2) = \bigvee_{f\in\mathcal{F}} |f(x_1)-f(x_2)| \qquad (\text{for } \mathcal{F}\subseteq[0,1]^X)$$

$$\gamma_{\mathrm{B}}(d) = \{f\in[0,1]^X \mid \forall x_1,x_2\in X : |f(x_1)-f(x_2)|\le d(x_1,x_2)\} \quad (\text{for } d\in \mathit{PMet}(X)).$$

That is, $\alpha_{\mathrm{B}}(\mathcal{F})$ is the least metric on X such that all functions in $\mathcal{F}$ are non-expansive wrt. the Euclidean metric on $[0,1]$, while γ_{B} returns all the non-expansive functions wrt. $d\in \mathit{PMet}(X)$.

Next we introduce a family of modalities $(\bigcirc_a f)_{a\in A}$ in the style of [6]:

$$\bigcirc_a f(x) = \bigvee\{\overline{D_a}(b)\wedge f(x') \mid x \xrightarrow{b} x'\}, \quad \text{where } D_a(b)=d_A(b,a) \text{ and } \overline{D_a}(b)=1-D_a(b).$$

We consider the (continuous) logic function $\log_{\mathrm{B}}\colon \mathcal{P}([0,1]^X)\to\mathcal{P}([0,1]^X)$ that maps a set $\mathcal{F}\subseteq[0,1]^X$ of functions to the set $\bigcup_{a\in A}\bigcirc_a[\mathrm{cl}_f^{\wedge,\neg,\mathrm{sh}}(\mathcal{F})]$, where $\mathrm{cl}_f^{\wedge,\neg,\mathrm{sh}}$ closes $\mathcal{F}$ under finite meets, complements ($f\mapsto 1-f$), and constant shifts (and hence also under finite joins), which are all non-expansive operators (cf. Proposition 5.3). It should be noted that $\bigcirc_a$ is a quantitative generalization of the qualitative diamond modality in the following sense.

▶ **Proposition 5.1.** *If d_A is a discrete metric then $\bigcirc_a f(x)=1 \iff x\in\Diamond_a f^{-1}(\{1\})$.*

Following the development of Section 4.1, we establish the metric version of Lemma 4.3:

▶ **Lemma 5.2.** *Let $(X,\to,d_A)$ be a finitely branching metric transition system and $\mathcal{F}\subseteq[0,1]^X$ be a family of functions. Then for $c\in A$ we have $\bigcirc_c\left(\bigwedge_{f\in\mathcal{F}} f\right)=\bigwedge_{\substack{\mathcal{F}_0\subseteq\mathcal{F}\\ \mathcal{F}_0\ \text{finite}}}\bigcirc_c\left(\bigwedge_{f\in\mathcal{F}_0} f\right)$.*

In the quantitative case, the closures induced by the Galois connections had appealing characterizations via boolean operators. Here the closure $\mathrm{c_B}$ is obtained by post-composing the functions in $\mathcal{F}$ with *all* non-expansive operators. This is in fact a corollary of the McShane-Whitney extension theorem [20, 26].

▶ **Proposition 5.3.** *The closure $\mathrm{c_B}=\gamma_{\mathrm{B}}\circ\alpha_{\mathrm{B}}$ on $\mathcal{F}\subseteq[0,1]^X$ can be characterized as follows:*

$$\mathrm{c_B}(\mathcal{F})=\{\mathrm{op}\circ\langle\mathcal{F}\rangle \mid \mathrm{op}\colon[0,1]^{\mathcal{F}}\to[0,1] \text{ is non-expansive wrt. the sup-metric}\}.$$

Moreover, the co-closure $\alpha_{\mathrm{B}}\circ\gamma_{\mathrm{B}}$ is the identity.

The proof of the above proposition and the next two results are analogous to the corresponding results in the next section on simulation.

▶ **Proposition 5.4.** *For finitely branching transition systems, $\log_{\mathrm{B}}$ is $\mathrm{c_B}$-compatible.*

▶ **Theorem 5.5.** *The behaviour function $\mathrm{beh_B}$ on any $d\in\mathit{PMet}(X)$ is $\mathrm{beh_B}(d)=(d_A\otimes d)_H\circ(\delta\times\delta)$, which results exactly in bisimulation metrics as considered in [6]. Moreover, $\mathrm{beh_B}$ is continuous for finitely branching transition systems.*

It is well-known that the kernel of the bisimulation metric, i.e., the pairs of states with distance 0, is exactly bisimilarity [8].

