A Weyl Criterion for Finite-State Dimension and Applications

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──────── **Abstract** ────────

Finite-state dimension, introduced early in this century as a finite-state version of classical Hausdorff dimension, is a quantitative measure of the lower asymptotic *density of information* in an infinite sequence over a finite alphabet, as perceived by finite automata. Finite-state dimension is a robust concept that now has equivalent formulations in terms of finite-state gambling, lossless finite-state data compression, finite-state prediction, entropy rates, and automatic Kolmogorov complexity. The 1972 Schnorr-Stimm dichotomy theorem gave the first automata-theoretic characterization of normal sequences, which had been studied in analytic number theory since Borel defined them in 1909. This theorem implies, in present-day terminology, that a sequence (or a real number having this sequence as its base-b expansion) is normal if and only if it has finite-state dimension 1. One of the most powerful classical tools for investigating normal numbers is the 1916 Weyl's criterion, which characterizes normality in terms of exponential sums. Such sums are well studied objects with many connections to other aspects of analytic number theory, and this has made use of Weyl's criterion especially fruitful. This raises the question whether Weyl's criterion can be generalized from finite-state dimension 1 to arbitrary finite-state dimensions, thereby making it a quantitative tool for studying data compression, prediction, etc. i.e., *Can we characterize all compression ratios using exponential sums?*.

This paper does exactly this. We extend Weyl's criterion from a characterization of sequences with finite-state dimension 1 to a criterion that characterizes every finite-state dimension. This turns out *not* to be a routine generalization of the original Weyl criterion. Even though exponential sums may diverge for non-normal numbers, finite-state dimension can be characterized in terms of the *dimensions* of the *subsequence limits* of the exponential sums. In case the exponential sums are convergent, they converge to the Fourier coefficients of a probability measure whose *dimension* is precisely the finite-state dimension of the sequence.

This new and surprising connection helps us bring Fourier analytic techniques to bear in proofs in finite-state dimension, yielding a new perspective. We demonstrate the utility of our criterion by substantially improving known results about preservation of finite-state dimension under arithmetic. We strictly generalize the results by Aistleitner and Doty, Lutz and Nandakumar for finite-state dimensions under arithmetic operations. We use the method of exponential sums and our Weyl criterion to obtain the following new result: *If y is a number having finite-state strong dimension 0, then* $\dim_{FS}(x + qy) = \dim_{FS}(x)$ *and* $\mathrm{Dim}_{FS}(x + qy) = \mathrm{Dim}_{FS}(x)$ *for any* $x \in \mathbb{R}$ *and* $q \in \mathbb{Q}$. This generalization uses recent estimates obtained in the work of Hochman [17] regarding the entropy of convolutions of probability measures.

1   Introduction

Finite-state compressibility [34], or equivalently, finite-state dimension [10, 2, 5] is a quantification of the information rate in data as measured by finite-state automata. This formulation, initially motivated by practical constraints, has proved to be rich and mathematically robust, having several equivalent characterizations. In particular, the finite state-dimension of a sequence is equal to the compression ratio of the sequence using information lossless finite-state compressors ([10, 2]). Finite-state dimension has unexpected connections to areas such as number theory, information theory, and convex analysis [20, 12]. Schnorr and Stimm [28] establish a particularly significant connection by showing that a number is Borel normal in base b (see for example, [24, 6, 8]) if and only if its base b expansion has finite-state compressibility equal to 1, *i.e.*, is incompressible (see also: [3, 5, 13, 15]). Equivalently, a number $x \in [0, 1)$ is normal if and only if $\dim_{FS}(x)$, the finite-state dimension of x is equal to 1. A celebrated characterization of Borel normality in terms of exponential sums, provided by Weyl's criterion [32], has proved to be remarkably effective in the study of normality. Weyl's criterion on uniformly distributed sequences modulo 1 yields a characterization that a real number r is normal to base b if and only if for every integer k,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k(b^j r)} = 0. \tag{1}$$

This tool was used by Wall [31] in his pioneering thesis to show that normality is preserved under certain operations like selection of subsequences along arithmetic progressions, and multiplication with non-zero rationals. Weyl's criterion facilitates the application of tools from Fourier analysis in the study of Borel normality. Weyl's criterion is used in several important constructions of normal numbers including those given by Cassels [7], Erdös and Davenport [11] etc. The criterion was also instrumental in obtaining the construction of absolutely normal numbers given by Schmidt in [27].

The finite-state compression ratio/dimension of an arbitrary sequence is a quantity in [0,1]. The classical Weyl's criterion provides a characterization of numbers having finite-state dimension equal to 1 in terms of exponential sums. This leads us to the natural question - *Can we characterize arbitrary compression ratios using exponential sums?*. This question turns out to be highly non-trivial. It is not easy to generalize Weyl's criterion to study arbitrary finite-state compression ratios/dimension. The major conceptual hurdle arises from the fact that for non-normal numbers, the Weyl sum averages in (1) need not converge. The Weyl averages need not converge even when the finite-state dimension and the strong dimension of a sequence are equal.

We demonstrate this by explicitly constructing such a sequence in Lemma 19. Using a new construction method involving the controlled concatenation of two special sequences, we demonstrate the existence of a sequence $x \in \Sigma^\infty$ with non-convergent Weyl averages, while having finite-state dimension and strong dimension both equal to $\frac{1}{2}$. The proof that this constructed sequence satisfies the required properties uses new techniques, which might be

of independent interest. Due to the existence of such sequences, it is unclear how to *extract* the finite-state dimension of a sequence from non-convergent Weyl averages. Indeed, it was unclear whether any generalization of the Weyl's criterion to arbitrary finite-state dimensions even exists.

Our paper rescues this approach and gives such a characterization of arbitrary finite-state compressibility/dimension by introducing one important viewpoint, that turns out to be the major theoretical insight. Even when the exponential sums diverge, the theory of *weak convergence* of probability measures ([4]) enables us to consider the collection of all probability measures having Fourier coefficients equal to the subsequence limits of the Weyl averages. The *dimensions* of the measures in the set of subsequence weak limit measures gives a generalization of Weyl's criterion. For any x, let $\dim_{FS}(x)$ and $\mathrm{Dim}_{FS}(x)$ denote the finite-state dimension and finite-state strong dimension [2] of x respectively. We now informally state our Weyl's criterion for finite-state dimension.

▶ **Theorem** (Informal statement of Theorem 22). *Let $x \in [0, 1)$. If for any subsequence $\langle n_m \rangle_{m=0}^{\infty}$ of natural numbers, there exist complex numbers c_k such that for every $k \in \mathbb{Z}$, $\lim_{m\to\infty} \frac{1}{n_m} \sum_{j=0}^{n_m-1} e^{2\pi ik(b^j x)} = c_k$, then, there exists a probability measure μ on $[0, 1)$ such that for every k, $c_k = \int e^{2\pi iky} d\mu$. Let $\mathcal{W}_x$ be the collection of all such probability measures μ on $[0, 1)$ that can be obtained as the subsequence limits of Weyl averages. Then, $\dim_{FS}(x) = \inf_{\mu \in \mathcal{W}_x} H^-(\mu)$ and $\mathrm{Dim}_{FS}(x) = \sup_{\mu \in \mathcal{W}_x} H^+(\mu)$.*

The *correct* notion of *dimensions* of the subsequence weak limit measures in $\mathcal{W}_x$ which yields the finite-state dimensions of x turns out to be H^- and H^+, the *lower* and *upper average entropies* of μ as defined in [2] [1]. Therefore, this new characterization enables us to *extract* the finite-state compressibility/dimension by studying the behavior of the Weyl sum averages, thereby extending Weyl's criterion for normality to arbitrary finite-state dimensions.

An interesting special case of our criterion is when the exponential averages of a sequence are convergent. In this case, the averages $\langle c_k \rangle_{k \in \mathbb{Z}}$ are precisely the *Fourier coefficients* of a *unique* limiting measure, whose *dimension* is precisely the finite-state dimension of the sequence. This relates two different notions of dimension to each other. We give the informal statement of our criterion for this special case.

