Noncrossing Longest Paths and Cycles

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Abstract

Edge crossings in geometric graphs are sometimes undesirable as they could lead to unwanted situations such as collisions in motion planning and inconsistency in VLSI layout. Short geometric structures such as shortest perfect matchings, shortest spanning trees, shortest spanning paths, and shortest spanning cycles on a given point set are inherently noncrossing. However, the longest such structures need not be noncrossing. In fact, it is intuitive to expect many edge crossings in various geometric graphs that are longest.

Recently, Álvarez-Rebollar, Cravioto-Lagos, Marín, Solé-Pi, and Urrutia (Graphs and Combinatorics, 2024) constructed a set of points for which the longest perfect matching is noncrossing. They raised several challenging questions in this direction. In particular, they asked whether the longest spanning path, on any finite set of points in the plane, must have a pair of crossing edges. They also conjectured that the longest spanning cycle must have a pair of crossing edges.

In this paper, we give a negative answer to the question and also refute the conjecture. We present a framework for constructing arbitrarily large point sets for which the longest perfect matchings, the longest spanning paths, and the longest spanning cycles are noncrossing.

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1 Introduction

Traversing points in the plane by a polygonal path or cycle possessing a desired property has a rich background. For instance, the celebrated travelling salesperson problem asks for a polygonal path or cycle with minimum total edge length [9, 23, 25]. In recent years, there has been increased interest in paths and cycles with properties such as being noncrossing [2, 16], minimizing the longest edge length [6, 12, 22], maximizing the shortest edge length [7], minimizing the total or largest turning angle [1, 11, 18, 21], and minimizing the number of turns [13, 17, 26] to name a few. The longest cycle – the MaxTSP – is NP-hard in Euclidean spaces of dimension ≥ 3, but the complexity of the planar MaxTSP is unknown [20, 10]. Paths and cycles that have combinations of these properties have also attracted attention. For example, simultaneously being noncrossing and having maximum total edge length [3, 19] is difficult to satisfy: to achieve a larger length we typically introduce more crossings.

Edge crossings in geometric graphs are usually undesirable as they have the potential of creating unwanted situations such as collisions in motion planning and inconsistency in VLSI layout. They are also undesirable in the context of graph drawing and network visualization as they make drawings more difficult to read and use. Short geometric structures such as shortest perfect matchings, shortest spanning trees, shortest spanning paths, and shortest spanning cycles are inherently noncrossing. This property, however, does not necessarily hold if the structure is not shortest. For long structures such as longest perfect matchings, longest spanning trees, longest spanning paths, and longest spanning cycles – the other end of the spectrum – it seems natural to expect many crossings. Counting crossings in geometric graphs and finding geometric structures with a minimum or maximum number of crossings are active research areas in discrete geometry. The study of this type of problem attracted more attention after the work of Aronov et al. [8] in 1994, who showed that any set of n points in the plane in general position admits a *crossing family* (a set of pairwise intersecting segments) of size $\Omega(\sqrt{n})$. They also conjectured that the true lower bound is linear in n. The current best lower bound, $n^{1-o(1)}$, was established by Pach et al. [24] in 2019.

The noncrossing property of shortest structures is mainly ensured by the triangle inequality. The triangle inequality, as noted by Alon et al. [3], also implies that the longest structures often have crossings because a structure usually gets longer by creating more crossings. Alon et al. [3] studied the problem of finding longest noncrossing structures (such as matchings, paths, or trees). Some of their initial results have been improved and extended by Dumitrescu and Tóth [19] (for matchings, paths, and cycles), by Biniaz et al. [14] and by Cabello et al. [15] (for trees). Along this direction, one might wonder whether a longest structure (defined on an arbitrarily large point set) is necessarily crossing. This was explicitly asked by Álvarez-Rebollar et al. [4]. Among other interesting results, they presented arbitrarily large planar point sets for which the longest perfect matching is noncrossing. They asked the following question and proposed the following conjecture:

▶ **Question 1** (Álvarez-Rebollar et al. [4]). *For every sufficiently large planar point set, must the longest spanning path have two edges that cross each other?*

▶ **Conjecture 1** (Álvarez-Rebollar et al. [4]). *The longest spanning cycle on every sufficiently large set of points in the plane has a pair of crossing edges.*

The “sufficiently large” condition in the question and conjecture makes sense, as otherwise one can take any 3 points in general position, or any 4 points that are not in a convex position – for such point sets, all spanning paths and cycles are noncrossing.

In the other direction, one might wonder about maximizing the number of crossings in cycles. Here, we would like to highlight another result of Álvarez-Rebollar et al. [4, 5]. Let $C(n)$ be the largest number such that any set of n points in the plane admits a spanning cycle with at least $C(n)$ pairs of crossing edges. Álvarez-Rebollar et al. [4, 5] established the following lower and upper bounds: $n^2/12 - O(n) < C(n) < 5n^2/18 - O(n)$. In other words, any set of n points in the plane admits a spanning cycle with at least $n^2/12 - O(n)$ crossings, and there is a family of point sets that does not admit any cycle with more than $5n^2/18 - O(n)$ crossings.

1.1 Our contributions

In this paper, we provide negative answers to both Question 1 and Conjecture 1. For any integer $n \geq 1$ we present a set of n points in the plane for which the longest spanning path is unique and noncrossing. Similarly, for any integer $n \geq 4$, we present a set of n points in the plane for which the longest spanning cycle is unique and noncrossing. To build such point sets, we use the following framework: First, we choose a set P of points on the x-axis for which the longest structure may not be unique. Then, we assign new y-coordinates to points in P to obtain a new point set P' for which the longest structure corresponds to one in P and is also unique and noncrossing. In Section 6, we present some structural properties of longest paths and cycles, which may be of independent interest.

1.2 Preliminaries

All point sets considered in this paper are in the Euclidean plane. A *geometric graph* is a graph with vertices represented by points and edges represented by line segments between the points. Let P be a finite point set. A *spanning path* for P is a path drawn with straight-line edges such that every point in P lies at a vertex of the path and every vertex of the path lies at a point in P. A *spanning cycle* is defined analogously. In other words, a spanning path is a Hamiltonian path in the complete geometric graph on P, and a spanning cycle is a Hamiltonian cycle in this graph.

Consider two line segments, each connecting a pair of points in P. If the interiors of the segments intersect, then we say that they *cross*; this configuration is called a *crossing*. A path or a cycle is called *noncrossing* if its edges do not cross each other. We denote the undirected edge between two points p and q by pq, the directed edge from p towards q by (p, q), and the Euclidean distance between p and q by $|pq|$. The *length* of a geometric graph G is the sum of the lengths of its edges, and we denote it by $|G|$.

