Modification Problems Toward Proper (Helly) Circular-Arc Graphs

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Abstract

We present a $9^k \cdot n^{O(1)}$-time algorithm for the proper circular-arc vertex deletion problem, resolving an open problem of van 't Hof and Villanger [Algorithmica 2013] and Crespelle et al. [Computer Science Review 2023]. Our structural study also implies parameterized algorithms for modification problems toward proper Helly circular-arc graphs.

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1 Introduction

A graph is a *circular-arc graph* if its vertices can be assigned to arcs on a circle such that there is an edge between two vertices if and only if their corresponding arcs intersect. If none of the arcs properly contains another, then the graph is a *proper circular-arc graph*. See Figure 1 for two examples of proper circular-arc graphs. Proper circular-arc graphs "form an important subclass of the class of all claw-free graphs," and their study has been an important step towards finding "a structural characterization of all claw-free graphs" [6]. The structures and recognition of proper circular-arc graphs have been well studied and well understood [19, 8].

Another and earlier motivation for studying (proper) circular-arc graphs is from their relation with *(proper) interval graphs*, i.e., intersection graphs of intervals on the real line. The *intersection graph* of a family of sets has a vertex for each set and an edge between two vertices if and only if their the sets they represent have a nonempty intersection. It is easy to see that each (proper) interval graph is a (proper) circular-arc graph, and the connection of these classes has been used in both structural and algorithmic studies of these classes. Indeed, the first linear-time recognition algorithm for proper circular-arc graphs is based on a general observation of both proper circular-arc graphs and proper interval graphs [8]. Neither graph in Figure 1 is a proper interval graph, but removing any vertex from Figure 1(a), or any vertex but v_5 from Figure 1(b) leaves a proper interval graph.

Let $\mathcal{G}$ be a hereditary (closed under taking induced subgraphs) graph class. Given a graph G and an integer k, the $\mathcal{G}$ vertex deletion problem asks whether we can remove k vertices from G to make a graph in $\mathcal{G}$. It is known that the $\mathcal{G}$ vertex deletion problem is NP-hard

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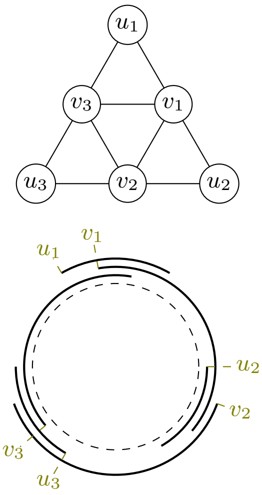
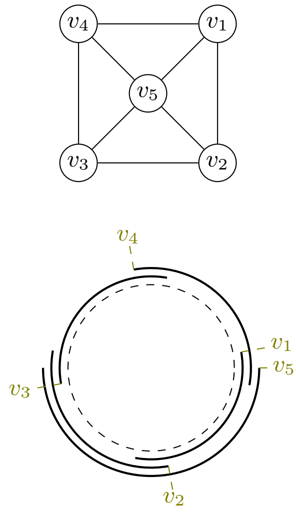


(a) S_3.



(b) W_4.

Figure 1 Two proper circular-arc graphs and their arc models.

when $\mathcal{G}$ is nontrivial (i.e., having infinite members and infinite non-members) [15]. These problems have been intensively studied in the framework of parameterized computation. Suppose that the input graph has n vertices and m edges. We say that a graph problem is *fixed-parameter tractable (FPT)* if there is an algorithm solving it in time $f(k) \cdot n^{O(1)}$, where f is a computable function depending only on k [9]. For example, it is well known that the proper interval vertex deletion problem is FPT[20, 3]. In the algorithm of van 't Hof and Villanger [20], the kinship between proper circular-arc graphs and proper interval graphs plays a crucial role. They showed that it suffices to destroy all the small forbidden induced subgraphs, and then the graph is already a proper circular-arc graph, on which the proper interval vertex deletion problem can be solved in linear time. They asked whether the proper circular-arc vertex deletion problem is FPT as well, and this open problem was recently raised again by Crespelle et al. [7]. We answer this question affirmatively.

▶ **Theorem 1.** *The proper circular-arc vertex deletion problem can be solved in time* $9^k \cdot n^{O(1)}$.

A major difference between the class of proper interval graphs and the class of proper circular-arc graphs is that the later class is not closed under disjoint union. This can be easily observed from their models: while we can always put intervals for two different components side by side, no such accommodation is possible for two sets of arcs if one set of them covers the whole circle. As a matter of fact, if a proper circular-arc graph is not connected, it has to be a proper interval graph. (The same remark applies to the relation between circular-arc graphs and interval graphs.)

If a proper circular-arc graph contains a hole of length at least five, then its property is quite similar to a proper interval graph. What is difficult is when a few arcs cover the whole circle in an arc model. For such a graph, it is more convenient to study its complement. Indeed, when characterizing proper circular-arc graphs, Tucker [19] actually listed the forbidden induced subgraphs of the complement class. He also observed that if the complement $\overline{G}$ of a proper circular-arc graph G is not connected, then $\overline{G}$ is bipartite. *Permutation graphs* are the intersection graphs of line segments between two parallel lines, and *bipartite permutation graphs* are those permutation graphs that are bipartite. Bipartite permutation graphs are also known as proper interval bigraphs and unit interval bigraphs [14]. It is well known that a co-bipartite graph H is a proper circular-arc graph if and only if $\overline{H}$ is a bipartite permutation graph.