▶ **Theorem** (Informal statement of Theorem 23). *Let $x \in [0, 1)$. If there exist complex numbers c_k for $k \in \mathbb{Z}$ such that $\frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi ik(b^j x)} \to c_k$ as $n \to \infty$, then, there exists a unique measure μ on $[0, 1)$ such that for every k, $c_k = \int e^{2\pi iky} d\mu$. Furthermore, $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x) = H^-(\mu) = H^+(\mu)$.*

Our results also show that in case there is a unique weak limit measure, the exponential sums (1) converge for every $k \in \mathbb{Z}$. These give the first known relations between Fourier coefficients and finite-state compressibility/dimension. The proof of Weyl's criterion for finite-state dimension is not a routine generalization of the available proofs of Weyl's criterion for normality (see [32, 14], [30]) and requires several facts from the theory of weak convergence of probability measures and new relationships involving the exponential sums, the *dimensions* of weak limit measures and the finite-state dimension of the given sequence. We overcome certain additional technical difficulties in working with two different topologies - the topology on the torus $\mathbb{T}$ where Fourier coefficients uniquely determine a measure, and another, Cantor space, which is required for studying combinatorial properties of sequences, like normality.

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[1] These are analogues of the well-known *Rényi upper and lower dimensions* of measures as defined in [25]. See the remark following Definition 5.

1.1 Applications of our criterion

We illustrate how this framework can be applied in sections 6. These results justify that this framework pioneers a new, powerful, approach to data compression. It is not very surprising that when the Weyl averages converge, our criterion has applications. Importantly, even in situations where the Weyl averages do not converge, it is possible to derive non-trivial consequences. We apply our techniques to substantially improve known results about the preservation of finite-state dimension under arithmetic and combinatorial operations.

Doty, Lutz and Nandakumar [12] show that if x is any real and q is any non-zero rational, then the finite-state dimensions and strong dimensions of x, qx and $x + q$ are equal. When x is normal, a generalization is obtained by Aistleitner [1], which can be described as follows. Let y be any real such that the asymptotic density of zeroes in its expansion is one. Then, for any rational q, we have $x + qy$ is normal. We generalize these results by allowing both the following conditions simultaneously,

1. x is allowed to be any real, obtaining a result for all finite-state dimensions rather than only for normals as in Aistleitner [1] and

2. y is allowed to be any real with finite-state strong dimension 0 which satisfies a natural independence condition. This generalizes both the restrictions in Doty, Lutz and Nandakumar [12] and Aistleitner [1],

and show that for any rational $q \in \mathbb{Q}$, $\dim_{FS}(x + qy) = \dim_{FS}(x)$ and $\mathrm{Dim}_{FS}(x + qy) = \mathrm{Dim}_{FS}(x)$ if x and y are *independent* (see Definition 30) and $\mathrm{Dim}_{FS}(y) = 0$.

Using our Weyl criterion along with the results in Hochman [17], we obtain the following. Let x and y be real numbers in $\mathbb{T}$ such that x and y are independent. Then for any $d, e \in \mathbb{Z}$, $\dim_{FS}(dx + ey) \geq \max\{\dim_{FS}(dx), \dim_{FS}(ey)\}$ and $\dim_{FS}(dx + ey) \leq \dim_{FS}(dx) + \mathrm{Dim}_{FS}(ey)$. Similarly, $\mathrm{Dim}_{FS}(dx + ey) \geq \max\{\mathrm{Dim}_{FS}(dx), \mathrm{Dim}_{FS}(ey)\}$ and $\mathrm{Dim}_{FS}(dx + ey) \leq \mathrm{Dim}_{FS}(dx) + \mathrm{Dim}_{FS}(ey)$. Our main results are consequences of these inequalities.

There are several known techniques for explicit constructions of normal numbers (see [20, 6]), but constructions of those with finite-state dimension $s \in [0, 1)$ follow two techniques: first, to start with a normal sequence, and to *dilute* it with an appropriate fraction of simple patterns, as we did in Section 4, and second, to start with a coin with bias p such that $-p \log_2 p - (1 - p) \log_2(1 - p) = s$, and consider any typical sequence drawn from this distribution (see also [23]). We note that our Weyl criterion along with techniques from Mance and Madritsch [22] yields new methods for the explicit construction of numbers having a specified finite-state dimension.

Lossless data compression is practically significant, and theoretically sophisticated. We show how one of the major tools of modern mathematics, Fourier analysis, can be brought to bear to study compressibility of individual data sequences. We hope that our criterion will facilitate the application of more powerful Fourier analytic tools in future works involving finite-state compression/dimension.

After the preliminary sections, section 3 gives Weyl's criterion on Cantor space using weak convergence of measures. Next, we show the necessity and the sufficiency of passing to *subsequences* of sequences of measures in order to generalize Weyl's criterion for finite-state dimension. In section 6 we show the applications of our Weyl criterion to yield new, general results regarding the preservation of finite-state dimension under arithmetic and combinatorial operations.

2   Preliminaries

For any natural number $b > 1$, Σ_b denotes the alphabet $\{0, 1, 2, \ldots b - 1\}$. Throughout this paper, we work with base 2, but our results generalize to all bases. We use Σ to denote the binary alphabet Σ_2. We denote the set of finite binary strings by Σ^* and the set of infinite sequences by Σ^∞. For any $w \in \Sigma^*$, let C_w be the set of infinite sequences with w as a prefix, called a *cylinder*. For any sequence $x = x_0 x_1 x_2 \ldots$ in Σ^∞, we denote the substring $x_i x_{i+1} \ldots x_j$ of x, by x_i^j. The Borel σ-algebra generated by the set of all cylinder sets is denoted by $\mathcal{B}(\Sigma^\infty)$. Let $\mathbb{T}$ denote the one-dimensional torus or unit circle. i.e, $\mathbb{T}$ is the unit interval $[0, 1)$ with the metric $d(r, s) = \min\{|r - s|, 1 - |r - s|\}$. $\mathbb{T}$ is a compact metric space. The Borel σ-algebra generated by all open sets in $\mathbb{T}$ is denoted by $\mathcal{B}(\mathbb{T})$. For any base b, let v_b be the *evaluation map* which maps any $x \in \Sigma^\infty$ to its *value* in $\mathbb{T}$ which is $\sum_{i=0}^{\infty} \frac{x_i}{b^{i+1}}$ mod 1. We use the simplified notation v to denote the base 2 evaluation map v_2. Let T be the left shift transformation $T(x_0 x_1 x_2 \ldots) = x_1 x_2 x_3 \ldots$ on Σ^∞. For any base b and $w \in \Sigma_b^*$, let I_w^b denote the interval $\left[v_b(w0^\infty), v_b(w0^\infty) + b^{-|w|} \right)$ in $\mathbb{T}$. We use the simplified notation I_w to refer to I_w^2. Let $\mathbb{D}$ be the set of all dyadic rationals in $\mathbb{T}$. It is easy to see that $v : \Sigma^\infty \to \mathbb{T}$ has a well-defined inverse, denoted v^{-1}, over $\mathbb{T} \setminus \mathbb{D}$. For any measure μ on $\mathbb{T}$ (or Σ^∞), we refer to the collection of complex numbers $\int e^{2\pi i k y} d\mu$ where k ranges over $\mathbb{Z}$ as the *Fourier coefficients of measure* μ. For measures over Σ^∞, the function $e^{2\pi i k y}$ inside the integral is replaced with $e^{2\pi i k v(y)}$. For every measure μ on $\mathbb{T}$, we define the corresponding lifted measure on Σ^∞ as follows.

