Another Hamiltonian Cycle in Bipartite Pfaffian Graphs

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Abstract

Finding a Hamiltonian cycle in a given graph is computationally challenging, and in general remains so even when one is further given one Hamiltonian cycle in the graph and asked to find another. In fact, no significantly faster algorithms are known for finding another Hamiltonian cycle than for finding a first one even in the setting where another Hamiltonian cycle is structurally guaranteed to exist, such as for odd-degree graphs. We identify a graph class – the bipartite Pfaffian graphs of minimum degree three – where it is NP-complete to decide whether a given graph in the class is Hamiltonian, but when presented with a Hamiltonian cycle as part of the input, another Hamiltonian cycle can be found efficiently.

We prove that Thomason's lollipop method [Ann. Discrete Math., 1978], a well-known algorithm for finding another Hamiltonian cycle, runs in a linear number of steps in cubic bipartite Pfaffian graphs. This was conjectured for cubic bipartite planar graphs by Haddadan [MSc thesis, Waterloo, 2015]; in contrast, examples are known of both cubic bipartite graphs and cubic planar graphs where the lollipop method takes exponential time.

Beyond the reach of the lollipop method, we address a slightly more general graph class and present two algorithms, one running in linear-time and one operating in logarithmic space, that take as input (i) a bipartite Pfaffian graph G of minimum degree three, (ii) a Hamiltonian cycle H in G, and (iii) an edge e in H, and output at least three other Hamiltonian cycles through the edge e in G.

We also present further improved algorithms for finding optimal traveling salesperson tours and counting Hamiltonian cycles in bipartite planar graphs with running times that are not achieved yet in general planar graphs.

Our technique also has purely graph-theoretical consequences; for example, we show that every cubic bipartite Pfaffian graph has either zero or at least six distinct Hamiltonian cycles; the latter case is tight for the cube graph.

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1 Introduction

Finding a Hamiltonian cycle in a given undirected graph is a well-known, well-researched, and hard problem. This paper studies the question whether knowledge of one Hamiltonian cycle helps in finding another one. More precisely, the Another Hamiltonian Cycle problem asks, given as input (i) a graph[1] G, (ii) a Hamiltonian cycle H in G, and (iii) an edge $e \in E(H)$, to find another Hamiltonian cycle $H' \neq H$ in G with $e \in E(H')$.

Our interest is in the class of bipartite *Pfaffian*[2] graphs, a superclass of the bipartite planar graphs. This focus is partly motivated by the fact that both in cubic bipartite graphs and cubic planar graphs, a well-known general algorithm for finding another Hamiltonian cycle, Thomason's lollipop method [27], requires exponential time in the worst case, cf. Section 1.1. Also, the problem of deciding if the graph contains a Hamiltonian cycle at all remains NP-hard in the family of cubic bipartite planar graphs, as proved by Akiyama, Nishizeki, and Saito [1].

As our main result, we show that Another Hamiltonian Cycle admits both a linear-time algorithm as well as a logarithmic-space algorithm in bipartite Pfaffian graphs of minimum degree three. Further restricted to cubic bipartite Pfaffian graphs, we prove that Thomason's lollipop method runs in a linear number of steps and can be implemented to run in linear time. This is to our knowledge a first example of a nontrivial graph class where Another Hamiltonian Cycle is efficiently solvable; such an example was solicited by Kintali [18]. By trivial we here intend a graph class in which Hamiltonicity detection is NP-hard but is artificially constructed to ensure a simple local rerouting of any Hamiltonian cycle. Rather, in our case the global properties of bipartiteness, Pfaffianity, and the everywhere-local property of minimum-degree three, interplay to enable an efficient algorithm. Without the minimum-degree constraint, the problem is NP-hard (see the full version [4]).

Our techniques have also purely graph-theoretic consequences. We show that every cubic bipartite Pfaffian graph has at least three other Hamiltonian cycles through any edge of a Hamiltonian cycle. All three Hamiltonian cycles can be found in linear time. We also show that such Hamiltonian graphs must have at least six Hamiltonian cycles. The 8-vertex cube graph (⧈), the canonical example in this class, is an extremal example to both results. It has precisely six distinct Hamiltonian cycles with every graph edge in exactly four of them.

1.1 Motivation and earlier work

While a graph need not be Hamiltonian, and a Hamiltonian graph need not admit another Hamiltonian cycle, there exist graph families with Hamiltonian members where another Hamiltonian cycle is always known to exist. Perhaps the most prominent such family are the odd-degree graphs, which via Smith's Theorem (see [30]) have an even number of Hamiltonian cycles through any given edge. Thomason [27] gave a constructive proof by describing an algorithm that solves for another Hamiltonian cycle in cubic graphs. The algorithm is often called Thomason's lollipop method as it transforms a Hamiltonian cycle to another one by a sequence of lollipop graphs, see Section 3.3 for a precise description of the algorithm. Dropping the requirement that the Hamiltonian cycle should go through a specific edge,

1 We tacitly work with undirected simple loopless graphs unless mentioned otherwise, as well as assume knowledge of standard graph-theoretic terminology [32]. Our conventions with graphs can be found in Section 1.6.

2 We postpone a precise definition and motivation of Pfaffian graphs to Section 1.5. Planar graphs are Pfaffian.

Bosák [5] proved that every cubic bipartite graph has an even number of Hamiltonian cycles. Thomassen [28, 29] showed that no bipartite graph in which every vertex in one of the two parts of the bipartition has degree at least three, has a unique Hamiltonian cycle. A famous conjecture due to Sheehan [26] claims that no 4-regular graph can have a unique Hamiltonian cycle; Thomassen's result proves Sheehan's conjecture for bipartite graphs.

Papadimitriou [24] popularized the Another Hamiltonian Cycle problem and Thomason's algorithm by introducing the complexity class PPA and showed the containment of the problem in odd-degree graphs; completeness for PPA remains open. It is also open whether the problem can be solved in polynomial time – indeed, the drawback of Thomason's algorithm is that it may run for a very long time; Krawczyk [19] and Cameron [8] showed that Thomason's algorithm requires exponential time for a family of cubic planar graphs. Later this was shown by Zhong [33] also for cubic bipartite graphs. The best bound to date is the recent result of Briański and Szady [6], which shows that there are cubic 3-connected planar graphs on n vertices, in which Thomason's lollipop algorithm runs in $\Omega(1.18^n)$ time.

The above papers reason about Thomason's algorithm specifically, but it may of course be other algorithms that solve the problem more efficiently. Some progress in this direction was provided by Bazgan, Santha, and Tuza [2] that showed that one given a cubic graph on n vertices and one of its Hamiltonian cycles can find another cycle of length $(1-\epsilon)n$ for any fixed constant $\epsilon > 0$ in polynomial time. Deligkas, Mertzios, Spirakis, and Zamaraev [12] derived an exponential-time polynomial-space deterministic algorithm that given a cubic graph along with one of its Hamiltonian cycles finds another Hamiltonian cycle; the algorithm is shown to be faster than the fastest known exponential-time polynomial-space deterministic algorithm for finding a Hamiltonian cycle in cubic graphs.

1.2 Main results for bipartite Pfaffian graphs

Let us now review our main results for bipartite Pfaffian graphs and the underlying techniques in more detail. Our main theorems are as follows.

▶ **Theorem 1** (Main; Linear–time Another Hamiltonian Cycle in minimum degree three). *There exists a deterministic linear-time algorithm that, given as input (i) a bipartite Pfaffian graph G with minimum degree three, (ii) a Hamiltonian cycle H in G, and (iii) an edge $e \in E(H)$, outputs a Hamiltonian cycle $H' \neq H$ in G with $e \in E(H')$.*

▶ **Theorem 2** (Main; Logarithmic–space Another Hamiltonian Cycle in minimum degree three). *There exists a deterministic logarithmic-space algorithm that, given as input (i) a bipartite Pfaffian graph G with minimum degree three, (ii) a Hamiltonian cycle H in G, and (iii) an edge $e \in E(H)$, outputs a Hamiltonian cycle $H' \neq H$ in G with $e \in E(H')$.*

The framework underlying our main theorems can be used to prove an upper bound on the number of steps needed for Thomason's lollipop method to terminate.

▶ **Theorem 3** (Thomason's lollipop method in cubic bipartite Pfaffian graphs). *Thomason's lollipop method starting from any Hamiltonian cycle H and any edge $e \in E(H)$ in an n-vertex cubic bipartite Pfaffian graph G, terminates after at most n steps.*

It was conjectured in a master thesis at Waterloo by Haddadan [14] that Thomason's lollipop method runs in a linear number of steps in cubic bipartite planar graphs. It was there also proven to hold for the subfamily of such graphs that does not have the wheel graph on six vertices as a minor. However, as the author himself points out, finding a first Hamiltonian cycle in this limited graph family does not seem intractable. We are not aware of any other papers providing a polynomial time bound on Thomason's lollipop method in any graph class.

Our framework also can be used to prove the following structural results for Hamiltonian cycles. Their proofs appear in the full version [4] of this paper.

▶ **Corollary 4** (Non-uniqueness in minimum degree three). *For every bipartite Pfaffian graph G of minimum degree three and for every edge $e \in E(G)$, it holds that G has either zero or at least four distinct Hamiltonian cycles H with $e \in E(H)$.*

The cube graph is a cubic bipartite Pfaffian graph with six distinct Hamiltonian cycles. We show that no Hamiltonian graph in this class can have fewer Hamiltonian cycles.

▶ **Corollary 5** (Cubic tight lower bound). *Every cubic bipartite Pfaffian graph has either zero or at least six distinct Hamiltonian cycles.*

Chia and Ong [9] (in the paragraph after Theorem 10) asked whether there exists cubic bipartite planar graphs with exactly four Hamiltonian cycles. The above Corollary rules out that possibility.

