The Threshold Problem for Hypergeometric Sequences with Quadratic Parameters

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Abstract

Hypergeometric sequences are rational-valued sequences that satisfy first-order linear recurrence relations with polynomial coefficients; that is, $\langle u_n \rangle_{n=0}^{\infty}$ is hypergeometric if it satisfies a first-order linear recurrence of the form $p(n)u_{n+1} = q(n)u_n$ with polynomial coefficients $p, q \in \mathbb{Z}[x]$ and $u_0 \in \mathbb{Q}$.

In this paper, we consider the Threshold Problem for hypergeometric sequences: given a hypergeometric sequence $\langle u_n \rangle_{n=0}^{\infty}$ and a threshold $t \in \mathbb{Q}$, determine whether $u_n \geq t$ for each $n \in \mathbb{N}_0$. We establish decidability for the Threshold Problem under the assumption that the coefficients p and q are monic polynomials whose roots lie in an imaginary quadratic extension of $\mathbb{Q}$. We also establish conditional decidability results; for example, under the assumption that the coefficients p and q are monic polynomials whose roots lie in any number of quadratic extensions of $\mathbb{Q}$, the Threshold Problem is decidable subject to the truth of Schanuel's conjecture. Finally, we show how our approach both recovers and extends some of the recent decidability results on the Membership Problem for hypergeometric sequences with quadratic parameters.

1 Introduction

Background

The *Threshold Problem* is a fundamental open decision problem in automated verification that asks to determine whether every term in a recursively defined sequence is bounded from below by a given value (commonly, the *threshold*). The Threshold Problem appears under many guises across the computational and mathematical sciences with applications in fields as diverse as software verification, probabilistic model checking, combinatorics, and formal languages (we refer the interested reader to the discussion in [26] and the references therein).

The inputs for the Threshold Problem are a recursively defined sequence $\langle u_n \rangle_{n=0}^{\infty} \subseteq \mathbb{Q}$ and a threshold $t \in \mathbb{Q}$. (Hereafter, we shall use tuple notation $(\langle u_n \rangle_{n=0}^{\infty}, t)$ as shorthand for a given problem instance.) Threshold then asks to determine whether $u_n \geq t$ for each $n \in \mathbb{N}_0$. Arguably, the variant of the Threshold Problem that has received the most attention in automated verification is the Positivity Problem for *C-finite sequences* (those sequences that obey a linear recurrence relation with constant coefficients). Therein, Positivity sets as a threshold $t = 0$ and so asks whether every term in a C-finite sequence is non-negative.

Herein we consider the hypergeometric subclass of *P-finite sequences* (those sequences that satisfy a linear recurrence relation with polynomial coefficients) [11]. Recall that a *hypergeometric sequence* is a rational-valued first-order linear recurrence sequence with polynomial coefficients; that is to say, a sequence $\langle u_n \rangle_{n=0}^{\infty} \subseteq \mathbb{Q}$ that satisfies a relation of the form

$$p(n)u_{n+1} = q(n)u_n \tag{1}$$

where $p, q \in \mathbb{Z}[x]$ and $p(x)$ has no non-negative integer zeros. By the latter assumption on $p(x)$, the recurrence relation (1) uniquely defines an infinite sequence of rational numbers once the initial value $u_0 \in \mathbb{Q}$ is specified. For a hypergeometric sequence $\langle u_n \rangle_{n=0}^{\infty}$ satisfying (1), we call the roots of the polynomial pq the sequence's *parameters*. Hypergeometric sequences and their associated generating functions, the hypergeometric series, are commonplace in fields such as numerical analysis and analytic combinatorics [8, 11].

In this paper, we consider the Threshold Problem for hypergeometric sequences. Naïvely, we might construe that decidability of the Threshold Problem in this setting is easily settled. Consider an instance of the Threshold Problem $(\langle u_n \rangle_{n=0}^{\infty}, t)$. Without loss of generality, we can assume that $\langle u_n \rangle_{n=0}^{\infty}$ either diverges to infinity or converges to a non-zero limit (further explanation behind this assumption is given in the Preliminaries). Suppose that $\langle u_n \rangle_{n=0}^{\infty}$ converges to a limit not equal to t. From the form of the recurrence relation in (1), we can compute a bound B such that if $n > B$ then $u_n > t$ or $u_n < t$. Similar deductions handle the case that $\langle u_n \rangle_{n=0}^{\infty}$ diverges to infinity. In the case that the limit of $\langle u_n \rangle_{n=0}^{\infty}$ is the threshold t, we can compute a similar bound based on the fact that the convergence to t is eventually monotonic. It follows that, in each case, the Threshold Problem reduces to exhaustively checking whether $u_n \geq t$ for each $n \in \{0, 1, \ldots, B\}$. Unfortunately this reasoning does not suffice to decide the Threshold Problem. Indeed, we do not know how to decide whether a generic hypergeometric sequence converges to a given rational limit. Further, such convergence questions are intricately linked to open problems concerning algebraic relations for the gamma function (we give further details below).

Contributions

Our primary contributions are:
(a) The Threshold Problem for hypergeometric sequences whose polynomial coefficients are monic and split over an imaginary quadratic field are decidable (Theorem 15).
(b) The Threshold Problem for hypergeometric sequences whose polynomial coefficients are monic and each irreducible factor of pq is either linear or quadratic is decidable subject to the truth of Schanuel's conjecture (Corollary 18).

We delay a formal statement of Schanuel's conjecture to the Preliminaries. For our conditional decidability results, we note that only termination is conditional on Schanuel's conjecture and that correctness of our procedure is unconditional (Remark 24). Corollary 18 follows from the more general result:

(c) The Threshold Problem for hypergeometric sequences whose monic polynomial coefficients possess Property S is decidable subject to the truth of Schanuel's conjecture (Theorem 17 in Subsection 4.1).

Polynomials with Property S (Subsection 4.1) lead to classes of hypergeometric sequences with unnested radical and cyclotomic parameters.

Our secondary contribution concerns the *Membership Problem* for hypergeometric sequences. Given a hypergeometric sequence $\langle u_n \rangle_{n=0}^{\infty}$ and target $t \in \mathbb{Q}$, Membership asks to determine whether there is an $n \in \mathbb{N}_0$ for which $u_n = t$.

(d) For classes of hypergeometric sequences where we establish (un)conditional decidability of the Threshold Problem, we also obtain (un)conditional decidability of the Membership Problem. This contribution is a straightforward corollary of the following observation: for hypergeometric sequences, decidability of the Membership Problem reduces to that of the Threshold Problem (Proposition 7).

We note this secondary contribution both recovers and extends some of the recent results in work by Kenison et al. [15]. For the avoidance of doubt, we break new ground for the Membership Problem. For example, we have conditional decidability of the Membership Problem for hypergeometric sequences with unnested radical and cyclotomic parameters (both classes fall outside of the remit of the previous works). A concrete subclass is given by those hypergeometric sequences whose polynomial coefficients are of the form $(x^2 - \ell_1)(x^2 - \ell_2) \cdots (x^2 - \ell_d)$ where $\ell_1, \ldots, \ell_d \in \mathbb{Z}$.

Approach

As previously mentioned, an obstacle that prevents us from settling decidability of the Threshold Problem for hypergeometric sequences is determining whether a given hypergeometric sequence converges to some rational limit.

To each hypergeometric sequence $\langle u_n \rangle_{n=0}^{\infty}$ satisfying (1), we associate the *shift quotient* $r(x) := q(x)/p(x) \in \mathbb{Q}(x)$. It is clear that the terms of $\langle u_n \rangle_{n=0}^{\infty}$ are given by a sequence of partial products such that the nth term is given thus: $u_n = u_0 \cdot \prod_{k=0}^{n} r(k)$. Without loss of generality, we can a normalise a sequence with $u_0 \neq 0$ by assuming that $u_0 = 1$. (We note that the Threshold Problem is trivial to decide when $u_0 = 0$.) Thus our consideration of the Threshold Problem reduces to analysing the sequence of partial products $\langle \prod_{k=0}^{n} r(k) \rangle_{n=0}^{\infty}$. In all problem instances where the Threshold Problem is not trivial to determine, we employ a classical theorem in analysis (Theorem 5) that permits us to write the limit of the sequence $\langle \prod_{k=0}^{n} r(k) \rangle_{n=0}^{\infty}$ as a quotient of two finite products involving the gamma function. Thus the Threshold Problem for hypergeometric sequences reduces to testing an equality between gamma products. Our novel approach leverages algebraic and transcendental properties to settle such equality tests. For example, we frequently employ the algebraic independence of transcendental constants π and e^{π} (a celebrated consequence of Nesterenko's work on modular functions [20]).

There is a large corpus of research connecting hypergeometric sequences and the gamma function (often under the guise of infinite product identities [1, 5, 7]). This is particularly relevant for our approach (as described above) and sets us apart from previous papers on the Membership Problem in this setting [15, 22]. As a nod to the wider appeal of our approach, let us consider two examples from the literature on numeric and symbolic computation.