▶ **Definition 1** (Lift $\hat{\mu}$ of a measure μ on $\mathbb{T}$). *If μ is a measure on $\mathbb{T}$, then we define the* lift *$\hat{\mu}$ of μ to be the unique measure on Σ^∞ satisfying $\hat{\mu}(C_w) = \mu(I_w)$ for every string $w \in \Sigma^*$.* [2]

▶ **Definition 2.** *Let $x \in \Sigma^*$ have length n. We define the* sliding count probability *of $w \in \Sigma^*$ in x denoted $P(x, w)$, and the* disjoint block probability *of w in x, denoted $P^d(x, w)$, as follows.*

$$P(x, w) = \frac{|\{i \in [0, n - |w|] : x_i^{i+|w|-1} = w\}|}{n - |w| + 1} \ \text{ and } \ P^d(x, w) = \frac{|\{i \in [0, \frac{n}{|w|}) : x_{|w|i}^{|w|(i+1)-1} = w\}|}{n/|w|}$$

Now, we define normal sequences in Σ^∞ and normal numbers on $\mathbb{T}$.

▶ **Definition 3.** *A sequence $x \in \Sigma^\infty$ is* normal *if for every $w \in \Sigma^*$, $\lim_{n\to\infty} P(x_0^{n-1}, w) = 2^{-|w|}$. $r \in \mathbb{T}$ is* normal *if and only if $r \notin \mathbb{D}$ and $v^{-1}(r)$ is a normal sequence in Σ^∞.*

Equivalently, we can formulate normality using disjoint probabilities [20]. The following is the block entropy characterization of finite-state dimension from [5], which we use instead of the original formulation using s-gales (see [10],[2],[21]).

▶ **Definition 4** ([10, 5]). *For a given block length l, we define the* sliding block entropy *over x_0^{n-1} as $H_l(x_0^{n-1}) = -\frac{1}{l} \sum_{w \in \Sigma^l} P(x_0^{n-1}, w) \log(P(x_0^{n-1}, w))$. The* finite-state dimension *of $x \in \Sigma^\infty$, denoted $\dim_{FS}(x)$, and* finite-state strong dimension *of x, denoted $\mathrm{Dim}_{FS}(x)$, are defined as follows. $\dim_{FS}(x) = \inf_l \liminf_{n\to\infty} H_l(x_0^{n-1})$ and $\mathrm{Dim}_{FS}(x) = \inf_l \limsup_{n\to\infty} H_l(x_0^{n-1})$* [3]

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[2] The uniqueness of $\hat{\mu}$ follows from routine measure theoretic arguments
[3] The fact that $\dim_{FS}(x)$ and $\mathrm{Dim}_{FS}(x)$ are equivalent to the lower and upper finite-state compressibilities of x using lossless finite-state compressors, follows immediately from the results in [34] and [10].

Disjoint block entropy H_l^d is defined similarly by replacing P with P^d. Bourke, Hitchcock and Vinodchandran [5], based on the work of Ziv and Lempel [34], demonstrated the entropy characterization of finite-state dimension using H_l^d instead of H_l. Kozachinskiy and Shen ([19]) proved that the finite-state dimension of a sequence can be equivalently defined using sliding block entropies (as in Definition 4) instead of disjoint block entropies. It is clear from the definition that, for any $x \in \Sigma^\infty$, $\dim_{FS}(x) \leq \mathrm{Dim}_{FS}(x)$. Any x with $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x)$ is called a *regular sequence*. Upper and lower average entropies were defined in [2] for measures constructed out of infinite bias sequences. We extend these notions to the set of all measures on Σ^∞ below.

▶ **Definition 5.** *For any probability measure μ on Σ^∞, let $\mathbf{H}_n(\mu) = -\sum_{w \in \Sigma^n} \mu(C_w) \log(\mu(C_w))$. The* upper average entropy *of μ, denoted $H^+(\mu)$, and its* lower average entropy, *denoted $H^-(\mu)$, are respectively the limit superior and the limit inferior as n tends to ∞ of $H_n(\mu)/n$.*

Upper and lower average entropies are the Cantor space analogues of *Rényi upper and lower dimensions* of measures on [0,1) which were originally defined for measures on the real line in [25]. For any $x \in \mathbb{T}$ (or $x \in \Sigma^\infty$), let δ_x denote the Dirac measure at x. i.e, $\delta_x(A) = 1$ if $x \in A$ and 0 otherwise for every $A \in \mathcal{B}(\mathbb{T})$ (or $A \in \mathcal{B}(\Sigma^\infty)$). Given a sequence $\langle x_n \rangle_{n=0}^\infty$ of numbers in $\mathbb{T}$(or Σ^∞), we investigate the behavior of exponential averages $\frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k x_j}$ by studying the weak convergence of sequences of averages of Dirac measures.

▶ **Definition 6.** *Given a sequence $\langle x_n \rangle_{n=0}^\infty$ in $\mathbb{T}$ (or elements in Σ^∞), we say that $\langle \nu_n \rangle_{n=1}^\infty$ is the sequence of averages of Dirac measures over $\mathbb{T}$ (or over Σ^∞) constructed out of the sequence $\langle x_n \rangle_{n=0}^\infty$ if, $\nu_n = n^{-1} \sum_{i=0}^{n-1} \delta_{x_i}$ for each $n \in \mathbb{N}$.*

3   Weyl's criterion and weak convergence

Schnorr and Stimm [28] (see also [3, 5]) showed a central connection between normal numbers and finite-state compressibility, or equivalently, finite-state dimension: a sequence $x \in \Sigma^\infty$ is normal if and only if its finite-state dimension is 1. Any $x \in \Sigma^\infty$ has finite-state dimension (equivalently, finite-state compressibility) between 0 and 1. In this sense, finite-state dimension is a generalization of the notion of normality. Another celebrated characterization of normality, in terms of exponential sums, was provided by Weyl in 1916. This characterization has resisted attempts at generalization. In the present section, we show that the theory of weak convergence of measures yields a generalization of Weyl's characterization for arbitrary dimensions. We demonstrate the utility of this new characterization to finite-state compressibilty/finite-state dimension, in subsequent sections. Weyl criterion for normal numbers on $\mathbb{T}$ is the following.

▶ **Theorem 7** (Weyl's criterion [32])**.** *A number $r \in \mathbb{T}$ is normal if and only if for every $k \in \mathbb{Z}$, $\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k (2^j r)} = 0$.*

The insight in this theorem is the connection between a number x being normal, and the concept of the collection of its shifts being uniformly distributed in the unit interval. It is the latter concept which leads to the cancellation of the exponential sums of all orders. The following is a formulation of this criterion on Cantor space, which we require in our work.

▶ **Theorem 8** (Weyl's criterion on Σ^∞)**.** *A sequence $x \in \Sigma^\infty$ is a normal sequence if and only if for every $k \in \mathbb{Z}$, $\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k (v(T^j x))} = 0$.*

The key to generalizing Weyl's criterion to sequences with finite-state dimension less than 1 is to characterize convergence of subsequences of exponential sums using weak convergence of probability measures on Σ^∞ (see Billingsley [4]). Over $\mathbb{T}$, this equivalent characterization is well-known (see Section 4.4 from [14]). Obtaining the same equivalence over Σ^∞ involves some technical hurdles due to the fact that continuous functions over Σ^∞ need not have a uniform approximation using trigonometric polynomials. In order to overcome these, we need to carefully study the relationship between the convergence of Weyl averages and weak convergence over Σ^∞. We develop these relationships in the following lemmas. At the end of this section we characterize Theorem 8 in terms of weak convergence of a sequence of measures over Σ^∞.