Remarks. One consequence of Theorem 1 is that if the Another Hamiltonian Cycle problem in general odd degree graphs is PPA-complete as hypothesized by Papadimitriou [24, Open Problem (4)], then any proof cannot carry over to cubic bipartite planar graphs unless also PPA = FP. Our result also seems related to another well-known conjecture, namely Barnette's conjecture (cf. Tutte [31, Unsolved Problem V]), which states that every cubic 3-connected bipartite planar graph (*Barnette graph*) has a Hamiltonian cycle. Gorsky, Steiner, and Wiederrecht [13] recently extended the conjecture by showing that if Barnette's conjecture is true, it also holds that every cubic bipartite 3-connected Pfaffian graph has a Hamiltonian cycle. It is known that if Barnette's conjecture is true, then there is a Hamiltonian cycle through every edge in every such graph, see Kelmans [17]. Moreover, it was indirectly shown by Holton, Manvel, and McKay [15] that any Barnette graph larger than the smallest such graph – namely the cube graph – can be reduced to a smaller Barnette graph in such a way that if the smaller graph has a Hamiltonian cycle through every edge, then the larger one must also have a Hamiltonian cycle. This means that if it was possible given any single Hamiltonian cycle in a Barnette graph to generate a Hamiltonian cycle through any specific edge *not* on the initial cycle, then Barnette's conjecture would be constructively true. We remark that our algorithm is not known to be able to do this, not even indirectly by applying it several times in a chain of Hamiltonian cycle transformations. We do note however, that our algorithms in Theorems 1 and 2 not only make sure the edge $e = \{s, t\}$ is part of both Hamiltonian cycles, they also retain the other edge incident to s on H; this can be observed by Lemma 10, that is, no edge incident to s is changed by the algorithms since it is not on the alternating cycle we use. In particular this makes it possible given H and any given edge $f \in E(G)$ *not* on H, to find another Hamiltonian cycle H' such that f is also not part of H'.

1.3 Overview of techniques

At the heart of our algorithms and structural results is what we believe to be a new framework for efficiently witnessing a Hamiltonian cycle H through an arbitrary *anchor* edge e in a bipartite Pfaffian graph G. We associate a (not necessarily proper) two-coloring $\chi_H : V(G) \to \{0, 1\}$ to H that is unique to H (but dependent on a fixed but arbitrary Pfaffian orientation of G as well as e) and that defines a unique *acyclic* Hamiltonian[3] orientation of

[3] In precise terms, the orientation is a directed acyclic graph that contains as a directed subgraph a directed Hamiltonian *path* from one end of e to the other.

$G \setminus e$. We will refer to such an χ_H as a *good coloring.* This acyclicity in particular enables the unique recovery of H in linear time by standard topological sorting when given χ_H as advice. The reader may want to consult Figure 1 (on page 9) for an advance illustration at this point; the framework itself is developed in Section 2.

We also show that it suffices to know χ_H in only one of the parts of a bipartition of G to efficiently extend to a Hamiltonian cycle, which is not necessarily equal to H however. More precisely, we show that a coloring λ of one of the parts leads to an auxiliary bipartite graph F_λ whose perfect matchings correspond to the good colorings χ_H extending λ, which in turn each define a unique Hamiltonian cycle H. We refer to Figure 2 (on page 11) for an advance illustration of this setting.

Finally, when G has minimum degree three, we observe that we can use F_λ to efficiently switch from one Hamiltonian cycle H in G (described by a perfect matching M_H in F_λ) to another Hamiltonian cycle $H' \neq H$ in G by switching along an alternating cycle in F_λ which can be discovered through a directed cycle in an auxiliary directed graph $D_{\lambda,H}$. Moreover, we show that this construction and discovery can be executed in deterministic linear time – we refer to Figure 3 (on page 14) for an advance illustration; the switching construction itself is developed in Section 3.

1.4 Further results

Our framework for witnessing Hamiltonicity should be contrasted with the Cut&Count approach for detecting Hamiltonian cycles by Cygan, Nederlof, Pilipczuk, Pilipczuk, Van Rooij, and Wojtaszczyk [10], which reduces the Hamiltonian cycle problem to a local problem by showing that a cycle cover of the input graph is Hamiltonian if and only if the number of the exponentially many vertex partitions that are consistent (defined in a certain local way) with it is *odd.* This approach therefore necessarily reduces the original decision problem to a parity counting problem, which has several disadvantages that seem inherent to the approach, including the need for randomization and a running time factor that is pseudo-polynomial in the integer weights for edge-weighted problem variants. Our framework shows that for bipartite Pfaffian graphs there is a more natural way to witness that a cycle cover is Hamiltonian using only a *single* vertex partition; that is, χ_H.

We explore the algorithmic consequences of our technique in the context of counting Hamiltonian cycles and in the context of the Traveling Salesperson Problem (TSP) in bipartite Pfaffian/planar graphs. For reasons of space, we postpone a detailed statement of these results, a discussion of pertinent earlier work, as well as the proofs, to the full version [4] of this paper.

1.5 Pfaffian graphs

Let us now define and motivate Pfaffian graphs in more detail. An *orientation* of a graph G replaces every edge $\{u, v\} \in E(G)$ with either the directed arc (u, v) or the directed arc (v, u), thereby obtaining a directed graph $\vec{G}$. A cycle C in G is *central* if the graph $G \setminus V(C)$ admits a perfect matching. We say that an orientation of a cycle is *consistent* if it is strongly connected. An orientation $\vec{G}$ of G is *Pfaffian* if for every central cycle C in G it holds that both consistent orientations of C have an odd number of arcs in common with $\vec{G}$. A graph is *Pfaffian* if it admits a Pfaffian orientation.

The bipartite Pfaffian graphs are most famous as the graph class in which Pólya's permanent problem has a solution, the bipartite graphs in which one can compute the number of perfect matchings efficiently by reduction to a matrix determinant, see e.g. Robertson, Seymour, and Thomas [25] and McCuaig [22]. This brings us to one of our motivations to study the complexity of the detection (and counting) of Hamiltonian cycles on this graph class: Previous algorithms for Hamiltonian cycles (such as the one by Björklund [3]) use determinant-based methods previously designed for counting matchings (modulo 2) in polynomial time; given this close connection between the two problems it is natural to ask whether Pfaffianity can be exploited for detecting and counting Hamiltonian cycles, similarly as for counting perfect matchings.

The bipartite Pfaffian graphs were characterized by Little [21] as those graphs G that do not have a vertex set $U \subseteq V(G)$ such that $G \setminus U$ has a perfect matching and the induced subgraph $G[U]$ admits an even subdivision of $K_{3,3}$ as a subgraph. McCuaig [22] and Robertson, Seymour, and Thomas [25] gave a structural characterization of bipartite Pfaffian graphs and the latter also outlined an $O(n^3)$ time algorithm for their recognition; this algorithm can also produce a Pfaffian orientation when one exists.

A general Pfaffian graph, as opposed to a bipartite one, can be very dense as observed by Norine [23]: there is an infinite family of n-vertex Pfaffian graphs with $\Omega(n^2)$ edges. This construction in particular poses obstacles to find characterizations as the ones mentioned above for bipartite graphs. Indeed, it is not known how to efficiently recognize a general Pfaffian graph.

The most famous Pfaffian graphs are the *planar* ones, graphs whose vertices can be embedded in the plane with straight lines connecting the vertices of every edge without line crossings except at endpoints. That planar graphs are Pfaffian was discovered by Kasteleyn [16]; there is also a linear-time algorithm that finds a Pfaffian orientation given a planar graph by Little [20].

1.6 Conventions and organization

We assume knowledge of standard graph-theoretic terminology; see e.g. West [32]. Graphs in this paper are undirected unless otherwise mentioned; this in particular also applies to subgraphs such as paths, cycles, and Hamiltonian cycles. No graph or directed graph in this paper has loops or multiple edges. For a graph or directed graph G, we write $V(G)$ for the vertex set of G and $E(G)$ for the edge set of G. We identify the *edges* of a graph with two-subsets $\{u, w\}$ where u and w are distinct vertices. We call the edges of a directed graph *arcs* in what follows, and identify each arc with a two-tuple (u, w) where u and w are distinct vertices. We recall our conventions with orientations and Pfaffian graphs from Section 1.5.

We work with Iverson's bracket notation – for a logical proposition P, we define

$$[\![P]\!] = \begin{cases} 1 & \text{if } P \text{ is true}\,; \\ 0 & \text{if } P \text{ is false}\,. \end{cases}$$

The rest of this paper is organized as follows. Section 2 presents our novel witnessing technique for Hamiltonian cycles in bipartite Pfaffian graphs. We prove our main theorems and their indirect structural corollaries in Section 3. Proofs of Lemmas, Theorems, and Corollaries that are omitted due to space constraints can be found in the full version [4] of this paper. In the full version we also prove the NP-hardness of Another Hamiltonian Cycle in bipartite Pfaffian graphs without constrained vertex degrees, and state, motivate, and prove further results on counting Hamiltonian cycles as well as on TSP.

2 Hamiltonian cycles in bipartite Pfaffian graphs

This section presents what we believe to be a novel technique to efficiently witness Hamiltonian cycles in bipartite Pfaffian graphs via (not necessarily proper) two-colorings of the vertices. We also show how to efficiently construct a Pfaffian orientation from a known Hamiltonian cycle in a bipartite Pfaffian graph, as well as show how to efficiently find a witness by extending a given partial witness defined on only one of the parts of a bipartition.

Throughout this section G is an n-vertex bipartite Pfaffian graph and $\vec{G}$ is a fixed but otherwise arbitrary Pfaffian orientation of G. Since we are interested in whether G is Hamiltonian, without loss of generality we may assume that n is even and $n \geq 4$ in what follows.

