Proofs and Refutations for Intuitionistic and Second-Order Logic

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Abstract

The λ^{PRK}-calculus is a typed λ-calculus that exploits the duality between the notions of proof and refutation to provide a computational interpretation for classical propositional logic. In this work, we extend λ^{PRK} to encompass classical second-order logic, by incorporating parametric polymorphism and existential types. The system is shown to enjoy good computational properties, such as type preservation, confluence, and strong normalization, which is established by means of a reducibility argument. We identify a syntactic restriction on proofs that characterizes exactly the intuitionistic fragment of second-order λ^{PRK}, and we study canonicity results.

2012 ACM Subject Classification Theory of computation → Proof theory; Theory of computation → Type theory

Keywords and phrases lambda-calculus, propositions-as-types, classical logic, proof normalization

Digital Object Identifier 10.4230/LIPIcs.CSL.2023.9

Related Version *Extended Version*: `https://arxiv.org/abs/2210.07274` [5]

Funding This work was partially supported by project grant PUNQ 2247/22.

1 Introduction

Constructivism in logic is closely related with the notion of *algorithm* in computer science. The reason is that a constructive proof of existence of a mathematical object fulfilling certain properties should provide an *effective construction* of such an object. For example, a constructive proof of $\forall x \in \mathbb{N}. \exists y \in \mathbb{N}. P(x, y)$ may be understood as an algorithm that takes as input a natural number x and produces as output a natural number y that verifies $P(x, y)$. The close relationship that exists between *proofs* and *computer programs*, and between *logical propositions* and *program specifications* (or *types*), can be taken to its maximum consequences in the form of the **propositions-as-types correspondence**.

This correspondence has given rise to a broad and active area of research, guided by the principle that *each proof-theoretical notion has a computational counterpart and vice-versa*. These ideas allow logic and computer science to feed back on each other, and they have been extended to such settings as *first-order logic* [15, 30, 8], *second-order logic* [21, 42], *linear logic* [22], *modal logic* [6, 14] and *classical logic* [23, 10, 3, 37]. The question of what kind of computational system would constitute a reasonable counterpart for **classical logic**, from the point of view of the propositions-as-types correspondence, is far from being definitely settled. This work is part of the quest for a satisfactory answer to this problem.

The proofs and refutations calculus (λ^{PRK}). Until the late 1980s, it was widely thought that it was not possible to extend the propositions-as-types correspondence to encompass classical logic. This view changed when Griffin [23] remarked that the classical principle of double negation elimination ($\neg\neg A \to A$) can be understood as the typing rule for a control operator $\mathcal{C}$, closely related to Felleisen's $\mathcal{C}$ operator [19] and to SCHEME's `call/cc`. Since then, many other calculi for classical logic have been proposed. Significant examples are Parigot's $\lambda\mu$ [37], Barbanera and Berardi's calculus [3], and Curien and Herbelin's $\bar{\lambda}\mu\tilde{\mu}$ calculus [10].

The starting point of this paper is the logical system PRK, introduced recently by the authors [4] and extending Nelson's constructive negation [34]. In PRK, propositions become classified along two dimensions: their *sign*, which may be positive or negative, and their *strength*, which may be strong or weak. This results into four possible *modes*:

	Positive	Negative
Strong	A^+	A^-
Weak	$A^\oplus$	$A^\ominus$

Positive ($A^+/A^\oplus$) and negative ($A^-/A^\ominus$) propositions correspond to affirmations and denials. Strong (A^+/A^-) and weak ($A^\oplus/A^\ominus$) propositions impose restrictions on the shape of canonical proofs: a canonical proof of a strong affirmation (A^+) must always be constructed with an introduction rule for the corresponding logical connective, whereas a canonical proof of a weak affirmation ($A^\oplus$) must always proceed by *reductio ad absurdum*, by assuming the weak denial $A^\ominus$ and proving the strong affirmation A^+.

We summarize some important characteristics of PRK. First, PRK is a **refinement** of classical logic: $A_1, \ldots, A_n \vdash B$ holds in classical propositional logic if and only if $A_1^\oplus, \ldots, A_n^\oplus \vdash B^\oplus$ holds in PRK. In fact PRK is "finer" than classical logic: for example, the law of excluded middle holds *weakly*, *i.e.* $(A \vee \neg A)^\oplus$ is valid in PRK, whereas it does not hold *strongly*, *i.e.* $(A \vee \neg A)^+$ is not valid (in general) in PRK. Second, the λ^{PRK}-calculus, which results from assigning proof terms to PRK proofs and endowing it with rewrite rules, turns out to be **confluent** and **strongly normalizing**, besides enjoying **subject reduction**. Third, as a result, PRK enjoys **canonicity**: a proof of a sequent $\vdash P$ without assumptions can always be normalized to a *canonical* proof, headed by an introduction rule.

Contributions and structure of this paper. The PRK logical system of [4] only treats three propositional connectives: conjunction, disjunction, and negation.

- In Section 2, we **extend the $\lambda^{\mathbf{PRK}}$ calculus** to propositional second-order logic. We incorporate **second-order universal and existential quantification**, as well as two propositional connectives, implication and co-implication. The system is shown to **refine classical second-order logic**, and to enjoy good computational properties: **subject reduction** and **confluence**. This extension increases the expressivity of the system, allowing to encode inductive datatypes such as natural numbers, lists, and trees.
- In Section 3, we study **Böhm–Berarducci encodings**, that is, we study how the logical connectives of second-order λ^{PRK} may be encoded in terms of universal quantification and implication only ($\{\forall, \to\}$). The encoding turns out to be only partially satisfactory: it simulates proof normalization for an introduction rule followed by an elimination rule in the *positive* case but, unfortunately, not in the negative case.
- In Section 4 we prove **strong normalization** for the second-order λ^{PRK}-calculus. This is the most technically challenging part of the work. In [4], normalization of the propositional fragment of λ^{PRK} is attained by means of a translation to System F with non-strictly positive recursion. This technique does not carry over to the second-order case. To prove strong normalization, we use a variant of Girard's technique of reducibility candidates and, in particular, we resort to a non-trivial adaptation of Mendler's proof of strong normalization for System F with non-strictly positive recursion [31].

- In Section 5, we define a subsystem of second-order λ^{PRK}, called λ^{PRJ}, by imposing a syntactic restriction on terms. We show that λ^{PRJ} **refines second-order intuitionistic logic**, in the sense that λ^{PRJ} is a conservative extension of second-order intuitionistic logic and, conversely, second-order intuitionistic logic can be embedded in λ^{PRJ}.
- In Section 6 we formulate **canonicity** results for λ^{PRK}. In particular, we strengthen the canonicity results of [4] to show that an explicit witness can be extracted from a proof of P.
- Finally, in Section 7 we conclude and we discuss some related and future work.

2 Second-Order Proofs and Refutations

In this section we define a second-order extension of λ^{PRK}, including its syntax, typing rules, and rewriting rules. We show that the system enjoys **subject reduction**, it is **confluent**, and it **refines classical second-order logic** (Thm. 3).

Syntax of types. We assume given a denumerable set of *type variables* $\alpha, \beta, \gamma, \ldots$. The sets of *pure types* $(A, B, \ldots)$ and *types* $(P, Q, \ldots)$ are given by:

$$A ::= \alpha \mid A \wedge A \mid A \vee A \mid A \to A \mid A \ltimes A \mid \neg A \mid \forall \alpha.\, A \mid \exists \alpha.\, A \qquad P ::= A^{+} \mid A^{-} \mid A^{\oplus} \mid A^{\ominus}$$

where $A \ltimes B$ represents *co-implication*, the dual connective to implication, to be understood (roughly) as $\neg A \wedge B$. The four modes represent strong affirmation (A^{+}), strong denial (A^{-}), weak affirmation ($A^{\oplus}$), and weak denial ($A^{\ominus}$). Note that *modes* $({}^{+}, {}^{-}, {}^{\oplus}, {}^{\ominus})$ can only decorate the root of a type, *i.e.* they cannot be nested.

Sometimes one may be interested in *fragments* of the system. For instance, the λ^{PRK}-calculus of [4] corresponds to the $\{\wedge, \vee, \neg\}$ fragment. In this paper we are usually interested in the full $\{\wedge, \vee, \to, \ltimes, \neg, \forall, \exists\}$ fragment. As long as there is little danger of confusion we still speak of λ^{PRK} without further qualifications.

Syntax of terms. Terms of λ^{PRK} are given by the following grammar. The letter i ranges over $\{1, 2\}$. Some terms are decorated with either "$+$" or "$-$". In the grammar we write "$\pm$" to stand for either "$+$" or "$-$".

$$\begin{array}{rlll|ll}
t, s, \ldots ::= & x^{P} & \text{variable} & & t \bowtie_{P} s & \text{absurdity} \\
\mid & \bigcirc^{\pm}_{(x:P)}.\, t & {}^{\oplus}/{}^{\ominus} \text{ introduction} & & t \bullet^{\pm} s & {}^{\oplus}/{}^{\ominus} \text{ elimination} \\
\mid & \langle t, s \rangle^{\pm} & \wedge^{+}/\vee^{-} \text{ introduction} & & \pi^{\pm}_{i}(t) & \wedge^{+}/\vee^{-} \text{ elimination} \\
\mid & \mathsf{in}^{\pm}_{i}(t) & \vee^{+}/\wedge^{-} \text{ introduction} & & \delta^{\pm} t\, [_{x:P}.s][_{y:Q}.u] & \vee^{+}/\wedge^{-} \text{ elimination} \\
\mid & \lambda^{\pm}_{(x:P)}.t & \to^{+}/\ltimes^{-} \text{ introduction} & & t @^{\pm} s & \to^{+}/\ltimes^{-} \text{ elimination} \\
\mid & (t\, ;^{\pm} s) & \ltimes^{+}/\to^{-} \text{ introduction} & & \varrho^{\pm} t[_{x:P;y:Q}.s] & \ltimes^{+}/\to^{-} \text{ elimination} \\
\mid & \mathsf{N}^{\pm} t & \neg^{+}/\neg^{-} \text{ introduction} & & \mathsf{M}^{\pm} t & \neg^{+}/\neg^{-} \text{ elimination} \\
\mid & \lambda^{\pm}_{\alpha}.\, t & \forall^{+}/\exists^{-} \text{ introduction} & & t @^{\pm} A & \forall^{+}/\exists^{-} \text{ elimination} \\
\mid & \langle A, t \rangle^{\pm} & \exists^{+}/\forall^{-} \text{ introduction} & & \nabla^{\pm}\, t[_{(\alpha, x:P)}.s] & \exists^{+}/\forall^{-} \text{ elimination}
\end{array}$$

The notions of free and bound occurrences of variables are defined as expected, with the typographical convention that subscripted variable occurrences are binding. Terms are considered up to α-renaming of bound variables. We write $\mathsf{fv}(t)$ for the set of free variables of t and $\mathsf{ftv}(t)$ for the set of type variables occurring free in t. By $t\{x := s\}$ we mean the capture-avoiding substitution of the free occurrences of x in t by s.