▶ **Definition 9.** *A sequence $\langle \nu_n \rangle_{n \in \mathbb{N}}$ of probability measures on a metric space (X, d) converges weakly to a probability μ on (X, d), denoted $\nu_n \Rightarrow \mu$, if for every bounded continuous function $f : X \to \mathbb{C}$, we have $\lim_{n \to \infty} \int f d\nu_n = \int f d\mu$.*

If a sequence of measures $\langle \nu_n \rangle_{n \in \mathbb{N}}$ on a metric space (X, d) has a weak limit measure, then the weak limit must be unique (see Theorem 1.2 from [4]). Since $\mathbb{T}$ and $\Sigma^{\infty\,4}$ are compact metric spaces, using Prokhorov's Theorem (see Theorem 5.1 from [4]) we get that any sequence of measures $\langle \nu_n \rangle_{n \in \mathbb{N}}$ on $\mathbb{T}$ (or Σ^∞), has a measure μ on $\mathbb{T}$ (or Σ^∞) and a subsequence $\langle \nu_{n_m} \rangle_{m \in \mathbb{N}}$ such that $\nu_{n_m} \Rightarrow \mu$. We first establish a relationship between weak convergence of measures on $\mathbb{T}$ and the convergence of measures of dyadic intervals in $\mathbb{T}$. Since the set of all finite unions of dyadic intervals in $\mathbb{T}$ is closed under finite intersections, we obtain the following lemma using Theorem 2.2 from [4].

▶ **Lemma 10.** *If for every dyadic interval I in $\mathbb{T}$, $\lim_{n \to \infty} \nu_n(I) = \mu(I)$, then $\nu_n \Rightarrow \mu$.*

The Portmanteau theorem (Theorem 2.1 from [4]) gives the following partial converse.

▶ **Lemma 11.** *Let $\nu_n \Rightarrow \mu$. Then $\lim_{n \to \infty} \nu_n(I) = \mu(I)$ for dyadic interval $I = [d_1, d_2)$ if $\mu(\{d_1\}) = \mu(\{d_2\}) = 0$.*

We characterize convergence of exponential sums in terms of weak convergence of probability measures, first on $\mathbb{T}$ and then on the Cantor space Σ^∞. Unlike Theorem 7, the result on Σ^∞ does not follow immediately from that on $\mathbb{T}$. On $\mathbb{T}$, the following theorem holds due to Prokhorov theorem, the fact that continuous functions on $\mathbb{T}$ can be approximated uniformly using trigonometric polynomials, and that Fourier coefficients of measures over $\mathbb{T}$ are unique due to Bochner's theorem (see Theorem 4.19 from [16]).

▶ **Theorem 12.** *Let $r \in \mathbb{T}$ and let $\langle \nu_n \rangle_{n=1}^\infty$ be the sequence of averages of Dirac measures constructed out of $\langle 2^n r \bmod 1 \rangle_{n=0}^\infty$. Let $\langle n_m \rangle_{m \in \mathbb{N}}$ be any subsequence of natural numbers. Then for every $k \in \mathbb{Z}$, there is a $c_k \in \mathbb{C}$ such that $\lim_{m \to \infty} \frac{1}{n_m} \sum_{j=0}^{n_m - 1} e^{2\pi i k(2^j r)} = c_k$ if and only if there is a unique measure μ such that $\nu_{n_m} \Rightarrow \mu$. Furthermore, if any of the above conditions are true, then $c_k = \int e^{2\pi i k y} d\mu$ for every $k \in \mathbb{Z}$ and μ is the unique measure on $\mathbb{T}$ having Fourier coefficients $\langle c_k \rangle_{k \in \mathbb{Z}}$.*

We require an analogue of this theorem for Cantor space. But the proof above cannot be adapted because on Cantor space, there are continuous functions which cannot be approximated uniformly using trigonometric polynomials. For example, consider χ_{C_0}. Observe that $\chi_{C_0}(0^\infty) = 1 \neq 0 = \chi_{C_0}(1^\infty)$. But since $v(0^\infty) = v(1^\infty)$, every trigonometric polynomial

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[4] The metric on Σ^∞ is $d(x, y) = 2^{-\min\{i \mid x_i \neq y_i\}}$.

has the same value on 0^∞ and 1^∞. However, we recover the analogue by handling dyadic rational sequences and other sequences in separate cases. Since the set of all finite unions of cylinder sets in Σ^∞ is closed under finite intersections and since the characteristic functions of cylinder sets are continuous on the Cantor space, we get the following analogue of Lemma 10 and 11 using Theorem 2.2 from [4].

▶ **Lemma 13.** *For a sequence of measures $\langle \nu_n \rangle_{n \in \mathbb{N}}$ on Σ^∞, $\nu_n \Rightarrow \mu$ iff $\lim_{n \to \infty} \nu_n(C_w) = \mu(C_w)$ for every $w \in \Sigma^*$.*

In the following theorems we relate the convergence of measures of cylinder sets to the convergence of Weyl averages on the Cantor space using Theorem 12 and Lemma 13. We state these theorems for convergence along any subsequence, since we require these more general results for studying the subsequence limits of Weyl averages.

▶ **Theorem 14.** *Let $x \in \Sigma^\infty$ and $\langle \nu_n \rangle_{n=1}^\infty$ be the sequence of averages of Dirac measures on Σ^∞ constructed out of $\langle T^n x \rangle_{n=0}^\infty$. Let $\langle n_m \rangle_{m \in \mathbb{N}}$ be any subsequence of natural numbers. If $\lim_{m \to \infty} \nu_{n_m}(C_w) = \mu(C_w)$ for every $w \in \Sigma^*$, then for every $k \in \mathbb{Z}$ we have $\lim_{m \to \infty} \frac{1}{n_m} \sum_{j=0}^{n_m - 1} e^{2\pi i k v(T^j x)} = \int e^{2\pi i k v(y)} d\mu$.*

Observe that $n_m^{-1} \sum_{j=0}^{n_m - 1} e^{2\pi i k v(T^j x)} = \int e^{2\pi i k v(y)} d\nu_{n_m}$. Hence, the above claim follows from Lemma 13 and the definition of weak convergence since for every $k \in \mathbb{Z}$, $e^{2\pi i k v(y)}$ is a continuous function on Σ^∞[5]. While Fourier coefficients uniquely determine measures over $\mathbb{T}$, Bochner's Theorem does not hold over Σ^∞. For example let $\mu_1 = \delta_{0^\infty}$ and let $\mu_2 = \delta_{1^\infty}$. Then $\mu_1 \neq \mu_2$, but it is easy to verify that for any $k \in \mathbb{Z}$, $\int e^{2\pi i k v(y)} d\mu_1 = e^{2\pi i k v(0^\infty)} = 1 = e^{2\pi i k v(1^\infty)} = \int e^{2\pi i k v(y)} d\mu_2$. The following lemma leads to a converse of Theorem 14.

▶ **Lemma 15.** *Let $x \in \Sigma^\infty$ such that $v(x) \notin \mathbb{D}$ and let $\langle \nu'_n \rangle_{n=1}^\infty$ be the sequence of averages of Dirac measures on $\mathbb{T}$ constructed out of the sequence $\langle 2^n v(x) \bmod 1 \rangle_{n=0}^\infty$. Let d be any non-zero dyadic rational. If $\nu'_{n_m} \Rightarrow \mu'$ for some subsequence of natural numbers $\langle n_m \rangle_{m \in \mathbb{N}}$, then $\mu'(\{d\}) = 0$.*

Using the above results we obtain the following partial converse of Theorem 14.

▶ **Theorem 16.** *Let $x \in \Sigma^\infty$ and let $\langle n_m \rangle_{m \in \mathbb{N}}$ be any subsequence of natural numbers. Let $\langle c_k \rangle_{k \in \mathbb{Z}}$ be complex numbers such that $\lim_{m \to \infty} \frac{1}{n_m} \sum_{j=0}^{n_m - 1} e^{2\pi i k v(T^j x)} = c_k$ for every $k \in \mathbb{Z}$. Then there exists a unique measure μ on $\mathbb{T}$ having Fourier coefficients $\langle c_k \rangle_{k \in \mathbb{Z}}$ and $\lim_{m \to \infty} \nu_{n_m}(C_w) = \hat{\mu}(C_w)$ for every $w \in \Sigma^*$ such that $w \neq 1^{|w|}$ and $w \neq 0^{|w|}$.*

For any $x \in \Sigma^\infty$, let $\langle \nu_n \rangle_{n=1}^\infty$ be the sequence of averages of Dirac measures on Σ^∞ constructed out of the sequence $\langle T^n x \rangle_{n=0}^\infty$. Now, for any $A \in \mathcal{B}(\Sigma^\infty)$, $\nu_n(A)$ is the proportion of elements in the finite sequence $x, Tx, T^2 x, \ldots T^{n-1} x$ which falls inside the set A. From this remark, and the definitions of ν_n and the sliding count probability P, the following lemma follows easily.