▶ **Example 1** (Numerical evaluation of the Kepler–Bouwkamp constant [5, Section 4])**.** This example closely follows work by Chamberland and Straub [5]. Those authors demonstrate the use of hypergeometric sequences to efficiently approximate certain numerical constants given by infinite products. One such example is the Kepler–Bouwkamp constant $\prod_{k=3}^{\infty} \cos(\pi/k)$. The convergence of this infinite product is notoriously slow: the error bound between the approximation $\prod_{k=3}^{10^4} \cos(\pi/k)$ and the Kepler–Bouwkamp constant is 10^{-4}.

Let us consider a Padé approximation that uses hypergeometric sequences with quadratic parameters. We recall that the $[2, 2]$-*Padé approximant* of a function $f(x)$ is the rational function $q(x)/p(x)$ where $p, q \in \mathbb{Z}[x]$ are quadratic polynomials for which the Maclaurin series of $q(x)/p(x)$ agrees with that of $f(x)$ up to order 4. For example, the $[2, 2]$-Padé approximant of $\cos(x)$ is

$$r_2(x) := \frac{12 - 5x^2}{12 + x^2} = \cos(x) + O(x^6).$$

Now consider

$$\prod_{k=3}^{\infty} r_2(\pi/k) = \prod_{k=3}^{\infty} \frac{12k^2 - 5\pi^2}{12k^2 + \pi^2} = \frac{\Gamma(3 - \frac{i}{6}\sqrt{3}\pi)\Gamma(3 + \frac{i}{6}\sqrt{3}\pi)}{\Gamma(3 - \frac{1}{6}\sqrt{15}\pi)\Gamma(3 + \frac{1}{6}\sqrt{15}\pi)}.$$

Here the evaluation as a quotient of two gamma products follows from Theorem 5. We pass this evaluation to any modern computer algebra system and determine that the error between $\prod_{k=3}^{\infty} r_2(\pi/k)$ and the Kepler–Bouwkamp constant is bounded by 10^{-3}.

Loosely speaking, the $[2, 2]$-Padé approximant leverages a hypergeometric sequence with quadratic parameters.[1] Higher-order approximants will give a closer numerical estimate via hypergeometric sequences with higher-degree parameters.

▶ **Example 2** (Evaluation of a gamma product with cyclotomic parameters). Here we include a simple (yet concrete) example [4, pages 4–6] of the difficulty in determining whether a hypergeometric sequence converges to a rational limit. Consider

$$\prod_{k=2}^{\infty} \frac{k^5 - 1}{k^5 + 1} = \frac{2 \cdot \Gamma(-\omega_{10})\Gamma(\omega_{10}^2)\Gamma(-\omega_{10}^3)\Gamma(\omega_{10}^4)}{5 \cdot \Gamma(\omega_{10})\Gamma(-\omega_{10}^2)\Gamma(\omega_{10}^3)\Gamma(-\omega_{10}^4)} \tag{2}$$

where $\omega_{10} = e^{2\pi i/10}$ and, once again, the right-hand side is derived from Theorem 5. As noted by the authors of [4], it is not known whether the limit in (2) is even algebraic.

Whilst the state of the art cannot generally handle the evaluation of expressions given by gamma products, many works in the literature have established identities for restricted classes of products (a non-exhaustive list includes [5, 18, 21, 27, 34, 35]). Our connection to such interests stems from our approach herein: our reduction-step leaves us to determine whether the ratio of two gamma products (as above) is rational.

Related Work

Membership for Hypergeometric Sequences.   Two recent works consider the Membership Problem for hypergeometric sequences [15, 22]. In both of these works, the authors use p-adic techniques and divisibility arguments (in stark contrast to the approach herein). It is worth noting that such techniques seem appropriate only for the Membership Problem and not the Threshold Problem.

The authors of [22] establish decidability of the Membership Problem for the class of hypergeometric sequences with rational parameters. Closer to our setting, the authors of [15] establish decidability of the Membership Problem for the class of hypergeometric sequences whose polynomial coefficients (as in (1)) are both monic and split over a quadratic field. By comparison to [15], we establish decidability of not only the Membership Problem, but also the Threshold Problem (Theorem 15) for hypergeometric sequences whose monic polynomial coefficients split over an imaginary quadratic field. We note our result for the Membership Problem is weaker for sequences whose monic polynomial coefficients split over a real quadratic field: in this setting we are limited to conditional decidability (Proposition 7 and Corollary 18).

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[1] The quadratics $12k^2 - 5\pi^2$ and $12k^2 + \pi^2$ do not have rational coefficients, but in Appendix B we demonstrate how our approach handles decidability of the Threshold Problem in this example.

Positivity for P-finite sequences. Identities for P-finite sequences are frequent in the literature; however, as noted by Kauers and Pillwein, "in contrast,...almost no algorithms are available for inequalities" in this setting [12]. Determining whether the terms of a P-finite sequence are non-negative has garnered much attention in recent works (see, for example, [10, 12, 13, 28]). On the one hand, these works handle higher-order P-finite sequences than the hypergeometric sequences we consider. On the other hand, the algorithms described in the above studies are restricted in their applicability (placing syntactic restrictions on the polynomial coefficients) and termination is not guaranteed for all initial values. Indeed, genericity of initial conditions (in the sense that, the growth rate of a recurrence sequence is determined by a positive dominant eigenvalue) is required for the algorithms in [10, 12]. Additionally, determining whether the initial conditions of a given sequence are generic is an open problem (even at low orders) [13].

Positivity for C-finite sequences. It is easily seen that the Threshold Problem for C-finite sequences reduces to the Positivity Problem for C-finite sequences. Recall that *Positivity* asks to determine whether all terms in a sequence lie above the threshold zero (so the variant of the Threshold Problem where $t = 0$). This reduction is straightforward: to determine whether $u_n \geq t$ for each $n \in \mathbb{N}_0$ we can equivalently ask whether $v_n := u_n - t \geq 0$ for each $n \in \mathbb{N}_0$. Observe that $\langle v_n \rangle_{n=0}^{\infty}$ is C-finite since it is given by the difference of two C-finite sequences $\langle u_n \rangle_{n=0}^{\infty}$ and $\langle t \rangle_{n=0}^{\infty}$ and we are done.

Decidability of the Positivity Problem for C-finite sequences is considered a challenging open problem. Further, Positivity and its variants have garnered much research interest in recent works [9, 14, 23, 24, 25, 26]. Akin to our focus on restricted classes of hypergeometric sequences herein, the authors of [14] and [23] consider restricted classes of C-finite sequences: both of those works place restrictions on the algebraic properties of the associated recurrence relations.

Structure and Outline

This paper is structured as follows. In the next section we gather together relevant preliminary material. In Sections 3 and 4, we establish (un)conditional decidability of the Threshold Problem for classes of hypergeometric sequences. In one sense, Section 3 gives an overview of our approach in the setting of hypergeometric sequences with quadratic parameters. In Section 4, we introduce the class of polynomials with Property S (Subsection 4.1) and then show that the Threshold Problem for hypergeometric sequences whose monic polynomial coefficients possess Property S is decidable subject to the truth of Schanuel's conjecture (Theorem 17 in Subsection 4.2). We make suggestions for future avenues of research in the conclusion (Section 5). Proofs omitted from the main text are given in Appendix A. Appendix B contains a worked example related to the Kepler–Bouwkamp (Example 1) and demonstrates an application of Schanuel's conjecture.

2 Preliminaries

The Gamma Function. The approach herein relies on transcendence theory for the gamma function Γ where

$$\Gamma(z) = \int_0^{\infty} x^{z-1} \mathrm{e}^{-x} \, dx \quad \text{for } z \in \mathbb{C} \text{ with } \mathrm{Re}(z) > 0.$$

It is possible to analytically extend the domain of Γ to the whole complex plane minus the non-positive integers where the function has simple poles. We briefly recall standard results for the gamma function. Further details and historical accounts are given in a number of sources (cf. [2, 37]).

The standard relations for the gamma function give the functional identities: the *recurrence* (or *translation*) *property* $\Gamma(z+1) = z\Gamma(z)$ for $z \notin \mathbb{Z}$ and the *reflection property* $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ for $z \notin \mathbb{Z}$. In the domain of the gamma function, repeated application of the translation property leads to the following "rising factorial" identity. For $n \in \{1, 2, \ldots\}$, we have

$$\frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1)\cdots(z+n-1).$$

Similarly, the "falling factorial" identity is given by

$$\frac{\Gamma(z+1)}{\Gamma(z-n+1)} = z(z-1)\cdots(z-n+1).$$

The next technical lemma is derived from the aforementioned properties of the gamma function. We employ the notation $\frac{1}{2}\mathbb{Z}$ for the set of integers and half-integers.