2 Longest Paths and Cycles on the Real Line

In this section we characterize longest paths and cycles in dimension one. These observations play a pivotal role in our constructions in the plane (Sections 3 and 4). We say that an edge e *intersects a point* p if the intersection of e and p is not empty (the intersection could be an endpoint of e). For a sorted set of $2k+1$ numbers, the median is the number with rank $k+1$, and for a sorted set of $2k$ numbers, the median is the mean of the two numbers with ranks k and $k+1$.

▶ **Lemma 2.** *Let P be a set with an even number of points in $\mathbb{R}$, i.e., in dimension one. The endpoints of any longest spanning path on P lie on different sides of the median of P.*

Proof. Let $P = \{p_1, \dots, p_n\}$ and assume *w.l.o.g.* that 0 is the median of P (in particular, $0 \notin P$). Let H be a longest spanning path on P. Orient the edges of H to make it a directed path. Let p_s and p_e be the starting and ending points of H, respectively. For the sake of contradiction, assume that p_s and p_e have the same sign, which we may assume, due to symmetry, to be positive. Thus $p_s, p_e > 0$. Then, the sum of degrees of vertices in H to the left of the origin is 2 more than the sum of degrees of vertices to the right. Therefore, H must have a directed edge (p_a, p_b) where $p_a, p_b < 0$. If $p_b < p_a$, then by replacing (p_a, p_b) with the undirected edge $p_s p_b$ we obtain a longer undirected path; and if $p_b > p_a$ by replacing (p_a, p_b) with $p_e p_a$ we obtain a longer undirected path. Both cases lead to a contradiction. ◀

▶ **Lemma 3.** *Let P be a set with an even number of points in $\mathbb{R}$, i.e., in dimension one. Let H be a spanning path on P. Then H is a longest spanning path if and only if*

(i) *every edge of H intersects the median of P, and*

(ii) *the two endpoints of H are the two points closest to the median of P.*

Proof. Let $P = \{p_1, \dots, p_n\}$ so that $p_i < p_j$ for all $i < j \in \{1, \dots, n\}$, and assume *w.l.o.g.* that 0 is the median of P. Note that $0 \notin P$ since n is even. First, we prove by contradiction that if H is a longest spanning path, then (i) and (ii) hold.

Suppose that (i) does not hold. Orient the edges of H to make it a directed path. Let (p_a, p_b) be an edge of H that does not intersect the median. Due to symmetry, assume that $p_a, p_b < 0$. By Lemma 2, the endpoints of H lie on different sides of the median. This implies that both sides have the same sum of vertex degrees. Thus H must have an edge (p_c, p_d) such that $c, d > 0$. By replacing these edges with $p_a p_c$ and $p_b p_d$ we obtain an (undirected) spanning path that is longer than H because $|p_a - p_c| + |p_b - p_d| > |p_a - p_b| + |p_c - p_d|$. This contradicts H being a longest path.

Now suppose that (ii) does not hold: without loss of generality $p_{n/2}$ is not an endpoint of H. (The case for $p_{n/2+1}$ can be handled symmetrically). Then H has an endpoint p_a with $a < n/2$. Orient the edges of H so that the path is directed from p_a towards the other endpoint. Let $(p_{n/2}, p_b)$ be the outgoing edge from $p_{n/2}$. By part (i), we have $p_b \geq 0$. By removing $(p_{n/2}, p_b)$ we obtain two paths, and p_b is an endpoint on one of those paths. Next, join the paths with a new edge (p_a, p_b). Thus we obtain an (undirected) spanning path that is longer than H because $|p_a - p_b| > |p_{n/2} - p_b|$. This contradicts H being longest.

Finally, we prove that any spanning path H that satisfies (i) and (ii) is longest, using a direct proof. Consider a longest spanning path L on P. By the sufficiency proof, (i) and (ii) hold for L. This implies that the positive interval $[p_{n/2}, p_{n/2+1}]$ is contained in each of the $n-1$ edges, hence it contributes to the length of L with multiplicity $n - 1$. Similarly, for any $i \in \{2, \dots, n/2\}$ the positive interval $[p_{n/2+i-1}, p_{n/2+i}]$ contributes to the length of L by multiplicity $n - 2i$. A similar argument holds for negative intervals. See Figure 1. On the other hand, any spanning path (including H) that satisfies (i) and (ii) receives the exact same multiplicities from the corresponding intervals. Therefore H and L have the same length, and hence H is also a longest path. ◀

A statement similar to that of Lemma 3 can be proved for paths with an odd number of points (in this case one endpoint is the median itself and the other endpoint is the closest point to the median). However, we will not use this in our construction.

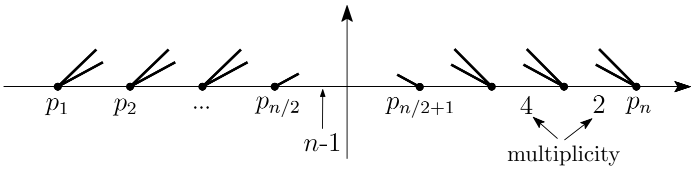

Figure 1 Illustration of a longest path for a point set on a line, for the case where the number of points, n, is even. Numbers below intervals $[p_{n/2+i}, p_{n/2+i+1}]$ represent the multiplicity of the contribution of the corresponding intervals to the length of the longest path.

▶ **Lemma 4.** *Let P be a finite set in $\mathbb{R}$, i.e., in dimension one.*

(i) *A spanning cycle on P is longest iff each of its edges intersects the median of P.*

(ii) *If P contains an odd number of points, then for any longest spanning cycle the two edges incident to the median lie on opposite sides of it.*

(iii) *Assume that P contains $n = 2k+1$ points and there is an interval I of length $h > 0$ between the leftmost $k+1$ and the rightmost k points. Then in any longest spanning cycle, $n-1 = 2k$ edges contain the interval I; and if a spanning cycle has fewer than $2k$ edges that contain I, then it is at least $2h$ shorter than a longest cycle.*

Proof. Let $P = \{p_1, \ldots, p_n\}$ so that $p_i < p_j$ for all $i < j \in \{1, \ldots, n\}$, and assume *w.l.o.g.* that 0 is the median of P. Note that $0 \notin P$ if n is even, and $p_{\lceil n/2 \rceil} = 0$ if n is odd.

First we prove the sufficiency of (i) by contradiction. Let C be a longest cycle on P, and orient its edges to obtain a directed cycle. Suppose, for the sake of contradiction, that the edge (p_a, p_b) of C does not intersect the median. We may assume *w.l.o.g.* that $p_a, p_b < 0$. The sum of vertex degrees strictly on the left and right side of the median are the same, and the edges that contain 0 in their interior contribute 1 to both sums. Consequently, C contains an edge (p_c, p_d) with $p_c, p_d > 0$; or (when n is odd) there are two edges incident to the median, say $(p_c, 0)$ and $(0, p_d)$ with $p_c, p_d > 0$. In the first case, we can replace edges (p_a, p_b) and (p_c, p_d) with (p_a, p_c) and (p_b, p_d). In the second case, replace (p_a, p_b) and $(p_c, 0)$ with (p_a, p_c) and $(p_b, 0)$. In both cases, we obtain a longer (undirected) spanning cycle, contradicting the maximality of C.