Let (G, k) be an instance to the proper circular-arc vertex deletion problem, and let V_- be a solution. If $G - V_-$ is not connected, then it is a proper interval graph; if $G - V_-$ is not co-connected, then it is the complement of a bipartite permutation graph. We can call the algorithm of Cao [3] and the algorithm of Bożyk et al. [1] to check whether such a set V_- exists, and we are done if it does. In the rest, we may assume that $G - V_-$ is neither a proper interval graph nor the complement of a bipartite graph, hence both connected and co-connected. For this purpose we may assume that G itself is both connected and co-connected; otherwise, there is a unique component C of G or $\overline{G}$ such that $V(G) \setminus V(C) \subseteq V_-$. Either the instance is trivially FPT, when $n = O(k)$, or it suffices to consider the largest component of G or $\overline{G}$.

The algorithm proceeds as follows. We can destroy all forbidden induced subgraphs of order at most seven by branching. Now G is free of small forbidden induced subgraphs and is both connected and co-connected. Our key observation is that if G is not already a proper circular-arc graph, then $\overline{G}$ must be bipartite. Note that any induced subgraph of a bipartite graph is bipartite, but we have assumed that $G - V_-$ is not the complement of a bipartite graph. Therefore, we are ready to directly return "yes" or "no."

Since the parameterized algorithm branches on a small set of vertices that intersects every solution, we can easily turn it into an approximation algorithm for the maximum proper circular-arc induced subgraph problem.

▶ **Theorem 2.** *There is a polynomial-time approximation algorithm of approximation ratio* 9 *for the minimization version of the proper circular-arc vertex deletion problem.*

Proper circular-arc graphs also arise naturally when we consider the clique graph (the intersection graph of maximal cliques of a host graph) of a circular-arc graph. The complicated structures of circular-arc graphs are mainly due to the lack of the so-called *Helly* property: every set of pairwise intersecting arcs has a common intersection. For example, neither model in Figure 1 is Helly: the set $\{v_1, v_2, v_3\}$ in (a) and the set $\{v_3, v_4, v_5\}$ in (b) violate the Helly property. A graph is a *Helly circular-arc graph* if it admits an arc model that is Helly. Since every interval model is Helly, all interval graphs are Helly circular-arc graphs. It is well known that the clique graph of an interval graph, with at most n maximal cliques, is a proper interval graph [13]. The same upper bound holds for the number of maximal cliques in a Helly circular-arc graph, and the clique graph of a Helly circular-arc graph is always a proper circular-arc graph [10]. Let us remark parenthetically that a circular-arc graph may have an exponential number of maximal cliques, e.g., the complement of the union of p disjoint edges, which has $2p$ vertices, each of degree $2p - 2$.

The class of proper Helly circular-arc graphs is sandwiched between proper circular-arc graphs and proper interval graphs. This observation has been crucial for the algorithms for modification problems toward proper interval graphs [3]. A graph is a *proper Helly circular-arc graph* if it has an arc model that is both proper and Helly. A word of caution is worth on the definition of proper Helly circular-arc graphs. One graph might admit two arc models, one being proper and the other Helly, but no arc model that is both proper and Helly. For example, both models in Figure 1 are proper but neither is Helly, and it is not difficult to make Helly arc models for S_3 and W_4, but, as the reader may easily verify, neither of them admits an arc model that is both proper and Helly. Therefore, the class of proper Helly circular-arc graphs does not contain all those graphs being both proper circular-arc graphs and Helly circular-arc graphs, but a proper subclass of it. Indeed, a proper circular-arc graph is a proper Helly circular-arc graph if and only if it is $\{S_3, W_4\}$-free [16]. Another characterization of proper Helly circular-arc graphs is that they are precisely those graphs whose clique matrices have the circular-ones property for both rows and columns [17].

We then consider modification problems toward proper Helly circular-arc graphs. For this class we also consider the edge deletion and completion problems (a proof of their NP-completeness was provided in the full version). The edge deletion (resp., completion) problem asks whether we can delete (resp., add) at most k edges to a graph to make it satisfy certain properties. Again, we start by destroying all small forbidden induced subgraphs, up to six vertices. We show that a connected graph free of such induced subgraphs is already a proper Helly circular-arc graph. For the vertex deletion problem, either we remove all but one component, or we remove vertices to get a proper interval graph. The edge deletion problem is even simpler: if the graph is not connected, we cannot make it connected by deleting edges. Thus, depending on whether the graph is connected, either we are already done, or we are solving the proper interval edge deletion problem. This idea can even solve the general deletion problem that allows k_1 vertex deletions and k_2 edge deletions. The situation is quite different for the completion problem. We are happy if we can add at most k edges to make the input graph a proper interval graph. Otherwise, we have to make a connected proper Helly circular-arc graph. After we have dealt with all the small forbidden induced subgraphs, the only nontrivial case is when there is a large component, which contains a long hole H, and several small components. We need to "attach" these small components to vertices on H. Since these operations are local, we can find a solution by dynamic programming. Thus, all three problems are FPT, and they can be done in linear FPT time. Again, the parameterized algorithm for the vertex deletion problem can be easily turned into an approximation algorithm.