▶ **Lemma 3.** *Let $\rho \in \frac{1}{2}\mathbb{Z}$. Suppose that $w \in \mathbb{C}$ is an algebraic number such that both $\rho + w$ and $\rho - w$ lie in the domain of the gamma function and $w \notin \frac{1}{2}\mathbb{Z}$. Up to multiplication by an algebraic number, we have the following equalities:*

$$\Gamma(\rho + w)\Gamma(\rho - w) = \begin{cases} \dfrac{2\pi i e^{\pi w i}}{w(1 - e^{2\pi w i})} & \text{if } \rho \text{ is an integer, or} \\[2ex] \dfrac{2\pi e^{\pi w i}}{e^{2\pi w i} + 1} & \text{if } \rho \text{ is a half-integer.} \end{cases}$$

Proof. Let us apply the rising and falling factorial identities (as appropriate to the sign of ρ). Then, up to multiplication by an algebraic number, we have the following equalities:

$$\Gamma(\rho + w)\Gamma(\rho - w) = \begin{cases} \Gamma(w)\Gamma(-w) & \text{if } \rho \text{ is an integer, or} \\ \Gamma(1/2 + w)\Gamma(1/2 - w) & \text{if } \rho \text{ is a half-integer.} \end{cases}$$

Consider the first of the two cases above. The reflection and recurrence formulas lead to

$$\Gamma(w)\Gamma(-w) = \frac{\Gamma(w)\Gamma(1-w)}{-w} = \frac{\pi}{-w\sin(\pi w)} = -\frac{2\pi i}{w(e^{\pi w i} - e^{-\pi w i})}.$$

For the second case, we employ the cosine variant of Euler's reflection formula to obtain

$$\Gamma(1/2 + w)\Gamma(1/2 - w) = \frac{\pi}{\cos(\pi w)} = \frac{2\pi}{e^{\pi w i} + e^{-\pi w i}}.$$

The equalities in the statement of the lemma quickly follow.   ◄

Decidability and Reduction Results.   Recall that a rational-valued sequence $\langle u_n \rangle_{n=0}^{\infty}$ is *hypergeometric* if it satisfies a first-order recurrence relation of the form (1) with polynomial coefficients $p, q \in \mathbb{Z}[x]$. Due to space restrictions, we omit the proofs of Lemma 4, Proposition 6, and Proposition 7 from the main text. Each proof is included in Appendix A.

The following straightforward lemma appears in previous works [13, 22].

▶ **Lemma 4.** *Consider the class of hypergeometric sequences $\langle u_n \rangle_n$ whose shift quotients $r(n)$ either diverge to $\pm\infty$ or converge to a limit ℓ with $|\ell| \neq 1$. For this class, the Membership and Threshold Problems are both decidable.*

Thus to decide the Membership and Threshold Problems for hypergeometric sequences, we need only consider the sequences whose shift quotients $r(n)$ converge to ± 1 as $n \to \infty$. We say an infinite product $\prod_{k=0}^{\infty} r(k)$ *converges* if the sequence of partial products converges to a finite non-zero limit (otherwise the product is said to diverge). Recall the following classical theorem ([37, §12] and [5]).

▶ **Theorem 5.** *Consider the rational function*

$$r(k) := \frac{c(k + \alpha_1) \cdots (k + \alpha_m)}{(k + \beta_1) \cdots (k + \beta_{m'})}$$

where we suppose that each $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_{m'}$ is a complex number that is neither zero nor a negative integer. The infinite product $\prod_{k=0}^{\infty} r(k)$ converges to a finite non-zero limit only if $c = 1$, $m = m'$, and $\sum_j \alpha_j = \sum_j \beta_j$. Further, the value of the limit is given by

$$\prod_{k=0}^{\infty} r(k) = \prod_{j=1}^{m} \frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}.$$

With Theorem 5 in mind, it is useful to introduce the following terminology for shift quotients. We call a rational function $r(k)$ (as above) *harmonious* if $r(k)$ satisfies the assumptions $c = 1$, $m = m'$, and $\sum_j \alpha_j = \sum_j \beta_j$. From Theorem 5, it is immediately apparent that a hypergeometric sequence $\langle u_n \rangle_n$ with shift quotient r converges to a finite non-zero limit only if r is harmonious.

Related to the assumptions in Theorem 5 (and Proposition 6 below), when considering Membership and Threshold we can assume without loss of generality that the roots $\alpha_1, \ldots, \alpha_m$ of the coefficient q (as in (1)) are neither zero nor negative integers. For otherwise, a hypergeometric sequence eventually hits zero and is identically zero thereafter. In [22], Nosan et al. establish Propositions 6 and 7 for rational parameters. The proof of Proposition 6 given in Appendix A is all but identical to the proof of Proposition 2 in [22].

▶ **Proposition 6.** *Let $\langle u_n \rangle_n$ be a hypergeometric sequence whose shift quotient is given by a ratio of two polynomials with real coefficients. For such sequences, the Membership and Threshold Problems are both Turing-reducible to the following decision problem. Given $d \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_d \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ (the roots of some $P(x) \in \mathbb{R}[x]$), and $\beta_1, \ldots, \beta_d \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ (the roots of some $Q(x) \in \mathbb{R}[x]$), determine whether*

$$\frac{\Gamma(\beta_1) \cdots \Gamma(\beta_d)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_d)} = t$$

for $t \in \mathbb{Q} \setminus 0$.

To be absolutely clear, the fact that $t \in \mathbb{Q}$ is non-zero in Proposition 6 follows directly from the infinite product in Theorem 5 converging to a non-zero limit.

Our (un)conditional decidability results for the Membership Problem follow from the next proposition. This proposition can be deduced from the work in [22] (a straightforward proof is given in Appendix A).

▶ **Proposition 7.** *For hypergeometric sequences, decidability of the Membership Problem Turing-reduces to that of the Threshold Problem.*

Number Fields. We recall standard results for quadratic fields below (cf. [33, Chapter 3]). A number field K is *quadratic* if $[K : \mathbb{Q}] = 2$. A field K is quadratic if and only if there is a square-free integer d such that $K = \mathbb{Q}(\sqrt{d})$. Further, a quadratic field $\mathbb{Q}(\sqrt{d})$ is *imaginary* if $d < 0$.

▶ **Theorem 8.** *Suppose that $d \in \mathbb{Z}$ is square-free. Then the algebraic integers of $\mathbb{Q}(\sqrt{d})$ are given by $\mathbb{Z}[\sqrt{d}]$ if $d \not\equiv 1 \pmod 4$ or $\mathbb{Z}[1/2 + \sqrt{d}/2]$ if $d \equiv 1 \pmod 4$.*

We include the following straightforward lemma for ease of reference.

▶ **Lemma 9.** *Let $\mathbb{L}/\mathbb{Q}$ be a finite Galois extension. Suppose that $\mathcal{P} \in \mathbb{L}(X_1, \ldots, X_m)$ is a polynomial such that $\mathcal{P}(s_1, \ldots, s_m) = 0$ with $(s_1, \ldots, s_m) \in \mathbb{C}^m$. Then there is a polynomial $\mathcal{Q} \in \mathbb{Q}(X_1, \ldots, X_m)$ such that $\mathcal{Q}(s_1, \ldots, s_m) = 0$.*

Proof. Let $\mathcal{P} = \sum_{(t_1, \ldots, t_m)} c_{(t_1, \ldots, t_m)} X_1^{t_1} X_2^{t_2} \cdots X_m^{t_m}$ and for each $\sigma \in G$ (the Galois group of $\mathbb{L}/\mathbb{Q}$) let

$$\sigma(\mathcal{P}) = \sum_{(t_1, \ldots, t_m)} \sigma(c_{(t_1, \ldots, t_m)}) X_1^{t_1} X_2^{t_2} \cdots X_m^{t_m}.$$

Let $\mathcal{Q} = \mathrm{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{P}) := \prod_{\sigma \in G} \sigma(\mathcal{P})$. It is clear that each of the coefficients of the polynomial $\mathcal{Q}$ is rational since the coefficients are invariant under the action of the group G. Further,

$$\mathcal{Q}(s_1, \ldots, s_m) = \mathcal{P}(s_1, \ldots, s_m) \prod_{\sigma \in G \backslash e_G} \sigma(\mathcal{P})(s_1, \ldots, s_m) = 0,$$

as desired. ◀

Transcendental Number Theory. The transcendence degree of a field extension is a measure of the size of the extension. In fact, for finitely generated extensions of $\mathbb{L}/\mathbb{Q}$ (such as those that we consider), the transcendence degree indicates the largest cardinality of an algebraically independent subset of $\mathbb{L}$ over $\mathbb{Q}$. For a field extension $\mathbb{L}/\mathbb{Q}$, a subset $\{\xi_1, \ldots, \xi_n\} \subset \mathbb{L}$ is *algebraically independent* over $\mathbb{Q}$ if for each polynomial $P(X_1, \ldots, X_n) \in \mathbb{Q}[X_1, \ldots, X_n]$ we have that $P(\xi_1, \ldots, \xi_n) = 0$ only if P is identically zero.

It is useful to recall the Gelfond–Schneider Theorem that establishes the transcendentality of α^β for algebraic numbers α and β except for the cases where $\alpha = 0, 1$ or β is rational.

Schanuel's conjecture is a unifying prediction in transcendental number theory. If Schanuel's conjecture is true, then it generalises several of the principal results in transcendental number theory such as: the Gelfond–Schneider Theorem, the Lindemann–Weierstrass Theorem, and Baker's theorem (cf. [16, 3, 36]). The conjecture makes the following prediction: for $\xi_1, \ldots, \xi_n$ rationally linearly independent complex numbers, there is a subset of $\{\xi_1, \ldots, \xi_n, \mathrm{e}^{\xi_1}, \ldots, \mathrm{e}^{\xi_n}\}$ of size at least n that is algebraically independent over $\mathbb{Q}$.