The necessity of (i) can be proved by a counting argument similar to that of Lemma 3-(i).

Now, we prove (ii) by contradiction. Without loss of generality, let $0 \in P$ be the median of P. Suppose that the median is incident to two edges $(p_c, 0)$ and $(0, p_d)$ with $p_c, p_d > 0$. Then, there is a point in P to the right of 0 incident to an edge of C that does not contain 0 in its interior. Denote this edge by (p_a, p_b), where $p_a, p_b < 0$. We can replace edges (p_a, p_b) and $(p_c, 0)$ with (p_a, p_c) and $(p_b, 0)$ to obtain a longer spanning cycle, contradicting the maximality of C.

To prove the first part of (iii), note that if $n = 2k+1$, then the median is the $(k+1)$-st point of P, that we denote by p_0. Let C be a longest cycle on P. It is implied from (i) and (ii) that exactly one edge of C (which is incident to p_0) does not contain I. The remaining $n-1 = 2k$ edges contain I.

For the second claim in (iii), let C be a spanning cycle on P in which fewer than $2k$ edges contain I. Orient the edges of C to obtain a directed cycle. The sum of degrees of the leftmost $k+1$ (resp., rightmost k) vertices is $2k+2$ (resp., $2k$), and the edges containing I have fewer than $2k$ left (resp., right) endpoints. Consequently, the leftmost $k+1$ (resp., rightmost k) points in P induce at least two edges (resp., one edge) of C. Therefore, C contains two edges, (p_a, p_b) and (p_c, p_d), such that p_a, p_b are to the left of I and p_b, p_d are to the right of I. We can replace these two edges with (p_a, p_c) and (p_b, p_d), to obtain a spanning cycle C' that traverses I two more times than C. In particular, we have $|C'| \geq |C| + 2\,|I| = |C| + 2h$, hence $|C| \leq |C'| - 2h \leq |C_{\max}| - 2h$, where $C_{\max}$ is a longest cycle on P. ◀

3 Noncrossing Longest Paths

Let $n \geq 1$ be an integer. In this section, we construct n points for which the longest spanning path is unique and noncrossing. This can be easily observed for $n < 5$: For example, for $n = 4$, any spanning path of the vertices of a triangle and a point in the interior is noncrossing. Thus, we will now assume that $n \geq 5$. In Section 2, we uncovered some structural properties of longest paths for n points on a line. Here we show how to construct a 2-dimensional point set starting with n points on the x-axis and then assigning y-coordinates to the points. We show that the longest path is unique and noncrossing. We describe our construction for the case where n is even; the construction for the case where n is odd follows with some minor changes. The following theorem summarizes our result in this section.

▶ **Theorem 5.** *For every integer $n \geq 1$ there exists a set of n points in the plane for which the longest spanning path is unique and noncrossing.*

In Section 3.1 we give an overview of our construction for an even number of points. The details and proofs are given in Section 3.2. The case of odd paths is considered in Section 3.3.

3.1 A path with an even number of points: An overview

For $k \geq 3$, consider a set P of $n = 2k$ points p_i on the x-axis such that $p_1 = (0, 0)$ and $p_i = (i, 0)$ for $i = -1, \pm 2, \ldots, \pm k$, as illustrated in Figure 2(a). Our construction would work even if we set $p_1 = (1, 0)$; however, for a reason that will become clear in Section 3.3, we set p_1 differently. The longest spanning path for this point set is not unique. In fact, Lemma 3 implies that any spanning path with endpoints p_1 and p_{-1} and with all edges crossing the y-axis is a longest path. Conversely, any longest path must have endpoints p_1 and p_{-1}, and its edges must cross the y-axis. Let $\mathcal{H}$ be the set of these paths. Let P' be the point set obtained by assigning to each point p_i a y-coordinate y_i such that, as illustrated in Figure 2(b), the following holds:

$$\frac{1}{8k} = y_1 \gg y_{-2} \gg y_2 \gg y_{-3} \gg y_3 \gg \cdots \gg y_{-k} \gg y_k \gg y_{-1} = 0.$$

The value y_1 is much larger than y_{-2}, which is in turn much larger than y_2 and so on. Notice that the largest y-coordinate y_1 is $1/8k$ which is much smaller than 1. Due to the small y-coordinates, a longest path H' on P' corresponds to a path $H \in \mathcal{H}$. The length of H' is roughly the length of H plus a very small value $\Delta(H')$, which depends on the new y-coordinates. Let e_1 be the only edge of H' incident to p_1. Since p_1 has a very large y-coordinate compared to other points, the contribution of e_1 to $\Delta(H')$ is larger than the contribution of other edges. The contribution of e_1 is maximized if it connects p_1 to the nearest plausible neighbor, which is p_{-2}; this can be observed from Figure 2(b). Therefore $e_1 = p_1p_{-2}$. By a similar argument, p_{-2} gets connected to p_2, and so on. It follows that the path H' is unique and it is $p_1, p_{-2}, p_2, p_{-3}, p_3, \ldots, p_{-k}, p_k, p_{-1}$. This path is y-monotone, and hence noncrossing; see Figure 2(b).

Note. Figures 2(a) and 2(b) are not to scale. The y-coordinates should be small enough so that all points lie almost on the x-axis (We exaggerated the y-coordinates to facilitate readability). Moreover, if we orient the path from p_1 towards p_{-1}, then the extension of every directed edge intersects all edges that follow.

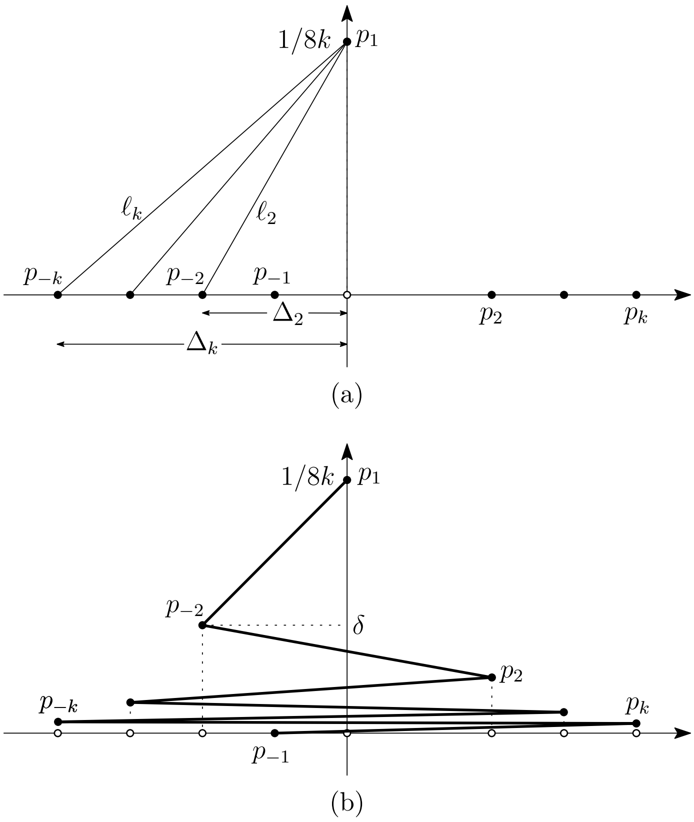

Figure 2 Illustration of the construction of a longest path for $2k$ points. The figure is not to scale as the real y-coordinates are very small so that the points lie almost on the x-axis. (a) Lifting p_1 to the y-coordinate $1/8k$. (b) The final longest path.