▶ **Theorem 3.** *For modification problems toward proper Helly circular-arc graphs, there are*
- *an $O(6^k \cdot (m+n))$-time algorithm for the vertex deletion problem;*
- *an $O(8^k \cdot (m+n))$-time algorithm for the edge deletion problem;*
- *an $O(14^{k_1+k_2} \cdot (m+n))$-time algorithm for the deletion problem; and*
- *a $k^{O(k)} \cdot (m+n)$-time algorithm for the completion problem.*

Moreover, there is an $O(nm+n^2)$-time approximation algorithm of approximation ratio 6 *for the minimization version of the proper Helly circular-arc vertex deletion problem.*

Somewhat surprisingly, modification problems toward circular-arc graphs and its subclasses have not received sufficient attention. We hope our work will inspire more study in this direction. Apart from the two classes in the present paper, the next interesting class is the class of normal Helly circular-arc graphs, a super class of proper Helly circular-arc graphs. They have played crucial roles in solving modification problems to interval graphs [5, 2]. Also related and probably simpler are the modification problems toward unit (Helly) circular-arc graphs. It is well known that a graph is a proper interval graph if and only if it is a unit interval graph. However, there are proper (Helly) circular-arc graphs that are not unit (Helly) circular-arc graphs, e.g., the graph obtained from an even hole of length at least eight by adding edges to connect consecutive even-numbered vertices.

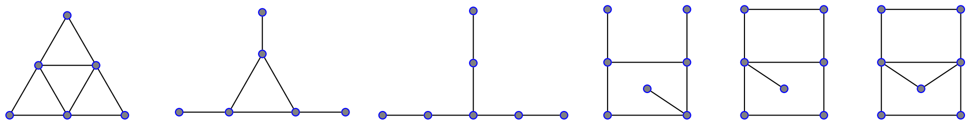
2 Preliminaries

All graphs discussed in this paper are undirected, simple, and finite. The vertex set and edge set of a graph G is denoted by, respectively, $V(G)$ and $E(G)$. Let $n = |V(G)|$ and $m = |E(G)|$. A *walk* in a graph G is a sequence of vertices and edges in the form of v_0,

v_0v_1, v_1, v_1v_2, ..., v_ℓ. Since the edges are determined by the vertices, such a walk can be denoted by $v_0v_1 \dots v_\ell$ unambiguously. We say that this walk *connects* v_0 and v_ℓ, which are the *ends* of this walk, and refer to it as a v_0–v_ℓ walk. The *length* of a walk is the number of occurrences of edges it contains, and ℓ in the previous example. A walk is *closed* if $\ell > 1$ and $v_0 = v_\ell$. A walk is a *path* if all its vertices are distinct. A path is *nontrivial* if it contains at least three vertices. A closed walk of length ℓ is a *cycle* if it visits precisely ℓ vertices; i.e., no repeated vertices except the two ends. The length of a cycle C is denoted as $|C|$. For simplicity, we denote a cycle of length ℓ as $v_1v_2 \cdots v_\ell$ instead of $v_1v_2 \cdots v_\ell v_1$. The indices are understood to be modulo ℓ; e.g., $v_0 = v_\ell$ and $h_{-1} = h_{\ell-1}$. A *hole* is an induced cycle of length at least four. A walk, path, cycle, or hole is *odd* (resp., *even*) if its length is odd (resp., even). For $\ell \geq 3$, we use C_ℓ to denote an induced cycle on ℓ vertices; if we add a new vertex to a C_ℓ and make it adjacent to no or all vertices on the cycle, then we end with a C_ℓ^* or W_ℓ, respectively.

The complement graph $\overline{G}$ of a graph G is defined on the same vertex set $V(G)$, where a pair of vertices u and v is adjacent in $\overline{G}$ if and only if $uv \notin E(G)$; e.g., $\overline{C_5^*}$ is W_5. The graph $\overline{C_3^*}$ is also called a *claw*. A graph G is *connected* if every pair of vertices is connected by a path, and *co-connected* if $\overline{G}$ is connected.

A *circular-arc graph* is the intersection graph of a set of arcs on a circle. The set of arcs is called an *arc model* of this graph. In this paper, all arcs are closed. An arc model is *proper* if no arc in it properly contains another arc. A graph is a *proper circular-arc graph* if it has a proper arc model. In case there is a point of the circle avoided by all the arcs in an arc model, we can cut the circle and straighten all the arcs into line segments. Such a graph is an *interval graph*, i.e., the intersection graph of a set of closed intervals on the real line, and the set of intervals is an *interval model* of this graph. Proper interval graphs are defined analogously. Clearly, any (proper) interval model can be viewed as a (proper) arc model leaving some point uncovered, and hence all (proper) interval graphs are always (proper) circular-arc graphs.



(a) S_3 (tent). **(b)** $\overline{S_3}$ (net). **(c)** F_1 (long claw). **(d)** F_2. **(e)** F_3. **(f)** F_4.

Figure 2 Some small forbidden induced graphs.

Let F be a fixed graph. We say that a graph G is *F-free* if G does not contain F as an induced subgraph. For a set $\mathcal{F}$ of graphs, a graph G is *$\mathcal{F}$-free* if G is F-free for every $F \in \mathcal{F}$. If every $F \in \mathcal{F}$ is minimal, i.e., not containing any $F' \in \mathcal{F}$ as a proper induced subgraph, then $\mathcal{F}$ comprises the (minimal) *forbidden induced subgraphs* of this class. See Figure 2 for some of the forbidden induced subgraphs considered in the present paper. We use S_3^* to denote the graph obtained by adding an isolated vertex to S_3.