Variables are annotated with their type, which we usually omit. Sometimes we also omit the types of bound variables if they are clear from the context, as well as the name of unused bound variables, writing "_" instead. For example, if $x \notin \mathsf{fv}(t)$ we may write $\bigcirc^{+}_{\_}.\, t$ rather than $\bigcirc^+_{(x:A^-)}.\,t$. Application-like operators are assumed to be left-associative; for example, $t @^+ s \bullet^+ u @^+ A$ stands for $((t @^+ s) \bullet^+ u) @^+ A$. In a term of the form $\bigcirc^{\pm}_{(x:P)}.\,t$, the variable x is called the *counterfactual*, and more specifically a *negative counterfactual* in a term of the form $\bigcirc^+_{(x:A^\ominus)}.\,t$. In a term of the form $t \bullet^{\pm} s$, we call t the *subject* and s the *argument*. We write $\mathtt{C}$ for arbitrary term *contexts*, *i.e.* terms with a single free occurrence of a distinguished variable $\square$ called a *hole*, and $\mathtt{C}\langle t\rangle$ for the variable-capturing substitution of the hole of $\mathtt{C}$ by t.

The λ^{PRK} type system. A *typing context*, ranged over by $\Gamma, \Delta, \ldots$, is a finite assignment of variables to types, written as $x_1 : P_1, \ldots, x_n : P_n$. We write $\mathsf{dom}(\Gamma)$ for the *domain* of Γ, *i.e.* the finite set $\{x_1, \ldots, x_n\}$. Typing judgments in λ^{PRK} are of the form $\Gamma \vdash t : P$, meaning that t has type P under the context Γ.

We write $\Gamma \vdash_{\mathrm{PRK}} t : P$ if the typing judgment $\Gamma \vdash t : P$ is derivable in λ^{PRK}. When we wish to emphasize the logical point of view, we may write sequents as $P_1, \ldots, P_n \vdash Q$, and we may write $P_1, \ldots, P_n \vdash_{\mathrm{PRK}} Q$ to mean that there exists a term t such that $x_1 : P_1, \ldots, x_n : P_n \vdash_{\mathrm{PRK}} t : Q$. Derivable judgments are given inductively by the typing rules below.

Basic rules

$$\frac{}{\Gamma, x : P \vdash x : P}\,\textsc{Ax} \qquad \frac{\Gamma \vdash t : A^+ \quad \Gamma \vdash s : A^-}{\Gamma \vdash t \bowtie_P s : P}\,\textsc{Abs} \qquad \frac{\Gamma, x : A^\ominus \vdash t : A^+}{\Gamma \vdash \bigcirc^+_{x:A^\ominus}.\,t : A^\oplus}\,\mathrm{I}^+_\circ$$

$$\frac{\Gamma, x : A^\oplus \vdash t : A^-}{\Gamma \vdash \bigcirc^+_{x:A^\oplus}.\,t : A^\ominus}\,\mathrm{I}^-_\circ \qquad \frac{\Gamma \vdash t : A^\oplus \quad \Gamma \vdash s : A^\ominus}{\Gamma \vdash t \bullet^+ s : A^+}\,\mathrm{E}^+_\circ \qquad \frac{\Gamma \vdash t : A^\ominus \quad \Gamma \vdash s : A^\oplus}{\Gamma \vdash t \bullet^- s : A^-}\,\mathrm{E}^-_\circ$$

Conjunction and disjunction

$$\frac{\Gamma \vdash t : A^\oplus \quad \Gamma \vdash s : B^\oplus}{\Gamma \vdash \langle t, s\rangle^+ : (A \land B)^+}\,\mathrm{I}^+_\land \qquad \frac{\Gamma \vdash t : A^\ominus \quad \Gamma \vdash s : B^\ominus}{\Gamma \vdash \langle t, s\rangle^- : (A \lor B)^-}\,\mathrm{I}^-_\lor \qquad \frac{\Gamma \vdash t : (A_1 \land A_2)^+}{\Gamma \vdash \pi^+_i(t) : A_i{}^\oplus}\,\mathrm{E}^+_{\land i}$$

$$\frac{\Gamma \vdash t : (A_1 \lor A_2)^-}{\Gamma \vdash \pi^-_i(t) : A_i{}^\ominus}\,\mathrm{E}^-_{\lor i} \qquad \frac{\Gamma \vdash t : A_i{}^\oplus \quad i \in \{1,2\}}{\Gamma \vdash \mathsf{in}^+_i(t) : (A_1 \lor A_2)^+}\,\mathrm{I}^+_{\lor i} \qquad \frac{\Gamma \vdash t : A_i{}^\ominus \quad i \in \{1,2\}}{\Gamma \vdash \mathsf{in}^-_i(t) : (A_1 \land A_2)^-}\,\mathrm{I}^-_{\land i}$$

$$\frac{\Gamma \vdash t : (A \lor B)^+ \quad \Gamma, x : A^\oplus \vdash s : P \quad \Gamma, y : B^\oplus \vdash u : P}{\Gamma \vdash \delta^+ t\,[_{x:A^\oplus}.s][_{y:B^\oplus}.u] : P}\,\mathrm{E}^+_\lor$$

$$\frac{\Gamma \vdash t : (A \land B)^- \quad \Gamma, x : A^\ominus \vdash s : P \quad \Gamma, y : B^\ominus \vdash u : P}{\Gamma \vdash \delta^- t\,[_{x:A^\ominus}.s][_{y:B^\ominus}.u] : P}\,\mathrm{E}^-_\land$$

Implication and co-implication

$$\frac{\Gamma, x : A^\oplus \vdash t : B^\oplus}{\Gamma \vdash \lambda^+_{x:A^\oplus}.t : (A \to B)^+}\,\mathrm{I}^+_\to \qquad \frac{\Gamma, x : A^\ominus \vdash t : B^\ominus}{\Gamma \vdash \lambda^-_{x:A^\ominus}.t : (A \ltimes B)^-}\,\mathrm{I}^-_\ltimes \qquad \frac{\Gamma \vdash t : (A \to B)^+ \quad \Gamma \vdash s : A^\oplus}{\Gamma \vdash t @^+ s : B^\oplus}\,\mathrm{E}^+_\to$$

$$\frac{\Gamma \vdash t : (A \ltimes B)^- \quad \Gamma \vdash s : A^\ominus}{\Gamma \vdash t @^- s : B^\ominus}\,\mathrm{E}^-_\ltimes \qquad \frac{\Gamma \vdash t : A^\ominus \quad \Gamma \vdash s : B^\oplus}{\Gamma \vdash (t \,;^+ s) : (A \ltimes B)^+}\,\mathrm{I}^+_\ltimes \qquad \frac{\Gamma \vdash t : A^\oplus \quad \Gamma \vdash s : B^\ominus}{\Gamma \vdash (t \,;^- s) : (A \to B)^-}\,\mathrm{I}^-_\to$$

$$\frac{\Gamma \vdash t : (A \ltimes B)^+ \quad \Gamma, x : A^\ominus, y : B^\oplus \vdash s : P}{\Gamma \vdash \varrho^+ t[_{x:A^\ominus;y:B^\oplus}.s] : P}\,\mathrm{E}^+_\ltimes \qquad \frac{\Gamma \vdash t : (A \to B)^- \quad \Gamma, x : A^\oplus, y : B^\ominus \vdash s : P}{\Gamma \vdash \varrho^- t[_{x:A^\oplus;y:B^\ominus}.s] : P}\,\mathrm{E}^-_\to$$

Negation

$$\frac{\Gamma \vdash t : A^{\ominus}}{\Gamma \vdash \mathsf{N}^{+}t : (\neg A)^{+}}\,\mathrm{I}^{+}_{\neg} \qquad \frac{\Gamma \vdash t : A^{\oplus}}{\Gamma \vdash \mathsf{N}^{-}t : (\neg A)^{-}}\,\mathrm{I}^{-}_{\neg} \qquad \frac{\Gamma \vdash t : (\neg A)^{+}}{\Gamma \vdash \mathsf{M}^{+}t : A^{\ominus}}\,\mathrm{E}^{+}_{\neg} \qquad \frac{\Gamma \vdash t : (\neg A)^{-}}{\Gamma \vdash \mathsf{M}^{-}t : A^{\oplus}}\,\mathrm{E}^{-}_{\neg}$$

Second-order quantification

$$\frac{\Gamma \vdash t : A^{\oplus} \qquad \alpha \notin \mathsf{ftv}(\Gamma)}{\Gamma \vdash \lambda^{+}_{\alpha}.\,t : (\forall\alpha.\,A)^{+}}\,\mathrm{I}^{+}_{\forall} \qquad \frac{\Gamma \vdash t : A^{\ominus} \qquad \alpha \notin \mathsf{ftv}(\Gamma)}{\Gamma \vdash \lambda^{-}_{\alpha}.\,t : (\exists\alpha.\,A)^{-}}\,\mathrm{I}^{-}_{\exists} \qquad \frac{\Gamma \vdash t : (\forall\alpha.\,B)^{+}}{\Gamma \vdash t@^{+}A : B^{\oplus}\{\alpha := A\}}\,\mathrm{E}^{+}_{\forall}$$

$$\frac{\Gamma \vdash t : (\exists\alpha.\,B)^{-}}{\Gamma \vdash t@^{-}A : B^{\ominus}\{\alpha := A\}}\,\mathrm{E}^{-}_{\exists} \qquad \frac{\Gamma \vdash t : B^{\oplus}\{\alpha := A\}}{\Gamma \vdash \langle A, t\rangle^{+} : (\exists\alpha.\,B)^{+}}\,\mathrm{I}^{+}_{\exists} \qquad \frac{\Gamma \vdash t : B^{\ominus}\{\alpha := A\}}{\Gamma \vdash \langle A, t\rangle^{-} : (\forall\alpha.\,B)^{-}}\,\mathrm{I}^{-}_{\forall}$$

$$\frac{\Gamma \vdash t : (\exists\alpha.\,A)^{+} \qquad \Gamma, x : A^{\oplus} \vdash s : P \qquad \alpha \notin \mathsf{ftv}(\Gamma, P)}{\Gamma \vdash \nabla^{+}\,t[_{(\alpha,x:A^{\oplus})}.s] : P}\,\mathrm{E}^{+}_{\exists}$$

$$\frac{\Gamma \vdash t : (\forall\alpha.\,A)^{-} \qquad \Gamma, x : A^{\ominus} \vdash s : P \qquad \alpha \notin \mathsf{ftv}(\Gamma, P)}{\Gamma \vdash \nabla^{-}\,t[_{(\alpha,x:A^{\ominus})}.s] : P}\,\mathrm{E}^{-}_{\forall}$$

The typing rules may be informally explained as follows. AX is the standard axiom. The *absurdity rule* (ABS) allows to derive any conclusion from a strong proof and a strong refutation of A. Introduction and elimination rules for weak affirmation and denial ($\mathrm{I}^{\pm}_{\circ}$, $\mathrm{E}^{\pm}_{\circ}$) follow the principle that a weak affirmation $A^{\oplus}$ behaves like an implication "$A^{\ominus} \to A^{+}$". Indeed, $\mathrm{I}^{+}_{\circ}$ and $\mathrm{E}^{+}_{\circ}$ have the same structure as the introduction and the elimination rule for an implication "$A^{\ominus} \to A^{+}$", where $\bigcirc^{+}_{x}.\,t$ and $t \bullet^{+} s$ are akin to λ-abstraction and application. The intuition behind this is that $A^{\oplus}$ is the type of weak proofs of a proposition A, where a weak proof proceeds by *reductio ad absurdum*, assuming a weak refutation ($A^{\ominus}$), and providing a *strong* proof (A^{+}). Dually, a weak denial $A^{\ominus}$ behaves like "$A^{\oplus} \to A^{-}$".