Select an arbitrary edge $e \in E(G)$ and call it the *anchor* edge.

2.1 Preliminaries: The structure of Pfaffian orientations

We start by recalling the known structure of Pfaffian orientations of G. Namely, de Carvalho, Lucchesi, and Murty [11] observed that all Pfaffian orientations of G are obtainable from each other by reversals of arcs across vertex cuts. More precisely, for any Pfaffian orientation $\vec{G}$ and any vertex $u \in V(G)$, it holds that reversing the arcs incident to u in $\vec{G}$ results in another Pfaffian orientation; furthermore, every Pfaffian orientation of G can be obtained by starting with an arbitrary Pfaffian orientation of G and repeating such operations for different vertices [11].

2.2 The two-coloring defined by an anchored Hamiltonian cycle

We are interested in characterising each Hamiltonian cycle H in G that traverses the selected anchor edge e – we say that such an H is *anchored* – using a function $\chi_H : V(G) \to \{0,1\}$ that is unique[4] to H and from which we will (in the next subsection) see H can be efficiently constructed.

Towards this end, let us study the Pfaffian orientation $\vec{G}$ at the anchor e. Let $(s,t) \in E(\vec{G})$ be the arc in $\vec{G}$ whose underlying edge in G is the anchor edge $e = \{s,t\}$. Construct from the Pfaffian orientation $\vec{G}$ a new orientation $\vec{G}_e$ of G that is otherwise identical to $\vec{G}$ except that the arc (s,t) has been replaced with the arc (t,s). That is, by definition we have $(t,s) \in E(\vec{G}_e)$.

Now consider an arbitrary anchored Hamiltonian cycle H in G. Since $e \in E(H)$, there is a unique consistent orientation $\vec{H}$ of H such that $(t,s) \in E(\vec{H})$. Let us write $v_0, v_1, \ldots, v_{n-1}$ for the vertices of G indexed in the directed $\vec{H}$-path order from s to t; that is,

$$v_0 = s\,, \quad v_{n-1} = t\,, \quad \text{and} \quad (v_i, v_{(i+1) \bmod n}) \in E(\vec{H}) \quad \text{for all } i = 0,1,\ldots,n-1\,. \tag{1}$$

Associate with H the (not necessarily proper) vertex-coloring function $\chi_H : V(G) \to \{0,1\}$ defined by setting

$$\begin{aligned} &\chi_H(v_0) = 0 \quad \text{and} \\ &\chi_H(v_{i+1}) \equiv \chi_H(v_i) + [\![(v_i, v_{i+1}) \in E(\vec{G}_e)]\!] \pmod 2 \quad \text{for all } i = 0,1,\ldots,n-2\,. \end{aligned} \tag{2}$$

[4] Unique but not canonical; as we will see, the function χ_H will depend not only on the Hamiltonian cycle H but also on the choice of our assumed fixed but arbitrary Pfaffian orientation $\vec{G}$ of G.

Because $\vec{G}$ is a Pfaffian orientation and H is a central cycle of G, we have

$$|E(\vec{H}) \cap E(\vec{G})| = \sum_{i=0}^{n-1} [\![(v_i, v_{(i+1) \bmod n}) \in E(\vec{G})]\!] \equiv 1 \pmod 2 . \tag{3}$$

Since $\vec{G}$ and $\vec{G}_e$ differ only in the orientation of $e \in E(H)$, from (3) we immediately have

$$|E(\vec{H}) \cap E(\vec{G}_e)| = \sum_{i=0}^{n-1} [\![(v_i, v_{(i+1) \bmod n}) \in E(\vec{G}_e)]\!] \equiv 0 \pmod 2 . \tag{4}$$

We thus conclude

$$\begin{aligned} \chi_H(v_0) \overset{(2)}{=} 0 \overset{(4)}{\equiv} |E(\vec{H}) \cap E(\vec{G}_e)| &= \sum_{i=0}^{n-2} [\![(v_i, v_{i+1}) \in E(\vec{G}_e)]\!] + [\![(v_{n-1}, v_0) \in E(\vec{G}_e)]\!] \\ &\overset{(2)}{\equiv} \chi_H(v_{n-1}) + [\![(v_{n-1}, v_0) \in E(\vec{G}_e)]\!] \pmod 2 . \end{aligned} \tag{5}$$

Since by definition of $\vec{G}_e$ we have $(t, s) \in E(\vec{G}_e)$, from (1) and (5) we conclude that $\chi_H(t) = 1$. Furthermore, from (2) and (5) we have for all $(u, w) \in E(\vec{H})$ that

$$\chi_H(w) \equiv \chi_H(u) + [\![(u, w) \in E(\vec{G}_e)]\!] \pmod 2 . \tag{6}$$

That is, each arc $(u, w) \in E(\vec{H})$ is χ_H-monochromatic (i.e. both endpoints are assigned the same value by χ_H) if and only if $(w, u) \in E(\vec{G}_e)$.

2.3 The orientation induced by a good coloring

Suppose now that we do not know anything about the (anchored) Hamiltonian cycles of G, if any, and have access only to the Pfaffian orientation $\vec{G}$ and the orientation $\vec{G}_e$; the latter is easily obtainable from $\vec{G}$, cf. Section 2.2.

Consider an arbitrary vertex coloring $\chi : V(G) \to \{0, 1\}$ with $\chi(s) = 0$ and $\chi(t) = 1$. The last observation in Section 2.2 suggests that we should explore reversing exactly the χ-monochromatic arcs in $\vec{G}_e$. Let us make this formal as follows. Let the orientation $\vec{G}_e^{\chi}$ *induced* by the coloring χ be the unique orientation of G that for each edge $\{u, w\} \in E(G)$ satisfies

$$(u, w) \in E(\vec{G}_e^{\chi}) \quad \text{if and only if} \quad \chi(w) \equiv \chi(u) + [\![(u, w) \in E(\vec{G}_e)]\!] \pmod 2 . \tag{7}$$

To witness the serendipity of (7), suppose that G admits an anchored Hamiltonian cycle H; it follows immediately from (6) and (7) that $E(\vec{H}) \subseteq E(\vec{G}_e^{\chi_H})$. Thus, if we know only the coloring χ_H but not H, we can search for $E(\vec{H})$ in $E(\vec{G}_e^{\chi_H})$; let us next analyse this situation in more detail from the standpoint of our arbitrary χ.

Call the coloring χ *good* if there exists an anchored Hamiltonian cycle H in G with $\chi = \chi_H$; otherwise call χ *bad.* The following lemma shows that the orientation $\vec{G}_e^{\chi}$ for a good χ enables *linear-time* and unique algorithmic recovery of H by standard longest-path search in a directed acyclic graph (DAG); in fact, mere topological sorting suffices, as is apparent from the proof. See also Figure 1 for an illustration of the concepts involved in a Hamiltonian planar graph.

▶ **Lemma 6** (Acyclic Hamiltonicity of good-coloring-induced orientations). *Let χ be good. Then, the orientation $\vec{G}_e^{\chi} \setminus (t, s)$ of $G \setminus e$ is acyclic with the unique source vertex s and the unique sink vertex t. Moreover, the longest directed path in $\vec{G}_e^{\chi} \setminus (t, s)$ is unique and a directed Hamiltonian path.*

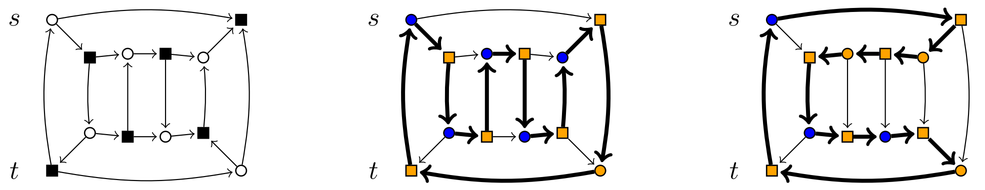

Figure 1 Illustration of orientations induced by good colorings. Left: an undirected bipartite planar graph G drawn in one of its orientations $\vec{G}_e$ with $e = \{s,t\}$, one arc reversal away from a Pfaffian orientation $\vec{G}$. Middle and right: two vertex colorings χ_H and coloring-induced orientations $\vec{G}_e^{\chi_H}$ for two different Hamiltonian cycles H, with the arcs of $\vec{H}$ drawn in bold in each case. Observe that every monochromatic arc reverses its orientation with respect to $\vec{G}_e$, whereas bichromatic arcs keep their orientation. Observe also that the removal of the arc (t,s) from $\vec{G}_e^{\chi_H}$ leaves an acyclic Hamiltonian directed graph, whereby the directed Hamiltonian path and hence H can be found, for example, by topological sorting; cf. Lemma 6.

Proof. Since χ is good, there exists an anchored Hamiltonian cycle H with $\chi = \chi_H$. Furthermore, we can follow the notational conventions in Section 2.2 with respect to this H, including the vertex-indexing $v_0, v_1, \ldots, v_{n-1}$ for G and (1) in particular. From $E(\vec{H}) \subseteq E(\vec{G}_e^{\chi})$ we thus conclude that the sequence $v_0, v_1, \ldots, v_{n-1}$ defines a longest directed path (which is also a directed Hamiltonian path from s to t) in the directed graph $\vec{G}_e^{\chi} \setminus (t,s)$. It follows immediately that s is the only possible source vertex and t is the only possible sink vertex in $\vec{G}_e^{\chi} \setminus (t,s)$.