▶ **Theorem 4** ([19]). *A graph is a proper circular-arc graph if and only if it is free of S_3^*, C_ℓ^* with $\ell \geq 4$, as well as the complements of S_3, F_1, F_2, F_3, F_4, $C_{2\ell+2}$, and $C_{2\ell-1}^*$ with $\ell \geq 2$.*

Neither the class of circular-arc graphs nor the class of proper circular-arc graphs is closed under taking disjoint unions. Indeed, if a (proper) circular-arc graph G is not a (proper) interval graph, then in any model of G, the union of the arcs covers the whole circle. Such a graph is necessarily connected.

▶ **Proposition 5** (Folklore). *If a proper circular-arc graph G is not connected, then G is a proper interval graph.*

Proper circular-arc graphs have three infinite families of forbidden induced subgraphs, namely, $\{C_\ell^* \mid \ell \geq 4\}$, $\{\overline{C_{2\ell+2}} \mid \ell \geq 2\}$, and $\{\overline{C_{2\ell-1}^*} \mid \ell \geq 2\}$ by Theorem 4. The first of them can be ignored for connected graphs.

▶ **Lemma 6.** *Let G be a connected graph. If G does not contain the complement of C_3^* or the complement of S_3, then G is $\{C_\ell^* \mid \ell \geq 5\}$-free.*

Proof. Suppose for contradiction that there exist an induced cycle C and a vertex v in G with $|C| \geq 5$ and $V(C) \cap N(v) = \emptyset$. Since G is connected, we can find a shortest path from v to C. Let the last three vertices on this path be x, y, and z; note that z is on C and x is nonadjacent to any vertex on C. We may number the vertices on C such that $C = v_1 v_2 \cdots v_{|C|}$ and $z = v_2$. If y is adjacent to only v_2 on C, then $\{v_1, v_2, v_3, y\}$ induces a claw. If y is also adjacent to both v_1 and v_3, then $\{v_1, v_3, x, y\}$ induces a claw, and it is similar if y is adjacent to any three consecutive vertices on C. Otherwise, y is adjacent to precisely one of v_1 and v_3. Without loss of generality, assume that y is adjacent to v_3 but not v_1. Note that y is not adjacent to v_4 either, and then $\{v_1, v_2, v_3, v_4, x, y\}$ induces a copy of the complement of S_3. ◀

An arc model is *Helly* if every set of pairwise intersecting arcs has a nonempty common intersection. A circular-arc graph is *proper Helly* if it has an arc model that is both proper and Helly.

▶ **Theorem 7** ([17]). *A proper circular-arc graph is a proper Helly circular-arc graph if and only if it contains no W_4 or S_3.*

Note that S_3^* contains S_3, while all the complements of F_1, F_2, F_3, F_4, and $\{C_{2\ell}, C_{2\ell-1}^* \mid \ell \geq 4\}$ contain W_4. The following corollaries follow from Theorem 7, together with Theorem 4 and Lemma 6, respectively.

▶ **Corollary 8** ([17]). *A graph is a proper Helly circular-arc graph if and only if it contains no $\overline{C_3^*}$, $S_3, \overline{S_3}$, W_4, W_5, $\overline{C_6}$, or C_ℓ^* for $\ell \geq 4$.*

▶ **Corollary 9.** *Let G be a connected graph. If G does not contain $\overline{C_3^*}$, C_4^*, $S_3, \overline{S_3}$, W_4, W_5, or $\overline{C_6}$, then G is a proper Helly circular-arc graph.*

Recall that proper interval graphs are precisely $\{\overline{C_3^*}, S_3, \overline{S_3}, C_\ell \mid \ell \geq 4\}$-free graphs [18, 21].

▶ **Corollary 10.** *Let G be a proper Helly circular-arc graph. Then G is a proper interval graph if and only if G does not contain any holes.*

The following can be viewed as a constructive version of Corollary 9.[1]

▶ **Proposition 11** ($\star$). *Let G be a connected graph. In $O(m+n)$ time we can either detect an induced subgraph in $\{\overline{C_3^*}, C_4^*, S_3, \overline{S_3}, W_4, W_5, \overline{C_6}\}$, or build a proper and Helly arc model for G.*

A graph is a *permutation graph* if its vertices can be assigned to line segments between two parallel lines such that there is an edge between two vertices if and only if their corresponding segments intersect. The class of permutation graphs has a large number of forbidden induced subgraphs [11]. Fortunately, most of them contain an odd cycle, and thus the structures of forbidden induced subgraphs of bipartite permutation graphs are far simpler.

[1] Proofs of statements marked with $\star$ are given in the full version (attached).

▶ **Theorem 12** ([11]). *A graph is a bipartite permutation graph if and only if it is free of F_1, F_2, F_3, C_3, and C_ℓ with $\ell \geq 5$.*

We correlate proper circular-arc graphs and bipartite permutation graphs.

▶ **Theorem 13** (Folklore). *The following are equivalent on a graph G:*
i) *G is a proper circular-arc graph and $\overline{G}$ is bipartite; and*
ii) *$\overline{G}$ is a bipartite permutation graph.*

The following is complement to Proposition 5 in a sense. Note that (proper) circular-arc graphs that are co-bipartite have played crucial roles in understanding these graph classes [19].

▶ **Proposition 14** ($\star$). *Let G be a proper circular-arc graph. If $\overline{G}$ is not connected, then $\overline{G}$ is a bipartite permutation graph.*

3 Deletions to proper Helly circular-arc graphs

We first study the proper Helly circular-arc vertex deletion problem. We may assume without loss of generality that the input graph cannot be made a proper interval graph by removing k vertices. Therefore, the resulting graph after removing any k-solution is connected by Proposition 5. An FPT algorithm is immediate from Corollary 9: after destroying all the copies of $\overline{C_3^*}$, C_4^*, S_3, $\overline{S_3}$, W_4, W_5, and $\overline{C_6}$ in G by standard branching, we return all vertices except those in a maximum-order component. A similar (and simpler) approach works for the proper Helly circular-arc edge deletion problem. The focus of the following proof is thus on efficient implementations.