3.2 A path with an even number of points: Details

Recall the set P of $2k$ points, $k \geq 3$, on the x-axis, described in the previous section and illustrated in Figure 2(a). We say that an edge e *intersects the y-axis* if the intersection of e and the y-axis is not empty (the intersection could be an endpoint of e). The longest paths for points on a line were characterized in Lemma 3. Denote by $\mathcal{H}$ the set of all longest spanning paths on P.

▶ **Lemma 6.** *Let $0 \leq \varepsilon \leq \frac{1}{8k}$ be a real number. Suppose that every point in P is perturbed by a distance of at most ε. Let P' be the new point set after perturbation. Then, the order of the points along any longest path for P' is the same as the order of the points along some path in $\mathcal{H}$.*

Proof. The length of any path on P is an integer. Therefore, any path in $\mathcal{H}$ is at least 1 unit longer than any path not in $\mathcal{H}$.

Let H' be any longest path on P'. The difference between its length and the length of any path in $\mathcal{H}$ is at most $(2k-1) \cdot 2\varepsilon$ because H' has $2k-1$ edges, each edge has 2 endpoints, and each endpoint is at distance at most ε from its corresponding point in P. Since $\varepsilon \leq \frac{1}{8k}$ the difference is less than $1/2$. Therefore, H' cannot correspond to a path that is not in $\mathcal{H}$, so H' corresponds to a path in $\mathcal{H}$ with the same order of points. ◀

Our plan is to assign new y-coordinates to the points of P to obtain a point set P' for which the longest path is y-monotone and unique. The new y-coordinates will be at most $\frac{1}{8k}$, and thus, by Lemma 6, the longest path H' of P' will correspond to a path in $\mathcal{H}$. We will make H' correspond to the path $p_1, p_{-2}, p_2, p_{-3}, p_3, \ldots, p_k, p_{-1}$, which is in $\mathcal{H}$ (by Lemma 3) and depicted in Figure 2(b). We assign to each point p_i the y-coordinate y_i such that the following holds:

$$y_1 \gg y_{-2} \gg y_2 \gg y_{-3} \gg y_3 \gg \cdots \gg y_{-k} \gg y_k \gg y_{-1}.$$

We set $y_1 = \frac{1}{8k}$, $y_{-1} = 0$, and use the following lemma to identify the remaining y-coordinates.

▶ **Lemma 7.** *There exists a real number δ, with $0 < \delta < y_1$, such that if $0 \leq y_i \leq \delta$ for each $i \neq 1$ then the longest path on P' connects p_1 to p_{-2}.*

Proof. Since each y_i is at most $1/8k$, Lemma 6 implies that any longest path H' on P' corresponds to a path H in $\mathcal{H}$. Due to small y-coordinates, we have $|H'| = |H| + \Delta(H')$ for some small value $\Delta(H') \geq 0$ which depends on the new y-coordinates. Specifically, we have

$$\begin{aligned} |H'| &= \sum_{(p_i,p_j)\in E(H')} |p_ip_j| = \sum_{(p_i,p_j)\in E(H')} \sqrt{|i-j|^2 + |y_i - y_j|^2} \\ &= |H| + \sum_{(p_i,p_j)\in E(H')} \left(\sqrt{|i-j|^2 + |y_i - y_j|^2} - |i-j|\right) = |H| + \Delta(H'), \end{aligned}$$

where $0 \leq \sqrt{|i-j|^2 + |y_i - y_j|^2} - |i-j| \leq |y_i - y_j| \leq \max\{y_i, y_j\}$.

Recall from Lemma 3 that p_1 is an endpoint of any longest path in $\mathcal{H}$. Moreover, p_1 is connected to a point (different from p_{-1}) to the left of the y-axis. For $j \in \{2, \ldots, k\}$ let ℓ_j be the Euclidean distance between p_1 and the point $(-j, 0)$, and let Δ_j be the difference of their x-coordinates as in Figure 2(a). The contribution of p_1p_{-j} to $|H'|$ would be at least $\ell_j - \delta$ (when p_{-j} has y-coordinate δ) and at most ℓ_j (when p_{-j} has y-coordinate 0). The contribution of the corresponding edge to $|H|$ would be Δ_j. Hence the contribution of p_1p_{-j} to $\Delta(H')$ would be at least $\ell_j - \delta - \Delta_j$ and at most $\ell_j - \Delta_j$. An easy calculation shows that

$$\ell_2 - \Delta_2 > \ell_3 - \Delta_3 > \cdots > \ell_k - \Delta_k;$$

this is also implied by the fact that $\Delta_{i+1} - \Delta_i = 1$ while $\ell_{i+1} - \ell_i < 1$. If we set $\delta < (\ell_2 - \Delta_2) - (\ell_3 - \Delta_3)$, then the contribution of p_1p_{-2} to $\Delta(H')$ is at least

$$\ell_2 - \delta - \Delta_2 > \ell_2 - \Delta_2 - ((\ell_2 - \Delta_2) - (\ell_3 - \Delta_3)) = \ell_3 - \Delta_3,$$

which is larger than the contribution of any other plausible edge p_1p_{-j}. Since the y-coordinates of all other points are less than δ, any other edge of H' contributes less than δ to $\Delta(H')$. By setting

$$\delta = \frac{(\ell_2 - \Delta_2) - (\ell_3 - \Delta_3)}{2k - 1},$$

the contribution of p_1p_{-2} exceeds the sum of the contributions of the remaining $2k-2$ edges of H'. Thus, for this choice of δ the longest path H' connects p_1 to p_{-2}. ◀

By Lemma 7, we have a specific value δ such that the longest path includes edge (p_1, p_{-2}). Now we set $y_{-2} = \delta$ and repeat the arguments of Lemma 7, with y_{-2} and p_{-2} (instead of y_1 and p_1). This implies that the next edge of the longest path will connect p_{-2} to p_2. Repeating this $2k-5$ more times, we obtain the unique longest path $p_1, p_{-2}, p_2, p_{-3}, p_3, \ldots, p_k, p_{-1}$, as in Figure 2(b); in each of the last two steps, there is only one remaining plausible edge (namely, $p_{-k}p_k$ from p_{-k}, and p_kp_{-1} from p_k). This path is y-monotone and hence is noncrossing.