5.2 Directed Simulation Metrics

In this section, we will treat simulation distance. Our logical and behavioural universes are $\mathbb{L}=(\mathcal{P}([0,1]^X),\subseteq)$ and $\mathbb{B}=(\mathit{DPMet}(X),\le)$ with

$$\alpha_{\mathrm{S}}(\mathcal{F})(x_1,x_2) = \bigvee_{f\in\mathcal{F}} (f(x_1)\ominus f(x_2)) \qquad (\text{for } \mathcal{F}\subseteq[0,1]^X)$$

$$\gamma_{\mathrm{S}}(d) = \{f\in[0,1]^X \mid \forall x_1,x_2\in X : f(x_1)\ominus f(x_2)\le d(x_1,x_2)\} \quad (\text{for } d\in \mathit{DPMet}(X)).$$

Now our (continuous) logic function $\log_S: \mathcal{P}([0,1]^X) \to \mathcal{P}([0,1]^X)$ is the mapping $\mathcal{F} \mapsto \bigcup_{c\in A} \bigcirc_c[\mathrm{cl}_f^{\wedge,\mathrm{sh}}(\mathcal{F})]$, where $\mathrm{cl}_f^{\wedge,\mathrm{sh}}$ closes $\mathcal{F}$ under finite meets and constant shifts (whose necessity is argued in Example 5.9). To characterize the closure c_S we use a directed version of the McShane-Whitney extension theorem [20, 26] (a special case of enriched Kan extensions).

▶ **Proposition 5.6.** *The closure $c_S = \gamma_S \circ \alpha_S$ on $\mathcal{F}$ is the set given in Proposition 5.3 except that* op *is non-expansive wrt. the directed sup-metric. The co-closure $\alpha_S \circ \gamma_S$ is the identity.*

In order to show c_S-compatibility of $\log_S$, we first derive an alternative characterization of the closure in terms of certain normal form given below. Note that a similar statement holds in the context of bisimulation pseudo-metric when we replace the closure c_S by c_B.

▶ **Proposition 5.7** (Normal Form). *Let $\mathcal{F} \subseteq [0,1]^X$ with $1 \in \mathcal{F}$ and $f \in c_S(\mathcal{F})$. Then there is a family of functions $\{f^{\varepsilon}_{(x,y)} \mid \varepsilon > 0$ and $x, y \in X\}$, where each function $f^{\varepsilon}_{(x,y)}$ is a constant shift of a function in $\mathcal{F}$, such that $f = \bigvee_{\varepsilon>0} \bigwedge_{x\in X} \bigvee_{y\in X} f^{\varepsilon}_{(x,y)}$.*

These results enable us to show that the logic function is indeed compatible with closure, a prerequisite for being able to derive the corresponding behaviour function.

▶ **Proposition 5.8.** *For finitely branching transition systems, $\log_S$ is c_S-compatible.*

▶ **Example 5.9.** *We show that adding shifts to the logic function is necessary to obtain compatibility. Consider the metric transition system $(\{x, y, x_1, y_1\}, \{x \xrightarrow{1} x_1, y \xrightarrow{0} y_1\}, d_A)$ with d_A is an Euclidean metric over the alphabet $A = [0,1]$.*

Assume that $\mathcal{F} = \{f\}$ with $f(x) = f(y) = f(x_1) = 1/2$, $f(y_1) = 0$, where the pseudo-metric $d = \alpha_S(\mathcal{F})$ has distance 0 for the states x, y, x_1 and it yields distance $1/2$ between y_1 and all other states. Then it is easy to see that g with $g(x) = g(y) = g(x_1) = 1$, $g(y_1) = 1/2$ is contained in $c_S(\mathcal{F})$, since it is non-expansive wrt. d. Then $\bigcirc_1 g \in \log_S(c_S(\mathcal{F}))$ and

$$\bigcirc_1 g(x) \ominus \bigcirc_1 g(y) = (\overline{D_1}(1) \wedge g(x_1)) \ominus (\overline{D_1}(0) \wedge g(y_1)) = (1 \wedge 1) \ominus (0 \wedge 1/2) = 1.$$

In order for compatibility to hold, $\bigcirc_1 g$ must be contained in $c_S(\log_S(\mathcal{F}))$, i.e., it has to be non-expansive wrt. $\alpha_S(\log_S(\mathcal{F}))$. If the logic function does not use shifts, it only closes $\mathcal{F}$ under finite meets and joins, which results in f, 0 (empty join), 1 (empty meet). For all $r \in [0,1], \bar{f} \in c_S(\mathcal{F})$, we have $\bigcirc_r \bar{f}(x) \ominus \bigcirc_r \bar{f}(y) < \bigcirc_1 g(x) \ominus \bigcirc_1 g(y)$, which means $\bigcirc_1 g \notin c_S(\log_S(\mathcal{F}))$. In particular,

$$\bigcirc_r f(x) \ominus \bigcirc_r f(y) = (\overline{D_r}(1) \wedge f(x_1)) \ominus (\overline{D_r}(0) \wedge f(y_1)) = (\overline{D_r}(1) \wedge 1/2) \ominus (\overline{D_r}(0) \wedge 0) \leq 1/2.$$