The remaining rules are introduction and elimination rules for positive and negative strong connectives. These rules come in dual pairs: *for each rule for a connective with positive sign there is a symmetric rule for the dual connective with negative sign.* For example, the introduction rule for positive conjunction ($\mathrm{I}^{+}_{\wedge}$) states that to strongly prove $A \wedge B$ it suffices to weakly prove A and weakly prove B. Dually, the introduction rule for negative disjunction ($\mathrm{I}^{-}_{\vee}$) states that to strongly refute $A \vee B$ it suffices to weakly refute A and weakly refute B.

The introduction and elimination rules for most logical connectives ($\wedge$, $\vee$, $\to$, $\ltimes$, $\forall$, $\exists$) are mechanically derived from the standard natural deduction rules following this methodology, taking in account that the dual pairs of connectives are $(\wedge, \vee)$, $(\to, \ltimes)$, and $(\forall, \exists)$. In general, *introduction rules have weak premises and strong conclusions*, whereas *elimination rules have strong premises and weak conclusions.*

The typing rules for conjunction ($\mathrm{I}^{+}_{\wedge}, \mathrm{E}^{+}_{\wedge i}, \mathrm{I}^{-}_{\wedge i}, \mathrm{E}^{-}_{\wedge}$), disjunction ($\mathrm{I}^{+}_{\vee i}, \mathrm{E}^{+}_{\vee}, \mathrm{I}^{-}_{\vee}, \mathrm{E}^{-}_{\vee i}$), positive implication ($\mathrm{I}^{+}_{\to}, \mathrm{E}^{+}_{\to}$), negative co-implication ($\mathrm{I}^{-}_{\ltimes}, \mathrm{E}^{-}_{\ltimes}$), universal ($\mathrm{I}^{+}_{\forall}, \mathrm{E}^{+}_{\forall}, \mathrm{I}^{-}_{\forall}, \mathrm{E}^{-}_{\forall}$) and existential quantification ($\mathrm{I}^{+}_{\exists}, \mathrm{E}^{+}_{\exists}, \mathrm{I}^{-}_{\exists}, \mathrm{E}^{-}_{\exists}$) are typical, so for instance $\langle t, s\rangle^{\pm}$ forms a pair, $\pi^{\pm}_{i}(t)$ is the i-th projection, $\mathsf{in}^{\pm}_{i}(t)$ is the i-th injection into a disjoint union type, $\delta^{\pm} t\,[_{x}.s][_{y}.u]$ is a pattern matching construct, and so on. The rules differ from usual typed λ-calculi only in the signs and strengths that decorate premises and conclusions.

The typing rules for positive co-implication ($\mathrm{I}^{+}_{\ltimes}, \mathrm{E}^{+}_{\ltimes}$), sometimes called *subtraction* [9], and negative implication ($\mathrm{I}^{-}_{\to}, \mathrm{E}^{-}_{\to}$) follow the rough interpretation of $A \ltimes B$ as $\neg A \wedge B$, so a strong proof of a co-implication $A \ltimes B$ is given by a pair $(t\,;^{+} s)$ comprising a weak refutation of A and a weak proof of B. Dually, a strong refutation of $A \to B$ is given by a pair $(t\,;^{-} s)$ comprising a weak proof of A and a weak refutation of B. The eliminators $\varrho^{\pm} t[_{x;y}.s]$ are presented as generalized elimination rules, in multiplicative style.

The typing rules for negation ($\mathrm{I}^+_\neg, \mathrm{E}^+_\neg, \mathrm{I}^-_\neg, \mathrm{E}^-_\neg$) express that to strongly prove $\neg A$ is the same as to weakly refute A, and dually for strong refutations of $\neg A$.

▶ **Example 1.** Let $\top \stackrel{\text{def}}{=} \forall\alpha.\,(\alpha \to \alpha)$ and $\bot \stackrel{\text{def}}{=} \forall\alpha.\,\alpha$. Recall from [4] that the weak non-contradiction principle $\Gamma \vdash (A \wedge \neg A)^\ominus$ holds in λ^{PRK}. Then $\vdash \top^\oplus$ and $\bot^\oplus \vdash A^\oplus$ hold, where A stands for any pure type:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\top^\ominus, (\alpha \to \alpha)^\ominus, \alpha^\oplus \vdash \alpha^\oplus}\,\mathrm{Ax}}{\top^\ominus, (\alpha \to \alpha)^\ominus \vdash (\alpha \to \alpha)^+}\,\mathrm{I}^+_\to}{\top^\ominus \vdash (\alpha \to \alpha)^\oplus}\,\mathrm{I}^+_\circ}{\top^\ominus \vdash \top^+}\,\mathrm{I}^+_\forall}{\vdash \top^\oplus}\,\mathrm{I}^+_\circ
\qquad
\dfrac{\dfrac{}{\bot^\oplus \vdash \bot^\oplus}\,\mathrm{Ax} \quad \dfrac{\dfrac{\dfrac{\text{(By weak non-contradiction.)}}{\bot^\oplus, \bot^\oplus \vdash (B \wedge \neg B)^\ominus}}{\bot^\oplus, \bot^\oplus \vdash \bot^-}\,\mathrm{I}^-_\forall}{\bot^\oplus \vdash \bot^\ominus}\,\mathrm{I}^-_\circ}{\bot^\oplus \vdash A^\oplus}\,\mathrm{A{\scriptsize BS}}'$$

The $\boldsymbol{\lambda}^{\mathsf{PRK}}$-calculus. The opposite type $P^\sim$ of a given type P is defined by flipping the sign, *i.e.* $(A^\oplus)^\sim \stackrel{\text{def}}{=} A^\ominus$; $(A^\ominus)^\sim \stackrel{\text{def}}{=} A^\oplus$; $(A^+)^\sim \stackrel{\text{def}}{=} A^-$; and $(A^-)^\sim \stackrel{\text{def}}{=} A^+$. If $\Gamma \vdash_{\mathrm{PRK}} t : P$ and $\Gamma \vdash_{\mathrm{PRK}} s : P^\sim$ then a term $t \bowtie_Q s$ may be constructed such that $\Gamma \vdash_{\mathrm{PRK}} t \bowtie_Q s : Q$, as follows:

$$\begin{array}{rlrl}
t \bowtie_Q s \stackrel{\text{def}}{=} t \blacktriangleright\!\!\blacktriangleleft_Q\, s & \text{if } P = A^+ & t \bowtie_Q s \stackrel{\text{def}}{=} s \blacktriangleright\!\!\blacktriangleleft_Q\, t & \text{if } P = A^- \\
t \bowtie_Q s \stackrel{\text{def}}{=} (t \bullet^+ s) \blacktriangleright\!\!\blacktriangleleft_Q (s \bullet^- t) & \text{if } P = A^\oplus & t \bowtie_Q s \stackrel{\text{def}}{=} (s \bullet^+ t) \blacktriangleright\!\!\blacktriangleleft_Q (t \bullet^- s) & \text{if } P = A^\ominus
\end{array}$$

We endow typable PRK terms with a notion of reduction, defining the $\lambda^{\mathbf{PRK}}$**-calculus** by a binary rewriting relation $\to$ on typable PRK terms, given by the rewriting rules below, and closed by compatibility under arbitrary contexts. Rules are presented following the convention that, if many occurrences of "$\pm$" appear in the same expression, they are all supposed to stand for the same sign:

$$\begin{array}{rlrl}
(\bigcirc^\pm_x.\,t) \bullet^\pm s & \xrightarrow{\beta^+_\circ\,/\,\beta^-_\circ} t\{x := s\} & & \\
\pi^\pm_i(\langle t_1, t_2\rangle^\pm) & \xrightarrow{\beta^+_\wedge\,/\,\beta^-_\vee} t_i & \delta^\pm(\mathsf{in}^\pm_i(t))\,[_x.s_1][_x.s_2] & \xrightarrow{\beta^+_\vee\,/\,\beta^-_\wedge} s_i\{x := t\} \\
(\lambda^\pm_x.t)@^\pm s & \xrightarrow{\beta^+_\to\,/\,\beta^-_\ltimes} t\{x := s\} & \varrho^\pm(t\,;^\pm s)[_{x;y}.u] & \xrightarrow{\beta^+_\ltimes\,/\,\beta^-_\to} u\{x := t\}\{y := s\} \\
\mathsf{M}^\pm(\mathsf{N}^\pm t) & \xrightarrow{\beta^+_\neg\,/\,\beta^-_\neg} t & & \\
(\lambda^\pm_\alpha.\,t)@^\pm A & \xrightarrow{\beta^+_\forall\,/\,\beta^-_\exists} t\{\alpha := A\} & \nabla^\pm\,\langle A, t\rangle^\pm[_{(\alpha,x)}.s] & \xrightarrow{\beta^+_\exists\,/\,\beta^-_\forall} s\{\alpha := A\}\{x := t\} \\
\langle t_1, t_2\rangle^+ \blacktriangleright\!\!\blacktriangleleft \mathsf{in}^-_i(s) & \xrightarrow{\blacktriangleright\!\blacktriangleleft_\wedge} \quad t_i \bowtie s & \mathsf{in}^+_i(t) \blacktriangleright\!\!\blacktriangleleft \langle s_1, s_2\rangle^- & \xrightarrow{\blacktriangleright\!\blacktriangleleft_\vee} \quad t \bowtie s_i \\
\lambda^+_x.t \blacktriangleright\!\!\blacktriangleleft (s\,;^- u) & \xrightarrow{\blacktriangleright\!\blacktriangleleft_\to} \quad t\{x := s\} \bowtie u & (t\,;^+ s) \blacktriangleright\!\!\blacktriangleleft \lambda^-_x.u & \xrightarrow{\blacktriangleright\!\blacktriangleleft_\ltimes} \quad s \bowtie u\{x := t\} \\
(\mathsf{N}^+ t) \blacktriangleright\!\!\blacktriangleleft (\mathsf{N}^- s) & \xrightarrow{\blacktriangleright\!\blacktriangleleft_\neg} \quad t \bowtie s & & \\
(\lambda^+_\alpha.\,t) \blacktriangleright\!\!\blacktriangleleft \langle A, s\rangle^- & \xrightarrow{\blacktriangleright\!\blacktriangleleft_\forall} \quad t\{\alpha := A\} \bowtie s & \langle A, t\rangle^+ \blacktriangleright\!\!\blacktriangleleft (\lambda^-_\alpha.\,s) & \xrightarrow{\blacktriangleright\!\blacktriangleleft_\exists} \quad t \bowtie s\{\alpha := A\}
\end{array}$$

The λ^{PRK}-calculus has two kinds of rules: "β" rules, akin to proof normalization rules in natural deduction, and "$\blacktriangleright\!\!\blacktriangleleft$" rules, akin to cut elimination rules in sequent calculus. The $\beta^+_\circ\,/\,\beta^-_\circ$ rules are exactly like the standard β-rule of the λ-calculus, with the difference that the abstraction $\bigcirc^+_{x:A^\ominus}.\,t$ is not an introduction of an implication "$A^\ominus \to A^+$" but rather the introduction of a weak affirmation $A^\oplus$. The $\beta^+_\to\,/\,\beta^-_\ltimes$ rules also describe a similar behavior, where $\lambda^+_{x:A^\oplus}.t$ is of type $(A \to B)^\oplus$. The remaining β rules are straightforward, encoding projection ($\beta^+_\wedge\,/\,\beta^-_\vee$), pattern matching ($\beta^+_\vee\,/\,\beta^-_\wedge$), etcetera.