3.3 A path with an odd number of points

In this section, we obtain a noncrossing longest path with an odd number of points. Here is the place where we use the coordinate $(0,0)$ of the point $p_1 \in P$. We show that our construction for even paths leads to a construction for odd paths by simply removing p_1. Thus we do not need to repeat the lemmas of Section 3.2 for the odd case.

We claim that if we remove the point p_1 from the path H' constructed on P' in the previous section, the remaining path, i.e., $H'' = p_{-2}, p_2, p_{-3}, p_3, \dots, p_k, p_{-1}$, is the longest path for the remaining $2k-1$ points. By construction, $|H'| = |H''| + |p_1 p_{-2}|$. Assume, for the sake of proof by contradiction, that the longest path L for the remaining points is longer than H''. Among the two endpoints of L, let p_i be an endpoint that is not p_{-1}. Due to our choices of the x- and y-coordinates we have $|p_1 p_i| \geq |p_1 p_{-2}|$. Therefore the concatenation of L and $p_1 p_i$ would give a path on P' of length $|L| + |p_1 p_{-2}|$ which is larger than $|H'|$. This contradicts H' being the longest path on P'.

4 Noncrossing Longest Cycles

Let $n \geq 3$ be an integer. In this section, we construct a set of n points for which the longest spanning cycle is unique and noncrossing. For $n = 3$, every spanning cycle is noncrossng. For $n = 4$, we take three vertices of a triangle and a point in the interior. Thus, we assume that $n \geq 5$.

▶ **Theorem 8.** *For every integer $n \geq 3$ there exists a set of n points in the plane for which the longest spanning cycle is unique and noncrossing.*

In Section 4.1 we give an overview of our construction for an even number of points. The details and proofs are given in Section 4.2. For an odd number of points we sketch a construction in Section 4.3.

4.1 A cycle with an even number of points: An overview

Let $n \geq 6$ be an even number. Then either $n = 4k$ or $n = 4k-2$ for some integer k. To simplify the indexing (of points and y-coordinates) in our construction, from now on we assume that $n = 4k-2$. Let P be a set of n points, consisting of $2k$ points $p_i = (i, 0)$ for $i = \pm 1, \pm 2, \dots, \pm k$ and $2k-2$ points $p'_i = (i+\epsilon, 0)$ for $i = -1, \pm 2, \dots, \pm(k-1), k$, where $\epsilon > 0$ is a small value to be determined; see Figure 3. (The construction for $n = 4k$ is similar; it consists of P and two additional points $p_{k+1} = (k+1, 0)$ and $p'_{-k} = (-k+\epsilon, 0)$.) Our construction for cycles is somewhat similar to that of paths in the sense that our cycle consists of two y-monotone interior-disjoint paths between p_1 and p_{-k} (or between p_1 and p_{k+1} when n is a multiple of 4). Although the main idea sounds simple, the noncrossing property of the longest cycle is not straightforward and involves a more detailed analysis.

Lemma 4 implies that a spanning cycle on P is longest if and only if each of its edges intersects the y-axis. Let $\mathcal{C}$ be the set of all longest spanning cycles on P. As illustrated in Figure 3, we obtain a point set P' by assigning to each point p_i and p'_i the respective y-coordinates y_i and y'_i such that:

$$\frac{1}{16k} = y_1 \gg y_{-1} \gg y'_2 \gg y_{-2} \gg y'_3 \gg \cdots \gg y'_k \gg y_{-k} = 0.$$

For each $i \in \{2, 3, \dots, k\}$ we choose y_i such that p_i lies just below (almost on) the segment $p'_{-i+1} p'_i$, and for each $i \in \{-1, -2, \dots, -(k-1)\}$ we choose y'_i such that p'_i lies just below (almost on) the segment $p_{-i} p_i$.

Due to the small y-coordinates, any longest cycle C' on P' corresponds to a cycle $C \in \mathcal{C}$. Moreover $|C'| = |C| + \Delta(C')$ for some small value $\Delta(C')$ which depends on the new y-coordinates. Since p_1 has the largest y-coordinate, the contribution of the two edges of C' that are incident to p_1 (say e_1 and e_2) is maximized when they are connected to the nearest plausible neighbors which are p_{-1} and p'_{-1}. We will choose the y-coordinates in such a way that the contribution of e_1 and e_2 is larger than the sum of the contributions of the remaining edges of the cycle. Thus C' must connect p_1 to p_{-1} and p'_{-1}. Similarly, by a suitable choice of y-coordinates, we enforce C' to connect p_{-1} and p'_{-1} to the nearest plausible neighbors which are p_2 and p'_2, and so on. By repeating this process, the longest cycle C' would be the concatenation of two paths $p_1, p_{-1}, p_2, p_{-2}, \ldots, p_{-k}$ and $p_1, p'_{-1}, p'_2, p'_{-2}, \ldots, p'_k, p_{-k}$.

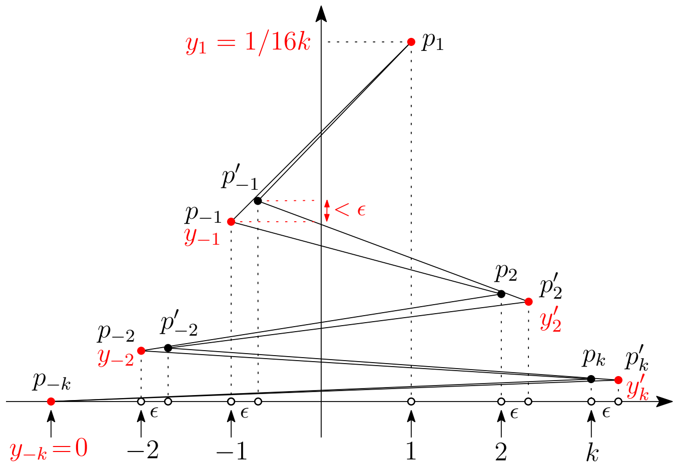

Figure 3 Illustration of the construction of a longest cycle for $4k-2$ points. The figure is not to scale. The y-coordinates should be small enough so that all points lie almost on the x-axis.

4.2 A cycle with an even number of points: Details

Recall the point set P from the previous section (the y-coordinates and the value of $\epsilon > 0$ will be determined in this section). The longest cycles for points on a line were characterized in Lemma 4. Let $\mathcal{C}$ be the set of all longest cycles on P.