▶ **Theorem 5.10.** *The behaviour function* beh_S *can be characterized as* $\mathrm{beh}_S(d) = (d_A \otimes d)_{\overrightarrow{H}} \circ (\delta \times \delta)$ *for any $d \in \mathit{DPMet}(X)$. In particular, $\alpha_S(\mu \log_S) = \mu\, \mathrm{beh}_S$ is the directed similarity metric of [6]. Moreover,* beh_S *is continuous for finitely branching transition systems.*

Every metric transition system can be viewed as a classical one by forgetting the metric on the labels. In addition, we can first compute the simulation metric of the quantitative system and then discretize the values to obtain qualitative simulation equivalence.

▶ **Proposition 5.11.** *Consider the Galois connection $\alpha \dashv \gamma$ with $\alpha: \mathit{DPMet}(X) \to \mathit{Pre}(X)$ and $\gamma: \mathit{Pre}(X) \to \mathit{DPMet}(X)$ given by the maps $\alpha(d) = \{(x,y) \mid d(x,y) = 0\}$, $\gamma(R) = 1 - \chi_R$. If the transition system is finitely branching, then $\alpha \circ \mathrm{beh}_S(d) = \mathrm{beh}_s \circ \alpha(d)$ for every $d \in \mathit{DPMet}(X)$. In particular $\mu\, \mathrm{beh}_s = \alpha(\mu\, \mathrm{beh}_S)$ due to Theorem 3.2.*

We conclude this section by the observation that the characterization in Theorem 5.10 allows us to eliminate constant shifts from the simulation logic.

▶ **Corollary 5.12.** *Let* $\log' \colon \mathcal{P}([0,1]^X) \to \mathcal{P}([0,1]^X)$ *be the variant of* $\log_{\mathrm{S}}$*, where we do not close under constant shifts. If the transition system is finitely branching, then* $\log'$ *is still sound and expressive for simulation, that means* $\alpha_{\mathrm{S}}(\mu\ \log') = \mu\ \mathrm{beh}_{\mathrm{S}}$.

5.3 Directed Trace Metrics

In this section, we treat the directed version of the (decorated) trace distance whose fixpoint characterization is novel and, at the same time, the most complex scenario considered in this paper. We will sometimes omit the adjective "directed".

Based on the qualitative case of trace equivalence (Section 4.3), we fix the logical and behavioural universes to be the lattices $\mathbb{L} = (\mathcal{P}([0,1]^X), \subseteq)$ and $\mathbb{B} = (\mathit{DPMet}(\mathcal{P}(X)), \leq)$ with

$$\alpha_{\mathrm{T}}(\mathcal{F})(X_1, X_2) = \bigvee_{f\in\mathcal{F}} (\tilde{f}(X_1) \ominus \tilde{f}(X_2)) \qquad \text{(for } \mathcal{F} \subseteq [0,1]^X)$$

$$\gamma_{\mathrm{T}}(d) = \{f \in [0,1]^X \mid \tilde{f} \text{ is non-expansive wrt. } d\} \qquad \text{(for } d \in \mathit{DPMet}(\mathcal{P}X)).$$

It is easy to see that $\alpha_{\mathrm{T}}(\mathcal{F})$ is always join-preserving in its first argument. Notice that we could have defined $\mathbb{L}$ as those functions in $[0,1]^{\mathcal{P}(X)}$ that are join-preserving. As a result, one expects that the closure c_{T} may close a set $\mathcal{F}$ under all non-expansive and join-preserving operators. However, this is unfortunately not true as witnessed by the following counterexample. This points to the more fundamental problem that there is no McShane-Whitney type result for non-expansive, join-preserving operators: a non-expansive, join-preserving operator defined on some subset does not necessarily have an extension to the whole space which is both non-expansive and join-preserving.