The $\blacktriangleright\!\!\blacktriangleleft$ rules simplify an absurdity ($t \blacktriangleright\!\!\blacktriangleleft s$) as much as possible, but they are never able to get rid of the absurdity. Indeed, note that the right-hand side of all the $\blacktriangleright\!\!\blacktriangleleft$ rules include the generalized absurdity operator ($\bowtie$), which is in turn defined in terms of the absurdity operator

($\bowtie$). Recall, for example, that if $t : A^{\oplus}$ and $s : A^{\ominus}$ then $t \bowtie s = (t \bullet^{+} s) \bowtie (s \bullet^{-} t)$. This means that an instance of the absurdity is a proof which provides *no relevant information*. For example, if $P = (B \wedge C)^{+}$ then $\pi_1^{+}(t \bowtie_P s)$ is a well-formed term of type $B^{\oplus}$; but the argument of the projection is a term $t \bowtie_P s$ which may never become a pair $\langle p, q\rangle^{+}$. This means that the normal form will be a "stuck" term of the form $\pi_1^{+}(t' \bowtie_P s')$.

▶ **Example 2** (Reduction in λ^{PRK}). Let $\Gamma \vdash_{\mathrm{PRK}} t : A^{\oplus}$ and $\Gamma \vdash_{\mathrm{PRK}} s : A^{\ominus}$. Then:

$$\begin{array}{ll}
 & (\lambda^{+}_{\alpha}.\, \circ^{+}_{_:(\alpha\to\alpha)^{\ominus}}.\, \lambda^{+}_{x:\alpha^{\oplus}}.x) \bowtie \langle A, \circ^{-}_{_:(\alpha\to\alpha)^{\oplus}}.\,(t\,;^{-} s)\rangle^{-} \\
\xrightarrow{\bowtie_{\forall}} & (\circ^{+}_{_:(A\to A)^{\ominus}}.\, \lambda^{+}_{x:A^{\oplus}}.x) \bowtie (\circ^{-}_{_:(A\to A)^{\oplus}}.\,(t\,;^{-} s)) \\
= & ((\circ^{+}_{_:(A\to A)^{\ominus}}.\, \lambda^{+}_{x:A^{\oplus}}.x) \bullet^{+} (\circ^{-}_{_:(A\to A)^{\oplus}}.\,(t\,;^{-} s))) \bowtie \\
 & ((\circ^{-}_{_:(A\to A)^{\oplus}}.\,(t\,;^{-} s)) \bullet^{-} (\circ^{+}_{_:(A\to A)^{\ominus}}.\, \lambda^{+}_{x:A^{\oplus}}.x)) \\
\xrightarrow{\beta^{\pm}_{\circ}}{}^{*} & (\lambda^{+}_{x:A^{\oplus}}.x) \bowtie (t\,;^{-} s) \xrightarrow{\bowtie_{\to}} t \bowtie s = (t \bullet^{+} s) \bowtie (s \bullet^{-} t)
\end{array}$$

An η-like rewriting rule can also be incorporated to λ^{PRK} as done in [4, Thm. 37], declaring that $\circ^{\pm}_{x}.\,(t \bullet^{\pm} x) \xrightarrow{\eta_{\circ}} t$ if $x \notin \mathsf{fv}(t)$. The λ^{PRK}-calculus (with or without the $\eta_{\circ}$ rule) enjoys the following properties:

▶ **Theorem 3.**
1. Subject reduction. *If* $\Gamma \vdash_{\mathrm{PRK}} t : P$ *and* $t \to s$*, then* $\Gamma \vdash_{\mathrm{PRK}} s : P$.
2. Confluence. *The* λ^{PRK}*-calculus has the Church–Rosser property.*
3. Classical refinement. $A_1^{\oplus}, \ldots, A_n^{\oplus} \vdash B^{\oplus}$ *holds in* λ^{PRK} *if and only if* $A_1, \ldots, A_n \vdash B$ *holds in the classical second-order natural deduction system* NK.

Proof. Subject reduction is a straightforward extension of [4, Prop. 24], with minor adaptations to account for implication, co-implication, and second-order quantification. Confluence follows from the fact that λ^{PRK} can be modeled as an orthogonal higher-order rewriting system in the sense of Nipkow [35]. Classical refinement is an extension of Prop. 38 and Thm. 39 from [4]; this theorem has two parts:

- The "only if" direction ($A_1^{\oplus}, \ldots, A_n^{\oplus} \vdash_{\mathrm{PRK}} B^{\oplus}$ implies $A_1, \ldots, A_n \vdash_{\mathsf{NK}} B$) means that PRK is a *conservative extension* of classical second-order logic. To prove this statement, we generalize the statement as follows: if $P_1, \ldots, P_n \vdash_{\mathrm{PRK}} Q$ then $\iota(P_1), \ldots, \iota(P_n) \vdash_{\mathsf{NK}} \iota(Q)$, where $\iota(A^{\oplus}) = \iota(A^{+}) = A$ and $\iota(A^{\ominus}) = \iota(A^{-}) = \neg A$. This can be shown by a straightforward induction on the derivation of the first judgment.
- The "if" direction ($A_1, \ldots, A_n \vdash_{\mathsf{NK}} B$ implies $A_1^{\oplus}, \ldots, A_n^{\oplus} \vdash_{\mathrm{PRK}} B^{\oplus}$) means that classical logic can be *embedded* into PRK. The essence of the proof is showing that all the inference rules of classical second-order natural deduction are admissible in λ^{PRK}, taking the weak affirmation of all propositions (*i.e.* decorating all formulas with "$\oplus$"). Some cases are subtle, especially elimination rules. Here we show the introduction and elimination rules for quantifiers:
  1. *Universal introduction.* Let $\Gamma \vdash_{\mathrm{PRK}} t : (\forall\alpha.\, B)^{\oplus}$, and define $t\, @^{\mathcal{C}} A$ as the following term: $\circ^{+}_{(x:(B\{\alpha:=A\})^{\ominus})}.\,((t \bullet^{+} \circ^{+}_{(\_:(\forall\alpha.\, B)^{\oplus})}.\, \langle A, x\rangle^{-})@^{+}A \bullet^{+} x)$. Then we have that $\Gamma \vdash_{\mathrm{PRK}} t\, @^{\mathcal{C}} A : B\{\alpha := A\}^{\oplus}$.
  2. *Universal elimination.* Let $\Gamma \vdash_{\mathrm{PRK}} t : (\forall\alpha.\, B)^{\oplus}$. Define $t\, @^{\mathcal{C}} A$ as the following term: $\circ^{+}_{(x:(B\{\alpha:=A\})^{\ominus})}.\,((t \bullet^{+} \circ^{+}_{(\_:(\forall\alpha.\, B)^{\oplus})}.\, \langle A, x\rangle^{-})@^{+}A \bullet^{+} x)$. Then we have that $\Gamma \vdash_{\mathrm{PRK}} t\, @^{\mathcal{C}} A : B\{\alpha := A\}^{\oplus}$.
  3. *Existential introduction.* Let $\Gamma \vdash_{\mathrm{PRK}} t : (B\{\alpha := A\})^{\oplus}$. Define $\langle A, t\rangle^{\mathcal{C}}$ as the following term: $\circ^{+}_{(\_:(\exists\alpha.\, B)^{\ominus})}.\, \langle A, t\rangle^{+}$. Then we have that $\Gamma \vdash_{\mathrm{PRK}} \langle A, t\rangle^{\mathcal{C}} : (\exists\alpha.\, B)^{\oplus}$.

4. *Existential elimination.* Let $\Gamma \vdash_{\text{PRK}} t : (\exists\alpha.\, A)^{\oplus}$ and $\Gamma, x : A^{\oplus} \vdash_{\text{PRK}} s : B^{\oplus}$ with $\alpha \notin \mathsf{ftv}(\Gamma, P)$. Define $\nabla^{\mathcal{C}}\, t[_{(\alpha,x)}.s]$ as the following term: $\bigcirc^{+}_{(y:B^{\ominus})}.\,(\nabla^{+}\, t'[_{(\alpha,x)}.s] \bullet^{+} y)$ where $t' \stackrel{\text{def}}{=} t \bullet^{+} \bigcirc^{-}_{(\_:(\exists\alpha.\,A)^{\oplus})}.\,\lambda^{-}_{\alpha}.\,\bigcirc^{-}_{(x:A^{\oplus})}.\,(s \bowtie_{A^{-}} y)$. Then $\Gamma \vdash_{\text{PRK}} \nabla^{\mathcal{C}}\, t[_{(\alpha,x)}.s] : B^{\oplus}$. ◀

3 Böhm–Berarducci Encodings

It is well-known that, in System F, logical connectives such as $\top$, $\bot$, $\wedge$, $\vee$, $\exists$, as well as inductive data types, can be represented using only $\forall$ and $\to$ by means of their *Böhm–Berarducci encodings* [7], which can be understood as *universal properties* or *structural induction principles*. Böhm–Berarducci encodings can be reproduced in λ^{PRK}. In the following subsections we study the encoding of connectives in terms of universal quantification and implication.

The encoding of conjunction, for instance, can be taken to be $A \wedge B \stackrel{\text{def}}{=} \forall\alpha.\,((A \to B \to \alpha) \to \alpha)$. Then **positive typing rules for conjunction**, analogous to $\mathrm{I}^{+}_{\wedge}$ and $\mathrm{E}^{+}_{\wedge i}$ are derivable, and their constructions simulate the $\beta^{+}_{\wedge}$ rule. Indeed, let $X := (A_1 \to A_2 \to A_i) \to \alpha$ and $Y := A_1 \to A_2 \to \alpha$. Moreover, let $X_i := (A_1 \to A_2 \to A_i) \to A_i$ and $Y_i := A_1 \to A_2 \to A_i$. Given $\Gamma \vdash t_1 : A_1{}^{\oplus}$ and $\Gamma \vdash t_2 : A_2{}^{\oplus}$ and $\Gamma \vdash s : (A_1 \wedge A_2)^{+}$, define:

- $\langle t_1, t_2\rangle^{+} \stackrel{\text{def}}{=} \lambda^{+}_{\alpha}.\,\bigcirc^{+}_{(\_:X^{\ominus})}.\,\lambda^{+}_{(x:Y^{\oplus})}.\bigcirc^{+}_{(y:\alpha^{\ominus})}.\,x \bullet^{+} (\bigcirc^{-}_{(\_:Y^{\oplus})}.\,(t_1 \,;^{-} u))@^{+}t_1 \bullet^{+} u@^{+}t_2 \bullet^{+} y$,
  where $u \stackrel{\text{def}}{=} \bigcirc^{-}_{(\_:(B\to\alpha)^{\oplus})}.\,(t_2 \,;^{-} y)$.
- $\pi^{+}_{i}(s) \stackrel{\text{def}}{=} \bigcirc^{+}_{(x:A_i{}^{\ominus})}.\,(t_1@^{+}A_i \bullet^{+} (\bigcirc^{+}_{(\_:X_i{}^{\oplus})}.\,(r \,;^{-} x))@^{+}r \bullet x)$,
  where $r \stackrel{\text{def}}{=} \bigcirc^{+}_{(\_:Y_i{}^{\ominus})}.\,\lambda^{+}_{(y_1:A_1{}^{\oplus})}.\bigcirc^{+}_{(\_:(A_2\to A_i)^{\ominus})}.\,\lambda^{+}_{(y_2:A_2{}^{\oplus})}.y_i$.