▶ **Theorem 15.** *The proper Helly circular-arc vertex deletion problem and the proper Helly circular-arc edge deletion problem can be solved in time $O(6^k \cdot (m+n))$ and $O(10^k \cdot (m+n))$, respectively.*

Proof. Let (G, k) be an instance of proper Helly circular-arc vertex deletion. Our algorithm proceeds as follows. We start by calling the algorithm of Cao [3] to check whether there is a set V_- of at most k vertices such that $G - V_-$ is a proper interval graph. If the set is found, then we return "yes." In the rest, we look for a solution V_- such that $G - V_-$ is *not* a proper interval graph. By Proposition 5, (note that a proper Helly circular-arc graph is a proper circular-arc graph,) $G - V_-$ is connected.

For the general case, the algorithm solves the problem by making recursive calls to itself; we return "no" directly for a recursive call in which $k < 0$. For each component C of G, we call the algorithm of Proposition 11. If a subgraph induced by $F \subseteq V(G)$ is found, then the algorithm calls itself $|F|$ times, each with a new instance $(G - v, k - 1)$ for some $v \in F$. Since we need to delete at least one vertex from F, the original instance (G, k) is a yes-instance if and only if at least one of the instances $(G - v, k - 1)$ is a yes-instance. Now that G is free of $\overline{C_3^*}$, C_4^*, S_3, $\overline{S_3}$, W_4, W_5, and $\overline{C_6}$, every component of G is a proper Helly circular-arc subgraph (Corollary 9). We find a component C of G that has the maximum order. We return "yes" if $|V(C)| \geq n - k$, when $V(G) \setminus V(C)$ is a solution, or "no" otherwise. Since each of $\overline{C_3^*}, C_4^*, S_3, \overline{S_3}, W_4, W_5$, and $\overline{C_6}$ has at most 6 vertices, at most 6 recursive calls are made, all with parameter value $k - 1$. By Proposition 11, each recursive call can be made in $O(m + n)$ time. Therefore, the total running time is $O(6^k \cdot (m + n))$.

The algorithm for the edge deletion problem is even simpler. Again, we start by calling the algorithm for proper interval edge deletion problem [3], which takes time $O(4^k \cdot (m + n))$. We proceed only when the answer is "no." In the recursive calls for the general case, we always return “no” whenever k is negative or G becomes disconnected; note that a disconnected graph cannot be made connected by edge deletions. We call the algorithm of Proposition 11, and return “yes” if G is already a proper Helly circular-arc graph. Otherwise, an induced subgraph F is found. The algorithm calls itself $|E(F)|$ times, each with a new instance $(G - uv, k - 1)$ for some edge uv in $G[F]$. By Proposition 11, each recursive call can be made in $O(m + n)$ time. Therefore, the total running time is $O(10^k \cdot (m + n))$, where 10 is the number of edges in a W_5. ◀

It is straightforward to adapt an approximation algorithm for the proper Helly circular-arc vertex deletion problem from the parameterized algorithm in Theorem 15. From Theorem 15 we can easily derive an FPT algorithm for the combined deletion problem toward proper Helly circular-arc graphs, which allows k_1 vertex deletions and k_2 edge deletions. We can fill in the gap between the constants in Theorems 15 and 3. Only S_3, W_5, and $\overline{C_6}$ have more that eight edges. For an S_3, either we delete one edge between two degree-four vertices, or we have to delete both edges incident to a degree-2 vertex. For the other two cases, the ideas are similar. The details are deferred to the full version.

4 Proper Helly circular-arc completion

Compared to the deletion problems, the completion problem toward proper Helly circular-arc graphs is significantly more difficult. For all the deletion problems, we can always assume that the graph is connected, and then by Corollary 9, we are only concerned with small forbidden induced subgraphs. Since adding edges may make a graph connected, we cannot assume connected input graphs for the completion problem.

Every hole in a proper Helly circular-arc graph is a dominating set of the graph, and we can be more specific on the intersection between a hole and the neighborhood of any vertex.

▶ **Proposition 16** ($\star$). *Let H be a hole in a proper Helly circular-arc graph. Every vertex in this graph has at least two neighbors on H.*

It is well known that the maximal cliques of an interval graph can be arranged as a path. Gavril [12] showed that the maximal cliques of a Helly circular-arc graph can be arranged as a circle. This implies that a Helly circular-arc graph has a linear number of maximal cliques.

▶ **Theorem 17** ([12]). *A graph G is a Helly circular-arc graph if and only if its maximal cliques can be arranged as a circle so that for every vertex v in G, the maximal cliques containing v are consecutive.*

We use a *clique cycle* to denote the circular arrangement of maximal cliques specified in Theorem 17, and a *clique path* is defined analogously. In a clique path, we call the first and the last cliques *end cliques*. Note that a clique path can always be viewed as a clique cycle, while if two consecutive cliques of a clique cycle are disjoint, then it can be viewed as a clique path.