▶ **Lemma 17.** *Let w be any finite string in Σ^* and let $l = |w|$. Let x be any element in Σ^∞. If $\langle \nu_n \rangle_{n=1}^\infty$ is the sequence of averages of Dirac measures over Σ^∞ constructed out of the sequence $\langle T^n x \rangle_{n=0}^\infty$. Then for any n, $\nu_n(C_w) = P(x_0^{n+l-2}, w)$.*

We now give a new characterization of Weyl's criterion on Cantor Space (Theorem 8) in terms of weak convergence of measures.

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[5] This follows easily by observing that the valuation map $v : \Sigma^\infty \to \mathbb{T}$ is a continuous function on Σ^∞.

▶ **Theorem 18** (Weyl's criterion on Σ^∞ and weak convergence). *Let $x \in \Sigma^\infty$, and $\langle \nu_n \rangle_{n=1}^\infty$ be the sequence of averages of Dirac measures constructed out of $\langle T^n x \rangle_{n=0}^\infty$, and μ be the uniform measure on Σ^∞. Then the following are equivalent.*

1. *x is normal.*
2. *For every $w \in \Sigma^*$, the sliding block frequency $P(x_0^{n-1}, w) \to 2^{-|w|}$ as $n \to \infty$.*
3. *For every $k \in \mathbb{Z}$, $\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k v(T^j x)} = 0$.*
4. *$\nu_n \Rightarrow \mu$.*

4   Divergence of exponential sums for non-normal numbers

Weyl's criterion says that when $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x) = 1$ the averages of the exponential sums for every k converges to 0. However for x with $\dim_{FS}(x) < 1$, the situation is different. It is easy to construct a sequence a with $\dim_{FS}(a) < 1$ and a $k \in \mathbb{Z}$ such that the sequence of Weyl averages with parameter k do not converge. It is natural to ask if the condition $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x)$ is sufficient to guarantee convergence of the exponential sum averages. But we construct an x with $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x) = \frac{1}{2}$ such that for some k, the sequence $\langle \sum_{j=0}^{n-1} e^{2\pi i k (v(T^j x))}/n \rangle_{n=1}^\infty$ diverges. Entropy rates converging to a limit does not imply that the empirical probability measures converge to a limiting distribution, and it is the latter notion which is necessary for exponential sums to converge.

▶ **Lemma 19.** *There exists $x \in \Sigma^\infty$ with $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x) = \frac{1}{2}$ such that for some $k \in \mathbb{Z}$, the sequence $\langle \sum_{j=0}^{n-1} e^{2\pi i k (v(T^j x))}/n \rangle_{n=1}^\infty$ is not convergent.*

Generalizing the construction of diluted sequences in [10], we define an x with $v(x) \in \mathbb{T} \setminus \mathbb{D}$ and $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x) = 1/2$, but where for some $k \in \mathbb{Z}$, the sequences of Weyl sum averages diverge. The idea of dilution is as follows. Let $y \in \Sigma^\infty$ be normal. Define $a \in \Sigma^\infty$ by $a_{2n} = 0$, $a_{2n+1} = y_n$, $n \in \mathbb{N}$. Then $\dim_{FS}(a) = \mathrm{Dim}_{FS}(a) = 1/2$. Note that $b \in \Sigma^\infty$ defined by $b_{4n} = b_{4n+3} = 0$, and $b_{4n+1} = y_{2n}$, $b_{4n+2} = y_{2n+1}$, $n \in \mathbb{N}$ is also a regular sequence with $\dim_{FS}(b) = \mathrm{Dim}_{FS}(b) = 1/2$. But, the sliding block frequency of 01 in a is $1/4$, whereas it is $3/16$ in b. We leverage the existence of such distinct sequences with equal dimension. The disjoint blocks of x alternate between the above two patterns in a controlled manner to satisfy the following conditions.

1. $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x) = 1/2$
2. There is an increasing sequence of indices $\langle n_i \rangle_{i=1}^\infty$ such that $\lim_{i \to \infty} P(x_0^{n_i - 1}, 01) = 1/4$.
3. There is an increasing sequence of indices $\langle n_i \rangle_{i=1}^\infty$ such that $\lim_{j \to \infty} P(x_0^{n_j - 1}, 01) = 3/16$.

Let $\langle \nu_n \rangle_{n=1}^\infty$ be the sequence of averages of Dirac measures constructed out of $\langle T^n x \rangle_{n=0}^\infty$, and $\langle \nu'_n \rangle_{n=1}^\infty$, those from $\langle 2^n v(x) \bmod 1 \rangle_{n=0}^\infty$. Assume that $\langle n^{-1} \sum_{j=0}^{n-1} e^{2\pi i k (v(T^j x))} \rangle_{n=1}^\infty$ converge for every $k \in \mathbb{Z}$. Using the same steps in the proof of Theorem 16, we get that $\nu'_n \Rightarrow \mu'$ where μ' is the unique measure on $\mathbb{T}$ having Fourier coefficients equal to the limits of the Weyl averages. Since $v(x) \in \mathbb{T} \setminus \mathbb{D}$, Theorem 16 implies that $\nu(C_{01})$ is convergent. Using Lemma 17, we infer that $\lim_{n \to \infty} P(x_0^{n-1}, 01)$ exists. But, we know from conditions 2 and 3 that $P(x_0^{n-1}, 01)$ is not convergent. Hence, we arrive at a contradiction. Therefore, for some $k \in \mathbb{Z}$, the Weyl averages $\langle n^{-1} \sum_{j=0}^{n-1} e^{2\pi i k (v(T^j x))} \rangle_{n=1}^\infty$ diverge. The above construction is easily adapted to show that for any rational number $p/q \in (0,1)$, there exists $x \in \Sigma^\infty$ with $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x) = p/q$ such that some Weyl average of x diverges.

Weyl's criterion for finite-state dimension

We saw in Lemma 19 that Weyl averages may diverge for x having finite-state dimension less than 1, even if x is regular. Hence, it is necessary for us to deal with divergent Weyl averages and obtain their relationship with the finite-state dimension of x. We know from Theorem 18 that Weyl's criterion for normality (Theorem 8) is equivalently expressed in terms of weak convergence of a sequence of measures over Σ^∞. In section 5.1, we generalize the weak convergence formulation to handle arbitrary finite state dimension. Applying this, in section 5.2, we generalize the exponential sum formulation.

5.1   Weak convergence and finite-state dimension

We know from Theorem 18 that $x \in \Sigma^\infty$ is normal (equivalently, $\dim_{FS}(x) = 1$) if and only if $\nu_n \to \mu$, where μ is the uniform distribution over Σ^∞. In this subsection we give a generalization of this formulation of Weyl's criterion which applies for x having any finite-state dimension. Lemma 19 and Theorem 14 together imply that ν_n's need not be weakly convergent even if x is guaranteed to be regular. However, studying the subsequence limits of $\langle \nu_n \rangle_{n=1}^\infty$ gives us the following generalization of Weyl's criterion for arbitrary $x \in \Sigma^\infty$.