Then $\Gamma \vdash \langle t_1, t_2\rangle^{+} : (A_1 \wedge A_2)^{+}$ and $\Gamma \vdash \pi^{+}_{i}(s) : A_i{}^{\oplus}$ and it can be easily checked that $\pi^{+}_{i}(\langle t_1, t_2\rangle^{+}) \to^{*} t_i$ (using $\eta_{\circ}$).

On the other hand, **negative typing rules for conjunction**, analogous to $\mathrm{I}^{-}_{\wedge i}$ and a weak variant of $\mathrm{E}^{-}_{\wedge}$ can also be derived. First note that, given terms $\Gamma \vdash p : (A \to B)^{\oplus}$ and $\Gamma \vdash q : A^{\oplus}$, a term $p \,@^{\mathcal{C}}\, q$ may be defined in such a way that $\Gamma \vdash_{\text{PRK}} p \,@^{\mathcal{C}}\, q : B^{\oplus}$. An explicit construction is $p \,@^{\mathcal{C}}\, q \stackrel{\text{def}}{=} \bigcirc^{+}_{(x:B^{\ominus})}.\,(p \bullet^{+} (\bigcirc^{-}_{(\_:(A\to B)^{\oplus})}.\,(q \,;^{-} x))@^{+}q \bullet^{+} x)$. Then we may encode $\mathrm{I}^{-}_{\wedge i}$ and a weak variant of $\mathrm{E}^{-}_{\wedge}$ as follows:

- $\mathsf{in}^{-}_{i}(t) \stackrel{\text{def}}{=} \langle A_i, \bigcirc^{-}_{(\_:X_i{}^{\oplus})}.\,(r \,;^{-} t)\rangle^{-}$,
  where $r \stackrel{\text{def}}{=} \bigcirc^{+}_{(\_:Y_i{}^{\ominus})}.\,\lambda^{+}_{(y_1:A_1{}^{\oplus})}.\bigcirc^{+}_{(\_:(A_2\to A_i)^{\ominus})}.\,\lambda^{+}_{(y_2:A_2{}^{\oplus})}.y_i$.
- $\delta^{-} t\,[_{a_1}.s_1][_{a_2}.s_2] \stackrel{\text{def}}{=} \nabla^{-}\, t[_{(\alpha,x:X^{\ominus})}.\bigcirc^{-}_{(c:C^{\oplus})}.\,s'_1 \bullet^{-} c]$,
  where $s'_1 \stackrel{\text{def}}{=} s_1\{a_1 := \bigcirc^{-}_{(z_1:A_1{}^{\oplus})}.\,(s'_2 \bowtie_{A^{-}} c)\}$, and $s'_2 \stackrel{\text{def}}{=} s_2\{a_2 := \bigcirc^{-}_{(z_2:A_2{}^{\oplus})}.\,(u \bowtie_{A^{-}} x)\}$,
  and $u \stackrel{\text{def}}{=} \bigcirc^{+}_{\_:X^{\ominus}}.\,\lambda^{+}_{y:Y^{\oplus}}.(y \,@^{\mathcal{C}}\, z_1 \,@^{\mathcal{C}}\, z_2)$.

Note that $\Gamma \vdash \mathsf{in}^{-}_{i}(t) : (A_1 \wedge A_2)^{-}$ and $\Gamma \vdash \delta^{-} t\,[_{a_1}.s_1][_{a_2}.s_2] : C^{\ominus}$. However, unfortunately, it is **not** the case that $\delta^{-}\mathsf{in}^{-}_{i}(t)\,[_{a_1}.s_1][_{a_2}.s_2] \to^{*} s_i\{a_i := t\}$; in fact the computation becomes stuck.

In general, these kinds of encodings are able to simulate reduction for the positive half of the system but not for the negative half[1]. This seems to suggest that λ^{PRK} cannot be fully simulated by the $\{\forall, \to\}$ fragment, although we do not know of a proof of this fact and there might exist other encodings which allow simulating the full λ^{PRK} calculus.

[1] Naturally, one may consider dual encodings in terms of $\exists$ and $\ltimes$, for example $A \vee B = \exists\alpha.\,((A \ltimes B \ltimes \alpha) \ltimes \alpha)$, which behave well only for the negative half of the system.

4 Normalization of Second-Order λ^{PRK}

In this section we construct a **reducibility model** for λ^{PRK} and we prove **adequacy** (Thm. 5) of the model, from which **strong normalization** of second-order λ^{PRK} follows. We only discuss the proof of strong normalization for the calculus without the $\eta_\circ$ rule[2].

In [4], strong normalization for the propositional fragment of λ^{PRK} is shown via a translation to System F extended with recursive type constraints enjoying a (non-strict) positivity condition. This technique does not seem to extend to the second-order case. The problem is that the translation given in [4] is *not closed under type substitution.* More precisely, if we denote the translation by $\{\!\!\{-\}\!\!\}$, an equality such that $\{\!\!\{t\{\alpha := A\}\}\!\!\} = \{\!\!\{t\}\!\!\}\{\alpha := \{\!\!\{A\}\!\!\}\}$ does not hold in general, making the proof fail.

Our proof of strong normalization is based on an adaptation of Girard's technique of reducibility candidates. Specifically, we adapt Mendler's proof of strong normalization for the extended System F given in [31]. We begin by defining an *untyped* version of λ^{PRK}:

The untyped λ^{PRK}-calculus ($\lambda^{\mathsf{PRK}}_{\mathbf{U}}$). By $\mathbf{U}$ we denote the set of *untyped terms*, given by the following grammar:

$$
\begin{array}{rcl}
a, b, c, \dots & ::= & x \mid a \bowtie b \mid \langle a, b\rangle \mid \pi_i(a) \mid \mathsf{in}_i(a) \mid \delta a\,[_x.b][_y.c] \mid \lambda_x.\,a \mid a@b \mid (a\,;b) \mid \varrho a[_{x;y}.b] \\
& \mid & \mathsf{N}a \mid \mathsf{M}a \mid \lambda_\diamond.\,a \mid a@\diamond \mid \langle\diamond, a\rangle \mid \nabla\, a[_{(\diamond,x)}.b]
\end{array}
$$

The reduction relation $\to_{\mathbf{U}} \subseteq \mathbf{U} \times \mathbf{U}$ of the $\lambda^{\mathrm{PRK}}_{\mathbf{U}}$-calculus is defined by the following reduction rules, closed by compatibility under arbitrary reduction contexts:

$$
\begin{array}{rcl@{\qquad}rcl}
\pi_i(\langle a_1, a_2\rangle) & \to_{\mathbf{U}} & a_i & \delta(\mathsf{in}_i(a))\,[_x.b_1][_x.b_2] & \to_{\mathbf{U}} & b_i\{x := a\} \\
(\lambda_x.\,a)@b & \to_{\mathbf{U}} & a\{x := b\} & \varrho(a_1\,;a_2)[_{x;y}.b] & \to_{\mathbf{U}} & b\{x := a_1\}\{y := a_2\} \\
\mathsf{M}(\mathsf{N}a) & \to_{\mathbf{U}} & a & & & \\
(\lambda_\diamond.\,a)@\diamond & \to_{\mathbf{U}} & a & \nabla\,\langle\diamond, a\rangle[_{(\diamond,x)}.b] & \to_{\mathbf{U}} & b\{x := a\} \\
\langle a_1, a_2\rangle \bowtie \mathsf{in}_i(b) & \to_{\mathbf{U}} & a_i \bowtie b & \mathsf{in}_i(a) \bowtie \langle b_1, b_2\rangle & \to_{\mathbf{U}} & a \bowtie b_i \\
\lambda_x.\,a \bowtie (b\,;c) & \to_{\mathbf{U}} & a\{x := b\} \bowtie c & (a\,;b) \bowtie \lambda_x.\,c & \to_{\mathbf{U}} & b \bowtie c\{x := a\} \\
(\mathsf{N}a) \bowtie (\mathsf{N}b) & \to_{\mathbf{U}} & b \bowtie a & & & \\
\lambda_\diamond.\,a \bowtie \langle\diamond, b\rangle & \to_{\mathbf{U}} & a \bowtie b & \langle\diamond, a\rangle \bowtie \lambda_\diamond.\,b & \to_{\mathbf{U}} & a \bowtie b
\end{array}
$$

where $a \bowtie b \stackrel{\mathrm{def}}{=} (a@b) \bowtie (b@a)$. The set $\mathbf{CAN} \subseteq \mathbf{U}$ of *canonical terms* is the set of terms built with a constructor, *i.e.* of any of the forms: $\langle a, b\rangle$, $\mathsf{in}_i(a)$, $\lambda_x.\,a$, $(a\,;b)$, $\mathsf{N}a$, $\lambda_\diamond.\,a$, $\langle\diamond, a\rangle$.

Note that the untyped calculus $\lambda^{\mathrm{PRK}}_{\mathbf{U}}$ is obtained from λ^{PRK} by erasing all signs and type annotations from terms, replacing types and type variables by a placeholder "$\diamond$" in introductors and eliminators for quantifiers, and identifying[3] "weak" abstraction and application ($\bigcirc_x.\,t$ and $t \bullet s$) with regular abstraction and application ($\lambda_x.\,t$ and $t@s$). It is easy to note that the $\to_{\mathbf{U}}$ reduction is confluent, observing that it can be modeled as an orthogonal higher-order rewriting system [35].

One difficult aspect of the strong normalization proof is that terms of type $A^{\oplus}$ behave as functions "$A^{\ominus} \to A^{+}$" and, dually, terms of $A^{\ominus}$ behave as functions "$A^{\oplus} \to A^{-}$". Consequently, sets of *reducible terms* cannot be defined by straightforward recursion, as this would lead to a non-well-founded mutual dependency between reducible terms of types $A^{\oplus}$ and $A^{\ominus}$. To address this difficulty, we follow Mendler's approach of taking fixed points in the *complete lattice of reducibility candidates.*

[2] Strong normalization for the full λ^{PRK}-calculus with the $\eta_\circ$ rule comes out as a relatively easy corollary by postponing $\eta_\circ$ steps (see *e.g.* [4, Theorem 37] for a similar result).

[3] This identification is not essential, but just a matter of syntactic economy.

4.1 A Reducibility Model for λ^{PRK}

We begin by recalling a few standard notions from order theory. A *complete lattice* is a partially ordered set $(A, \leq)$ such that every subset $B \subseteq A$ has a least upper bound and a greatest lower bound, denoted respectively by $\bigvee B$ and $\bigwedge B$. Then (see [12, Thm. 2.35]):

▶ **Theorem 4** (Knaster–Tarski fixed point theorem). *If $(A, \leq)$ is a complete lattice and $f : A \to A$ is an order-preserving map,* i.e. $a \leq a' \implies f(a) \leq f(a')$*, then f has a* least fixed point *and a* greatest fixed point*, given respectively by: $\boldsymbol{\mu}(f) = \bigwedge\{a \in A \mid f(a) \leq a\}$ and $\boldsymbol{\nu}(f) = \bigvee\{a \in A \mid a \leq f(a)\}$.*

We write $\boldsymbol{\mu}(\xi.f(\xi))$ for $\boldsymbol{\mu}(f)$ and $\boldsymbol{\nu}(\xi.f(\xi))$ for $\boldsymbol{\nu}(f)$.