Proper interval graphs are precisely claw-free interval graphs, which can be restated as a graph is a proper interval graph if and only if it is claw-free and has a clique path. One may thus expect that a graph is a proper Helly circular-arc graph if and only if it is claw-free and has a clique cycle. As we have mentioned, however, S_3 is a Helly circular-arc graph and hence has a clique cycle, but it is not a proper Helly circular-arc graph even though it is claw-free. The following statement can be directly observed from forbidden induced subgraphs of the class of proper Helly circular-arc graphs and of the class of normal Helly circular-arc graphs; see also Lin et al. [17, Theorem 9].[2]

[2] An arc model is known to be normal and Helly if no set of three or fewer arcs covers the circle [12, 4].

▶ **Lemma 18** ([17]). *A graph is a proper Helly circular-arc graph if and only if it is claw-free and it has an arc model in which no set of three or fewer arcs covers the circle.*

If a proper Helly circular-arc graph G is not an interval graph, then it has a hole (Corollary 10). The structure of every local part of G is very like a proper interval graph when the hole is long enough (the length at least six). With the removal of two maximal cliques with no edge in between from G, the hole is separated into two sub-paths. Since every remaining vertex is adjacent to one of the two sub-paths, the remaining graph has precisely two components.

▶ **Lemma 19** ($\star$). *Let G be a proper Helly circular-arc graph that is not an interval graph. Let A_1 and A_2 be two maximal cliques of G with no edge between them, and let B_1 and B_2 be the vertex sets of the two components of $G - (A_1 \cup A_2)$. Let G_1 be any proper interval graph on $B_1 \cup A_1 \cup A_2$ in which A_1 and A_2 are the end cliques, and $N_{G_1}(A_i) \cap B_1 = N_G(A_i) \cap B_1$ for $i = 1, 2$. Replacing $G[B_1 \cup A_1 \cup A_2]$ with G_1 gives another proper Helly circular-arc graph.*

For the completion problem, we may again assume that the input graph G is free of $\overline{C_3^*}$, C_4^*, $S_3, \overline{S_3}, W_4, W_5$, and $\overline{C_6}$. We are done if G is already a proper Helly circular-arc graph. In particular, this is the case when G is a proper interval graph or when G is connected (Corollary 9). Thus, we may assume that G is not connected and it is not a proper interval graph. There must be a hole in G (Corollary 10), and we add either a chord of this hole, or an edge between this hole and every vertex in other components. If there is a hole of length of no more than $16k + 16$, then there are only $O(k^2)$ such choices, and we can branch on adding one of them. In the rest, every hole is longer than $16k + 16$ (hence at least half of the vertices on H have no neighbors incident to a k-solution). Let H be such a hole, and let G_0 be the component of G that contains H. Note that G_0 is a proper Helly circular-arc graph (Corollary 9). After adding k or fewer edges, if the resulting graph is a proper Helly circular-arc graph, then there must be a hole of length greater than k in the subgraph induced by $V(H)$. Thus, for every vertex x in $V(G) \setminus V(G_0)$, at least two edges must be added between x and H (Proposition 16). We can return "no" if $|V(G_0)| < n - \frac{k}{2}$. Other components have fewer than k vertices while any hole is longer than $16k + 16$, and thus they are already proper interval subgraphs. They are accordingly called *small components.*

We say that a vertex x is *touched* by a solution E_+ if x is an endpoint of an edge in E_+, and a set X of vertices is *touched* if at least one vertex in X is touched. All vertices in $V(G) \setminus V(G_0)$ are touched, and we are more concerned with touched vertices in G_0.

▶ **Proposition 20** ($\star$). *Let E_+ be a solution to G. If a maximal clique K of G is untouched by E_+, then K is a maximal clique of $G + E_+$.*

Recall that a clique cycle of a proper Helly circular-arc graph can be found in linear time [4]. We may fix a clique cycle $\langle K_1, K_2, \ldots, K_\ell \rangle$ of G_0, denoted by $\mathcal{K}$, and assume that H and $\mathcal{K}$ are numbered such that no neighbor of v_1 or $v_{|H|}$ is touched, and $\{v_1, v_{|H|}\} \subseteq K_1$, which is untouched. Note that G_0 has at least $|H|$ maximal cliques. Since H is longer than $16k + 16$, few of them are touched by a k-solution E_+. By Lemma 19 and Proposition 20, these untouched maximal cliques serve as "isolators" of the modifications.

We can guess another untouched maximal clique K_p of G_0 that is disjoint from and nonadjacent to K_1. By Proposition 20, K_1 and K_p are both maximal cliques of a proper Helly circular-arc graph $G + E_+$. Since K_p is disjoint from and nonadjacent to K_1, it follows that $G + E_+ - (K_1 \cup K_p)$ is not connected. Then H is broken into two paths in $G - (K_1 \cup K_p)$. Recall that every vertex in $V(G) \setminus V(G_0)$ needs to be connected to a vertex on H. When the graph $G + E_+$ is not a proper interval graph, every small component is connected to exactly one of the two sub-paths of H. But then the resulting graph remains connected after all vertices in $K_1 \cup K_p$ removed, a contradiction. We can guess in 2^k time to which side each small component is attached. Then we need to add edges to make two proper interval graphs. However, we cannot call the proper interval completion problem to solve this task. For example, the subgraph induced by $\bigcup_{i=1}^{p} K_i$ together with the small components is already a proper interval graph. The trouble is how to make them connected while keeping K_1 and K_p the end cliques of the final clique path. The same holds for the other part of the problem.