▶ **Theorem 20.** *Let $x \in \Sigma^\infty$. Let $\langle \nu_n \rangle_{n=1}^\infty$ be the sequence of averages of Dirac measures on Σ^∞ constructed out of the sequence $\langle T^n x \rangle_{n=0}^\infty$. Let $\mathcal{W}_x$ be the collection of all subsequence weak limits of $\langle \nu_n \rangle_{n=1}^\infty$. i.e, $\mathcal{W}_x = \{\mu \mid \exists \langle n_m \rangle_{m=0}^\infty$ such that $\nu_{n_m} \Rightarrow \mu\}$. Then, $\dim_{FS}(x) = \inf_{\mu \in \mathcal{W}_x} H^-(\mu)$ and $\mathrm{Dim}_{FS}(x) = \sup_{\mu \in \mathcal{W}_x} H^+(\mu)$.*

The following is an equivalent version of Theorem 20 which we require in section 5.2. From the definition of lower average entropy, Theorem 20 shows that, $\dim_{FS}(x) = \inf_{\mu \in \mathcal{W}_x} \liminf_{l \to \infty} \mathbf{H}_l(\mu)/l$. This $\liminf$ can be replaced by an infimum.

▶ **Lemma 21.** $\dim_{FS}(x) = \inf_{\mu \in \mathcal{W}_x} \inf_l \mathbf{H}_l(\mu)/l$

5.2   Weyl averages and finite-state dimension

We now obtain the main result of the paper by relating subsequence limits of Weyl averages and finite-state dimension. In case the Weyl averages converge, we show that the sequence is regular. In particular, when the Weyl averages converge to 0, then the regular sequence is normal. We know from Lemma 19 that there exist regular sequences with non-convergent Weyl averages. In the absence of limits, we investigate the subsequence limits of Weyl averages in order to obtain a relationship with the finite-state dimension. If for some $x \in \Sigma^\infty$, there exist a sequence of natural numbers $\langle n_m \rangle_{m \in \mathbb{N}}$ and constants $\langle c_k \rangle_{k \in \mathbb{Z}}$ such that $\lim_{m \to \infty} \frac{1}{n_m} \sum_{j=0}^{n_m - 1} e^{2\pi i k(v(T^j x))} = c_k$. Then, using Theorem 16, we get that there exists a measure μ on $\mathbb{T}$ such that $c_k = \int e^{2\pi i k y} d\mu$ and $\lim_{m \to \infty} \nu_{n_m}(C_w) = \hat{\mu}(C_w)$ for every $w \neq 0^{|w|}$ and $w \neq 1^{|w|}$. But, $\nu_{n_m}(C_{0^l})$ and $\nu_{n_m}(C_{1^l})$ need not converge. Simple examples of such strings can be obtained by concatenating increasingly large runs of 0's and 1's in an alternating stage wise manner. However, the probabilities of the strings 0^l and 1^l have negligible effect on the finite-state dimension as l gets large. Using Theorem 20 we obtain the following.

▶ **Theorem 22** (Weyl's criterion for finite-state dimension). *Let $x \in \Sigma^\infty$. If for any $\langle n_m \rangle_{m=0}^\infty$ there exist constants c_k for $k \in \mathbb{Z}$ such that $\lim_{m \to \infty} \frac{1}{n_m} \sum_{j=0}^{n_m - 1} e^{2\pi i k(v(T^j x))} = c_k$, for every $k \in \mathbb{Z}$, then there exists a measure μ on $\mathbb{T}$ such that for every k, $c_k = \int e^{2\pi i k y} d\mu$. Let $\widehat{\mathcal{W}}_x$ be the collection of the lifted measures $\hat{\mu}$ on Σ^∞ for all μ on $\mathbb{T}$ that can be obtained as subsequence limits of Weyl averages. Then, $\dim_{FS}(x) = \inf\{H^-(\hat{\mu}) \mid \hat{\mu} \in \widehat{\mathcal{W}}_x\}$ and $\mathrm{Dim}_{FS}(x) = \sup\{H^+(\hat{\mu}) \mid \hat{\mu} \in \widehat{\mathcal{W}}_x\}$*

Proof of Theorem 22. If $v(x)$ is a dyadic rational in $\mathbb{T}$, then it can be easily verified that the Weyl averages are convergent to 1. The unique measure having all Fourier coefficients equal to 1 over $\mathbb{T}$ is δ_0. Since $\hat{\delta}_0 = \delta_{0\infty}$, it can be easily verified that $\dim_{FS}(x) = H^-(\delta_{0\infty}) = H^+(\delta_{0\infty}) = \mathrm{Dim}_{FS}(x) = 0$. Hence, we consider the case when $v(x)$ is not a dyadic rational. We first define analogues of finite-state dimension by avoiding the strings 0^l and 1^l for all l in calculating the sliding entropies. We define $\widetilde{H}_l(x_0^{n-1})$ to be the normalized sliding entropy over x_0^{n-1} as in the definition of $H_l(x_0^{n-1})$, except that the summation is taken over $\Sigma^l \backslash \{0^l, 1^l\}$ instead of Σ^l. Using this notion, we define $\widetilde{\dim}_{FS}(x) = \liminf_{l\to\infty} \liminf_{n\to\infty} \widetilde{H}_l(x_0^{n-1})$ and $\widetilde{\mathrm{Dim}}_{FS}(x) = \liminf_{l\to\infty} \limsup_{n\to\infty} \widetilde{H}_l(x_0^{n-1})$. Since, $\widetilde{H}_l(x_0^{n-1}) \leq H_l(x_0^{n-1}) \leq \widetilde{H}_l(x_0^{n-1}) + 2/l$, it can be shown using routine arguments that $\dim_{FS}(x) = \widetilde{\dim}_{FS}(x)$ and $\mathrm{Dim}_{FS}(x) = \widetilde{\mathrm{Dim}}_{FS}(x)$. Similarly we define $\widetilde{H}^+$ and $\widetilde{H}^-$ by reducing the range of the sum in the definition of $\mathbf{H}_l$ to $\Sigma^l \backslash \{0^l, 1^l\}$ instead of Σ^l. Using a similar argument as in the case of sliding entropy, it can be shown that $\widetilde{H}^+$ and $\widetilde{H}^-$ are the same as H^+ and H^- for any measure on Σ^∞. Let $\langle \nu_n \rangle_{n=1}^\infty$ be the sequence of averages of Dirac measures on Σ^∞ constructed out of the sequence $\langle T^n x \rangle_{n=0}^\infty$. Let $\mathcal{W}_x$ be the set of all weak limits of ν_n as constructed in Theorem 20. Since $v(x)$ is not a dyadic rational, using Prokhorov's theorem for weak convergence of $\mathbb{T}$ and weak convergence over Σ^∞, it can be shown that, $\inf_{\mu \in \mathcal{W}_x} H^-(\mu) = \inf_{\hat{\mu} \in \widehat{\mathcal{W}}_x} H^-(\hat{\mu})$ and $\sup_{\mu \in \mathcal{W}_x} H^+(\mu) = \sup_{\hat{\mu} \in \widehat{\mathcal{W}}_x} H^+(\hat{\mu})$. The claim now follows from Theorem 20. ◀

Hence, the finite-state dimension and finite-state strong dimension are related to the lower and upper average entropies of the subsequence limits of the Weyl averages. Using the above result, we get the following theorem in the case when the Weyl averages are convergent.

▶ **Theorem 23** (Weyl's criterion for convergent Weyl averages). *Let $x \in \Sigma^\infty$. If there exist $c_k \in \mathbb{C}$ for $k \in \mathbb{Z}$ such that $\frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k(v(T^j x))} \to c_k$ as $n \to \infty$, then, there exists a unique measure μ on $\mathbb{T}$ such that for every k, $c_k = \int e^{2\pi i k y} d\mu$. Furthermore, $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x) = H^-(\hat{\mu}) = H^+(\hat{\mu})$.*

As a special case, we derive Weyl's criterion for normality, i.e, for sequences x such that $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x) = 1$ as a special case of Theorem 20 and Theorem 23.