▶ **Example 5.13.** *Let* $X = \{x, y, z\}$ *and* $\mathcal{F} = \{f_1, f_2\} \subseteq [0,1]^X$*, where* f_1 *and* f_2 *are the mappings* $x, y \mapsto 1, z \mapsto 0$ *and* $x, z \mapsto 0, y \mapsto 1/2$*, respectively. Now consider a map* $g \colon X \to [0,1]$ *with* $g(x) = 1/2$*,* $g(y) = 1$ *and* $g(z) = 0$*. Then it is easily seen that* $g \in \mathrm{c}_{\mathrm{T}}(\mathcal{F})$*. However, we claim that there is no join-preserving and non-expansive operator* $\mathrm{op}\colon [0,1]^2 \to [0,1]$ *such that* $g = \mathrm{op} \circ \langle f_1, f_2\rangle$*. Assume otherwise that* $\mathrm{op}(f_1(u), f_2(u)) = g(x)$ *(for* $u \in X$*), which implies* $\mathrm{op}(1,0) = 1/2$*,* $\mathrm{op}(1, 1/2) = 1$*, and* $\mathrm{op}(0,0) = 0$*. Due to non-expansivity of* op *we conclude that* $\mathrm{op}(0, 1/2) \leq 1/2$*, which leads to the following contradiction:*

$$1 = \mathrm{op}(1, 1/2) = \mathrm{op}((1,0) \vee (0, 1/2)) = \mathrm{op}(1,0) \vee \mathrm{op}(0, 1/2) = 1/2.$$

As (continuous) logic function $\log_{\mathrm{T}} \colon \mathcal{P}([0,1]^X) \to \mathcal{P}([0,1]^X)$ we define $\log_{\mathrm{T}}(\mathcal{F}) = \bigcup_{a\in A} \bigcirc_a[\mathrm{cl}^{\mathrm{sh}}(\mathcal{F})] \cup \{1\}$, where $\mathrm{cl}^{\mathrm{sh}}$ closes a set of functions under constant shifts, as in Section 5.2, and 1 is the constant 1-function. Typically, operators of a "metric" logic ought to preserve non-expansiveness, which is not the case for the shift $f \mapsto f \oplus r$; since it might increase the distance of a non-empty set to the empty set. This is not problematic in our case, since the distance of $\emptyset$ to any other set is 1 anyway, induced by the constant operator 1. (Note that the empty join is 0.) We will show in Theorem 5.20 that our logic characterizes trace distance, i.e., $\alpha_{\mathrm{T}}(\mu \log_{\mathrm{T}}) = d_T$, where $d_T := (d_{\mathrm{Tr}})_{\overrightarrow{H}} \circ (\mathrm{Tr} \times \mathrm{Tr})$.

The co-closure, on the other hand, is straightforward to characterize.

▶ **Proposition 5.14.** *The co-closure* $\alpha_{\mathrm{T}} \circ \gamma_{\mathrm{T}}$ *maps a directed pseudo-metric* d *to the greatest directed pseudo-metric* d' *such that* $d' \leq d$ *and* d' *is join-preserving in its first argument.*

Next we turn our attention to the compatibility of our logic function. Here we have to work around the fact that the closure can not be easily characterized, neither in terms of operators nor in terms of a suitable normal form (cf. Proposition 5.7). Still, compatibility holds, even for transition systems of arbitrary branching type.

▶ **Lemma 5.15.** *Let $h\colon X \to [0,1]$, $c \in A$ and $Y \subseteq X$. Then it holds that*

$$\widetilde{\bigcirc_c h}(Y) = \bigwedge_{\Delta\subseteq\delta[Y]} \big(\widetilde{D_c}(\mathrm{lab}(\delta[Y]\setminus\Delta)) \vee \tilde{h}(\mathrm{tgt}(\Delta))\big)$$

▶ **Proposition 5.16.** *The logic function* $\log_{\mathrm{T}}$ *is* c_{T}*-compatible.*

Now we can characterize the behaviour function as follows. To the best of our knowledge, this is the first time that a fixpoint function for trace metrics on the powerset of the state space has been established. There is also a fixpoint characterization given in [8] although on an auxiliary lattice which serves as a memory.

▶ **Theorem 5.17.** *Let $d \in DPMet(\mathcal{P}(X))$ such that d is join preserving in its first argument and $d(X_1,\emptyset) = 1$ for every non-empty set $X_1 \subseteq X$. Then the behaviour function* $\mathrm{beh}_{\mathrm{T}}$ *can be characterized by the conditional equation:* $\mathrm{beh}_{\mathrm{T}}(d)(X_1,\emptyset) = 1$ *if $X_1 \neq \emptyset$ and otherwise*

$$\mathrm{beh}_{\mathrm{T}}(d)(X_1,X_2) = \bigvee_{(a,x')\in\delta[X_1]} \bigvee_{\Delta\subseteq\delta[X_2]} \Big(\bigwedge_{b\in\mathrm{lab}(\Delta)} d_A(a,b) \wedge d(\{x'\},\mathrm{tgt}(\delta[X_2]\setminus\Delta))\Big).$$

Moreover $\mathrm{beh}_{\mathrm{T}}$ *is continuous, independent of the branching type of the transition system.*

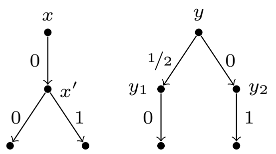
The special case of $X_1 \neq \emptyset$, $X_2 = \emptyset$ is an effect of the term $\{1\}$ in the logic function $\log_{\mathrm{T}}$.