Let us next show that $\vec{G}_e^{\chi} \setminus (t,s)$ is acyclic as a directed graph. To reach a contradiction, suppose that $\vec{D}$ is a directed cycle in $\vec{G}_e^{\chi} \setminus (t,s)$. Since $(t,s) = (v_{n-1}, v_0) \notin E(\vec{D})$ and $\vec{D}$ is a directed cycle with $V(\vec{D}) \subseteq \{v_0, v_1, \ldots, v_{n-1}\}$, there must exist $0 \le i < j \le n-1$ with at least two proper inequalities among the three such that $(v_j, v_i) \in E(\vec{D})$ and thus $(v_j, v_i) \in E(\vec{G}_e^{\chi})$. Let $\vec{C}$ be the directed cycle with $V(\vec{C}) = \{v_i, v_{i+1}, \ldots, v_j\}$ and $E(\vec{C}) = \{(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \ldots, (v_{j-1}, v_j), (v_j, v_i)\}$. In particular, $\vec{C} \neq \vec{H}$ since $(t,s) \notin E(\vec{C})$. Let C be the underlying undirected cycle of $\vec{C}$, and observe that C is a cycle of G. Since G is bipartite, C is even and has at least four vertices. Thus, the edges of $H \setminus V(C)$ contain a perfect matching of $G \setminus V(C)$, implying that C is central. Since C avoids e and $\vec{C}$ is a consistent orientation of C, we conclude by Pfaffianity that

$$|E(\vec{C}) \cap E(\vec{G}_e)| = |E(\vec{C}) \cap E(\vec{G})| \equiv 1 \pmod 2 .$$

But this is a contradiction since for all $(u,w) \in E(\vec{C})$ we have $(u,w) \in E(\vec{G}_e^{\chi})$, and thus by (7) it holds that $\chi(w) \equiv \chi(u) + [\![(u,w) \in E(\vec{G}_e)]\!] \pmod 2$; take the sum of these congruences over all arcs $(u,w) \in E(\vec{C})$ to conclude that $|E(\vec{C}) \cap E(\vec{G}_e)| \equiv 0 \pmod 2$, a contradiction. Thus, $\vec{G}_e^{\chi} \setminus (t,s)$ is acyclic as a directed graph.

From acyclicity it also immediately follows that s is a source vertex and t is a sink vertex of $\vec{G}_e^{\chi} \setminus (t,s)$; indeed, any arc into s or any arc out of t would complete a directed cycle together with an appropriate proper segment of the directed Hamiltonian path $s = v_0, v_1, \ldots, v_{n-1} = t$. This longest path (of n vertices) is also seen to be unique in $\vec{G}_e^{\chi} \setminus (t,s)$; indeed, the existence of any other such path would again imply an arc that would complete a directed cycle together with an appropriate proper segment of $v_0, v_1, \ldots, v_{n-1}$. ◀

An immediate corollary of the proof Lemma 6 is that there are at most 2^{n-2} Hamiltonian cycles through any fixed edge in an n-vertex bipartite Pfaffian graph. For comparison, there are planar graphs with at least 2.08^n Hamiltonian cycles, see [7].

2.4 Constructing a Pfaffian orientation from a Hamiltonian cycle

Next we address the task of constructing a Pfaffian orientation if we know one Hamiltonian cycle, with the intent of constructing possible further Hamiltonian cycles with the help of the Pfaffian orientation obtained.

▶ **Lemma 7** (Constructing a Pfaffian orientation from a Hamiltonian cycle). *There exists a linear-time algorithm that, given as input a bipartite Pfaffian graph G and a Hamiltonian cycle H in G, outputs a Pfaffian orientation $\vec{G}$ of G.* ⋆*See full version [4] for proof*⋆

2.5 Finding a good coloring

Let us now study the task of finding a good coloring χ given the bipartite Pfaffian graph G, the Pfaffian orientation $\vec{G}$, and the anchor edge e as input; also recall the conventions and further notation – in particular, the vertices s and t – from Sections 2.2 and 2.3. In this setting, a natural question to ask is how much one needs to reveal from a good coloring χ to enable efficient completion to a good coloring. We now show that it suffices to reveal χ in one of the parts of the bipartition of G by a reduction to bipartite perfect matching in an auxiliary bipartite graph.

More precisely, let the sets L ("left") and R ("right") form a partition of the vertices of G such that $s \in L$, $t \in R$, and every edge of G has one end in L and the other end in R.[5] Let $\lambda : L \to \{0,1\}$ with $\lambda(s) = 0$ be a given further input. Our task is to find whether there exists a good coloring $\chi : V(G) \to \{0,1\}$ with $\chi(\ell) = \lambda(\ell)$ for all $\ell \in L \subseteq V(G)$; that is, whether there exists a good coloring that extends the partial coloring λ.

Construct an auxiliary bipartite graph F_λ as follows. Let the vertex set $V(F_\lambda) = V(G) \times \{0,1\}$. To avoid notational confusion between arcs and vertices of F_λ, we will use bracketed notation $[u,k]$ for vertices of F_λ with $u \in V(G)$ and $k \in \{0,1\}$. The edge set $E(F_\lambda)$ is defined by the following rule. For all $\ell \in L$, $r \in R$, $p \in \{0,1\}$, and $\rho \in \{0,1\}$ with $\{\ell, r\} \in E(G)$, we have

$$\{[\ell,p],[r,\rho]\} \in E(F_\lambda) \tag{8}$$

if and only if both

$$\rho \equiv \lambda(\ell) + [\![(\ell,r) \in \vec{G}_e]\!] \pmod 2 \tag{9}$$

and

$$\ell \neq s \quad \text{or} \quad p \neq 0 \quad \text{or} \quad r = t\,. \tag{10}$$

For an edge $\{[\ell,p],[r,\rho]\} \in E(F_\lambda)$, we say that the edge $\{\ell,r\} \in E(G)$ is the *projection* of the edge (to G) and call p the *port* at ℓ and ρ the *parity* at r, stressing that port and parity have asymmetric roles in our construction even though both range in $\{0,1\}$.

Let us now start analysing the structure of F_λ in more detail. First, the parts $L \times \{0,1\}$ and $R \times \{0,1\}$ witness by (8) that F_λ is bipartite. In particular, F_λ has $2n$ vertices with $|L| = |R| = n/2$, where we recall that $n \geq 4$ is the number of vertices in G with n even. Second, recalling that $s \in L$ and $t \in R$, the constraint (10) effectively states that $[s,0]$ is adjacent only to $[t,0]$ in F_λ; indeed, recalling that $\lambda(s) = 0$ and $(s,t) \notin \vec{G}_e$, from (9) we

[5] This bipartition (L,R) of G is in fact unique unless G is not Hamiltonian. Moreover, (L,R) is computable in linear time from the given input.

have that $[s, 0]$ is not adjacent to $[t, 1]$. Third, for all $\ell \in L \setminus \{s\}$, we observe from (9) and (10), the latter being trivially true, that the vertices $[\ell, 0]$ and $[\ell, 1]$ have identical vertex neighborhoods in F_λ.

We are now ready for our next key lemma. Recall the coloring χ_H associated to an anchored Hamiltonian cycle H of G from Section 2.2. The first lemma shows that every perfect matching in F_λ gives rise to an anchored Hamiltonian cycle; different perfect matchings may give rise to the same anchored Hamiltonian cycle however. This structure is illustrated in Figure 2.

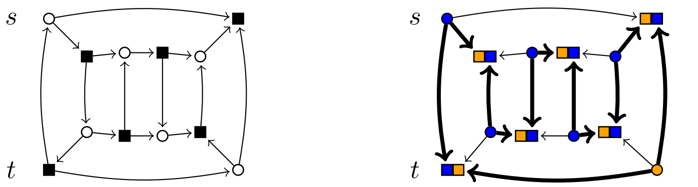

Figure 2 Illustration of a perfect matching in the graph F_λ. Left: The graph G drawn in one of its orientations $\vec{G}_e$ with $e = \{s, t\}$ and the bipartition (L, R) with vertices in L drawn as white circles and the vertices in R drawn as black boxes. Right: The graph F_λ and the coloring λ drawn as an oriented overlay of G. Observe that each vertex $r \in R$ has two copies $[r, \rho]$ in F_λ, one for each parity $\rho \in \{0, 1\}$, with a blue square indicating parity 0 and an orange square indicating parity 1. Although each vertex $\ell \in L$ has two copies $[\ell, p]$ in F_λ, one copy for each port $p \in \{0, 1\}$, we contract these two copies into one vertex (circle) in the drawing, and display for each vertex its color $\lambda(\ell) \in \{0, 1\}$ (blue or orange) instead. Each arc in the drawing is oriented from L to R and represents two edges of F_λ with opposite ports. A perfect matching M in F_λ is represented by the bold arcs. In particular, observe that each circle is incident to two bold arcs; these two bold arcs represent two edges in M with opposite ports. These opposite ports are otherwise arbitrary except for the edge of M that projects to $\{s, t\}$, which must have port 0. Also observe that from the drawn M it is visually intuitive how to obtain a Hamiltonian cycle in G corresponding to M by turning the bold arcs into the edges of a Hamiltonian cycle in G; this intuition is made rigorous in Lemma 8 by the Hamiltonian cycle $\mathrm{H}[M]$ of G obtained from M.