▶ **Theorem 24.** *Let $x \in \Sigma^\infty$. Then $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k(v(T^j x))} = 0$ for every $k \in \mathbb{Z}$ if and only $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x) = 1$.*

The conclusion of Theorem 23 says that $\dim_{FS}(x) = \mathrm{Dim}_{FS}(x)$. i.e, x is a regular sequence. Hence, Lemma 19 and Theorem 23 together yield the following.

▶ **Corollary 25.** *If for each $k \in \mathbb{Z}$, $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k(v(T^j x))} = c_k$ for a sequence of complex numbers $\langle c_k \rangle_{k \in \mathbb{Z}}$. Then, x is a regular sequence. But there exist regular sequences having non-convergent Weyl averages.*

6 Preservation of finite-state dimension under real arithmetic

In this section, we demonstrate the utility of our framework by proving the most general results yet regarding the preservation of finite-state dimension under arithmetic operations like addition with reals satisfying a natural independence condition, and multiplication with non-zero rationals. These results strictly generalize all known results regarding the preservation of finite-state dimension including those of Doty, Lutz and Nandakumar [12] and Aistleitner [1]. Our Weyl criterion plays a pivotal role in these extensions. We combine our Weyl criterion along with recent estimates by Hochman [17] for the entropy of convolution of probability measures. It is easier to analyze addition and multiplication as operations

over $\mathbb{T}$. Hence we first obtain an equivalent Weyl's criterion for finite-state dimension in terms of measures over $\mathbb{T}$. We now define the analogues of upper and lower average entropies for measures over $\mathbb{T}$. This turns out to be the notion of Rényi dimension as defined by Alfréd Rényi in [25]. Recall that for any m and $w \in \Sigma_m^n$, I_w^m denotes the interval $\left[v_m(w0^\infty), v_m(w0^\infty) + m^{-|w|}\right)$ in $\mathbb{T}$.

▶ **Definition 26** (Rényi Dimension). *For any probability measure μ on $\mathbb{T}$ and* partition factor m, *let* $\mathbf{H}_n^m(\mu) = -\sum_{w \in \Sigma_m^n} \mu(I_w^m) \log(\mu(I_w^m))$. *The* Rényi upper and lower dimensions *(see [25] and [33]) are defined as follows,* $\overline{\dim}_R^m(\mu) = \limsup\limits_{n \to \infty} \frac{\mathbf{H}_n^m(\mu)}{n \log m}$ *and* $\underline{\dim}_R^m(\mu) = \liminf\limits_{n \to \infty} \frac{\mathbf{H}_n^m(\mu)}{n \log m}$. *If* $\overline{\dim}_R^m(\mu) = \underline{\dim}_R^m(\mu)$ *then the* Rényi dimension *of μ is* $\dim_R^m(\mu) = \overline{\dim}_R^m(\mu) = \underline{\dim}_R^m(\mu)$.

From the above definition, it seems as if the notion of Rényi dimension is dependent on the choice of the partition factor m. However, Rényi upper and lower dimensions are quantities that are independent of the partition factor[6]. Hence, we suppress the partition factor m in the notations $\overline{\dim}_R^m(\mu)$, $\underline{\dim}_R^m(\mu)$ and $\dim_R^m(\mu)$ and use $\overline{\dim}_R(\mu)$, $\underline{\dim}_R(\mu)$ and $\dim_R(\mu)$ to refer to the corresponding quantities for a measure μ on $\mathbb{T}$. Now, we state an equivalent Weyl's criterion for finite-state dimension for $r \in \mathbb{T}$ in terms of weak limit measures over $\mathbb{T}$ and Rényi dimension of measures over $\mathbb{T}$.

▶ **Theorem 27** (Restatement of Weyl's criterion for finite-state dimension (Theorem 22)). *Let $r \in \mathbb{T}$. If for any $\langle n_m \rangle_{m=0}^\infty$ there exist c_k for $k \in \mathbb{Z}$ such that [7] $\lim_{m \to \infty} \frac{1}{n_m} \sum_{j=0}^{n_m - 1} e^{2\pi i k 2^j r} = c_k$ for every $k \in \mathbb{Z}$, then there exists a measure μ on $\mathbb{T}$ such that for every k, $c_k = \int e^{2\pi i k y} d\mu$. Let $\mathcal{W}_r$ be the collection of all μ on $\mathbb{T}$ that can be obtained as subsequence limits of Weyl averages. Then, $\dim_{FS}(r) = \inf\{\underline{\dim}_R(\mu) \mid \mu \in \mathcal{W}_r\}$ and $\mathrm{Dim}_{FS}(r) = \sup\{\overline{\dim}_R(\mu) \mid \mu \in \mathcal{W}_r\}$.*

D. D. Wall in his thesis [31] proved that *if $r \in [0, 1]$ and q is any non-zero rational number, then r is a normal number if and only if qr and $q + r$ are normal numbers.* Doty, Lutz and Nandakumar [12] generalized this result to arbitrary finite-state dimensions and proved that the finite-state dimension and finite-state strong dimension of any number are preserved under multiplication and addition with rational numbers.

▶ **Theorem 28** ([12]). *Let $r \in \mathbb{T}$ and q be any non-zero rational number. Then for any base b, $\dim_{FS}^b(r) = \dim_{FS}^b(q + r) = \dim_{FS}^b(qr)$ and $\mathrm{Dim}_{FS}^b(r) = \mathrm{Dim}_{FS}^b(q + r) = \mathrm{Dim}_{FS}^b(qr)$.*

In the above $\dim^b$ and Dim^b denotes the finite-state dimension and finite-state strong dimension of the number r calculated by considering the sequence representing the base-b expansion of r[8]. In the specific case of normal sequences, Wall's result has been generalized by Aistleitner in the following form. Let $\mathcal{C}$ be the set of reals $y = 0.y_0 y_1 \ldots$ such that the ratio $P(y_0^{n-1}, 0)$ goes to 1 as n tends to ∞. Then we have the following.

▶ **Theorem 29.** *If $y \in \mathcal{C}$, then for any normal $r \in \mathbb{T}$ and $q \in \mathbb{Q}$, the number $r + qy$ is normal.*

We strictly generalize all these above results by formulating a natural independence notion between two reals. We describe the framework below. Given strings x and y in Σ^∞ and strings $u, w \in \Sigma^\ell$ for some $\ell \geq 1$, we define the *joint occurrence count* of u and w

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[6] This important fact regarding Rényi dimension is a folklore result.

[7] The 2^j term in limit expression must be replaced with b^j while investigating the above criterion in any arbitrary base b

[8] For r having multiple base b expansions, this does not cause any ambiguity since in this case the finite-state dimensions of r are 0 with respect to any of the two possible expansions.

in x and y up to n as, $N_{u,w}(x_0^{n-1}, y_0^{n-1}) = |\{i \in [0, n-\ell] : x_i^{i+\ell-1} = u \text{ and } y_i^{i+\ell-1} = w\}|$. And, then the *joint occurrence probability* of u and w in x and y up to n is defined as $P_{u,w}(x_0^{n-1}, y_0^{n-1}) = \frac{N_{u,w}(x_0^{n-1}, y_0^{n-1})}{n-\ell+1}$.

Informally, we define two infinite strings x and y to be independent if for infinitely many lengths l, the occurrence probability distributions of l-length strings within x and y are *independent* in the limit. The straightforward formulation of independence between x and y is $\lim_{n\to\infty} P_{u,w}(x_0^{n-1}, y_0^{n-1}) = \lim_{n\to\infty} P(x_0^{n-1}, u) P(y_0^{n-1}, w)$. But these limits need not exist for general x and y. Hence, the more admissible and useful definition is the following.