Note that to our knowledge there is no straightforward way to compute the bisimilarity distance on the determinization (see Theorem 17 in [6]). Next, we explain the above fixpoint equation as a two-player game.

▶ Remark 5.18. Given two sets $X_1, X_2 \subseteq X$ and a threshold $\varepsilon \in [0,1]$, we want to check, whether $d_{\mathrm{T}}(X_1,X_2) \le \varepsilon$ with a game played by two players: Death D and Maiden M. First, D chooses a transition (a,x') of $\delta[X_1]$ and also stipulates a set $\Delta \subseteq \delta[X_2]$ of allowed transitions for M. Now M has two possibilities: she can either accept the set Δ presented by D, or she can reject it. If she rejects it, she can only reach states in $Y' := \mathrm{tgt}(\delta[X_2]\setminus\Delta)$ and whatever way D chooses to continue his trace from the state x', M must continue her trace from one of the states in Y'. The game therefore continues with the sets $\{x'\}$ and Y'. If, on the other hand, M chooses to accept the set presented by D, then, in trying to duplicate the trace begun by D with one of the transitions in Δ, she makes an error of at least $\bigwedge_{(b,y')\in\Delta} d_A(a,b)$. It is clear that M should only make this decision if she thinks that D can otherwise force a larger error in a later stage of the game. The game therefore ends and M wins iff $\bigwedge_{(b,y')\in\Delta} d_A(a,b) \le \varepsilon$.

▶ **Example 5.19.** *We compute the directed trace distance of the state x to the state y in the metric transition system over $A = [0,1]$ depicted in Figure 2 with $\mathrm{Tr}(x) = \{(0,0),(0,1)\}, \mathrm{Tr}(y) = \{(1/2,0),(0,1)\}$. There is only one outgoing transition from x and there are four possible choices for $\Delta \subseteq \delta(y)$. The corresponding terms are calculated in Figure 2, where we used that we already computed $d_{\mathrm{T}}(\{x'\},\{y_1\}) = d_{\mathrm{T}}(\{x'\},\{y_2\}) = d_{\mathrm{T}}(\{x'\},\emptyset) = 1$ and $d_{\mathrm{T}}(\{x'\},\{y_1,y_2\}) = 0$. Taking the maximum of the minima we see that $d_{\mathrm{T}}(\{x\},\{y\}) = 1/2$, which is indeed the Hausdorff distance between the two trace sets.*

In the case of the trace metric we can eliminate constant shifts from the logic without losing expressiveness. This is a consequence of Corollary 5.22, which we will prove later.



Δ	$\bigwedge_{(b,y')\in\Delta} d_A(a,b)$	$d_{\mathrm{T}}(\{x'\},\mathrm{tgt}(\delta(y)\setminus\Delta))$
$\emptyset$	1	0
$\{(1/2, y_1)\}$	$1/2$	1
$\{(0, y_2)\}$	0	1
$\delta(y)$	0	1

Figure 2 Computation of trace distance.

Decorated Trace Distances

Now we consider the quantitative generalization of decorated trace preorders. We follow a presentation similar to Theorem 4.12, wherein we characterize the least fixpoint of a behaviour function parameterized by a distance $d_0 \in \mathit{DPMet}(X)$, which is in turn induced by a set $\mathcal{G} \subseteq [0,1]^X$, corresponding to completed/failure/readiness traces.

▶ **Theorem 5.20.** *Let $d_0 \in \mathit{DPMet}(X)$ and consider the map $\mathrm{beh}_{d_0} \colon \mathit{DPMet}(\mathcal{P}(X)) \to \mathit{DPMet}(\mathcal{P}(X))$ defined as $\mathrm{beh}_{d_0}(d) = \mathrm{beh}_{\mathrm{T}}(d) \vee (d_0)_{\overrightarrow{H}}$, for any $d \in \mathit{DPMet}(\mathcal{P}(X))$. Then $\mu\,\mathrm{beh}_{d_0}(X_1, X_2)$ is characterized as the infimum of those $\varepsilon \in [0,1]$ that satisfy:*

$$\forall x_1 \in X_1, x_1' \in X, \sigma \in A^* \colon x_1 \xrightarrow{\sigma} x_1'$$
$$\implies \exists x_2 \in X_2, x_2' \in X, \tau \in A^* \colon x_2 \xrightarrow{\tau} x_2' \wedge d_{\mathrm{Tr}}(\sigma,\tau) \le \varepsilon \wedge d_0(x_1', x_2') \le \varepsilon.$$