▶ **Conjecture 10** (Schanuel). *Suppose that $\xi_1, \ldots, \xi_n \in \mathbb{C}$ are linearly independent over the rationals $\mathbb{Q}$. Then the transcendence degree of the field extension $\mathbb{Q}(\xi_1, \ldots, \xi_n, \mathrm{e}^{\xi_1}, \ldots, \mathrm{e}^{\xi_n})$ over $\mathbb{Q}$ is at least n.*

3 Hypergeometric Sequences with Quadratic Parameters

As an appetiser to the proofs of Theorems 15 and 17, we introduce our approach by first establishing decidability of the Threshold Problem for hypergeometric sequences with Gaussian integer parameters (Proposition 11 below). We also include a worked example in Example 12. Recall that the *Gaussian integers* $\mathbb{Z}[\mathrm{i}]$ are those complex numbers of the form $a + b\mathrm{i}$ for which $a, b \in \mathbb{Z}$.

▶ **Proposition 11.** *The Threshold Problem for hypergeometric sequences whose polynomial coefficients are monic and split over $\mathbb{Q}(\mathrm{i})$ are decidable.*

Proof. By Theorem 5 and Proposition 6, we need only consider hypergeometric sequences with harmonious shift quotients. Since the polynomial coefficients $p, q \in \mathbb{Z}[x]$ are monic, the roots and poles of the shift quotient are integers in $\mathbb{Q}(i)$; that is to say, they lie in $\mathbb{Z}[i]$. It follows from Lemma 3 that each such instance $(\langle u_n \rangle_n, t)$ of the Threshold Problem reduces to testing an equality of the form

$$\theta \pi^\ell \prod_m (\mathrm{e}^{b_m \pi} - \mathrm{e}^{-b_m \pi})^{\varepsilon_m} = t. \tag{3}$$

Here θ is rational and non-zero, $\ell \in \mathbb{Z}$, each pair $\Gamma(a_m + b_m \mathrm{i})\Gamma(a_m - b_m \mathrm{i})$ from Proposition 6 contributes a term $(\mathrm{e}^{b_m \pi} - \mathrm{e}^{-b_m \pi})^{\varepsilon_m}$ in the finite product, and $\varepsilon_m = \pm 1$.

We break the remainder of the proof into several subcases. Without loss of generality, we can assume that not all the roots and poles of r are rational integers, for otherwise testing (3) reduces to the decidable task of testing equality between two rational numbers.

We continue under the assumption that not all the roots and poles of r are rational integers. Let us now consider the product in (3). Up to multiplication by a rational, we can write the left-hand side of (3) in the form

$$\theta \pi^\ell \prod_m (\mathrm{e}^{b_m \pi} - \mathrm{e}^{-b_m \pi})^{\varepsilon_m} = \theta \pi^\ell \frac{f(\mathrm{e}^\pi)}{g(\mathrm{e}^\pi)}$$

where $f, g \in \mathbb{Q}[X]$ are non-trivial polynomials. Observe that $\mathrm{e}^\pi = (\mathrm{e}^{\pi \mathrm{i}})^{-\mathrm{i}} = (-1)^{-\mathrm{i}}$; thus, by the Gelfond–Schneider theorem, e^π is transcendental. We break the remainder of the proof into two cases depending on the rationality of $f(\mathrm{e}^\pi)/g(\mathrm{e}^\pi)$.

Suppose that $f(\mathrm{e}^\pi)/g(\mathrm{e}^\pi) \in \mathbb{Q}$. There are two further subcases to consider: if $\ell = 0$, then, once again, the equality test (3) reduces to deciding whether two rationals are equal; and if $\ell \neq 0$, then the equality test (3) reduces to testing whether π^ℓ is equal to a given rational number, which cannot hold for then π is necessarily algebraic.

All that remains is to consider the case where $f(\mathrm{e}^\pi)/g(\mathrm{e}^\pi) \notin \mathbb{Q}$, which we again split into two subcases. If $\ell = 0$, then it is trivial to see that (3) cannot hold as the right-hand side is rational. If $\ell \neq 0$ and we assume, for a contradiction, that (3) holds, then a simple rearrangement of (3) shows that there is a non-trivial polynomial $\mathcal{P} \in \mathbb{Q}[X, Y]$ such that $\mathcal{P}(\pi, \mathrm{e}^\pi) = 0$. This contradicts Nesterenko's theorem [20] that π and e^π are algebraically independent. We have dispatched each of the subcases and conclude the desired result. ◀

▶ **Example 12.** Suppose that $\langle u_n \rangle_{n=0}^\infty$ is the hypergeometric sequence defined by

$$u_n = \frac{n^2 - 4n + 5}{n^2 - 4n + 13} u_{n-1} \text{ with } u_0 = 1.$$

For the Threshold Problem, let us consider the problem instance $(\langle u_n \rangle_n, t)$ with $t \in \mathbb{Q}$. First, we evaluate the associated infinite product:

$$\prod_{k=0}^\infty \frac{k^2 - 4k + 5}{k^2 - 4k + 13} = \frac{\Gamma(-2 - 3\mathrm{i})\Gamma(-2 + 3\mathrm{i})}{\Gamma(-2 - \mathrm{i})\Gamma(-2 + \mathrm{i})} = \frac{\sinh(\pi)}{39 \sinh(3\pi)} = \frac{\mathrm{e}^{3\pi}(\mathrm{e}^{2\pi} - 1)}{39 \mathrm{e}^\pi(\mathrm{e}^{6\pi} - 1)}.$$

Second, as directed by the proof of Proposition 11, decidability of the Threshold Problem in this instance reduces to determining whether the following equality holds:

$$\frac{\mathrm{e}^{3\pi}(\mathrm{e}^{2\pi} - 1)}{39 \mathrm{e}^\pi(\mathrm{e}^{6\pi} - 1)} = t.$$

A simple rearrangement shows that if the above equality holds, then there is a non-trivial polynomial $p \in \mathbb{Q}[X]$ such that $p(\mathrm{e}^\pi) = 0$, from which we deduce that e^π is algebraic. However, by the Gelfond–Schneider theorem, e^π is transcendental. We have reached a contradiction and deduce that the aforementioned equality cannot hold.

Finally, as described in Proposition 6, the Threshold Problem reduces to an exhaustive search of a computable number of initial terms in the sequence $\langle u_n \rangle_n$. As an aside, by Proposition 7, we also obtain decidability of the Membership Problem for problem instances $(\langle u_n \rangle_n, t)$.

▶ **Remark 13.** Subject to appropriate changes and by employing Lemma 9, we can extend the result in Proposition 11 from instances of the Membership and Threshold Problems $(\langle u_n \rangle_n, t)$ with $u_0, t \in \mathbb{Q}$ to problem instances with $u_0, t \in \mathbb{L}(\pi, \mathrm{e}^\pi)$ where $\mathbb{L}$ is any finite Galois extension of $\mathbb{Q}$. This extension similarly holds for Theorem 15 (below).

▶ **Remark 14.** Analogous decision procedures to the proof of Proposition 11 also hold for other famous rings of integers: the Eisenstein, Kummer, and Kleinian integers. Recall that the *Eisenstein integers* are the elements of $\mathbb{Z}[\zeta_3] = \{a + b\zeta_3 : a, b \in \mathbb{Z}\}$ where $\zeta_3 := \mathrm{e}^{2\pi \mathrm{i}/3}$. Similarly, the *Kummer integers* are the elements of $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$. Finally, the *Kleinian integers* are the elements of $\mathbb{Z}[\mu] = \{a + b\mu : a, b \in \mathbb{Z}\}$ where $\mu = -1/2 + \sqrt{-7}/2$.

The claims for decidability in Remark 14, follow from the next theorem. We establish decidability of the Threshold Problem for hypergeometric sequences whose parameters are drawn from the ring of integers of an imaginary quadratic number field.

▶ **Theorem 15.** *The Threshold Problem for hypergeometric sequences whose polynomial coefficients are monic and split over an imaginary quadratic number field is decidable.*

Proof. Mutatis mutandis, the proof of Theorem 15 follows the approach in Proposition 11. For the sake of brevity, we shall indicate only the major changes to Proposition 11 here. Consider the ring of integers of an imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ where $-d \in \mathbb{N}$ is square-free. By Theorem 8, there are two cases to consider: first, when $d \not\equiv 1 \pmod 4$ and second, when $d \equiv 1 \pmod 4$.