Reducibility candidates. Let $\mathbf{SN} \subseteq \mathbf{U}$ denote the set of *strongly normalizing* terms, with respect to $\to_{\mathbf{U}}$. A set $\xi \subseteq \mathbf{SN}$ is *closed by reduction* if for every $a, b \in \mathbf{U}$ such that $a \in \xi$ and $a \to_{\mathbf{U}} b$, one has that $b \in \xi$. A set $\xi \subseteq \mathbf{SN}$ is *complete* if for every $a \in \mathbf{SN}$ the following holds:

$$(\forall b \in \mathbf{CAN}.\ ((a \to^*_{\mathbf{U}} b) \implies b \in \xi)) \qquad \text{implies} \qquad a \in \xi$$

A set $\xi \subseteq \mathbf{SN}$ is a *reducibility candidate* (or a r.c. for short) if it is closed by reduction and complete. We write $\mathbf{RC}$ for the set of all r.c.'s, that is, $\mathbf{RC} \stackrel{\text{def}}{=} \{\xi \subseteq \mathbf{SN} \mid \xi \text{ is a r.c.}\}$.

It is easy to see that reducibility candidates are non-empty. In particular, for every $\xi \in \mathbf{RC}$ we have that any variable $x \in \xi$ is strongly normalizing and it vacuously verifies the property $\forall c \in \mathbf{CAN}.\ ((x \to^*_{\mathbf{U}} c) \implies c \in \xi)$ so, since ξ is complete, we have that $x \in \xi$. Moreover, the set $\mathbf{RC}$ forms a complete lattice ordered by inclusion $\subseteq$. Following Mendler [31, Prop. 2], the greatest lower bound of $\{\xi_i\}_{i\in I}$ is given by the intersection $\cap_{i\in I}\xi_i$, and the bottom element is the set $\bot = \{a \in \mathbf{SN} \mid \nexists b \in \mathbf{CAN}.\ a \to^*_{\mathbf{U}} b\}$ of terminating terms that have no canonical reduct.

Operations on reducibility candidates. For each set of canonical terms $X \subseteq \mathbf{CAN}$, we define its *closure* $\mathbb{C}X$ as the set of all strongly normalizing terms whose canonical reducts are in X. More precisely, $\mathbb{C}X \stackrel{\text{def}}{=} \{a \in \mathbf{SN} \mid \forall b \in \mathbf{CAN}.((a \to^*_{\mathbf{U}} b) \implies b \in X)\}$. If ξ_1, ξ_2 are r.c.'s and if $\{\xi_i\}_{i\in I}$ is a set of r.c.'s, we define the following operations:

$$\begin{array}{rcl@{\qquad}rcl}
(\xi_1 \times \xi_2) & \stackrel{\text{def}}{=} & \mathbb{C}\{\langle a_1, a_2\rangle \mid a_1 \in \xi_1, a_2 \in \xi_2\} & (\xi_1 + \xi_2) & \stackrel{\text{def}}{=} & \mathbb{C}\{\mathsf{in}_i(a) \mid i \in \{1,2\}, a \in \xi_i\} \\
(\xi_1 \to \xi_2) & \stackrel{\text{def}}{=} & \{a \in \mathbf{SN} \mid \forall b \in \xi_1.\ a@b \in \xi_2\} & (\xi_1 \ltimes \xi_2) & \stackrel{\text{def}}{=} & \mathbb{C}\{(a_1 \,;\, a_2) \mid a_1 \in \xi_1, a_2 \in \xi_2\} \\
\sim\xi & \stackrel{\text{def}}{=} & \mathbb{C}\{\mathsf{N}a \mid a \in \xi\} & & & \\
\boldsymbol{\Pi}_{i\in I}\,\xi_i & \stackrel{\text{def}}{=} & \{a \in \mathbf{SN} \mid \forall i \in I.\ a@\diamond \in \xi_i\} & \boldsymbol{\Sigma}_{i\in I}\,\xi_i & \stackrel{\text{def}}{=} & \mathbb{C}\{\langle \diamond, a\rangle \mid \exists i \in I.\ a \in \xi_i\}
\end{array}$$

It can be checked that all these operations map r.c.'s to r.c.'s.
A straightforward observation is that the arrow operator is order-reversing on the left, *i.e.* that if $\xi_1 \subseteq \xi_1'$ then $(\xi_1' \to \xi_2) \subseteq (\xi_1 \to \xi_2)$.

Orthogonality. The idea of the normalization proof is, as usual, to associate, to each type P, a set of *reducible terms* $[\![P]\!] \in \mathbf{RC}$. The interpretation of a type variable, such as $[\![\alpha^+]\!]$ or $[\![\alpha^-]\!]$ shall be given by an *environment* ρ, mapping type variables to r.c.'s. However, the sets $[\![\alpha^+]\!]$ and $[\![\alpha^-]\!]$ *should not be chosen independently of each other*: we require them to be orthogonal in the following sense.

Two reducibility candidates $\xi_1, \xi_2 \in \mathbf{RC}$ are *orthogonal*, if for all $a_1 \in \xi_1$ and $a_2 \in \xi_2$ we have that $(a_1 \bowtie a_2) \in \mathbf{SN}$. We write $\perp\!\!\!\perp$ for the set of all pairs $(\xi_1, \xi_2) \in \mathbf{RC}^2$ such that ξ_1 and ξ_2 are orthogonal.

Reducible terms. The set of reducible terms is defined by induction on the following notion of *measure* $\#(-)$ of a type P, given by $\#(A^+) = \#(A^-) \stackrel{\text{def}}{=} 2|A|$ and $\#(A^\oplus) = \#(A^\ominus) \stackrel{\text{def}}{=} 2|A| + 1$, where $|A|$ denotes the *size*, *i.e.* the number of symbols, of the pure type A. Note for example that $\#((A \wedge B)^\oplus) > \#((A \wedge B)^+) > \#(A^\oplus)$.

An *environment* is a function ρ mapping each type variable $\alpha^\pm$, with either positive or negative sign, to a reducibility candidate $\rho(\alpha^\pm) \in \mathbf{RC}$, in such a way that $(\rho(\alpha^+), \rho(\alpha^-)) \in \perp\!\!\!\perp$. If $(\xi^+, \xi^-) \in \perp\!\!\!\perp$, we write $\rho[\alpha := \xi^+, \xi^-]$ for the environment ρ' that extends ρ in such a way that $\rho'(\alpha^+) = \xi^+$ and $\rho'(\alpha^-) = \xi^-$ and $\rho'(\beta^\pm) = \rho(\beta^\pm)$ for any other type variable $\beta \neq \alpha$.

Given an environment ρ, we define the set of *reducible terms* of type P under the environment ρ, written $[\![P]\!]_\rho$, by induction on the measure $\#(P)$ as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
[\![\alpha^+]\!]_\rho & \stackrel{\text{def}}{=} & \rho(\alpha^+) & [\![\alpha^-]\!]_\rho & \stackrel{\text{def}}{=} & \rho(\alpha^-) \\
[\![(A \wedge B)^+]\!]_\rho & \stackrel{\text{def}}{=} & [\![A^\oplus]\!]_\rho \times [\![B^\oplus]\!]_\rho & [\![(A \wedge B)^-]\!]_\rho & \stackrel{\text{def}}{=} & [\![A^\ominus]\!]_\rho + [\![B^\ominus]\!]_\rho \\
[\![(A \vee B)^+]\!]_\rho & \stackrel{\text{def}}{=} & [\![A^\oplus]\!]_\rho + [\![B^\oplus]\!]_\rho & [\![(A \vee B)^-]\!]_\rho & \stackrel{\text{def}}{=} & [\![A^\ominus]\!]_\rho \times [\![B^\ominus]\!]_\rho \\
[\![(A \to B)^+]\!]_\rho & \stackrel{\text{def}}{=} & [\![A^\oplus]\!]_\rho \to [\![B^\oplus]\!]_\rho & [\![(A \to B)^-]\!]_\rho & \stackrel{\text{def}}{=} & [\![A^\oplus]\!]_\rho \ltimes [\![B^\ominus]\!]_\rho \\
[\![(A \ltimes B)^+]\!]_\rho & \stackrel{\text{def}}{=} & [\![A^\ominus]\!]_\rho \ltimes [\![B^\oplus]\!]_\rho & [\![(A \ltimes B)^-]\!]_\rho & \stackrel{\text{def}}{=} & [\![A^\ominus]\!]_\rho \to [\![B^\ominus]\!]_\rho \\
[\![(\neg A)^+]\!]_\rho & \stackrel{\text{def}}{=} & \sim[\![A^\ominus]\!]_\rho & [\![(\neg A)^-]\!]_\rho & \stackrel{\text{def}}{=} & \sim[\![A^\oplus]\!]_\rho \\
[\![(\forall\alpha.\, A)^+]\!]_\rho & \stackrel{\text{def}}{=} & \mathbf{\Pi}_{(\xi^+,\xi^-) \in \perp\!\!\!\perp} [\![A^\oplus]\!]_{\rho[\alpha:=\xi^+,\xi^-]} & [\![(\forall\alpha.\, A)^-]\!]_\rho & \stackrel{\text{def}}{=} & \mathbf{\Sigma}_{(\xi^+,\xi^-) \in \perp\!\!\!\perp} [\![A^\ominus]\!]_{\rho[\alpha:=\xi^+,\xi^-]} \\
[\![(\exists\alpha.\, A)^+]\!]_\rho & \stackrel{\text{def}}{=} & \mathbf{\Sigma}_{(\xi^+,\xi^-) \in \perp\!\!\!\perp} [\![A^\oplus]\!]_{\rho[\alpha:=\xi^+,\xi^-]} & [\![(\exists\alpha.\, A)^-]\!]_\rho & \stackrel{\text{def}}{=} & \mathbf{\Pi}_{(\xi^+,\xi^-) \in \perp\!\!\!\perp} [\![A^\ominus]\!]_{\rho[\alpha:=\xi^+,\xi^-]} \\
[\![A^\oplus]\!]_\rho & \stackrel{\text{def}}{=} & \boldsymbol{\mu}(\xi.((\xi \to [\![A^-]\!]_\rho) \to [\![A^+]\!]_\rho)) & [\![A^\ominus]\!]_\rho & \stackrel{\text{def}}{=} & \boldsymbol{\nu}(\xi.((\xi \to [\![A^+]\!]_\rho) \to [\![A^-]\!]_\rho))
\end{array}
$$

It is straightforward to check for each type P and each environment ρ that $[\![P]\!]_\rho$ is a reducibility candidate, by induction on the measure $\#(P)$. In the case of $[\![A^\oplus]\!]_\rho$, by the Knaster–Tarski theorem (Thm. 4), to see that the least fixed point exists, it suffices to observe that the mapping $f(\xi) = ((\xi \to [\![A^-]\!]_\rho) \to [\![A^+]\!]_\rho)$ is order-preserving. The case of $[\![A^\ominus]\!]_\rho$ is similar.