When d_0 is the constant 0-metric, this results in the behaviour function $\mathrm{beh}_{\mathrm{T}}$ that characterizes trace distance. Next, we reformulate the result in terms of a set $\mathcal{G} \subseteq [0,1]^X$; this in turn helps in deriving the characterization of various decorated trace distances. We start by imposing a condition on such a set $\mathcal{G}$ that guarantees that $\alpha_{\mathrm{T}}(\mathcal{G})$ is the directed Hausdorff lifting of $d_0 = \alpha_{\mathrm{S}}(\mathcal{G})$ (cf. Section 5.2), which ensures that by Proposition 3.4 the enriched logic function induces a behaviour function as in the previous theorem.

▶ **Lemma 5.21.** *Let $\mathcal{G} \subseteq [0,1]^X$ such that $d_0 = \alpha_{\mathrm{S}}(\mathcal{G})$. Then $\alpha_{\mathrm{T}}(\mathcal{G}) = (d_0)_{\overrightarrow{H}}$ whenever*

$$\forall \varepsilon > 0, x \in X\ \exists g \in \mathcal{G} :\ g(x) = 1 \wedge \forall x' :\ g(x) \ominus g(x') \ge d_0(x,x') - \varepsilon.$$

▶ **Corollary 5.22.** *Assume that $\mathcal{G} \subseteq [0,1]^X$ satisfies the requirements of Lemma 5.21. Then $\alpha_{\mathrm{T}}(\mu(\log_{\mathrm{T}} \cup \mathcal{G})) = (d_{\mathrm{Tr}} \otimes d_0)_{\overrightarrow{H}} \circ (\hat{\delta} \times \hat{\delta})$. The same holds if the logic function $\log_{\mathrm{T}}$ is replaced by $\log'$ with $\log'(\mathcal{F}) = \bigcup_{a\in A} \bigcirc_a[\mathcal{F}] \cup \{1\}$ (without shifts).*

$\mathcal{G}$	$d_0(x,y)$	Behavioural distance
$\{f_A^{\mathrm{Ref}}\}$	$f_A^{\mathrm{Ref}}(x) \ominus f_A^{\mathrm{Ref}}(y)$	completed trace
$\{f_B^{\mathrm{Ref}}, \mid B \subseteq A\}$	$(d_{\mathrm{disc}})_{\overrightarrow{H}}(\mathrm{lab}(\delta(y)), \mathrm{lab}(\delta(x))$	(discrete) failures
$\mathrm{cl}^{\wedge,\mathrm{sh}}(\{g_a \mid a \in A\})$	$(d_A)_{\overrightarrow{H}}(\mathrm{lab}(\delta(y)), \mathrm{lab}(\delta(x)))$	(Hausdorff) failures
$\{f_B^{\mathrm{Ready}} \mid B \subseteq A\}$	$d_{\mathrm{disc}}(\mathrm{lab}(\delta(x)), \mathrm{lab}(\delta(y)))$	(discrete) readiness
$\mathrm{cl}^{\wedge,\mathrm{sh}}(\{g_a, 1-g_a \mid a \in A\})$	$(d_A)_H(\mathrm{lab}(\delta(x)), \mathrm{lab}(\delta(y)))$	(Hausdorff) readiness
$\mathrm{cl}^{\wedge,\mathrm{sh}}(\mu\ \log_{\mathrm{T}} \cup \neg(\mu\ \log_{\mathrm{T}}))$	$\overline{d_{\mathrm{T}}}(\{x\},\{y\})$	possible futures

Figure 3 Behavioural distances obtained from a logic of the form $\log_0(\mathcal{F}) = \log_{\mathrm{T}}(\mathcal{F}) \cup \mathcal{G}$, respectively a behaviour function of the form $\mathrm{beh}_0 = \mathrm{beh}_{\mathrm{T}} \vee (d_0)_{\overrightarrow{H}}$, where $d_0 = \alpha_{\mathrm{S}}(\mathcal{G})$.

These results supply fixpoint characterizations of several meaningful behavioural distances. In Figure 3 we summarize which primitive set of functions $\mathcal{G} \subseteq [0,1]^X$ has to be added to the trace logic in order to get (directed) metric versions of some decorated trace semantics considered in [25]: completed/failure/ready trace semantics.