▶ **Lemma 8** (Perfect matchings witness good extensions). *For every perfect matching M in F_λ, there exists a Hamiltonian cycle $\mathrm{H}[M]$ in G with $\chi_{\mathrm{H}[M]}(\ell) = \lambda(\ell)$ for all $\ell \in L$.*

Proof. Let M be an arbitrary perfect matching in F_λ. For $[\ell, p] \in L \times \{0, 1\}$ and $[r, \rho] \in L \times \{0, 1\}$, let us use functional notation $M([\ell, p]) = [r, \rho]$ or $M([r, \rho]) = [\ell, p]$ to signal that the vertices $[\ell, p]$ and $[r, \rho]$ are matched by M in F_λ. We construct the anchored Hamiltonian cycle $\mathrm{H}[M]$ as well as the coloring $\chi = \chi_{\mathrm{H}[M]}$ with $\chi(\ell) = \lambda(\ell)$ for all $\ell \in L$ by traversing all the vertices of F_λ in an order determined by M to yield the Hamiltonian cycle $\mathrm{H}[M]$. In particular, we will define $\mathrm{H}[M]$ in steps by introducing, one vertex at a time, a vertex order $v_0, v_1, \ldots, v_{n-1}$ for the vertices of G with $v_0 = s$, $v_{n-1} = t$, and $(v_i, v_{(i+1) \bmod n}) \in E(\overrightarrow{\mathrm{H}[M]}) \subseteq E(\vec{G}_e^\chi)$ for all $i = 0, 1, \ldots, n-1$.

Our traversal starts from the vertex $[\ell_0, p_0]$ of F_λ defined by $\ell_0 = s$ and $p_0 = 1$. The traversal then follows edges in the perfect matching M, changing parity (at vertices in $R \times \{0, 1\}$) and port (at vertices in $L \times \{0, 1\}$) to arrive at subsequent edges; for these changes, for $z \in \{0, 1\}$ it is convenient to write $\overline{z} = (z + 1) \bmod 2$ for notational brevity; that is, $\overline{0} = 1$ and $\overline{1} = 0$.

In precise terms, the traversal is as follows. Assuming we have defined $\ell_j \in L$ and $p_j \in \{0, 1\}$ for all $j \in \{0, 1, \ldots, i\}$ with $i \geq 0$, we proceed to define $\ell_{i+1} \in L$ and $p_{i+1} \in \{0, 1\}$ as follows. Set $v_{2i} = \ell_i$ and $\chi(\ell_i) = \lambda(\ell_i)$. Define $r_i \in R$ and $\rho_i \in \{0, 1\}$ by $M([\ell_i, p_i]) = [r_i, \rho_i]$.

Set $v_{2i+1} = r_i$ and $\chi(r_i) = \rho_i$. Define $\ell'_i \in L$ and $p'_i \in \{0,1\}$ by $M([r_i, \overline{\rho_i}]) = [\ell'_i, p'_i]$. Set $\ell_{i+1} = \ell'_i$ and $p_{i+1} = \overline{p'_i}$ as well as $\chi(\ell_{i+1}) = \lambda(\ell_{i+1})$ and $v_{2(i+1)} = \ell_{i+1}$. We continue this process for $i = 0, 1, \ldots$ and claim that eventually $\ell_{i+1} = \ell_0$ and $p_{i+1} = p_0$ with $i+1 = n/2$, at which point $\{v_0, v_1, \ldots, v_{n-1}\} = V(G)$ and $\mathrm{H}[M]$ is a Hamiltonian cycle in G with $\chi_{\mathrm{H}[M]} = \chi$ with $\chi(\ell) = \lambda(\ell)$ for all $\ell \in L$.

Let us now analyse the traversal process in more detail. First, we observe that $\ell_{i+1} \neq \ell_i$; indeed, suppose that $\ell_{i+1} = \ell_i$ and observe that the traversal step from ℓ_i to ℓ_{i+1} changes parity from ρ_i to $\bar{\rho}_i$ at r_i, yet from (9) we observe that every edge of F_λ that projects to the edge $\{\ell_i, r_i\} = \{\ell_{i+1}, r_i\}$ has the same parity at r_i, a contradiction. Next, let us observe that $(v_{2i}, v_{2i+1}) = (\ell_i, r_i) \in E(\vec{G}_e^\chi)$. Indeed, the identity is immediate, and membership holds by (7) and

$$\chi(r_i) = \rho_i \overset{(9)}{\equiv} \lambda(\ell_i) + [\![(\ell_i, r_i) \in \vec{G}_e]\!] = \chi(\ell_i) + [\![(\ell_i, r_i) \in \vec{G}_e]\!] \pmod 2 .$$

Let us then observe that $(v_{2i+1}, v_{2i+2}) = (r_i, \ell_{i+1}) \in E(\vec{G}_e^\chi)$. Again the identity is immediate, and membership holds by (7), the fact that $\vec{G}_e^\chi$ orients $\{\ell_{i+1}, r_i\} \in E(G)$ in one of two possible orientations, and

$$\chi(r_i) = \rho_i \neq \bar{\rho}_i \overset{(9)}{\equiv} \lambda(\ell_{i+1}) + [\![(\ell_{i+1}, r_i) \in \vec{G}_e]\!] = \chi(\ell_{i+1}) + [\![(\ell_{i+1}, r_i) \in \vec{G}_e]\!] \pmod 2 .$$

Next let us show that all the vertices ℓ_i and r_i traversed by the process are distinct, until $\ell_{i+1} = \ell_0$ for some $i \geq 1$, noting that the case $i = 0$ has already been excluded earlier. Suppose $\ell_1, \ell_2, \ldots, \ell_i$ are distinct; since M contains exactly two edges (of opposite parities) that project to edges incident to any fixed $r \in R$, we observe that these two edges of M have been each traversed once by the process for each $r_0, r_1, \ldots, r_i$ since $\ell_0, \ell_1, \ldots, \ell_i$ are distinct, implying that $r_0, r_1, \ldots, r_i$ are distinct, and thus that $v_0, v_1, \ldots, v_{2i+1}$ are distinct. So suppose that $\ell_{i+1} = \ell_j$ for some $0 \leq j \leq i$; also note that this must happen for some $i < |L| = n/2$. If $j \geq 1$, we have a contradiction since M contains exactly two edges (of opposite ports) that project to edges incident to any fixed $\ell \in L$, and for $\ell = \ell_j$ these two edges (projecting to $\{\ell_j, r_{j-1}\}$ and $\{\ell_j, r_j\}$) have already been traversed; so there is no edge in M that projects to $\{\ell_j, r_i\} = \{\ell_{i+1}, r_i\}$, a contradiction. So we must have $j = 0$. This implies in particular that $(v_k, v_{(k+1) \bmod (2i+2)}) \in E(\vec{G}_e^\chi)$ for all $k = 0, 1, \ldots, 2i+1$. Furthermore, since the edge of M that is incident to $[\ell_0, p_0] = [s, 1]$ has already been traversed, we have that the edge $\{[\ell_{i+1}, \overline{p_{i+1}}], [r_i, \bar{\rho}_i]\}$ in M must be the edge $\{[s, 0], [t, 0]\}$ (recall our analysis earlier that $[s, 0]$ is adjacent only to $[t, 0]$ in F_λ); thus we conclude that $\chi(t) = \chi(r_i) = \rho_i \neq \bar{\rho}_i = 0$; that is, $\chi(t) = 1$.

Let us next show that $i = n/2 - 1$. So to reach a contradiction, suppose that $i < n/2 - 1$. In particular, the edges of G underlying the arcs $(v_k, v_{(k+1) \bmod (2i+2)}) \in E(\vec{G}_e^\chi)$ for $k = 0, 1, \ldots, 2i+1$ trace a cycle of even length $2i + 2 < n$ in G. This leaves some of the vertices in G, and thus all corresponding vertices of F_λ regardless of port/parity, unvisited by the traversal process. By starting the traversal process again from an arbitrary unvisited vertex in $L \times \{0, 1\}$, we end up tracing a further even-length cycle in G, and repeating the process until all vertices of G are visited, we obtain a vertex-disjoint union of even-length cycles that together cover the vertices of G, as well as a coloring χ such that all the cycles (in their consistently oriented form as they were traversed) occur as directed subgraphs of $\vec{G}_e^\chi$. Since $2i + 2 < n$, this cycle cover thus contains a cycle C that does not contain the anchor edge e and whose consistent orientation $\vec{C}$ is a subgraph of $\vec{G}_e^\chi$; observing that C is central – indeed, use every other edge from each even cycle other than C in the cover to witness a perfect matching in $G \setminus V(C)$ – this leads to a contradiction via Pfaffianity by the same argument as was used in the proof of Lemma 6; thus, $i = n/2 - 1$.

Since $i = n/2 - 1$, it follows from $(v_k, v_{(k+1) \bmod n}) \in E(\vec{G}_e^{\chi})$ for all $k = 0, 1, \ldots, n-1$ and from Lemma 6 we conclude that $\chi = \chi_{\mathrm{H}[M]}$ for the anchored Hamiltonian cycle $\mathrm{H}[M]$ in G defined by $V(\mathrm{H}[M]) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(\mathrm{H}[M]) = \{\{v_k, v_{(k+1) \bmod n}\} : k = 0, 1, \ldots, n-1\}$. ◀

Conversely, we show that good extensions of λ are witnessed by perfect matchings in F_λ.

▶ **Lemma 9** (Good extensions witness perfect matchings). *For every anchored Hamiltonian cycle H in G with $\chi_H(\ell) = \lambda(\ell)$ for all $\ell \in L$, there exists a perfect matching M_H in F_λ with $\mathrm{H}[M_H] = H$.* ⋆*See full version [4] for proof*⋆

Thus, F_λ has a perfect matching if and only if λ has a good extension. Moreover, from the proofs of Lemma 8 and Lemma 9 we observe that the transformations $M \mapsto \mathrm{H}[M]$ and $H \mapsto M_H$ are computable in linear time. We also observe that for every anchored Hamiltonian cycle H in G there are exactly $2^{n/2-1}$ perfect matchings M in F_λ with $\mathrm{H}[M] = H$; these M are all obtainable from each other by transposing ports at zero or more vertices $\ell \in L \setminus \{s\}$.