We concentrate on the changes to the proof of Proposition 11 when $d \not\equiv 1 \pmod 4$. Like before, we can use the recurrence formula to write $\Gamma(a + b\sqrt{d}) = \theta\Gamma(b\sqrt{d})$ where $\theta \in \mathbb{N}$. Thus all that remains is to evaluate products $\Gamma(b\sqrt{d})\Gamma(-b\sqrt{d})$ of conjugate elements. By the reflection formula, we have

$$\Gamma(b\sqrt{d})\Gamma(-b\sqrt{d}) = -\frac{\pi}{b\sqrt{d}\sin(b\pi\sqrt{d})} = \frac{2\pi}{b\sqrt{-d}(\mathrm{e}^{\pi b\sqrt{-d}} - \mathrm{e}^{-\pi b\sqrt{-d}})}.$$

The important update is the product in (3). In our new setting, the product takes the form

$$\prod_m (\mathrm{e}^{\pi b\sqrt{-d}} - \mathrm{e}^{-\pi b\sqrt{-d}})^{\varepsilon_m}.$$

Observe that $\mathrm{e}^{\pi\sqrt{-d}}$ is transcendental (once again by Gelfond–Schneider) and that for each $-d \in \mathbb{N}$ the numbers π and $\mathrm{e}^{\pi\sqrt{-d}}$ are algebraically independent over $\mathbb{Q}$ [20, Corollary 6]. The rest of the proof in this case follows as before.

In the second case where $d \equiv 1 \pmod 4$ we must additionally deal with contributions of the form $\Gamma(b/2 + b\sqrt{d}/2)\Gamma(b/2 - b\sqrt{d}/2)$. This setting introduces cases where $2 \nmid b$, which we resolve by repeated application of the recurrence formula and the cosine variant of Euler's reflection formula. Indeed, we have

$$\Gamma(1/2 + b\sqrt{d}/2)\Gamma(1/2 - b\sqrt{d}/2) = \frac{\pi}{\cos(\pi b\sqrt{d}/2)} = \frac{2\pi}{\mathrm{e}^{\pi b\sqrt{-d}/2} + \mathrm{e}^{-\pi b\sqrt{-d}/2}},$$

and so we can construct an updated version of the product in (3). This update and analogous arguments for the transcendental properties of $e^{\pi\sqrt{-d}/2}$ let us conclude decidability in this case too. ◀

4 Conditional Decidability Subject to Schanuel's conjecture

In this section we shall give a generalisation (Theorem 17) of the decidability results in Section 3. The result applies to a strictly larger class of hypergeometric sequences; however, we sacrifice unconditional decidability. Briefly, termination of the decidability procedure in Theorem 17 is dependent on the truth of Schanuel's conjecture (further details are given in Remark 24). The motivation for studying reachability problems for this larger class arises from interest in the literature for limits of hypergeometric sequences with unnested radical and cyclotomic parameters (see Subsection 4.2).

Let us first describe the class of hypergeometric sequences for which we establish conditional decidability of the Membership and Threshold Problems. This will require us to introduce some notations and Property S below.

Consider $\mathcal{R}_f$ the multiset of roots of a given monic polynomial $f \in \mathbb{Z}[x]$ and $\mathcal{V}_f$ the multiset of irrational roots of f. We define the graph $\mathcal{G}_f := (\mathcal{V}_f, \mathcal{E}_f)$ with vertex set $\mathcal{V}_f$ and edge set $\mathcal{E}_f$. Here $\mathcal{E}_f$ is encoded using an adjacency matrix A (whose rows and columns are indexed by $\mathcal{V}_f$) as follows. For distinct $u, v \in \mathcal{V}_f$, let $A(u,v) = 1$ if $u = \rho + w$ and $v = \rho - w$ for some $\rho \in \frac{1}{2}\mathbb{Z}$ and w an algebraic number not in $\frac{1}{2}\mathbb{Z}$. Otherwise, let $A(u,v) = 0$.

▶ **Property S.** *We say that f has Property S if $\mathcal{G}_f$ admits a perfect matching.*

The motivation for introducing Property S is as follows: in combination with Lemma 3 and subject to the truth of Schanuel's conjecture, we can circumvent certain hard problems concerning the evaluation of gamma products if the input parameters possess certain symmetries. The main result of this section is a conditional decidability result for the class of hypergeometric sequences whose polynomial coefficients are monic and possess Property S.

▶ **Theorem 17.** *The Threshold Problem for hypergeometric sequences whose polynomial coefficients (as in (1)) are both monic and have Property S is decidable subject to the truth of Schanuel's conjecture.*

The remainder of this section is structured as follows. In Subsection 4.1 we list classes of polynomials with Property S and highlight straightforward corollaries (Corollaries 18, 22, and 23) of Theorem 17. In Subsection 4.2 we prove Theorem 17.

4.1 Polynomials with Property S

Even Polynomials. It is immediate that even polynomials with at least one irrational root possess Property S. Suppose that $f \in \mathbb{Z}[x]$ is a monic even polynomial. Then $f(-x) = f(x)$ and so $\mathcal{G}_f$ admits a perfect matching, as desired. More generally, we note that a horizontal translation of such a polynomial, say $\tilde{f}$, for which $\tilde{f}(\rho - x) = \tilde{f}(\rho + x)$ where $\rho \in \frac{1}{2}\mathbb{Z}$, also has Property S.

Quadratic Polynomials. Every irreducible monic quadratic polynomial in $\mathbb{Z}[x]$ possesses Property S. This observation is straightforward: consider an irreducible monic quadratic polynomial $x^2 + bx + c \in \mathbb{Z}[x]$. The roots of said quadratic satisfy $-\frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$. We note that $-\frac{b}{2} \in \frac{1}{2}\mathbb{Z}$ and $b^2 - 4c \neq 0$.

The following corollary is a straightforward consequence of Theorem 17: we have conditional decidability of the Threshold Problem for hypergeometric sequences whose parameters are drawn from the rings of integers of quadratic number fields.

▶ **Corollary 18.** *The Threshold Problem for hypergeometric sequences whose polynomial coefficients are monic with irreducible factors that are either linear or quadratic is decidable subject to the truth of Schanuel's conjecture.*

We note that the assumption in Corollary 18 permits us to draw sequence parameters from the integers of any number of quadratic fields in order to establish conditional decidability of both the Threshold and Membership Problems. This is in stark contrast to the work in [15] that establishes decidability of Membership for hypergeometric sequences whose parameters are drawn from the integers of a single quadratic field.

Algebraic Numbers with Rational Real Part.   Consider the class $\mathcal{C}$ of monic irreducible polynomials in $\mathbb{Q}[x]$ that possess a root with rational real part. Trivially, a linear polynomial with rational coefficients is always a member of $\mathcal{C}$ since the single root of the polynomial is rational. It is straightforward to see that an irreducible quadratic polynomial $x^2+bx+c \in \mathbb{Q}[x]$ is in $\mathcal{C}$ if and only if $b^2-4c < 0$. Dilcher, Noble, and Smyth [6] achieve a complete classification of the class $\mathcal{C}$ with the following result.

▶ **Theorem 19** ([6, Theorem 1]). *Let f be a polynomial of degree at least three. Then $f \in \mathcal{C}$ if and only if $f(x) = g((x - \rho)^2)$ for some $\rho \in \mathbb{Q}$ and monic irreducible $g \in \mathbb{Q}[X]$ that has a negative real root. In this case, f has a root with a rational real part ρ.*

The following corollary is key to understanding the connection to Property S and demonstrates the 2-fold rotational symmetry of the set of roots of a polynomial in $\mathcal{C}$.

▶ **Corollary 20** ([6, Corollary 2]). *Suppose that $f \in \mathcal{C}$ has degree at least three. The roots of f that have rational real part have the same real part ρ. Further, we have $f(\rho - x) = f(\rho + x)$.*

The trivial lemma below shows that if an algebraic integer has rational real part, then said real part lies in $\frac{1}{2}\mathbb{Z}$.

▶ **Lemma 21.** *Let α be an algebraic integer with $\mathrm{Re}(\alpha) \in \mathbb{Q}$. Then $\mathrm{Re}(\alpha) \in \frac{1}{2}\mathbb{Z}$.*

Proof.   Since $\mathrm{Re}(\alpha) = \frac{1}{2}(\alpha + \overline{\alpha})$, we deduce that $\alpha + \overline{\alpha} \in \mathbb{Q}$. Observe that both α and $\overline{\alpha}$ are algebraic integers and so $\alpha + \overline{\alpha}$ is too due to the closure of the ring of algebraic integers. We deduce that $\alpha + \overline{\alpha} \in \mathbb{Z}$, from which the desired result follows.   ◀

When we combine the result in Theorem 17 with the observations in Corollary 20 and Lemma 21 we obtain the following corollary.

▶ **Corollary 22.** *The Threshold Problem for hypergeometric sequences whose polynomial coefficients are monic with irreducible factors in $\mathcal{C}$ is decidable subject to the truth of Schanuel's conjecture.*

Unnested Radicals and Cyclotomic Polynomials.   We now highlight the class of hypergeometric sequences with parameters determined by unnested radicals and cyclotomic polynomials. The limits of such sequences are considered in both [29, pp. 753–757] and [7]. In the sequel we use Φ_d to denote the dth cyclotomic polynomial. We state and prove the following corollary of Theorem 17.