▶ **Corollary 5.23.** *Consider the following functions* $f_B^{\mathrm{Ref}}, f_B^{\mathrm{Ready}}, g_a \in [0,1]^X$*:*

$$f_B^{\mathrm{Ref}}(x) = \begin{cases} 1 & x \in \mathrm{Ref}(B) \\ 0 & \textit{otherwise} \end{cases} \qquad f_B^{\mathrm{Ready}}(x) = \begin{cases} 1 & x \in \mathrm{Ready}(B) \\ 0 & \textit{otherwise} \end{cases} \qquad g_a(x) = \bigwedge_{b \in \mathrm{lab}(\delta(x))} d_A(a,b).$$

Then by adding $\mathcal{G}$ *to* $\log_{\mathrm{T}}$ *results in the behaviour functions and distances as given in Figure 3.*

Note that the different versions of failures and readiness metrics correspond to different ways to measure the distance between the refuse/ready sets of two states. In the first version we take the discrete metric on $\mathcal{P}(A)$, and in the second version we take the Hausdorff lifting of d_A. In the qualitative setting, the two notions collapse. The Hausdorff versions are the ones to use if we want to recover the hierarchy of [25].

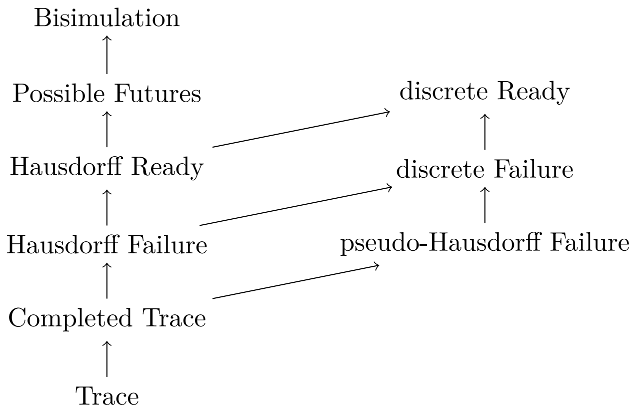


Figure 4 A spectrum of behavioural distances.

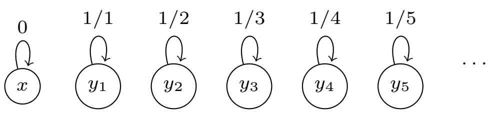
Consider also the pseudo-Hausdorff failure semantics arising from choosing a set $\mathcal{G}$ of predicates with $\alpha_S(\mathcal{G})(x,y) = d_0(x,y) = (d_A)_{\overrightarrow{H}}(A \backslash \mathrm{lab}(\delta(x)), A \backslash \mathrm{lab}(\delta(y)))$. In the qualitative setting this notion collapses with Hausdorff failure and discrete failure semantics, but in the metric setting the pseudo-Hausdorff failure distance is not even bounded by the bisimulation distance (see Figure 4). The inclusions shown in Figure 4 are obvious from comparing the corresponding metrics d_0 in Figure 3.

Again we conclude by comparing the qualitative and quantitative case.

▶ **Proposition 5.24.** *Consider the map* $\alpha\colon \mathit{DPMet}(\mathcal{P}(X)) \to \mathit{Pre}(\mathcal{P}(X))$ *given by* $\alpha(d) = \{(X_1,X_2) \mid d(X_1,X_2) = 0\}$*. If the set* A *of actions is finite, then* $\mu\,\mathrm{beh}_{\mathrm{t}} = \alpha(\overline{\mu\,\mathrm{beh}_{\mathrm{T}}})$*.*

The necessity of requiring finiteness of A is illustrated by Example 5.25.

▶ **Example 5.25.** *Consider the transition system depicted below:*



The trace distance of $X_1 = \{y_{i+1} \mid i \in \mathbb{N}\}$ *and* $X_2 = X_1 \cup \{x\}$ *is* $\overline{\mu\,\mathrm{beh}_{\mathrm{T}}}(X_1,X_2) = 0$*. However we do not have full trace inclusion, hence* $(X_1,X_2) \notin \mu\,\mathrm{beh}_{\mathrm{t}}$*.*

6 Concluding Remarks, Related and Future Work

We presented a recipe to construct (bi)simulation equivalence/distance and trace equivalence/distance (together with various forms of their decorated trace counterparts) as the least fixpoint of behaviour functions on the underlying lattice $\mathbb{B}$ of equivalences/distances. Furthermore, upon realising the relevant Galois connection $\alpha \dashv \gamma$ between the lattices $\mathbb{L}$ (modelling sets of predicates) and $\mathbb{B}$, we showed in each case that these behaviour functions arise naturally (i.e., $\mathrm{beh} = \alpha \circ \log \circ \gamma$) when the logic function log is compatible with the closure $\gamma \circ \alpha$. By doing so, we not only recover the fixpoint characterizations of the branching-time spectrum, but we also gave novel ones in the linear-time spectrum (like the trace distances and their variations: completed trace/failure/ready/possible futures).