3 Another Hamiltonian cycle in bipartite Pfaffian graphs

This section studies the problem of finding another Hamiltonian cycle when given as input (i) a bipartite Pfaffian graph G and (ii) a Hamiltonian cycle H in G. Recall from Lemma 7 that we can in linear time construct a Pfaffian orientation $\vec{G}$ from this input. In what follows we thus tacitly assume that such a $\vec{G}$ is available and fixed together with an arbitrary anchor edge $e \in E(H)$.

3.1 Linear-time solvability in minimum degree three

Our first objective in this section is our main theorem, which we restate below for convenience.

▶ **Theorem 1** (Main; Linear–time ANOTHER HAMILTONIAN CYCLE in minimum degree three). *There exists a deterministic linear-time algorithm that, given as input (i) a bipartite Pfaffian graph G with minimum degree three, (ii) a Hamiltonian cycle H in G, and (iii) an edge $e \in E(H)$, outputs a Hamiltonian cycle $H' \neq H$ in G with $e \in E(H')$.*

We now proceed to prove Theorem 1. Recall and assume the setting of Section 2.5. Observe that from the given input, we can in linear total time (a) find a Pfaffian orientation $\vec{G}$ using H, (b) compute the orientation $\vec{G}_e$, (c) compute the coloring χ_H, (d) compute the vertex bipartition (L, R) of G, (e) restrict χ_H to L to obtain the coloring λ, (f) construct the graph F_λ, as well as (g) construct the perfect matching M_H in F_λ.

Using M_H and F_λ, introduce the directed graph $D_{\lambda,H}$ with the vertex set $V(D_{\lambda,H}) = L$ and the arc set defined for all distinct $\ell, \ell' \in L$ by the rule $(\ell, \ell') \in E(D_{\lambda,H})$ if and only if there exist $p, p' \in \{0, 1\}$, $r \in R$, and $\rho \in \{0, 1\}$ such that

$$\{[\ell, p], [r, \rho]\} \in E(F_\lambda) \setminus M_H \quad \text{and} \quad \{[\ell', p'], [r, \rho]\} \in M_H\,. \tag{11}$$

That is, an arc $(\ell, \ell') \in E(D_{\lambda,H})$ indicates that (disregarding ports p and p') we can walk from ℓ to ℓ' in F_λ by traversing first an edge not in M_H, followed by an edge in M_H. We stress that the traversal (11) *preserves* the parity ρ for consecutive edges, whereas the traversal in the proof of Lemma 8 *changes* parity for consecutive edges; the latter also uses edges only in M_H.

We recall that G has minimum degree at least three; this enables us to find a Hamiltonian cycle other than H in G with the help of a directed cycle in $D_{\lambda,H}$ revealed in the following lemma.

▶ **Lemma 10** (Existence of an s-avoiding directed cycle in $D_{\lambda,H}$). *Suppose that every vertex of G has degree at least three. Then, the directed graph $D_{\lambda,H}$ contains at least one directed cycle that avoids the vertex s.* ⋆*See full version [4] for proof*⋆

By the previous lemma we thus know that $D_{\lambda,H}$ contains an s-avoiding directed cycle $\vec{Q}$ with $V(\vec{Q}) = \{\ell_0, \ell_1, \ldots, \ell_{k-1}\} \subseteq L$ and $(\ell_j, \ell_{(j+1) \bmod k}) \in E(\vec{Q})$ for each $j = 0, 1, \ldots, k-1$ and $k \geq 2$. We will use $\vec{Q}$ to construct from M_H another Hamiltonian cycle $H' \neq H$ in G. Figure 3 illustrates the construction.

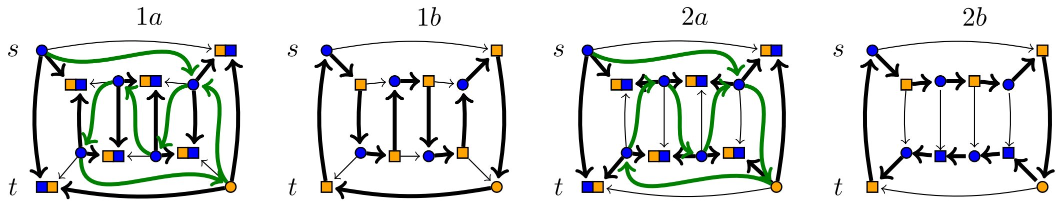

Figure 3 Obtaining another Hamiltonian cycle using a directed cycle in $D_{\lambda,H}$. $1a, 2a$: Two perfect matchings M_H in F_λ drawn as oriented overlays of G (cf. Figure 2), with further overlaying drawn in green and constituting the arcs of $D_{\lambda,H}$. $1b, 2b$: The corresponding two orientations $\vec{G}_e^{\chi_H}$ and colorings χ_H. In both cases we have that $D_{\lambda,H}$ contains a unique s-avoiding directed cycle $\vec{Q}$. Using $\vec{Q}$ in each case we can switch between the left and right Hamiltonian cycles in G. Note in particular that the two vertex colorings agree in L but differ in R.

From (11) applied to each arc of $\vec{Q}$ in turn we conclude that for $j = 0, 1, \ldots, k-1$ there exist $r_j \in R$ and $\rho_j, p_j \in \{0, 1\}$ with

$$\{[\ell_j, p_j], [r_j, \rho_j]\} \in E(F_\lambda) \setminus M_H \quad \text{and} \quad \{[\ell_{(j+1) \bmod k}, p'_j], [r_j, \rho_j]\} \in M_H\,. \tag{12}$$

We observe that the vertices $[r_j, \rho_j]$ for $j = 0, 1, \ldots, k-1$ are distinct because M_H is a perfect matching and $\ell_{(j+1) \bmod k}$ for $j = 0, 1, \ldots, k-1$ are distinct. In spite of this, the edges (12) for $j = 0, 1, 2, \ldots, k-1$ need *not* form a cycle in F_λ because we can have $p'_j \neq p_{(j+1) \bmod k}$. Here is where the fact that $\vec{Q}$ is s-avoiding pays off. Let $M = M_H$ and recall that the vertices $[\ell, 0]$ and $[\ell, 1]$ for each $\ell \in L \setminus \{s\}$ have identical vertex neighborhoods in F_λ. Thus, whenever we have $p'_j \neq p_{(j+1) \bmod k}$, we can modify M by transposing the vertices $[\ell_{(j+1) \bmod k}, 0]$ and $[\ell_{(j+1) \bmod k}, 1]$ in the edges of M; by the identical vertex neighborhoods property, the resulting M will still be a perfect matching in F_λ. Moreover, we have $H = \mathrm{H}[M_H] = \mathrm{H}[M]$; indeed, the traversal construction in Lemma 8 is insensitive to the specific values of the (opposite) ports. For $j = 0, 1, \ldots, k-1$ we thus now have

$$\{[\ell_j, p_j], [r_j, \rho_j]\} \in E(F_\lambda) \setminus M \quad \text{and} \quad \{[\ell_{(j+1) \bmod k}, p_{(j+1) \bmod k}], [r_j, \rho_j]\} \in M\,; \tag{13}$$

that is, these edges now form a $2k$-vertex cycle in F_λ. Let us write A for this cycle in F_λ.

Observe in particular from (13) that the edges of A alternate between edges in M and edges not in M. Thus, we have that the symmetric difference $M' = (M \setminus E(A)) \cup (E(A) \setminus M)$ is a perfect matching in F_λ. Furthermore, M' and M project to a different set of edges of G; indeed, from (13) we have that r_j changes adjacency from $\ell_{(j+1) \bmod k}$ in $\mathrm{H}[M]$ to ℓ_j in $\mathrm{H}[M']$ for $j = 0, 1, \ldots, k-1$. It follows that $H' = \mathrm{H}[M'] \neq \mathrm{H}[M] = H$. Thus, we have constructed a Hamiltonian cycle H' in G that is different from H. Moreover, this construction is computable in deterministic linear time. This completes the proof of Theorem 1. ◀

3.2 Logarithmic-space solvability in minimum degree three

We can implement the ideas of the previous section in an algorithm that uses little space. We consider as input a bipartite Pfaffian graph G of minimum degree three, a consistently oriented Hamiltonian cycle $\vec{H}$ in G, and an arc $(t, s) \in \vec{H}$. More precisely, we assume both graphs G and $\vec{H}$ are given in the input as a list of adjacency lists for each vertex. We seek to output a list of edges of another Hamiltonian cycle $H' \neq H$ with $\{s, t\} \in E(H')$. We assume that the vertices of G are represented as $O(\log n)$-bit integers in the input, where n is the number of vertices in G.

▶ **Theorem 2** (Main; Logarithmic–space Another Hamiltonian Cycle in minimum degree three). *There exists a deterministic logarithmic-space algorithm that, given as input (i) a bipartite Pfaffian graph G with minimum degree three, (ii) a Hamiltonian cycle H in G, and (iii) an edge $e \in E(H)$, outputs a Hamiltonian cycle $H' \neq H$ in G with $e \in E(H')$.*

⋆*See full version [4] for proof*⋆

3.3 Thomason's lollipop method in cubic bipartite Pfaffian graphs

In this section we prove that Thomason's lollipop method runs in linear time in cubic bipartite Pfaffian graphs. Let us first set up some preliminaries and then describe Thomason's lollipop method.

Let G be a Hamiltonian cubic graph and let H be a Hamiltonian cycle in G. Select an edge $e = \{s, t\}$ in the Hamiltonian cycle H. Let us call e the *anchor* edge. A *lollipop* is a connected graph with one vertex of degree one, one vertex of degree three, and all other vertices of degree two. All lollipops considered in what follows are subgraphs of G such that t is the unique degree-one vertex and e is the edge incident to t on the lollipop.