▶ **Corollary 23.** *The Threshold Problem for hypergeometric sequences whose polynomial coefficients have irreducible factors of the form $x^d - a$ with $d \in 2\mathbb{Z}$, or Φ_d with $d \in 4\mathbb{Z}$ is decidable subject to the truth of Schanuel's conjecture.*

Proof. In light of Theorem 17, it is sufficient to prove that non-linear polynomials of the form $x^d - a$ with $d \in 2\mathbb{Z}$, or Φ_d with $d \in 4\mathbb{Z}$ have Property S.

We first consider irreducible factors of the form $x^d - a \in \mathbb{Z}[x]$. The roots of $x^d - a$ are unnested radicals of the form $\sqrt[m]{a}\omega_m^j$ for $j \in \{0, \ldots, m-1\}$ where $\omega_d := \mathrm{e}^{2\pi i/d}$. Recall that $x^d - a \in \mathbb{Z}[x]$ is irreducible if a is not the Nth power of an element of $\mathbb{Q}$ for some $N > 1$ with $N \mid d$ (cf. [19]). When d is even, it follows that $x^d - a$ is even and so possesses Property S.

The cyclotomic polynomial Φ_d is irreducible and its roots are the primitive dth roots of unity. Under the assumption that d is a multiple of four, it is straightforward to show that Φ_d is even. It follows that for such d, Φ_d has Property S. ◀

We note with our approach we cannot lift the additional assumption that $d \in 4\mathbb{Z}$ for cyclotomic factors: the set of primitive 18th roots of unity $\{\omega_{18}, \omega_{18}^5, \omega_{18}^7, \omega_{18}^{11}, \omega_{18}^{13}, \omega_{18}^{17}\}$ show that Φ_{18} does not have Property S.

4.2 Proof of Theorem 17

We now prove our main result. A worked example, demonstrating our approach, is given in Appendix B.

▶ **Theorem 17.** *The Threshold Problem for hypergeometric sequences whose polynomial coefficients (as in (1)) are both monic and have Property S is decidable subject to the truth of Schanuel's conjecture.*

Proof. Let $\langle u_n \rangle_{n=0}^\infty$ be a hypergeometric sequence whose polynomial coefficients are both monic and satisfy Property S. We assume without loss of generality that the associated shift quotient $r(x) := q(x)/p(x)$ is harmonious. As previously noted, each instance of the Threshold Problem $(\langle u_n \rangle_n, t)$ with $t \in \mathbb{Q}$ reduces to checking an equality of the form

$$\Gamma(\beta_1) \cdots \Gamma(\beta_d) = t\Gamma(\alpha_1) \cdots \Gamma(\alpha_d) \tag{4}$$

where $\{\alpha_1, \ldots, \alpha_d\} =: \mathcal{R}_p$ and $\{\beta_1, \ldots, \beta_d\} =: \mathcal{R}_q$ are the multisets of the roots of the respective polynomial coefficients p and q. The proof is split into two parts: a reduction to an equality testing problem and a proof of decidability subject to the truth of Schanuel's conjecture.

Reduction to Equality Testing. Since pq is a monic polynomial, the rational elements of the multisets $\mathcal{R}_p$ and $\mathcal{R}_q$ are rational integers (so we can assume their contributions are absorbed into the rational parameter t in (4)). Further, under Property S there are perfect matchings on both of the graphs $\mathcal{G}_p$ and $\mathcal{G}_q$, which we denote by $\mathcal{M}_p$ and $\mathcal{M}_q$ respectively. Indeed, we recall that in $\mathcal{G}_p$ the edges in $\mathcal{M}_p$ are of the form $e(\alpha_-, \alpha_+)$ such that $\alpha_\pm = \rho_e \pm w_e$ where $\rho_e \in \frac{1}{2}\mathbb{Z}$ and w_e is an algebraic number not in $\frac{1}{2}\mathbb{Z}$ (and similarly for $\mathcal{G}_q$ and $\mathcal{M}_q$). In this setting, we repeatedly apply the recurrence formula and absorb the resulting algebraic factors into a single term θ in order to rewrite (4) as

$$\prod_{i \in \mathcal{M}_q} \Gamma(w_i)\Gamma(-w_i) = \theta \prod_{j \in \mathcal{M}_p} \Gamma(w_j)\Gamma(-w_j). \tag{5}$$

Consider the set of algebraic numbers $\{w_1, \ldots, w_M\}$ determined by the parameters in (5). We denote by $S' := \{s'_1, \ldots, s'_m\}$ a maximal subset of $\{w_1, \ldots, w_M\}$ for which the elements of $\{\pi, \pi\mathrm{i}\} \cup \pi S'$ are $\mathbb{Q}$-linearly independent (here $\pi S' := \{\pi s'_1, \ldots, \pi s'_m\}$). Then for each $k \in \{1, \ldots, M\}$, write w_k as a $\mathbb{Q}$-linear sum of elements in $\{s'_1, \ldots, s'_m\}$ so that

$$w_k = \frac{x_{k1}}{y_{k1}} s'_1 + \cdots + \frac{x_{km}}{y_{km}} s'_m.$$

We define $s_j := s'_j / \mathrm{lcm}(y_{1j}, y_{2j}, \ldots, y_{Mj})$ for each $j \in \{1, \ldots, m\}$. Now we can write each $w_k \in \{w_1, \ldots, w_M\}$ as a $\mathbb{Z}$-linear sum of elements in the normalised set $S := \{s_1, \ldots, s_m\}$.

We apply Lemma 3 to (5) and, by the preceding paragraph, determine that the problem of testing the equality in (5) reduces to that of determining whether a certain non-trivial polynomial with coefficients in $\mathbb{Q}(\theta)$ vanishes at a given point (this is analogous to the process in Proposition 11 and Example 12). More specifically, we want to test whether a given non-trivial polynomial $P \in \mathbb{Q}(\theta)[X_1, \ldots, X_{4m+4}]$ satisfies

$$P\left(\pi, \pi\mathrm{i}, \pi S, \pi S\mathrm{i}, \mathrm{e}^\pi, \mathrm{e}^{\pi\mathrm{i}}, \mathrm{e}^{\pi S}, \mathrm{e}^{\pi S\mathrm{i}}\right) = 0. \tag{6}$$

Here $\pi S\mathrm{i} := \{\pi s_1 \mathrm{i}, \ldots, \pi s_m \mathrm{i}\}$, $\mathrm{e}^{\pi S} := \{\mathrm{e}^{\pi s_1}, \ldots, \mathrm{e}^{\pi s_m}\}$, and likewise for $\mathrm{e}^{\pi S\mathrm{i}}$. Note that we need only consider a polynomial in $4m + 4$ variables as the parameters $\{w_1, \ldots, w_M\}$ of our problem instance are given by $\mathbb{Z}$-linear combinations of the elements of $S \cup S\mathrm{i}$. We claim that the equality in (6) cannot hold if Schanuel's conjecture is true. We prove this claim below.

Conditional Decidability Subject to Schanuel's conjecture. Consider the set

$$\mathfrak{S} := \left\{\pi, \pi\mathrm{i}, \pi S, \pi S\mathrm{i}, \mathrm{e}^\pi, \mathrm{e}^{\pi\mathrm{i}}, \mathrm{e}^{\pi S}, \mathrm{e}^{\pi S\mathrm{i}}\right\}$$

with cardinality $4m + 4$. We observe that the elements in the subset $\{\pi, \pi\mathrm{i}, \pi S, \pi S\mathrm{i}\} \subset \mathfrak{S}$ are $\mathbb{Q}$-linearly independent. It follows that if Schanuel's conjecture is true, then $\mathfrak{S}$ possesses a subset of cardinality at least $2m + 2$ whose elements are algebraically independent. By construction, this algebraically independent subset is necessarily $\{\pi, \mathrm{e}^\pi, \mathrm{e}^{\pi S}, \mathrm{e}^{\pi S\mathrm{i}}\}$ since the $2m + 2$ elements of $\{\pi, \pi\mathrm{i}, \pi S, \pi S\mathrm{i}\}$ are pairwise algebraically dependent and $\mathrm{e}^{\pi\mathrm{i}} = -1$.

We now rewrite the equality in (6) in terms of the (obvious) polynomial $\hat{P}$ that absorbs the algebraically dependent parameters of $S \cup S\mathrm{i}$ into the coefficients. That is to say, we employ a polynomial $\hat{P} \in \mathbb{L}(X_1, \ldots, X_{2m+2})$ where $\mathbb{L}$ is the Galois closure of the number field $\mathbb{Q}(\theta)(S, S\mathrm{i})$ and evaluate $\hat{P}$ on the algebraically independent subset $\{\pi, \mathrm{e}^\pi, \mathrm{e}^{\pi S}, \mathrm{e}^{\pi S\mathrm{i}}\}$ of $\mathfrak{S}$. It follows that the equality in (6) holds only if

$$\hat{P}(\pi, \mathrm{e}^\pi, \mathrm{e}^{\pi S}, \mathrm{e}^{\pi S\mathrm{i}}) = 0. \tag{7}$$

By Lemma 9, the equality in (7) holds only if there exists a non-trivial polynomial $Q \in \mathbb{Q}[X_1, \ldots, X_{2m+2}]$ such that $Q(\pi, \mathrm{e}^\pi, \mathrm{e}^{\pi S}, \mathrm{e}^{\pi S\mathrm{i}}) = 0$. Recall that if Schanuel's conjecture is true, then the elements of the set $\{\pi, \mathrm{e}^\pi, \mathrm{e}^{\pi S}, \mathrm{e}^{\pi S\mathrm{i}}\}$ are algebraically independent over $\mathbb{Q}$, from which we deduce that the preceding equality cannot hold.