Adequacy of the reducibility model. For each term t of λ^{PRK}, we define an untyped term $|t|$ of $\lambda^{\mathrm{PRK}}_{\mathbf{U}}$ via the obvious forgetful map. For instance $|\bigcirc^+_{(x:(\exists\alpha.\,\alpha)^\ominus)}.\, \langle B, z^{B^\oplus}\rangle^+| = \lambda_x.\, \langle \diamond, z \rangle$. Note that each reduction step $t \to s$ in λ^{PRK} is mapped to a reduction step $|t| \to_{\mathbf{U}} |s|$ in $\lambda^{\mathrm{PRK}}_{\mathbf{U}}$. Hence, if $|t|$ is strongly normalizing with respect to $\to_{\mathbf{U}}$, then t is strongly normalizing with respect to $\to$.

A *substitution* is a function σ mapping each variable to a term in $\mathbf{U}$. We write a^σ for the term that results from the capture-avoiding substitution of each free occurrence of each variable x in a by $\sigma(x)$. We say that the substitution σ is *adequate* for the typing context Γ under the environment ρ, and we write $\sigma \vDash_\rho \Gamma$, if for each type assignment $(x : P) \in \Gamma$ we have that $\sigma(x) \in [\![P]\!]_\rho$. We are finally able to state the key result:

▶ **Theorem 5** (Adequacy). *If* $\Gamma \vdash t : P$ *and* $\sigma \vDash_\rho \Gamma$ *then* $|t|^\sigma \in [\![P]\!]_\rho$.

The proof of the adequacy theorem relies on a number of auxiliary lemmas stating properties such as $[\![A^\oplus]\!]_\rho = [\![A^\ominus]\!]_\rho \to [\![A^+]\!]_\rho$ and $([\![A^+]\!]_\rho, [\![A^-]\!]_\rho) \in \perp\!\!\!\perp$. From this we obtain as an easy corollary that the λ^{PRK}-calculus is strongly normalizing, taking ρ as the environment that maps all type variables to the bottom reducibility candidate, and σ as the identity substitution.

5 Intuitionistic Proofs and Refutations

In natural deduction, it is well-known that classical logic can be obtained from the intuitionistic system by adding a single classical axiom, such as excluded middle or double negation elimination. In sequent calculus, it is well-known that intuitionistic logic can be obtained by restricting sequents $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ to have at most one formula on the right. As we have seen, λ^{PRK} refines classical logic. It is a natural question to ask what subsystem of λ^{PRK} corresponds to intuitionistic logic.

In this section we characterize a restricted subsystem of λ^{PRK} that corresponds to intuitionistic logic, called λ^{PRJ}, by imposing a syntactic restriction on the shape of λ^{PRK} proofs, that forbids certain specific patterns of reasoning. In particular, a variable x introduced by a positive weak introduction $\bigcirc^{+}_{x}.\,t$ can only occur free in t inside the arguments of weak eliminations. The main result in this section is that **$\lambda^{\mathbf{PRJ}}$ refines intuitionistic second-order logic** (Thm. 9).

As mentioned before, proofs of *strong* propositions in PRK must be constructive. However, this is only true for the toplevel logical connective in the formula. In general, a proof of A^{+} in PRK does not necessarily correspond to an intuitionistic proof of A. For example, a canonical proof of $(A \wedge B)^{+}$ is given by a proof of $A^{\oplus}$ and a proof of $B^{\oplus}$, but these subproofs may resort to classical reasoning principles.

The key to identify an intuitionistic subset of λ^{PRK} is to disallow inference rules which embody classical principles. One example is the $\mathrm{E}^{-}_{\neg}$ rule, which derives $A^{\oplus}$ from $(\neg A)^{-}$. This rule embodies the classical principle of double negation elimination $(\neg\neg A \to A)$. Another important example is the $\mathrm{I}^{+}_{\circ}$ rule, which derives $A^{\oplus}$ from $A^{\ominus} \vdash A^{+}$. This rule embodies the classical principle of *consequentia mirabilis* $((\neg A \to A) \to A)$.

The analysis of the $\mathrm{I}^{+}_{\circ}$ rule suggests that, in the intuitionistic fragment, $A^{\oplus}$ should not be identified with "$A^{\ominus} \to A^{+}$", but directly with A^{+}. One natural idea would be to impose an invariant over terms of the form $\bigcirc^{+}_{(x:A^{\ominus})}.\,t$, in such a way that the body t may have no free occurrences of the negative counterfactual x. With this invariant, all instances of the $\mathrm{I}^{+}_{\circ}$ rule are actually instances of the following variant of $\mathrm{I}^{+}_{\circ}$:

$$\frac{\Gamma \vdash t : A^{+} \qquad x \notin \mathsf{fv}(t)}{\Gamma \vdash \bigcirc^{+}_{(x:A^{\ominus})}.\,t : A^{\oplus}}\,\mathrm{I}^{+}_{\circ}(\text{variant})$$

This in turn means that, in an application $t \bullet^{+} s$, the argument s is *useless*. Indeed, if t becomes $\bigcirc^{+}_{(x:A^{\ominus})}.\,t'$, the invariant ensures that $x \notin \mathsf{fv}(t')$, so $(\bigcirc^{+}_{(x:A^{\ominus})}.\,t') \bullet^{+} s \to t'$, which does not depend on the specific choice of s.

Rather than completely forbidding classical reasoning principles, we relax this condition so that classical principles are allowed as long as they are useless, *i.e.* inside the argument of an application $t \bullet^{+} s$. Furthermore, the invariant over terms of the form $\bigcirc^{+}_{(x:A^{\ominus})}.\,t$, requesting that $x \notin \mathsf{fv}(t)$, can also be relaxed, in such a way that x is allowed to occur in t as long as all of its free occurrences are useless. Formally:

▶ **Definition 6** (Intuitionistic terms). *A subterm of a term t is said to be* **useless** *if it lies inside the argument of a positive weak elimination. More precisely, given a term $t = \mathtt{C}\langle s\rangle$, we say that the subterm s under the context $\mathtt{C}$ is useless if and only if there exist contexts $\mathtt{C}_1, \mathtt{C}_2$ and a term u such that $\mathtt{C}$ is of the form $\mathtt{C}_1\langle u \bullet^{+} \mathtt{C}_2\langle\Box\rangle\rangle$. A subterm of t is* **useful** *if it is not useless. A term is said to be* **intuitionistic** *if and only if the two following conditions hold:*

1. **Useless negative eliminations ($\mathbf{E}^-_\wedge$, $\mathbf{E}^-_\to$, $\mathbf{E}^-_\neg$, $\mathbf{E}^-_\forall$).** *There are no useful subterms of any of the following forms:* $\delta^- t\,[_{(x:A^\ominus)}.s][_{(y:B^\ominus)}.u]$, $\varrho^- t[_{(x:A^\oplus);(y:B^\ominus)}.s]$, $\mathsf{M}^- t$, $\nabla^-\, t[_{(\alpha,x)}.s]$.
2. **Useless negative counterfactuals.** *In every useful subterm of the form* $\bigcirc^+_{(x:A^\ominus)}.\,t$*, there are no useful occurrences of* x *in* t*.*

The λ^{PRJ} type system. The type system λ^{PRJ} is defined by imposing the restriction on λ^{PRK} that terms be intuitionistic. More precisely, we say that a judgment $\Gamma \vdash t : P$ holds in PRJ, and in this case we write $\Gamma \vdash_{\mathrm{PRJ}} t : P$, if the judgment holds in PRK and furthermore t is an intuitionistic term. We also write $P_1, \dots, P_n \vdash_{\mathrm{PRJ}} Q$ if there exists a term t such that $x_1 : P_1, \dots, x_n : P_n \vdash_{\mathrm{PRJ}} t : Q$.

▶ **Example 7.** The weak variant of the law of excluded middle, $(A \vee \neg A)^\oplus$, can be proven in PRK. For example, if we define $\pitchfork^+_A$:

$$\pitchfork^+_A \stackrel{\text{def}}{=} \bigcirc^+_{(x:(A\vee\neg A)^\ominus)}.\,\mathsf{in}_2^+(\bigcirc^+_{(y:\neg A^\ominus)}.\,\mathsf{N}^+\pi_1^-(x \bullet^- \Delta^+_{y,A}))$$
$$\text{where } \Delta^+_{y,A} \stackrel{\text{def}}{=} \bigcirc^+_{(\_:(A\vee\neg A)^\ominus)}.\,\mathsf{in}_1^+(\bigcirc^+_{(z:A^\ominus)}.\,(y \bowtie_{A^+} \bigcirc^+_{(\_:\neg A^\ominus)}.\,\mathsf{N}^+ z))$$

we can note that $\vdash_{\mathrm{PRK}} \pitchfork^+_A : (A \vee \neg A)^\oplus$ holds. However, $\pitchfork^+_A$ is not intuitionistic, due to the fact that *there is a useful occurrence of the negative counterfactual* x.

The intuitionistic fragment is stable by reduction:

▶ **Proposition 8** (Subject reduction for PRJ). *Let* $\Gamma \vdash_{\mathrm{PRJ}} t : P$ *and* $t \to s$*. Then* $\Gamma \vdash_{\mathrm{PRJ}} s : P$*.*

Proof. If X is a set of variables, we say that a term t is X*-intuitionistic* if it is intuitionistic and, furthermore, it has no *useful* free occurrences of variables in X. We write $\Gamma \vdash^X_{\mathrm{PRJ}} t : P$, if the judgment is derivable in PRK and t is X-intuitionistic. The statement of subject reduction is generalized as follows: $\Gamma \vdash^X_{\mathrm{PRJ}} t : P$ and $t \to s$, then $\Gamma \vdash^X_{\mathrm{PRJ}} s : P$.

The interesting case is the β rule for positive weak proofs, $(\bigcirc^+_{x:A^\ominus}.\,t) \bullet^+ s \xrightarrow{\beta^+_\circ} t\{x := s\}$. By hypothesis, $\Gamma \vdash^X_{\mathrm{PRJ}} (\bigcirc^+_{x:A^\ominus}.\,t) \bullet^+ s : A^+$. This judgment can only be derived from the $\mathrm{I}^+_\circ$ rule, so $\Gamma, x : A^\ominus \vdash^X_{\mathrm{PRJ}} t : A^+$. Moreover, x is a negative counterfactual, so there cannot be useful free occurrences of x in t, which means that $\Gamma, x : A^\ominus \vdash^{X\cup\{x\}}_{\mathrm{PRJ}} t : A^+$. On the other hand, s lies inside a positive application, so it is not necessarily X-intuitionistic, *i.e.* we only know $\Gamma \vdash_{\mathrm{PRK}} s : A^\ominus$. The key observation is that all the copies of s on the right-hand side $t\{x := s\}$ must be useless, because all the occurrences of x in t are useless. More precisely, from $\Gamma, x : A^\ominus \vdash^{X\cup\{x\}}_{\mathrm{PRJ}} t : A^+$ and $\Gamma \vdash_{\mathrm{PRK}} s : A^\ominus$ one concludes $\Gamma \vdash^X_{\mathrm{PRJ}} t\{x := s\} : P$ by induction on t. ◀

Reasoning principles in PRJ differ from those of PRK. For example, if P is a weak formula, a sequent $\Gamma, x : P \vdash_{\mathrm{PRK}} t : Q$ valid in PRK can always be *contraposed* to a sequent of the form $\Gamma, y : Q^\sim \vdash_{\mathrm{PRK}} t' : P^\sim$. The analogous of this contraposition principle in PRJ depends on the sign of P. If P is positive, *i.e.* $P = A^\oplus$ the sequent $\Gamma, x : A^\oplus \vdash_{\mathrm{PRJ}} t : Q$ can always be contraposed to $\Gamma, y : Q^\sim \vdash_{\mathrm{PRJ}} t' : A^\ominus$. But if P is negative, *i.e.* $P = A^\ominus$ the sequent $\Gamma, x : A^\ominus \vdash_{\mathrm{PRJ}} t : Q$ can only be contraposed to $\Gamma, y : Q^\sim \vdash_{\mathrm{PRJ}} t' : A^\oplus$ if there are no useful occurrences of x in t.