For each i with $1 < i < |H|$, let the maximal cliques containing v_i be $K_{\mathrm{from}(i)}, \ldots, K_{\mathrm{to}(i)}$; i.e., from($i$) and to($i$) are the smallest and, respectively, the largest indices. We define $K_{\mathrm{to}(1)}$ and $K_{\mathrm{from}(|H|)}$ analogously. Let r be the number of small components, denoted as $C_1, \ldots, C_r$. For each pair p, q of indices with $1 \leq p < q \leq |H|$, and each subset S of $[1..r]$, we check whether it is possible to add at most k edges to make $G[\bigcup_{i=\mathrm{to}(p)}^{\mathrm{from}(q)} K_i \cup \bigcup_{j \in S} C_j]$ a proper interval graph, under the condition that $K_{\mathrm{to}(p)}$ and $K_{\mathrm{from}(q)}$ are the end cliques and remain untouched. Let $\beta(S, p, q)$ denote the minimum cost if it is at most k, or ∞. We define $\beta(S, p, q)$ to be ∞ when $K_{\mathrm{to}(p)}$ and $K_{\mathrm{from}(q)}$ are not disjoint. Then $\beta([1..r], 1, |H|)$ is the value we need, which can be calculated as follows.

▶ **Proposition 21.** *The value of $\beta([1..r], 1, |H|)$ can be computed in $k^{O(k)}(n + m)$ time.*

Proof. First, for a and b with $a < b \leq a + 8k$ and $S \subseteq [1..r]$, we calculate $\beta(S, a, b)$ as follows. From each component C_j with $j \in S$, we take a vertex x, guess a vertex v_i with $a < i < b$, and add the edge xv_i. After that, the subgraph induced by $\bigcup_{i=\mathrm{to}(a)}^{\mathrm{from}(b)} K_i \cup \bigcup_{j \in S} C_j$ is connected. It will remain connected after adding edges. We then branch on adding edges to destroy induced subgraphs in $\{\overline{C_3^*}, S_3, \overline{S_3}\}$ and holes in the subgraph induced by $\bigcup_{i=\mathrm{to}(a)}^{\mathrm{from}(b)} K_i \cup \bigcup_{j \in S} C_j$, without adding any edges incident to $K_{\mathrm{to}(a)}$ or $K_{\mathrm{from}(b)}$.

Since G_0 is a proper Helly circular-arc graph, a vertex is adjacent to at most four vertices on H (there is a claw otherwise). A solution is incident to at most $2k$ vertices, and thus at most $8k$ vertices on H have touched neighbors. If v_{b-1} has a touched neighbor, then for some i with $2 \leq i \leq 8k + 2$, the vertex v_{b-i} has no touched neighbor. For $b - a > 8k$, by Lemma 19, we have

$$\beta(S, a, b) = \min_{\substack{1 \leq i \leq 8k+1 \\ S' \subseteq S}} \left(\beta(S \setminus S', a, b - i) + \beta(S', b - i, b)\right). \tag{1}$$

We can then use dynamic programming to calculate $\beta([1..r], 1, |H|)$ with (1). ◀

We are now ready to summarize the algorithm in Figure 3. The analysis is left to the full version.

5 Proper circular-arc vertex deletion

Since we will use properties of both the graph G and its complement, we beg the reader's attentiveness in reading this section. There are algorithms for the vertex deletion problem toward proper interval graphs and toward bipartite permutation graphs. We are henceforth focused on graphs that are both connected and co-connected. As usual, we can get rid of small forbidden induced subgraphs easily.

▶ **Definition 22.** *A graph is* reduced *if it is both connected and co-connected, and it contains no C_3^*, C_5^*, $\overline{C_4^*}$, C_6, S_3, $\overline{S_3^*}$, F_1, F_2, F_3, or F_4.*

1. **if** there exists an induced subgraph X in $\{\overline{C_3^*}, C_4^*, S_3, \overline{S_3}, W_4, W_5, \overline{C_6}\}$ **then**
   branch on adding missing edges of X; \\ *returns "no" if k becomes negative.*
2. **if** G is a proper Helly circular-arc graph **then return** "yes";
3. find a hole H of G and let G_0 be the component of G that contains H;
4. **if** $|H| \leq 16k + 16$ **then**
   branch on adding chords of H or edges between H and other components;
5. **if** $|V(G_0)| < n - \frac{k}{2}$ **then return** "no";
6. **if** $\beta([1..r], 1, |H|) \leq k$ **then return** "yes";
   else return "no."

Figure 3 The outline of the algorithm for the proper Helly circular-arc completion problem.

Similar to Proposition 11, one can make an algorithm for finding one of the subgraphs listed above when the input graph is not reduced. We omit details since it does not improve our main algorithm. The next lemma is complement and similar to Lemma 6.

▶ **Lemma 23.** *A reduced graph is $\{C_\ell^* \mid \ell \geq 7\}$-free.*

Proof. Let R be a reduced graph. Suppose for contradiction that there exist a hole H of length at least seven and a vertex that is nonadjacent to any vertex on H. Since R is connected, we can find a vertex x adjacent to H, and another vertex y that is adjacent to x but not to H. Let $H = v_1 v_2 \cdots v_{|H|}$. We argue first that x cannot be adjacent to two consecutive vertices on H. Suppose for contradiction that x is adjacent to both v_1 and v_2. Then v_1, v_2, and x form a triangle. Since R is free of C_3^*, it follows that x is adjacent to both v_4 and v_5. But then v_1, v_2, v_4, v_5, and x induce a $\overline{C_4^*}$ (note that v_5 and v_1 are nonadjacent because $\ell \geq 7$).