Subject to the truth of Schanuel's conjecture, we can determine equality tests of the above form. Thus we have conditional decidability of the Threshold Problem for the desired class of hypergeometric sequences. ◀

▶ **Remark 24.** We note that the equality test $\mathcal{Q}(\pi, \mathrm{e}^\pi, \mathrm{e}^{\pi S}, \mathrm{e}^{\pi S\mathrm{i}}) = 0$ can be realised as a proposition in the first-order theory of the reals with exponentiation. Macintyre and Wilkie [17] established decidability of said theory subject to the truth of Schanuel's conjecture. As noted in previous works, careful inspection of Macintyre and Wilkie's algorithm reveals

that correctness is independent of the truth of Schanuel's conjecture. Indeed, Schanuel's conjecture is only used to prove termination. Thus if we apply Macintyre and Wilkie's algorithm to determine whether the equality $\mathcal{Q}(\pi, e^{\pi}, e^{\pi S}, e^{\pi Si}) = 0$ holds and find the procedure terminates, then the output is certainly correct.

We note that Macintyre and Wilkie's algorithm terminates unless the inputs constitute a counterexample to Schanuel's conjecture. Thus, the process underlying the proof of Theorem 17 presents an interesting prospect in the sense described by Richardson in [31] (see also [32]) "A failure of the [process] to terminate would be even more interesting than [its] success."

▶ **Remark 25.** Recall Remark 13 where we extended our class of problem instances to include setups with $u_0, t \in \mathbb{L}(\pi, e^{\pi})$. Under the assumption that Schanuel's conjecture is true we can include a broader range of setups. Notice, for instance, that the algebraic independence of π and e is currently unknown; however, if Schanuel's conjecture is true, then it follows that π and e are algebraically independent. Thus, subject to the truth of Schanuel's conjecture, we can extend our results in Proposition 11, Theorem 15, and Theorem 17 to instances where $u_0, t \in \mathbb{L}(\pi, e)$. In fact, we can go further in this direction since the truth of Schanuel's conjecture implies the algebraic independence of the numbers e, e^{π}, e^e, π, π^{π}, π^e, 2^{π}, 2^e, $\log \pi$, $\log 2$, $\log 3$, $\log \log 2$, $(\log 2)^{\log 3}$, $2^{\sqrt{2}}$, and many more (cf. [30, Conjecture S_7]).

5 Conclusion

Summary. In this paper we establish (un)conditional decidability results for the Threshold Problem for hypergeometric sequences and, as a side-effect, (un)conditional decidability results for the Membership Problem for hypergeometric sequences. Previous works have considered the Membership Problem for hypergeometric sequences [13, 22]; however, the approach in those works cannot handle instances of the Threshold Problem. The novelty of our approach is the combination of a classical convergence result (Theorem 5) with results on the algebraic independence of common mathematical constants.

Obstacles. Let us illustrate an immediate obstacle to the methods herein. We cannot handle parameters drawn from biquadratic fields because the monic polynomials that split over such fields are not necessarily amenable to our approach. Recall that biquadratic fields are a particularly well-behaved class of quartic fields (such as $\mathbb{Q}(\sqrt{5}, \sqrt{13})$ and $\mathbb{Q}(\sqrt{21}, \sqrt{33})$). For example the minimal polynomial $x^4 - 5x^3 - 71x^2 + 120x + 1044$ of $(5 + 3\sqrt{5} + \sqrt{13} + 3\sqrt{65})/4 \in \mathbb{Q}(\sqrt{5}, \sqrt{13})$ does not have Property S. Similarly, the minimal polynomial $x^4 - x^3 - 16x^2 + 37x - 17$ of $(1 + \sqrt{21} + \sqrt{33} - \sqrt{77})/4 \in \mathbb{Q}(\sqrt{21}, \sqrt{33})$ does not have Property S. Both of these examples are taken from [38].

We also note that the class of sequences we can handle does not permit standard operations on the parameters such as addition. Consider, for example, that $\sqrt{2}$ and $\sqrt[4]{2}$ are both unnested radicals whose minimal polynomials satisfy Property S; however, the minimal polynomial $x^4 - 4x^2 - 8x + 2$ of $\sqrt{2} + \sqrt[4]{2}$ does not possess Property S.

It is not clear how to extend our approach to hypergeometric sequences with larger classes of parameters. For example, the parameters herein are all algebraic integers. Even in the restricted setting of hypergeometric sequences with rational parameters (as in the work of Nosan et al. [22]) it is beyond the state of the art to evaluate equalities between associated gamma products. Indeed, for $s \in \{1/6, 1/4, 1/3, 2/3, 3/4\}$ and $n \in \mathbb{N}$, it is known that $\Gamma(n + s)$ is a transcendental number and algebraically independent of π (cf. [36]). However, transcendence of the gamma function at other rational points is not known. It is notable that

for rational parameters determining equality between gamma products is decidable subject to the truth of the Rohrlich–Lang conjecture (which itself concerns multiplicative relations for the gamma function [16, 36]).

Directions for Future Work.   We give one class of hypergeometric sequences whose parameters link the related works. Consider the class of hypergeometric sequences whose parameters lie in $\mathbb{Q}(i)$. For such sequences, decidability of both the Membership and Threshold Problems is open.

The sequences in this class generalise the setting discussed here (and in work by Kenison et al. [15]) by removing the condition that the polynomial coefficients of the defining recurrence relation are both monic. Further, results in this direction would extend the discussion of the Membership Problem for hypergeometric sequences with rational parameters in work by Nosan et al. [22] as well as those sequences whose polynomial coefficients have irreducible factors in $\mathcal{C}$ (i.e., each irreducible factor has a root with rational real part) discussed herein.

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A Appendixed Proofs

▶ **Lemma 4.** *Consider the class of hypergeometric sequences $\langle u_n \rangle_n$ whose shift quotients $r(n)$ either diverge to $\pm\infty$ or converge to a limit ℓ with $|\ell| \neq 1$. For this class, the Membership and Threshold Problems are both decidable.*

Proof. When $r(n) \equiv 0$ decidability of both problems is trivial. So we assume that this is not the case for the remainder of the proof. Write the shift quotient $r(n) = c\frac{q(n)}{p(n)}$ where $p, q \in \mathbb{Q}[x]$ are monic polynomials and $c \in \mathbb{Q}$. Without loss of generality, we can assume that $c > 0$; for otherwise, sequence $\langle u_n \rangle_{n=0}^{\infty}$ is given by the interlacing of two hypergeometric sequences with this property. Let us assume that $r(n)$ diverges to $+\infty$. In this case it is easily seen that, for each $t \in \mathbb{Q}$, there exists a computable $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then $|u_n| = |u_0 \cdot \prod_{k=0}^{n} r(k)| > |t|$. Thus to determine the Threshold Problem in this instance, we need only determine the ultimate sign of $\langle u_n \rangle_{n=0}^{\infty}$ (which is straightforward). Moreover, we can compute a bound N_1 after which the sign of $\langle u_n \rangle_{n=0}^{\infty}$ is constant. Thus the Threshold Problem in such instances reduces to an exhaustive search that asks whether $u_n \geq t$ for each $n \in \{0, 1, \ldots, \max\{N_0, N_1\}\}$. Mutatis mutandis, decidability is similarly established for instances of the Threshold Problem where $r(n)$ converges to a limit ℓ with $|\ell| \neq 1$.

The argument for the Membership Problem is similar and given in full in [22]. ◀

▶ **Proposition 6.** *Let $\langle u_n \rangle_n$ be a hypergeometric sequence whose shift quotient is given by a ratio of two polynomials with real coefficients. For such sequences, the Membership and Threshold Problems are both Turing-reducible to the following decision problem. Given $d \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_d \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ (the roots of some $P(x) \in \mathbb{R}[x]$), and $\beta_1, \ldots, \beta_d \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ (the roots of some $Q(x) \in \mathbb{R}[x]$), determine whether*

$$\frac{\Gamma(\beta_1) \cdots \Gamma(\beta_d)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_d)} = t$$

for $t \in \mathbb{Q} \setminus 0$.

Proof. From Lemma 4, we need only consider cases where the associated shift quotient $r(k)$ converges to ± 1 and, by Theorem 5, we can assume without loss of generality that $r(k)$ is harmonious. We treat the case that the sequence of partial products $\langle \prod_{k=0}^{n} r(k) \rangle_n$ is eventually strictly increasing. The case where the sequence of partial products is eventually strictly decreasing follows *mutatis mutandis*.