The lollipop method is best described as operating on a family of Hamiltonian paths in G. We say that a Hamiltonian path in G that starts at the vertex t and continues via the anchor edge e is an e-*anchored* Hamiltonian path. Now recall that G is cubic, so the vertex t is adjacent to s (via the anchor edge e) and to two other vertices a and b. The lollipop method transforms a given e-anchored Hamiltonian path P_e that ends at either a or b into an e-anchored Hamiltonian path $P'_e \neq P_e$ that ends at either a or b. Observe in particular that both Hamiltonian paths P_e and P'_e can be completed into e-anchored Hamiltonian cycles by adding the missing edge $\{a, t\}$ or $\{b, t\}$ into the respective path.

The transformation from P_e to P'_e is via a sequence of *lollipop steps*. A lollipop step consists of adding one edge to an e-anchored Hamiltonian path and removing another one, so that another e-anchored Hamiltonian path is formed. More precisely, let Q be an e-anchored Hamiltonian path ending at some vertex u. Since G is cubic, u is adjacent to two other vertices, x and y, such that the edges $\{u, x\}$ and $\{u, y\}$ of G are not in Q. Assume that $x \neq t$. Add the edge $\{u, x\}$ into Q to obtain a lollipop Ω where the unique degree-three vertex is x. Now observe that among the three adjacent vertices to x there is a unique vertex $v \notin \{u, t\}$ such that both $\{v, x\} \in E(\Omega)$ and removing $\{v, x\}$ from Ω leaves an e-anchored Hamiltonian path Q' ending at v. The transformation from Q to Q' now constitutes one lollipop step. Observe also that lollipop steps are *reversible*; that is, we can go back to Q from Q' by performing a lollipop step starting from Q'.

The *lollipop state graph* $\mathcal{L}(G, s, t)$ has as its vertices the e-anchored Hamiltonian paths in G and two vertices are joined by an edge if and only if it is possible to transform between the e-anchored Hamiltonian paths by one lollipop step. We observe immediately that $\mathcal{L}(G, s, t)$ has no isolated vertices – indeed, from any vertex Q we can arrive at another vertex $Q' \neq Q$ by a lollipop step – and the degree-one vertices are exactly the e-anchored Hamiltonian paths

Q that end at a vertex u adjacent to t in G; that is, $u \in \{a, b\}$; moreover, all other vertices have degree two. Thus, we can transform from P_e to $P'_e \neq P_e$ by tracing a path in $\mathcal{L}(G, s, t)$ from P_e to P'_e.

We now proceed to prove an upper bound on the maximum length of a path in $\mathcal{L}(G, s, t)$ on a cubic bipartite Pfaffian graph G. An example of the lollipop method applied to a cubic bipartite planar graph using the terminology in the subsequent proof is given in Figure 4.

▶ **Theorem 3** (Thomason's lollipop method in cubic bipartite Pfaffian graphs)**.** *Thomason's lollipop method starting from any Hamiltonian cycle H and any edge $e \in E(H)$ in an n-vertex cubic bipartite Pfaffian graph G, terminates after at most n steps.*

Proof. To analyze the lollipop method in cubic bipartite Pfaffian graphs, let a cubic bipartite Pfaffian graph G, a Hamiltonian cycle H in G, and $e = \{s, t\} \in E(H)$ be given as input. This input enables us to work in the setting of Section 3.1; let the vertex bipartition (L, R) of G, the coloring λ of L, the graph F_λ, the perfect matching M_H in F_λ, and the directed graph $D_{\lambda,H}$ be constructed accordingly. Recall that $s \in L$ and $t \in R$.

The lollipop method starts by removing the edge $\{t, u\}$ with $u \neq s$ from H to obtain e-anchored Hamiltonian path P_e. Let $Q_0, Q_1, \ldots, Q_h$ be the sequence of e-anchored Hamiltonian paths traversed by consecutive lollipop steps in $\mathcal{L}(G, s, t)$ with $P_e = Q_0$ and $Q_h = P'_e$. We will show that P'_e ends at u and thus we can obtain a Hamiltonian cycle $H' \neq H$ by inserting the edge $\{t, u\}$ into P'_e. Moreover and crucially, we will show that $h \leq n$.

Our analysis of the lollipop method is based on the directed graph $D_{\lambda,H}$. We recommend consulting Figure 4 for intuition at this point. Recall from the proof of Lemma 10 that, in the directed graph $D_{\lambda,H}$, the vertex s has in-degree zero and every vertex has out-degree at least one. In particular, by traversing out-arcs from the vertex u in $D_{\lambda,H}$, and traversing the eventual directed cycle encountered, as well as traversing backwards to u from the directed cycle, in precise terms we observe that there exist vertices $w_0, w_1, \ldots, w_{d-1}$ with $(w_j, w_{(j+1) \bmod d}) \in E(D_{\lambda,H})$ for $j = 0, 1, \ldots, d-1$ as well as vertices $w'_0, w'_1, \ldots, w'_{d'}$ with $w_0 = w'_{d'}$, $w'_0 = u$, $\{w_0, w_1, \ldots, w_{d-1}\} \cap \{w'_0, w'_1, \ldots, w'_{d'-1}\} = \emptyset$, and $(w'_j, w'_{j+1}) \in E(D_{\lambda,H})$ for $j = 0, 1, \ldots, d'-1$. That is, the sequence $w'_0, w'_1, \ldots, w'_{d'}$ forms a directed path starting at the vertex $u = w'_0$ and ending at the vertex $w_{d'} = w_0$, which is on the directed cycle formed by the vertices $w_0, w_1, \ldots, w_{d-1}$ in $D_{\lambda,H}$; the directed cycle and the directed path intersect exactly at the vertex $w_{d'} = w_0$. In particular $d + d' \leq |L| = n/2$.

It will be convenient to introduce the following sequence of vertices visited on the traversal of $D_{\lambda,H}$ from u. For $i = 0, 1, \ldots, 2d' + d$, define

$$v_i = \begin{cases} w'_i & \text{for } i = 0, 1, \ldots, d'-1\,; \\ w_{i-d'} & \text{for } i = d', d'+1, \ldots, d'+d-1\,; \\ w'_{2d'+d-i} & \text{for } i = d'+d, d'+d+1, \ldots, 2d'+d\,. \end{cases} \tag{14}$$

We have $(v_i, v_{i+1}) \in E(D_{\lambda,H})$ for $i = 0, 1, \ldots, d' + d - 1$; these arcs are precisely the arcs traversed forward. We have $(v_{i+1}, v_i) \in E(D_{\lambda,H})$ for $i = d' + d, d' + d + 1, \ldots, 2d' + d - 1$; these arcs are precisely the arcs traversed backward. For an arc $(\ell, \ell') \in E(D_{\lambda,H})$, let us write $r(\ell, \ell')$, $\rho(\ell, \ell')$, and $p'(\ell, \ell')$, respectively, for the unique $r \in R$, $\rho \in \{0, 1\}$, and $p' \in \{0, 1\}$ such that (11) holds. Also, let us write $p(\ell, \ell')$ for the minimum $p \in \{0, 1\}$ such that that (11) holds.

Let M^- be a matching with $n-1$ edges in F_λ such that the vertex $[t, 1]$ is left unmatched by M^-; we call such matchings *almost perfect* – indeed, any perfect matching M in F_λ has n edges. Also observe that the other vertex left unmatched by M^- is $[\ell, p]$ for some $\ell \in L$ and $p \in \{0, 1\}$. Recall the parity-and-port-changing traversal of M in the proof of Lemma 8 resulting in the Hamiltonian cycle H$[M]$. Define a similar parity-and-port-changing traversal