▶ **Definition 30.** *Any two strings x and y in Σ^∞ are said to be* independent *if for infinitely many $\ell \geq 1$ and for every $u, w \in \Sigma^\ell$, $\lim_{n\to\infty} \left| P_{u,w}(x_0^{n-1}, y_0^{n-1}) - P(x_0^{n-1}, u) P(y_0^{n-1}, w) \right| = 0$.*

For any measures μ_1 and μ_2 on $\mathbb{T}$, let $\mu_1 * \mu_2$ denote the convolution of these two measures (see [26] or [16]). A basic intuition for our approach can be viewed as follows. A standard result in probability theory (see for example, Shiryaev [29], 2nd. edition, Section II.8) is that, if X and Y are two independent random variables, then the distribution of $X + Y$ is the convolution of the distributions of X and Y. Moreover, the Fourier coefficients of the convolution is the product of the Fourier coefficients of the individual distributions. Our result may be viewed as an analogous result using sequences. The following theorem gives an important connection between the exponential averages of the sum of independent reals and the exponential averages of the individual reals which is crucial in proving the main results in this section.

▶ **Theorem 31.** *If x and y are real numbers in $\mathbb{T}$ such that x and y are independent in the sense of condition 30, then for any integers d, e and $q \in \mathbb{Q}$,*

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k 2^j (dx+ey)} - \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k 2^j dx} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i k 2^j ey} \right| = 0.$$

The proofs of the following bounds on Rényi dimension of convolutions crucially employ results from Hochman [17] along with the data processing inequality ([9, 18]).

▶ **Lemma 32.** *For any measures μ_1 and μ_2 on $\mathbb{T}$,*
1. $\underline{\dim}_R(\mu_1 * \mu_2) \geq \max\{\underline{\dim}_R(\mu_1), \underline{\dim}_R(\mu_2)\}$ *and* $\underline{\dim}_R(\mu_1 * \mu_2) \leq \underline{\dim}_R(\mu_1) + \overline{\dim}_R(\mu_2)$
2. $\overline{\dim}_R(\mu_1 * \mu_2) \geq \max\{\overline{\dim}_R(\mu_1), \overline{\dim}_R(\mu_2)\}$ *and* $\overline{\dim}_R(\mu_1 * \mu_2) \leq \overline{\dim}_R(\mu_1) + \overline{\dim}_R(\mu_2)$

The following is our main result.

▶ **Theorem 33.** *Let x and y be real numbers in $\mathbb{T}$ such that x and y are independent in the sense of condition 30. Then for any $d, e \in \mathbb{Z}$,*
1. $\dim_{FS}(dx + ey) \geq \max\{\dim_{FS}(dx), \dim_{FS}(ey)\}$ *and* $\dim_{FS}(dx + ey) \leq \dim_{FS}(dx) + \mathrm{Dim}_{FS}(ey)$.
2. $\mathrm{Dim}_{FS}(dx + ey) \geq \max\{\mathrm{Dim}_{FS}(dx), \mathrm{Dim}_{FS}(ey)\}$ *and* $\mathrm{Dim}_{FS}(dx + ey) \leq \mathrm{Dim}_{FS}(dx) + \mathrm{Dim}_{FS}(ey)$.

The following technical lemmas that are consequences of Theorem 31 are required for proving Theorem 33.

▶ **Lemma 34.** *Let x and y be real numbers in $\mathbb{T}$ such that x and y are independent in the sense of condition 30 and let $d, e \in \mathbb{Z}$. Then, for any $\mu \in \mathcal{W}_{dx+ey}$ there exist $\mu_1 \in \mathcal{W}_{dx}$ and $\mu_2 \in \mathcal{W}_{ey}$ such that $\mu = \mu_1 * \mu_2$.*

▶ **Lemma 35.** *If x and y are real numbers in $\mathbb{T}$ such that x and y are independent in the sense of condition 30. Let $d, e \in \mathbb{Z}$ and $q \in \mathbb{Q}$. Then, for any $\mu_1 \in \mathcal{W}_{dx}$ there exist $\mu \in \mathcal{W}_{dx+ey}$ and $\mu_2 \in \mathcal{W}_{ey}$ such that $\mu = \mu_1 * \mu_2$.*

Now we prove Theorem 33.

Proof of Theorem 33. Consider any $\mu \in \mathcal{W}_{dx+ey}$. Using Lemma 34 we get that there exists $\mu_1 \in \mathcal{W}_{dx}$ and $\mu_2 \in \mathcal{W}_{ey}$ such that $\mu = \mu_1 * \mu_2$. Now, it follows from Lemma 32 that $\underline{\dim}_R(\mu) = \underline{\dim}_R(\mu_1 * \mu_2) \geq \underline{\dim}_R(\mu_1)$. On applying Theorem 27 for $dx \in \mathbb{T}$, we get $\underline{\dim}_R(\mu) \geq \dim_{FS}(dx)$. Since μ was arbitrary, applying Theorem 27 for $dx + ey \in \mathbb{T}$, we obtain $\dim_{FS}(dx + ey) \geq \dim_{FS}(dx)$. The proof of $\dim_{FS}(dx + ey) \geq \dim_{FS}(ey)$ is similar. This completes the proof of the first inequality. In order to show the second inequality, consider any $\mu_1 \in \mathcal{W}_{dx}$. Using Lemma 35, there exist $\mu \in \mathcal{W}_{dx+ey}$ and $\mu_2 \in \mathcal{W}_{ey}$ such that $\mu = \mu_1 * \mu_2$. Now using Lemma 32, it follows that $\underline{\dim}_R(\mu) = \underline{\dim}_R(\mu_1 * \mu_2) \leq \underline{\dim}_R(\mu_1) + \overline{\dim}_R(\mu_2)$. On applying Theorem 27 for the points $dx + ey \in \mathbb{T}$ and $ey \in \mathbb{T}$, we get $\dim_{FS}(dx + ey) \leq \underline{\dim}_R(\mu_1) + \text{Dim}_{FS}(ey)$. Since μ_1 was arbitrary, applying Theorem 27 for $dx \in \mathbb{T}$, we obtain $\dim_{FS}(dx + ey) \leq \dim_{FS}(dx) + \text{Dim}_{FS}(ey)$. 2 follows similarly. ◄

The following is an immediate corollary of the Theorem 33.

▶ **Corollary 36.** *If x and y are real numbers in $\mathbb{T}$ such that x and y are independent in the sense of condition 30, then for any $q \in \mathbb{Q}$,*
1. $\dim_{FS}(x+qy) \geq \max\{\dim_{FS}(x), \dim_{FS}(y)\}$ *and* $\dim_{FS}(x+qy) \leq \dim_{FS}(x) + \text{Dim}_{FS}(y)$.
2. $\text{Dim}_{FS}(x + qy) \geq \max\{\text{Dim}_{FS}(x), \text{Dim}_{FS}(y)\}$ *and* $\text{Dim}_{FS}(x + qy) \leq \text{Dim}_{FS}(x) + \text{Dim}_{FS}(y)$

On considering the case when $\text{Dim}_{FS}(y) = 0$, we obtain the following corollaries, generalizing earlier results by Doty, Lutz, Nandakumar [12] and Aistleitner [1], regarding the preservation of finite-state dimension under addition with an independent sequence having zero finite-state strong dimension.

▶ **Corollary 37.** *If x and y are real numbers in $\mathbb{T}$ such that x and y are independent in the sense of condition 30 with $\text{Dim}_{FS}(y) = 0$, then for any $q \in \mathbb{Q}$, $\dim_{FS}(x + qy) = \dim_{FS}(x)$ and $\text{Dim}_{FS}(x + qy) = \text{Dim}_{FS}(x)$.*

It is easy to verify that any string in $\mathcal{C}$ is independent of any other string $x \in \Sigma^\infty$. Thus we obtain the following generalization of Aistleitner's result to every dimension [1].

▶ **Corollary 38.** *If y is any real number in $\mathcal{C}$, then for any $x \in \mathbb{T}$ and $q \in \mathbb{Q}$, $\dim_{FS}(x+qy) = \dim_{FS}(x)$ and $\text{Dim}_{FS}(x + qy) = \text{Dim}_{FS}(x)$.*

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