Consider an instance $(\langle u_n \rangle_n, t)$ of the Threshold Problem with $r(k)$ as above and recall our running assumption that $u_0 = 1$. Let $\tau := \prod_{k=0}^{\infty} r(k)$. We assume that the sequence $\langle u_n \rangle_n$, whose terms $u_n = \prod_{k=0}^{n} r(k)$ are given by partial products, is eventually strictly increasing. Then there exists a computable $N \in \mathbb{N}$ such that $u_n < \tau$ for each $n \geq N$. There are two subcases to consider. First, if $\tau \leq t$ then it is clear that $u_n < t$ for each $n \geq N$ and so we return the answer `no` to the Threshold Problem. Second, if $\tau > t$ then there exists an $N_1 \in \mathbb{N}$ such that $u_n > t$ for each $n \geq N_1$. So decidability of Threshold in this instance reduces to an exhaustive check that asks whether $u_n \geq t$ for each $n \in \{0, 1, \ldots, N-1\}$.

All that remains is to decide whether $\tau \leq t$. It is clear that, by computing τ to sufficient precision, the problem of determining whether $\tau < t$ or $\tau > t$ is recursively enumerable. Thus we need only test whether the equality $\tau = t$ holds. By Theorem 5, we know that $\tau = \prod_{j=1}^{m} \Gamma(\beta_j)/\Gamma(\alpha_j)$, from which we deduce the desired result.

For the sake of brevity, we omit the argument for the reduction from the Membership Problem, which is near identical to the reasoning displayed above. ◀

▶ **Proposition 7.** *For hypergeometric sequences, decidability of the Membership Problem Turing-reduces to that of the Threshold Problem.*

Proof. By Theorem 5 and Proposition 6, we need only consider hypergeometric sequences with harmonious shift quotients. Thus, we continue under this assumption.

Let $(\langle u_n \rangle_n, t)$ be an instance of the Membership Problem as above and, in addition, assume that $\langle u_n \rangle_n$ is eventually decreasing. We note there is a computable bound $N_0 \in \mathbb{N}$ such that $u_{n+1} \leq u_n$ for all $n \geq N_0$. Now let τ be the limit of the the sequence $\langle u_n \rangle_n$. Either we have that $\tau = t$ or $\tau \neq t$. We note that the Membership Problem is decidable when $\tau \neq t$ and so it remains to test cases when $\tau = t$.

- Suppose that an oracle for the Threshold Problem returns the answer `yes` to the problem instance $(\langle u_n \rangle_{n=N_0}^{\infty}, \tau)$. Since $u_{n+1} < u_n$ for all $n \geq N_0$, we deduce that τ is not a member of the sequence $\langle u_n \rangle_{n=N_0}^{\infty}$. Thus all that remains is to test whether $\tau \in \{u_0, u_1, \ldots, u_{N_0-1}\}$.
- Suppose that an oracle for the Threshold Problem returns the answer `no` to the problem instance $(\langle u_n \rangle_{n=N_0}^{\infty}, \tau)$. Then there is a computable bound N_1 such that $u_n < \tau$ for all $n \geq N_1$. Thus we can decide the Membership Problem by testing whether $\tau \in \{u_0, u_1, \ldots, u_{N_1-1}\}$.

We note a similar argument to that given above holds for testing the Membership Problem for hypergeometric sequences that are eventually increasing. Thus we deduce the desired result: that decidability of the Membership Problem for hypergeometric sequences Turing-reduces to that of the Threshold Problem for hypergeometric sequences. ◀

B Threshold for the Kepler–Bouwkamp Constant Approximation

In this section we demonstrate that we can conditionally determine the Threshold Problem for instances $(\langle v_n \rangle_{n=3}^{\infty}, t)$ where sequence $\langle v_n \rangle_{n=3}^{\infty}$ is the recurrence sequence associated with the approximation of the Kepler–Bouwkamp Constant (Example 1) by the $[2, 2]$-Padé approximant of $\cos(\cdot)$.

Let us begin. Recall that the sequence $\langle v_n \rangle_{n=3}^{\infty}$ has terms given by $v_n := \prod_{k=3}^{n} \frac{12k^2 - 5\pi^2}{12k^2 + \pi^2}$. By Proposition 6, decidability of problem instance $(\langle v_n \rangle_{n=3}^{\infty}, t)$ reduces to determining whether

$$\prod_{k=3}^{\infty} \frac{12k^2 - 5\pi^2}{12k^2 + \pi^2} = \frac{\Gamma(3 - \frac{\mathrm{i}}{6}\sqrt{3}\pi)\Gamma(3 + \frac{\mathrm{i}}{6}\sqrt{3}\pi)}{\Gamma(3 - \frac{1}{6}\sqrt{15}\pi)\Gamma(3 + \frac{1}{6}\sqrt{15}\pi)} \tag{8}$$

is equal to t. (We note that the formulation as a gamma product is given to us by Theorem 5.)

We consider the numerator and denominator in turn. First, we use the translation and reflection properties of the gamma function to write the numerator of (8) as

$$\Gamma\left(3 - \frac{\sqrt{3}\pi i}{6}\right)\Gamma\left(3 + \frac{\sqrt{3}\pi i}{6}\right) = \Gamma\left(-\frac{\sqrt{3}\pi i}{6}\right)\Gamma\left(\frac{\sqrt{3}\pi i}{6}\right)\prod_{k=0}^{2}\left(k^2 + \frac{\pi^2}{12}\right)$$

$$= \frac{4\sqrt{3}}{e^{\frac{\pi^2}{2\sqrt{3}}} - e^{-\frac{\pi^2}{2\sqrt{3}}}}\prod_{k=0}^{2}\left(k^2 + \frac{\pi^2}{12}\right)$$

$$= \frac{4\sqrt{3}e^{\frac{\pi^2}{2\sqrt{3}}}}{e^{\frac{\pi^2}{\sqrt{3}}} - 1}\prod_{k=0}^{2}\left(k^2 + \frac{\pi^2}{12}\right). \tag{9}$$

Second, we likewise write the denominator of (8) as

$$\Gamma\left(3 - \frac{\sqrt{15}\pi}{6}\right)\Gamma\left(3 + \frac{\sqrt{15}\pi}{6}\right) = \Gamma\left(-\frac{\sqrt{15}\pi}{6}\right)\Gamma\left(\frac{\sqrt{15}\pi}{6}\right)\prod_{k=0}^{2}\left(k^2 - \frac{5\pi^2}{12}\right)$$

$$= -\frac{12i}{\sqrt{15}\left(e^{\frac{\sqrt{15}\pi^2 i}{6}} - e^{-\frac{\sqrt{15}\pi^2 i}{6}}\right)}\prod_{k=0}^{2}\left(k^2 - \frac{5\pi^2}{12}\right)$$

$$= -\frac{12ie^{\frac{\pi^2\sqrt{15}i}{6}}}{\sqrt{15}\left(e^{\frac{\pi^2\sqrt{15}i}{3}} - 1\right)}\prod_{k=0}^{2}\left(k^2 - \frac{5\pi^2}{12}\right). \tag{10}$$

Let us consider the decision problem at hand. Taken together, we are tasked to determine whether the ratio of (9) and (10) is equal to t. For this equality to hold, a simple rearrangement argument leads us to the following: there is a non-trivial polynomial $\mathcal{Q} \in \mathbb{Q}[x, y, z] = 0$ such that $\mathcal{Q}\left(\pi^2, e^{\frac{\pi^2}{\sqrt{3}}}, e^{\frac{\sqrt{15}\pi^2 i}{3}}\right) = 0$ (the fact that such a polynomial with rational coefficients exists is guaranteed by Lemma 9). However, we claim that no such polynomial $\mathcal{Q}$ exists if Schanuel's conjecture is true.

Let us prove the above claim. Consider the set

$$\mathfrak{S} := \left\{\frac{\pi^2}{\sqrt{3}}, \frac{\sqrt{15}\pi^2 i}{3}, \pi i, e^{\frac{\pi^2}{\sqrt{3}}}, e^{\frac{\sqrt{15}\pi^2 i}{3}}, e^{\pi i}\right\}.$$

We note that the elements of $\left\{\frac{\pi^2}{\sqrt{3}}, \frac{\sqrt{15}\pi^2 i}{3}, \pi i\right\}$ are $\mathbb{Q}$-linearly independent. Thus, Schanuel's conjecture predicts that there is a subset of $\mathfrak{S}$ of size at least 3 whose elements are algebraically independent. Since $\frac{\pi^2}{\sqrt{3}}$ and $\frac{\sqrt{15}\pi^2 i}{3}$ are algebraically dependent and $e^{\pi i} = -1$, we deduce that, subject to the truth of Schanuel's conjecture, the elements of $\left\{\pi^2, e^{\frac{\pi^2}{\sqrt{3}}}, e^{\frac{\sqrt{15}\pi^2 i}{3}}\right\}$ are algebraically independent. From the preceding work, we deduce that there is no non-trivial polynomial $\mathcal{Q}$ for which $\mathcal{Q}\left(\pi^2, e^{\frac{\pi^2}{\sqrt{3}}}, e^{\frac{\sqrt{15}\pi^2 i}{3}}\right) = 0$. It follows that the desired equality cannot hold. Thus we can conditionally decide the Threshold Problem for instances $(\langle v_n \rangle_{n=0}^{\infty}, t)$ with $t \in \mathbb{Q}$.