The following theorem is an analog of Thm. 3 for λ^{PRJ}.

▶ **Theorem 9** (Intuitionistic refinement). $A_1^\oplus, \dots, A_n^\oplus \vdash B^\oplus$ *holds in* λ^{PRJ} *if and only if* $A_1, \dots, A_n \vdash B$ *holds in intuitionistic second-order logic.*

6 Canonicity

In sequent calculus and natural deduction, an *indirect proof* (*e.g.* with cuts), can always be mechanically converted into a *canonical proof* (*e.g.* cut-free), in which the justification for the conclusion is immediately available, as is known from the works of Gentzen [20] and Prawitz [41]. Its philosophical importance is that the validity of an indirect proof can thus be justified by understanding it as a notation for describing a canonical proof. A practical consequence is that an explicit witness may be extracted from a proof of existence.

In this section, we formulate a **canonicity** result strengthening those of [4]. We start by introducing some nomenclature. *Neutral terms* $(e, \ldots)$ and *normal terms* $(f, \ldots)$ are given by $e ::= x \mid e \bowtie_P f \mid f \bowtie_P e \mid \pi_i^{\pm}(e) \mid \delta^{\pm} e\,[_x.f][_x.f] \mid e@^{\pm} f \mid \varrho^{\pm} e[_{x;y}.f] \mid \mathsf{M}^{\pm} e \mid e@^{\pm} A \mid \nabla^{\pm}\, e[_{(x,\alpha)}.f] \mid e \bullet^{\pm} f$ and $f ::= e \mid \langle f, f\rangle^{\pm} \mid \mathsf{in}_i^{\pm}(f) \mid \lambda_x^{\pm}.f \mid (f \,;^{\pm} f) \mid \mathsf{N}^{\pm} f \mid \lambda_\alpha^{\pm}.\, f \mid \langle A, f\rangle^{\pm} \mid \bigcirc_x^{\pm}.\, f$. Terms built with an introduction rule: $\langle t, s\rangle^{\pm}$, $\mathsf{in}_i^{\pm}(t)$, $\lambda_x^{\pm}.t$, $(t \,;^{\pm} s)$, $\mathsf{N}^{\pm} t$, $\lambda_\alpha^{\pm}.\, t$, $\langle A, t\rangle^{\pm}$, $\bigcirc_x^{\pm}.\, t$ are called *canonical*. Then:

▶ **Theorem 10** (Canonicity).

1. *If* $\vdash_{\mathrm{PRK}} t : P$*, then* t *reduces to a canonical normal form* f *such that* $\vdash_{\mathrm{PRK}} f : P$*.*
2. *If* $\vdash_{\mathrm{PRK}} t : P$*, where* P *is weak, then a canonical normal form* f *can be effectively found such that* $\vdash_{\mathrm{PRK}} \bigcirc_{(x:P^{\sim})}^{\pm}.\, f : P$

Note that this canonicity theorem applies to closed terms only, so there is no need to include commutative conversions, such as $\delta^{+} x\,[_y.t][_z.s] \bullet^{+} u \to \delta^{+} x\,[_y.t \bullet^{+} u][_z.s \bullet^{+} u]$, to unblock redexes. The preceding canonicity result extends and strengthens Thm. 35 of [4]. The first part of the theorem confirms the intuition, mentioned in the introduction, that canonical proofs of strong propositions are always constructed with an introduction rule for the corresponding logical connective, while canonical proofs of weak propositions proceed by *reductio ad absurdum*. For example, if $\vdash t : (A_1 \wedge A_2)^{+}$ is the *strong* proof of a conjunction, its normal form must be a pair $\langle t_1, t_2\rangle^{+}$ and we know that $\vdash t_i : A_i{}^{\oplus}$ must hold for $i \in \{1, 2\}$. On the other hand, if $\vdash s : (A_1 \wedge A_2)^{+}$ is the *weak* proof of a conjunction, we can only assure that its normal form is of the form $\bigcirc_{(x:(A_1 \wedge A_2)^{\ominus})}^{+}.\, s'$, where $x : (A_1 \wedge A_2)^{\ominus} \vdash s' : (A_1 \wedge A_2)^{+}$. The second part of the theorem provides the stronger guarantee that, in such case, one can compute a *canonical* s'' such that $x : (A_1 \wedge A_2)^{\ominus} \vdash s'' : (A_1 \wedge A_2)^{+}$, so in particular one has that $s'' = \langle s_1, s_2\rangle^{+}$ and that $x : (A_1 \wedge A_2)^{\ominus} \vdash s_i : A_i{}^{\oplus}$ for all $i \in \{1, 2\}$.

Canonicity can also be used to obtain a (weak) form of *disjunctive property*. In particular, from $\vdash t : (A_1 \vee A_2)^{\oplus}$ one can always find an $i \in \{1, 2\}$ and a term $\vdash \bigcirc_{(x:(A_1 \vee A_2)^{\ominus})}^{+}.\, \mathsf{in}_i^{+}(t') : (A_1 \vee A_2)^{\oplus}$ such that $x : (A_1 \vee A_2)^{\ominus} \vdash t' : A_i{}^{\oplus}$. Similarly, a (weak) form of *witness extraction* can be obtained: given $\vdash t : (\exists \alpha.\, A)^{\oplus}$ one can always find a term $\vdash \bigcirc_{(x:(\exists \alpha.\, A)^{\ominus})}^{+}.\, \langle B, t'\rangle^{+} : (\exists \alpha.\, A)^{\oplus}$ where $x : (\exists \alpha.\, A)^{\ominus} \vdash t' : A\{\alpha := B\}^{\oplus}$.

Furthermore, canonicity provides a purely syntactic proof of the consistency of PRK[4]. Note, for example, that if α is a base type, there is no canonical term t such that $\vdash_{\mathrm{PRK}} t : \alpha^{+}$.

7 Conclusion

In this paper we have extended the λ^{PRK}-calculus of [4] to incorporate implication and co-implication, as well as second-order quantifiers. From the logical point of view, this extension of λ^{PRK} **refines classical second order logic** (Thm. 3). From the computational point of view, it is **confluent** (Thm. 3) and **strongly normalizing** (Section 4). These

[4] Another way to prove consistency is using Thm. 3, noting that $\vdash_{\mathrm{PRK}} \alpha^{+}$ implies $\vdash_{\mathsf{NK}} \alpha$.

ingredients constitute a computational interpretation for second-order classical logic. We have identified a well-behaved subset of the system, called λ^{PRJ}, that **refines intuitionistic second-order logic** (Thm. 9). We have also formulated a **canonicity** (Thm. 10) result that strengthens results of previous works. One noteworthy property of λ^{PRK} is that both typing and reduction rules are fully symmetric with respect to the operation that flips signs and exchanges the roles of dual connectives, while still being confluent.

Related work. The "$\mathrm{I}^+_\circ$"/"$\mathrm{I}^-_\circ$" rules in λ^{PRK} encode a primitive variant of *consequentia mirabilis* ($(\neg A \to A) \to A$), while the "λ" rule in Barbanera and Berardi's calculus [3], as well as the "μ" rule in Parigot's $\lambda\mu$-calculus [37], encode a primitive variant of *double negation elimination* ($\neg\neg A \to A$). To prove A using double negation elimination one may assume $\neg A$ and then provide a proof of $\bot$. This proof *cannot be canonical*, as there are no introduction rules for the empty type. This motivates that we instead rely on *consequentia mirabilis*.

Strong normalization proofs are often based on reducibility candidates. Yamagata proves strong normalization for second-order formulations of classical calculi [48, 49], via reducibility candidates. Our proof is inspired by ideas known from the literature of logical relations and biorthogonality: for instance, the notions of *orthogonal* r.c.'s and *closure* of a r.c. can be traced back to Krivine's work on classical realizability [28], Pitts' $\top\top$-closed logical relations [39], and related notions (see *e.g.* [18]). The challenging aspect of λ^{PRK} is the mutually recursive dependency between $A^{\oplus}$ and $A^{\ominus}$, for which our key reference is Mendler's work [31].

The problem of finding a good calculus for classical logic has not been unquestionably settled. Current proof assistants based on type theory, such as Coq, allow classical reasoning by postulating axioms with no computational content, which breaks canonicity. An established classical calculus is Parigot's $\lambda\mu$, whose metatheory has been thoroughly developed; see for instance [16, 13, 29, 44, 45, 38, 27, 26]. One difference between $\lambda\mu$ and λ^{PRK} is that λ^{PRK} computational rules are based on the standard operation of substitution, while $\lambda\mu$ is based on an *ad hoc* substitution operator. Another difference is that the embedding of the λ-calculus into $\lambda\mu$ is an inclusion, whereas the embedding into λ^{PRK} is much more convoluted.

Another established classical calculus is Curien and Herbelin's [10] $\bar{\lambda}\mu\tilde{\mu}$, whose study is also quite mature; see for instance [40, 25, 17, 47, 11, 2, 1, 32]. One difference between $\bar{\lambda}\mu\tilde{\mu}$ and λ^{PRK} is that $\bar{\lambda}\mu\tilde{\mu}$ is not confluent unless a reduction strategy is fixed in the presence of a specific critical pair, whereas λ^{PRK} is orthogonal. Another difference is that $\bar{\lambda}\mu\tilde{\mu}$ is derived from a proof term assignment for classical sequent calculus, while λ^{PRK} is defined in natural deduction style with four forms of judgment (given by the modes A^+, A^-, $A^{\oplus}$, $A^{\ominus}$). Munch-Maccagnoni [33] proposes a classical calculus by *polarizing* Curien and Herbelin's calculus, in such a way that the reduction strategy becomes determined by the polarities.

As mentioned in the introduction, λ^{PRK} is related to Nelson's constructible falsity [34]. Parigot [36] studies *free deduction*, a system for classical logic in which natural deduction and sequent calculus can both be embedded. Rumfitt [43] proposes *bilateral* logical systems, in which assertion and denial judgments, with dual rules, are formulated. Zeilberger [50] studies a polarized logical system with proofs and refutations distinguishing between verificationist and pragmatist connectives. These systems, however, do not distinguish between weak and strong propositions, and they do not have rules analogous to $\mathrm{I}^{\pm}_\circ/\mathrm{E}^{\pm}_\circ$.

Future work. The merely *logical* correspondence between λ^{PRK} and other classical calculi is immediate, given the fact that λ^{PRK} refines second-order classical logic (Thm. 3). However, it is not obvious what their relation is from the *computational* point of view.

In order to be able to build programming languages and proof assistants based on the principles of λ^{PRK}, it would be convenient to study dependently typed extensions of the system. It is not *a priori* clear what such an extension would look like.

It is known that, in classical logic, *reductio ad absurdum* can be postponed, in such a way that it is used at most once, as the last rule in the derivation [46, 24]. It would be interesting to see if this result can be reproduced in PRK.

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