▶ **Lemma 9.** *Any cycle in $\mathcal{C}$ is at least 1 unit longer than any cycle not in $\mathcal{C}$.*

Proof. Consider any cycle D that is not in $\mathcal{C}$. Lemma 4 implies that D has an edge that does not intersect the y-axis. Orient the edges of D to make it a directed cycle. Since the number of points to the left of the y-axis is the same as the number of points to its right, D has two directed edges (p_a, p_b) and (p_c, p_d) such that $a, b \le -1$ and $c, d \ge 1$. By replacing these edges with p_ap_c and p_bp_d we obtain an (undirected) spanning cycle D' such that

$$|D'| - |D| = (|p_ap_c| + |p_bp_d|) - (|p_ap_b| + |p_cp_d|) \ge 2|p_1p'_{-1}| = 2(2-\epsilon) > 1.$$

Since the length of any cycle C in $\mathcal{C}$ is at least $|D'|$, we get $|C| > |D| + 1$. ◀

▶ **Lemma 10.** *Let $0 \le \varepsilon \le 1/16k$ be a real number. Suppose that every point of P is perturbed by a distance of at most ε. Then the order of the points along any longest cycle of the new point set is the same as the order of the points along some cycle in $\mathcal{C}$.*

Proof Sketch. The proof is similar to that of Lemma 6 and uses Lemma 9. The parameter ε is small enough such that the total change in the length of any spanning cycle on P is less than 1/2. Together with Lemma 9, this implies that any longest cycle on the perturbed points corresponds to a cycle in $\mathcal{C}$. ◀

To obtain P' we only need to describe the following y-coordinates:

$$y_1 \gg y_{-1} \gg y_2' \gg y_{-2} \gg y_3' \gg \cdots \gg y_k' \gg y_{-k}.$$

The y-coordinates of the remaining points would then follow as outlined in the previous section (more details are given after Lemma 12). We set $y_1 = \frac{1}{16k}$ and $y_{-k} = 0$. We use the following lemma (which can be proven similarly to Lemma 7) to assign the y-coordinates.

▶ **Lemma 11.** *There exists a real number δ, $\epsilon \leq \delta < y_1$, such that if $0 \leq y_i \leq \delta$ for $i \neq 1$ and $0 \leq y_i' \leq \delta$ for $i \neq -1$, then every longest cycle of P' connects p_1 to p_{-1} and p'_{-1}.*

Proof. Lemma 10 implies that any longest cycle C' on P' corresponds to a cycle C in $\mathcal{C}$. Due to small y-coordinates, we have $|C'| = |C| + \Delta(C')$ for some small value $\Delta(C') \geq 0$ which depends on the new y-coordinates. Lemma 4 implies that C' connects p_1 to two points to the left of the y-axis. Similar to Lemma 7, for $j \in \{1, \ldots, k\}$ define ℓ_j as the Euclidean distance between p_1 and the point $(-j, 0)$, and define Δ_j as the difference of their x-coordinates. Analogously, for $j \in \{1, \ldots, k-1\}$ define ℓ_j' and Δ_j' for p_1 and the point $(0, -j+\epsilon)$. Every edge that connects p_1 to a point to the left of the y-axis has the following contributions to $|C|$, $|C'|$ and $\Delta(C')$.

- For $j \in \{1, \ldots, k\}$ the contribution of $p_1 p_{-j}$ to $|C'|$ is at least $\ell_j - \delta$ and at most ℓ_j. The contribution of the corresponding edge to $|C|$ is Δ_j. Hence the contribution of $p_1 p_{-j}$ to $\Delta(C')$ is at least $\ell_j - \delta - \Delta_j$ and at most $\ell_j - \Delta_j$.
- For $j \in \{2, \ldots, k-1\}$ the contribution of $p_1 p'_{-j}$ to $|C'|$ is at least $\ell_j' - \delta$ and at most ℓ_j. The contribution of the corresponding edge to $|C|$ is Δ_j'. Thus the contribution of $p_1 p'_{-j}$ to $\Delta(C')$ is at least $\ell_j' - \delta - \Delta_j'$ and at most $\ell_j' - \Delta_j'$.
- The contribution of $p_1 p'_{-1}$ to $|C'|$ is at least $\ell_1' - \delta - \epsilon$ because the y-coordinate of p'_{-1} is at most $\delta + \epsilon$; to verify this observe that $y_{-1} \leq \delta$ and $y'_{-1} - y_{-1} < \epsilon$ because p'_{-1} is almost on $p_1 p_{-1}$ whose slope is less than 1; also see Figure 3 (recall that the figure is not to scale). The contribution of the corresponding edge to $|C|$ is Δ_1'. Therefore the contribution of $p_1 p'_{-1}$ to $\Delta(C')$ is at least $\ell_1' - \delta - \epsilon - \Delta_1'$ and at most $\ell_1' - \Delta_1'$.

Observe that

$$\ell_1' - \Delta_1' > \ell_1 - \Delta_1 > \ell_2' - \Delta_2' > \ell_2 - \Delta_2 > \cdots > \ell_k - \Delta_k.$$

If we set $\delta < \frac{1}{2}\left((\ell_1 - \Delta_1) - (\ell_2' - \Delta_2')\right)$, then the contributions of $p_1 p_{-1}$ and $p_1 p'_{-1}$ to $\Delta(C')$ would respectively be at least

$$\ell_1 - \delta - \Delta_1 > \ell_1 - 2\delta - \Delta_1 > \ell_1 - \Delta_1 - ((\ell_1 - \Delta_1) - (\ell_2' - \Delta_2')) = \ell_2' - \Delta_2', \text{ and}$$

$$\ell_1' - \delta - \epsilon - \Delta_1' \geq \ell_1' - 2\delta - \Delta_1' > \ell_1' - \Delta_1' - ((\ell_1 - \Delta_1) - (\ell_2' - \Delta_2')) > \ell_2' - \Delta_2',$$

which are larger than the contribution of any other edge $p_1 p_{-j}$ and $p_1 p'_{-j}$. By setting

$$\delta = \frac{1}{2} \frac{(\ell_1 - \Delta_1) - (\ell_2' - \Delta_2')}{4k - 2}$$

the contribution of each of $p_1 p_{-1}$ and $p_1 p'_{-1}$ would be even larger than the sum of the contributions of the remaining $4k-4$ edges of C'. Thus, for this choice of δ the longest cycle C' connects p_1 to p_{-1} and p'_{-1}. ◀

We choose δ as in the proof of Lemma 11, and set $y_{-1} = \delta$. Then we set y'_{-1} so that p'_{-1} lies just below (almost on) the segment p_1p_{-1}, as in Figure 3. Notice that $\delta < y'_{-1} < \delta+\epsilon = y_{-1}+\epsilon$. Then, by Lemma 11 the longest cycle connects p_1 to p_{-1} and p'_{-1}. By Lemma 4, the other edges incident to p_{-1} and p'_{-1} must cross the y-axis.

▶ **Lemma 12.** *There exists a real δ, $\epsilon \le \delta < y_{-1}$, such that if $0 \le y_i \le \delta$ for $i \ne -1, 1, 2$ and $0 \le y'_i \le \delta$ for $i \ne -1$, then every longest cycle of P' connects p_{-1} to p_2 and p'_{-1} to p'_2.*

Proof. Recall the longest cycle C' from the proof of Lemma 11. We choose δ small enough such that the contribution of each of $p_{-1}p_2$, $p_{-1}p'_2$, $p'_{-1}p_2$, and $p'_{-1}p'_2$ to $\Delta(C')$ is larger than the sum of the contributions of the remaining $4k{-}6$ edges of C'. This would force C' to connect p_{-1} and p'_{-1} to p_2 and p'_2.