Assume without loss of generality that x is adjacent to v_3. Note that x is adjacent to neither v_2 nor v_4. If x is adjacent to v_5 as well, then x is nonadjacent to v_6, and $\{v_2, \ldots, v_6, x, y\}$ induces an F_2. By symmetry, x cannot be adjacent to v_1 either. But then $\{v_1, \ldots, v_5, x, y\}$ induces an F_1. ◀

By definition, a reduced graph is C_3^*- and C_5^*-free.

▶ **Corollary 24.** *A reduced graph is $\{C_{2\ell+1}^* \mid \ell \geq 1\}$-free.*

By Theorem 4, the definition of reduced graphs, and Lemma 6, a reduced graph is the complement of a proper circular-arc graph if and only if it does not contain any even hole of length at least eight. We will therefore be focused on long even holes. The main structural statement characterizes reduced graphs that contain long even holes.

▶ **Lemma 25.** *If a reduced graph contains an even hole of length at least eight, then it is bipartite.*

Proof (sketch). Let $v_1 v_2 \cdots v_\ell$ be an even hole with $\ell \geq 8$ of a reduced graph R, and denote it by B. We prove the lemma with a sequence of claims.

1. No vertex on B participates in any triangle.
2. If some odd hole of R intersects B, then there exists an odd hole of R whose intersection with B is a nonempty sub-path of B.
3. If $V(C) \cap V(B)$ is consecutive for an odd induced cycle C, then $|V(C) \cap V(B)| \leq 4$.
4. No odd induced cycle can intersect B (no vertex on B is contained in any odd induced cycle).

Finally, we show that R does not contain any odd cycle at all. Let C be an odd induced cycle that is disjoint from B. First assume $C = x_1x_2x_3$. Since R is C_3^*-free, every vertex on B is adjacent to at least one vertex on C. Assume without loss of generality that x_1 has the largest number of neighbors on B, and let their indices be i_1, i_2, $\ldots$, i_p, sorted increasingly. Note that all of them have the same parity by the first claim. Since R is C_6-free, $i_{j+1} - i_j$ is either two or at least six for all $j = 1, \ldots, p-1$. Since $p \geq \lceil \frac{\ell}{3} \rceil$, there must be three consecutive ones with differences two; assume without loss of generality, that they are v_1, v_3, and v_5. If $\ell = 8$, then $x_1v_5v_6 \cdots v_8v_1$ has length six, and hence x_1 must be adjacent to v_7 as well; otherwise, x_1 has another neighbor on B because $p \geq \lceil \frac{\ell}{3} \rceil \geq 4$. This neighbor forms an F_3 with $\{x_1, v_1, v_2, \ldots, v_5\}$. Now that $|C| \geq 5$; let it be $x_1x_2 \cdots x_{|C|}$. We take $v_i \in N(x_1) \cap V(B)$ and $v_j \in N(x_3) \cap V(B)$. The sub-path $v_iv_{i+1} \cdots v_j$ forms an odd cycle with either $x_1x_2x_3$ (when $j - i$ is odd) or $x_3x_4 \cdots x_{|C|}x_1$ (when $j - i$ is even). From this odd cycle we can retrieve an induced odd cycle, which has to intersect both B and C (because both B and C themselves are induced cycles). This contradicts the fourth claim, and concludes the proof of this lemma. ◀

The following is immediate from Theorem 4, Lemma 6, and Lemma 25.

▶ **Corollary 26.** *If a reduced graph R is not bipartite, then R is the complement of a proper circular-arc graph.*

We are now ready to present the algorithm for the proper circular-arc vertex deletion problem in Figure 4. Let (G, k) be an instance to the problem, and we may assume without loss of generality that G does not contain any small forbidden induced subgraphs on at most seven vertices. If there is a set V_- of k vertices such that $G - V_-$ is a proper interval graph or $\overline{G} - V_-$ is a bipartite permutation graph, then we are done. Hence, we will look for a solution V_- such that $G - V_-$ is both connected and co-connected (Propositions 5 and 14).

For this purpose we may assume that G itself is connected and co-connected: if G is not connected, we can work on the components of G one by one, and it is similar for $\overline{G}$. Thus, $\overline{G}$ is a reduced graph. If $\overline{G}$ is not bipartite, then G is already a proper circular-arc graph (Corollary 26). Otherwise, $\overline{G}$ is bipartite, of which any induced subgraph of it is bipartite. In other words, if there exists a solution V_-, then $\overline{G} - V_-$ is a bipartite permutation graph, and this has been handled already.

1. **if** (G, k) is a yes-instance of proper interval vertex deletion **then**
   return “yes”;
2. **if** $(\overline{G}, k)$ is a yes-instance of bipartite permutation vertex deletion **then**
   return “yes”;
   \\ *We're looking for a solution V_- with both $G - V_-$ and $\overline{G} - V_-$ connected.*
3. **branch** on deleting vertices of small forbidden induced subgraphs;
4. $\mathcal{C} \leftarrow$ maximal vertex sets that are connected and co-connected;
5. **if** $G[C]$ is co-bipartite for all $C \in \mathcal{C}$ **then return** “no”;
6. $C \leftarrow$ a maximum set from $\mathcal{C}$ with $G[C]$ not co-bipartite;
7. **if** $|V(G) \setminus V(C)| \leq k$ **then return** “yes”;
   else return “no.”

Figure 4 The outline of the algorithm for proper circular-arc vertex deletion.

Again, it is quite straightforward to turn this algorithm into an approximation algorithm, and the proofs for Theorems 1 and 2 are left to the full version.

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