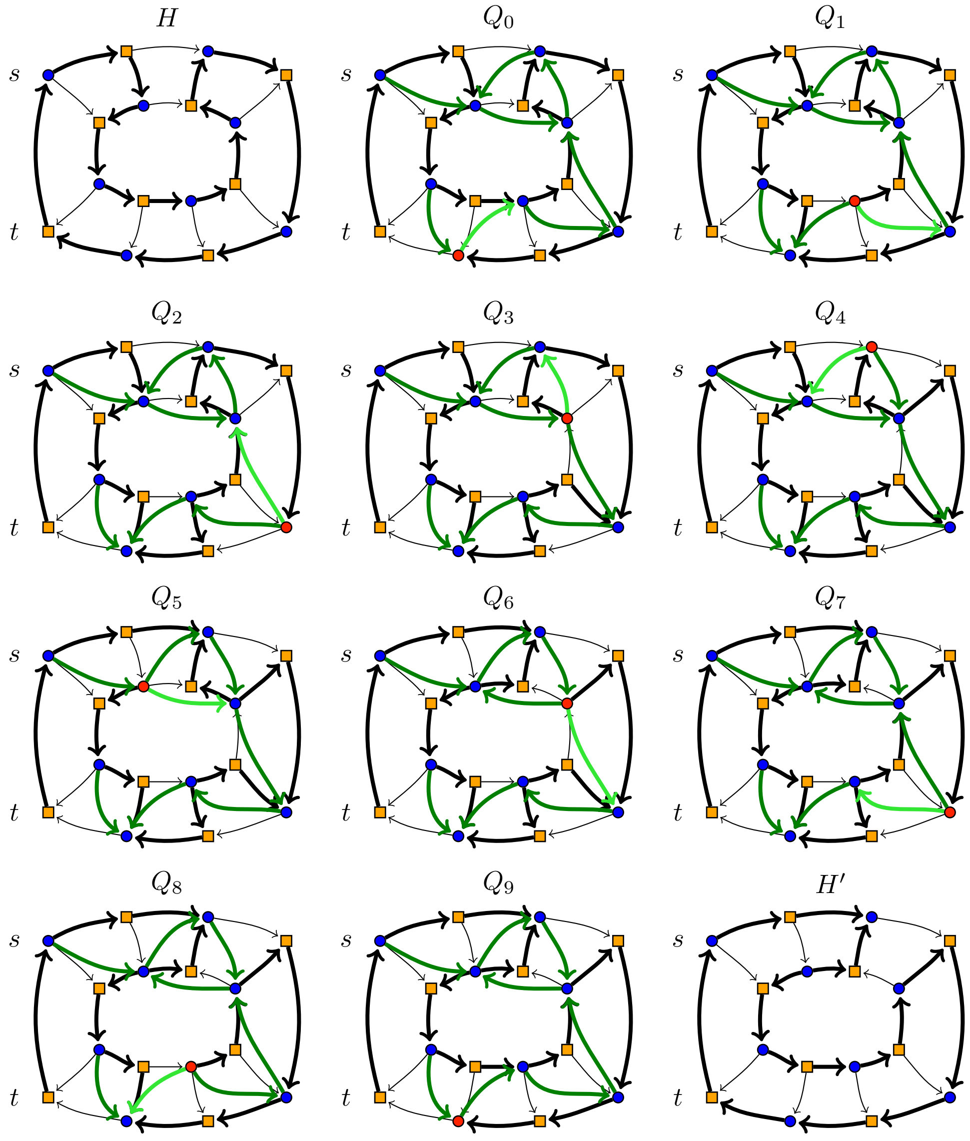

Figure 4 An example of the sequence of steps of Thomason's lollipop method in an n-vertex cubic bipartite planar graph viewed as a sequence of arc reversals in the directed graph $D_{\lambda,H}$. The given input G and H together with $e = \{s,t\}$ is displayed in the top left; the black arcs are oriented as in $\vec{G}_e$ obtained from Lemma 7 on input H. We display the initial Hamiltonian cycle H (top left) and the final Hamiltonian cycle H' (bottom right) obtained by the method, as well as the intermediate e-anchored Hamiltonian paths $Q_0, Q_1, \ldots, Q_9$ obtained in consecutive lollipop steps; the end-vertex of each Q_i is highlighted with red. The green arcs in Q_0 are the arcs of $D_{\lambda,H}$. Observe that each lollipop step from Q_i to Q_{i+1} can be understood as reversing the light-green arc in Q_i; the method terminates when the end-vertices of Q_0 and Q_{i+1} agree. By the structure of $D_{\lambda,H}$, we must have $i \leq n$; cf. Theorem 3.

of M^- by starting at the vertex $[\ell, \bar{p}]$ and observe by a similar argument as in the proof of Lemma 8 that this traversal defines an e-anchored Hamiltonian path $\mathrm{P}[M^-]$ from the vertex ℓ to the vertex t in G; in particular, observe that $\mathrm{P}[M^-]$ is e-anchored since by the structure of F_λ the almost perfect M^- must contain the edge $\{[s,0],[t,0]\}$.

We now proceed to characterize the e-anchored Hamiltonian paths $Q_0, Q_1, \ldots, Q_h$ using corresponding almost perfect matchings $M_0^-, M_1^-, \ldots, M_h^-$, and conclude that $h = 2d'+d \leq n$ in the process. For $i = 0, 1, \ldots, h$, let us write u_i for the end-vertex of Q_i other than t. Recalling that $Q_0 = P_e$ is constructed by deleting the edge $\{u,t\}$ from the Hamiltonian cycle H, let $p \in \{0,1\}$ be the port and $f \in E(F_\lambda)$ the edge with $f = \{[u,p],[t,1]\} \in M_H$. Take $M_0^- = M_H \setminus \{f\}$. In particular, we have $Q_0 = P_e = \mathrm{P}[M_0^-]$ and $u_0 = v_0 = u$. Let $p_0 = p$; we will fix values $p_i \in \{0,1\}$ for $i = 1, 2, \ldots, h$ as we progress in what follows.

We split the analysis into two ranges based on the parameter i. The first range corresponds to the forward-traversal of arcs in $D_{\lambda,H}$. For $i = 0, 1, \ldots, d'+d-1$, we say an almost perfect matching M^- has *property i* if

- **(i)** $[v_i, p_i]$ is left unmatched by M^-; and
- **(ii)** we have $\{[v_j, p(v_j, v_{j+1})], [r(v_j, v_{j+1}), \rho(v_j, v_{j+1})]\} \in M^-$ and $\{[r(v_j, v_{j+1}), \rho(v_j, v_{j+1})], [v_{j+1}, p'(v_j, v_{j+1})]\} \notin M^-$ for all $0 \leq j \leq i-1$; and
- **(iii)** we have $\{[v_j, p(v_j, v_{j+1})], [r(v_j, v_{j+1}), \rho(v_j, v_{j+1})]\} \notin M^-$ and $\{[r(v_j, v_{j+1}), \rho(v_j, v_{j+1})], [v_{j+1}, p'(v_j, v_{j+1})]\} \in M^-$ for all $i \leq j \leq d'+d$.

The second range corresponds to the backward-traversal of arcs in $D_{\lambda,H}$. For $i = d'+d, d'+d+1, \ldots, 2d'+d$, we we say an almost perfect matching M^- has *property i* if

- **(i')** $[v_i, p_i]$ is left unmatched by M^-; and
- **(ii')** we have $\{[v_j, p(v_j, v_{j+1})], [r(v_j, v_{j+1}), \rho(v_j, v_{j+1})]\} \in M^-$ and $\{[r(v_j, v_{j+1}), \rho(v_j, v_{j+1})], [v_{j+1}, p'(v_j, v_{j+1})]\} \notin M^-$ for all $d' \leq j \leq d'+d-1$ as well as for all $0 \leq j \leq 2d'+d-1-i$; and
- **(iii')** we have $\{[v_j, p(v_j, v_{j+1})], [r(v_j, v_{j+1}), \rho(v_j, v_{j+1})]\} \notin M^-$ and $\{[r(v_j, v_{j+1}), \rho(v_j, v_{j+1})], [v_{j+1}, p'(v_j, v_{j+1})]\} \in M^-$ for all $2d'+d-i \leq j \leq d'-1$.

From previous observations and (11) we have that M_0^- satisfies property 0.

Let us now analyse the lollipop step mapping Q_i to Q_{i+1} one value $i = 0, 1, \ldots, d'+d-1$ at a time. Suppose that there is an almost perfect matching M_i^- of F_λ that satisfies property i and that $Q_i = \mathrm{P}[M_i^-]$. In particular, we have $u_i = v_i$ by (i) and $Q_i = \mathrm{P}[M_i^-]$. We claim that the vertex $r(v_i, v_{i+1})$ is the unique degree-three vertex in the lollipop formed by the lollipop step transforming Q_i to Q_{i+1}. Observe by (iii) that $\{[r(v_i, v_{i+1}), \rho(v_i, v_{i+1})], [v_{i+1}, p'(v_i, v_{i+1})]\} \in M_i^-$, implying that $\{r(v_i, v_{i+1}), v_{i+1}\}$ is an edge in $Q_i = \mathrm{P}[M_i^-]$. Recalling that M_i^- is almost perfect, all vertices in $R \times \{0,1\}$ are matched, so $r(v_i, v_{i+1}) \in R$ is in fact adjacent to another vertex $\omega_i \neq v_{i+1}$ along an edge in $Q_i = \mathrm{P}[M_i^-]$. By (iii) and (11) we have $\{v_i, r(v_i, v_{i+1})\}$ is an edge in G but not in $Q_i = \mathrm{P}[M_i^-]$, and Q_i ends at v_i. Thus, $r(v_i, v_{i+1})$ is the unique degree-three vertex in the lollipop. Next, the lollipop step proceeds to delete an edge adjacent to the degree-three vertex $r(v_i, v_{i+1})$ in the lollipop. This edge is $\{v_{i+1}, r(v_i, v_{i+1})\}$ by the previous analysis. It follows that Q_{i+1} is obtained from Q_i by deleting $\{v_{i+1}, r(v_i, v_{i+1})\}$ and inserting $\{v_i, r(v_i, v_{i+1})\}$. Thus, Q_{i+1} ends at $u_{i+1} = v_{i+1}$. Define M_{i+1}^- by starting with M_i^- and deleting the edge $\{[r(v_i, v_{i+1}), \rho(v_i, v_{i+1})], [v_{i+1}, p'(v_i, v_{i+1})]\}$ as well as inserting the edge $\{[v_i, p(v_i, v_{i+1})], [r(v_i, v_{i+1}), \rho(v_i, v_{i+1})]\}$. Fix $p_{i+1} = p'(v_i, v_{i+1})$. From (i), (ii), and (iii) we have that M_{i+1}^- is an almost perfect matching that satisfies property $i+1$. Furthermore, $Q_{i+1} = \mathrm{P}[M_{i+1}^-]$.

The analysis of the lollipop step mapping Q_i to Q_{i+1} for $i = d'+d, d'+d+1, \ldots, 2d'+d-1$ is now similar, but relying on properties (i'), (ii'), (iii') instead. From the existence of an almost perfect matching M_i^- of F_λ that satisfies property i and $Q_i = \mathrm{P}[M_i^-]$, by a similar analysis we conclude that there exists an almost perfect matching M^-_{i+1} of F_λ that satisfies property $i+1$ and $Q_{i+1} = \mathrm{P}[M^-_{i+1}]$. Since $v_{2d'+d} = u$ and u is adjacent to t in G, from (i') we conclude in particular that $P'_e = Q_{2d'+d}$ and thus $h = 2d' + d$. Since $2d' + d \leq n$, we have shown that the lollipop method terminates in at most n lollipop steps. ◀

We note that the algorithm implicit in the proof not only uses at most a linear number of lollipop steps, but also can be implemented with the guidance of $D_{\lambda,H}$ to run in linear time.

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