By an argument similar to that of Lemma 11 we can find a parameter δ_1 that forces C' to connect p_{-1} to p_2 or p'_2 (δ_1, y_{-1}, p_{-1}, p_2, and p'_2 play the roles of δ, y_1, p_1, p'_{-1}, and p_{-1}, respectively). Similarly, we can find a parameter δ'_1 that forces C' to connect p'_{-1} to p_2 or p'_2 (where δ'_1, y'_{-1}, p'_{-1}, p_2, and p'_2 play the roles of δ, y_1, p_1, p'_{-1}, and p_{-1}, respectively). Then we choose $\delta = \min\{\delta_1, \delta'_1\}$.

Our choice of δ ensures that C' connects p_{-1} and p'_{-1} to p_2 and p'_2. Notice that p_{-1} and p'_{-1} cannot both connect to p_2 or to p'_2 because it closes the cycle. Thus C' must use $p_{-1}p_2$ and $p'_{-1}p'_2$ or $p_{-1}p'_2$ and $p'_{-1}p_2$. We show that C' uses $p_{-1}p_2$ and $p'_{-1}p'_2$. See Figure 4. Recall that p_2 is almost on the edge $p'_{-1}p'_2$, and hence $|p'_{-1}p'_2| \approx |p'_{-1}p_2| + |p_2p'_2|$. By the triangle inequality we get $|p_{-1}p_2| + |p_2p'_2| > |p_{-1}p'_2|$. Adding these two yields

$$|p_{-1}p_2| + |p'_{-1}p'_2| > |p_{-1}p'_2| + |p'_{-1}p_2|, \tag{1}$$

which means that C' connects p_{-1} to p_2 and p'_{-1} to p'_2. ◀

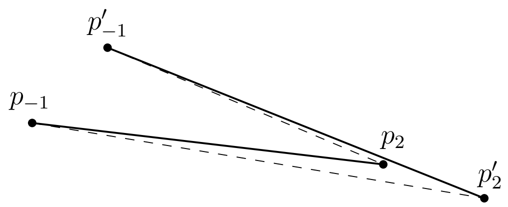

Figure 4 The longest cycle connects p_{-1} to p_2 and p'_{-1} to p'_2.

We choose our new δ as in the proof of Lemma 12, and set $y'_2 = \delta$. Now that the point p'_2 is fixed we can choose the y-coordinate of p_2 in the triangle $\triangle p_{-1}p'_{-1}p'_2$ and very close to the segment $p'_{-1}p'_2$ such that (1) holds. This forces the longest cycle to use $p_{-1}p_2$ and $p'_{-1}p'_2$. By repeatedly applying Lemma 12, the longest cycle will use the edges p_ip_{-i} and $p'_ip'_{-i}$ (for $i > 0$) and the edges p_ip_{-i+1} and $p'_ip'_{-i+1}$ (for negative $i < 0$). Therefore the longest cycle on P' is the concatenation of two paths: $p_1, p_{-1}, p_2, p_{-2}, \ldots, p_{-k}$ and $p_1, p'_{-1}, p'_2, p'_{-2}, \ldots, p'_k, p_{-k}$. This cycle is unique and noncrossing.

Each time we apply Lemma 12 we obtain a new value for δ. In each application we need δ to be greater than or equal to our fixed parameter ϵ. For this purpose, we choose ϵ to be the parameter δ that is obtained in the last application of Lemma 12, i.e., $\delta = y'_k$.

4.3 A cycle with an odd number of points: An overview

Our construction uses the longest paths of Section 3.2. First we observe that our path construction can be generalized to any set of x-coordinates.

▶ **Lemma 13.** *For every even integer $n \geq 4$, every set P of n real numbers, and every $\delta > 0$ such that the δ-neighborhood of the median of P does not contain any points in P, there exists a set P' of n points in the plane with the following properties:*

1. *the x-projection of P' is P;*
2. *all y-coordinates are in the interval $[0, \delta]$;*
3. *the x-projection of any longest path on P' is a longest path on P;*
4. *the longest spanning path on P' is unique and noncrossing; and*
5. *the y-coordinates of the two endpoints of the longest path are 0 and δ.*

Proof sketch. We choose the points in P' such that their x-coordinates are the same as the numbers in P and their y-coordinates are in $[0, \delta]$, and thus (1) and (2) follow.

By an argument similar to the proof of Lemma 3(i) one can show that the difference of lengths of a longest and a non-longest path on P is at least 2δ. Therefore Lemma 6 would imply that by choosing the y-coordinates in the interval $[0, 2\delta/8k]$, any longest path on P' corresponds to a longest path on P, and thus (3) follows. Items (4) and (5) follow by proper choices of y-coordinates similar to that of Lemma 7. ◀

We can now outline the construction; see Figure 5 for an illustration. Let $n = 2k{+}1$, for $k \geq 2$. We choose a set of x-coordinates as $P = \{-k, -(k{-}1)\epsilon, -(k{-}2)\epsilon, \ldots, -\epsilon, 0, 1, 2, \ldots, k\}$, where $\epsilon \in (0, 1/16k^2)$ will be specified later. Note that 0 is the median of P, and the set $A = \{-i \cdot \epsilon : i = 0, 1, \ldots, k-1\} \subset [-1/16k, 0]$ forms a small cluster. By Lemma 4(ii), all edges of any longest cycle on P intersect the y-axis; and Lemma 4(iii) implies the following.

▶ **Observation 14.** *The length of any cycle on P that connects p_{-k} to two points in A is at least 2 units shorter than a longest cycle on P.*

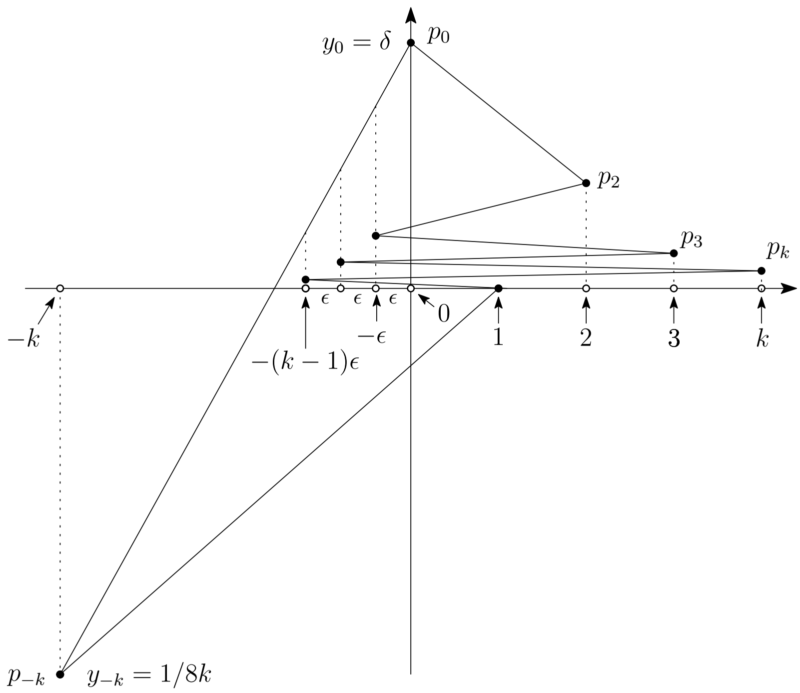

Figure 5 Illustration of the construction of a longest cycle for $2k{+}1$ points.