Related work

Our work is related to [6, 8], where the former establishes a logical characterization (using the syntax of LTL and μ-calculus) of bisimulation and trace distances, while the latter recasts a part of the classical linear-branching time spectrum to a quantitative one involving metrics, based on games. The fixpoint and logical characterizations of (decorated) trace distances were not present in both [6, 8]. In [8] the authors parameterize over various trace distances, which we are not, although this is an interesting direction for future work. By restricting to pointwise trace distance with discount one, we obtain corresponding notions for bisimilarity, trace and (Hausdorff) readiness. Note that [8] does not treat failures. Also, our game in Remark 5.18 is different from the games played in [8], since it is played locally on the powerset domain.

Coalgebraists familiar with fibrations/indexed categories [10] will recognize the Galois connection between the fibres of two indexed categories: one modelling the logical universe, the other behavioural universe on the state space of a coalgebra. Indeed, Klin in his PhD thesis [13] has explored this adjoint situation $\alpha_{\mathrm{b}} \dashv \gamma_{\mathrm{b}}$ (cf. Section 4.1); note that behavioural metrics were not treated in [13]. The two approaches diverge in the treatment of closures especially in the context of decorated traces. In this paper, closures are always induced as monad from the adjoint situation and to handle (decorated) trace equivalences we consider the adjoint situation $\alpha_{\mathrm{t}} \dashv \gamma_{\mathrm{t}}$ since the closure c_{b} is not sound w.r.t. (decorated) trace equivalence. In Klin's approach, on the contrary, the adjunction $\alpha_{\mathrm{b}} \dashv \gamma_{\mathrm{b}}$ used to characterize bisimilarity is fixed (even for decorated trace equivalences), but the notion of closure is left parametric [13, Definition 3.31]. Our new insight in the qualitative case is that the closure is naturally induced by the Galois connection and the characterization of fixpoint preservation is a fundamental ingredient.

We also point out the differences to the dual adjunction approach [12, 17, 18, 22] to coalgebraic modal logic. There the functor on the "logic universe" characterizes the *syntax* of the logics, while the semantics is given by a natural transformation. In [18] the approach is lifted to fibrations (in which the equivalence lives). Generalizing our approach however would lead to a situation where we obtain a fibred adjunction between two fibrations (for logic and behaviour) on the same category.

In [14] the approach of [18] is instantiated to a quantitative setting, without treating trace metrics. A central notion there is that of an approximating family, which, translated into our language, says that $\mathcal{F} \subseteq [0,1]^X$ is an approximating family iff $\forall f \in [0,1]^X : \alpha(\mathcal{F}) \geq \alpha(\{f\})$ implies $\alpha(\log(\mathcal{F})) \geq \alpha(\log(\{f\}))$, with log being restricted to applying modalities. If log is join-preserving, this is equivalent to $\log(c(\mathcal{F})) \subseteq c(\log(\mathcal{F}))$ (this is a direct consequence of Lemma 3.5), i.e., it is strongly related to compatibility.

Future work

Taking inspiration from the above, we want to generalize our work to the level of coalgebras with an approach based on fibrations, enabling us to treat other branching types, such as probabilistic branching. Note that the coalgebraic treatment of establishing Hennessy-Milner theorems in [11, 13] does not subsume the behavioural distances covered in this paper, while the qualitative spectrum has been generalized using graded monads [21]. We plan to develop fixpoint and logical characterizations of coalgebraic behavioural metrics [1, 15], which are generalizations of both bisimulation pseudo-metric and trace distance.

We are also interested in exploring connections with [16], a paper studying the question which formulas of Hennessy-Milner logic are preserved by quotienting through a behavioural equivalence.

Another direction is to consider behavioural equivalences (such as failure trace/ready trace equivalences and variations) that cannot be captured by our modular approach (i.e., by extending the logic functions $\log_{\mathrm{t}}/\log_{\mathrm{T}}$ with a *constant* function). We also want to characterize undirected trace distance directly without the symmetrization of directed trace distance.

Another line of research is to determine under which circumstances we can restrict to finitary operations, from which we deviate occasionally by closing under arbitrary meets or intersections. This should be feasible by restricting to finitely branching transition systems. Also, in the metric case, we plan to optimize the syntax by restricting shifts and modalities to rational numbers. Last, but not least, it will be interesting to work out the compatibility of $\log_{\mathrm{B}}$ for a weaker class of metric transition systems than those which are finitely branching.

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