Below, we will specify a y-coordinate for each element in P. This will result in the point set P' for which the longest spanning cycle is unique and noncrossing. We will denote by A' the set of points in P' corresponding to A.

It remains to specify the y-coordinates of the points in P' and the parameter ϵ. Let p_x denote the point in P' with x-coordinate $x \in P$. We first choose the y-coordinate for the leftmost point: Let $y_{-k} = -1/16k$; this is the only negative y-coordinate. We assume that $|y_i| \ll 1/16k$ for all other points. This ensures that the longest cycle on P' corresponds to a longest cycle on the 1-dimensional multiset where 0 represents the entire cluster A (cf. Lemma 4(iii) and Lemma 10). By Lemma 4(ii), for any longest cycle on P', the two edges incident to p_{-k} intersect the y-axis (i.e., the median). Furthermore, there is a threshold $\delta > 0$ such that if $0 \leq y_i \leq \delta$ for all remaining points, then p_{-k} must be adjacent to the two closest points on or to the right of the y-axis: That is, p_{-k} is adjacent to a point in cluster A' and to p_1 (cf. Observation 14 and Lemma 10). Next, we set $y_0 = \delta$ and find a threshold $\delta_1 \in (0, \delta)$ such that if $0 \leq y_i \leq \delta_1$ for all remaining points and $0 < \epsilon < \delta_1$, then the contribution of edge $p_{-k}p_0$ exceeds the sum of contributions of all remaining edges of a spanning cycle. Consequently, the longest cycle must include the edge $p_{-k}p_0$. Now both p_{-k} and p_0 are fixed, and we choose a sufficiently small $\epsilon \in (0, \delta_1)$ such that all remaining points in the cluster A' are below $p_{-k}p_0$ for all possible y-coordinates.

A longest cycle on P' comprises of $p_{-k}p_0$, $p_{-k}p_1$, and the longest path H' on $P' \setminus \{p_{-k}\}$ (from p_0 to p_1 cf. Lemma 3). By Lemma 13, we can choose y-coordinates for the remaining points such that H' is unique and noncrossing; and $y_1 = 0$. In particular, edge $p_{-k}p_1$ lies below the x-axis, hence below the entire path H'; and $P' \setminus \{p_{-k}, p_0\}$ lies below the supporting line of $p_{-k}p_0$. Consequently, the concatenation of $p_{-k}p_0$, $p_{-k}p_1$ and H' is noncrossing.

5 Noncrossing Longest Matchings

Álvarez-Rebollar et al. [4] showed that there exist point sets for which the longest perfect matchings are noncrossing. Their example is attributed to Kåra P. Villanger in a paper by Tverberg [27]. As illustrated in Figure 6, it consists of a set S of k segments with endpoints in $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$. The distance between any two points $a_i \in A$ and $b_j \in B$ is larger than the distance between any two points in A, or the distance between any two points in B. The points in B are roughly on a vertical line. Álvarez-Rebollar et al. [4] have provided a precise description of the construction along with a detailed proof that S is a longest matching for $A \cup B$.

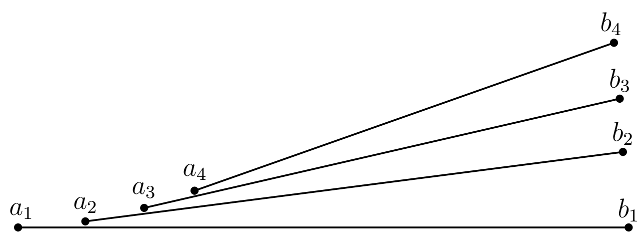

Figure 6 Villanger's configuration as illustrated in [4].

Here, we exhibit an alternative point set for which the longest perfect matching is noncrossing. Our construction follows the same framework as for paths and cycles. Let P be a set of $2k$ points $p_i = (i, 0)$ for $i = \pm 1, \pm 2, \ldots, \pm k$. One can verify that a perfect matching on P is longest if and only if all edges cross the y-axis. One such matching is

$M = \{p_{-i}p_i : i = 1, \dots, k\}$. Using ideas similar to those used for paths and cycles, one can assign to each p_i a new y-coordinate y_i to make M longest and noncrossing at the same time; see Figure 7. The new y-coordinates are of the following form: $y_1 \gg y_{-1} = y_2 \gg y_{-2} = y_3 \gg \cdots = y_k \gg y_{-k}$.

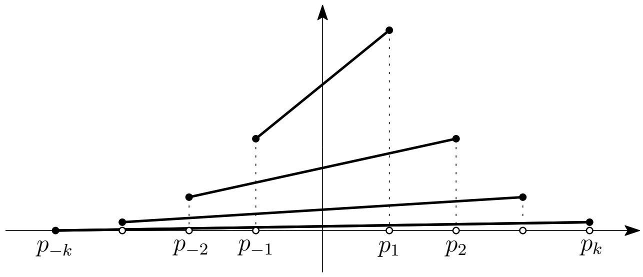

Figure 7 Illustration of our construction of a longest matching.

6 Some Properties of Longest Paths and Cycles

In this section we give some structural properties of longest paths and cycles, possibly of independent interest. We state these properties only for cycles, but they hold for paths as well. Two edges are in *convex position* if they are edges of their convex hull. Two directed edges in convex position have *the same orientation* if they are both directed clockwise or counterclockwise along their convex hull.

▶ **Observation 15.** *Suppose that we orient the edges of a longest cycle C to make it a directed cycle. Then C cannot have pair of non-adjacent edges that are in convex position and have the same orientation along their convex hull.*

To verify this, note that if C has two such edges, say e_1 and e_2, then flipping them (replacing e_1 and e_2 by the two diagonals of the convex hull of e_1 and e_2) would produce a longer undirected cycle as in Figure 8(a). Since e_1 and e_2 have the same orientation along their convex hull, the flip does not break the cycle into two components. If every directed simple polygon S contained a pair of non-adjacent edges in convex position with the same orientation along their convex hull, Observation 15 would imply Conjecture 1. However, some simple polygons do not have edges that can be flipped in this way; see e.g., Figure 8(b).

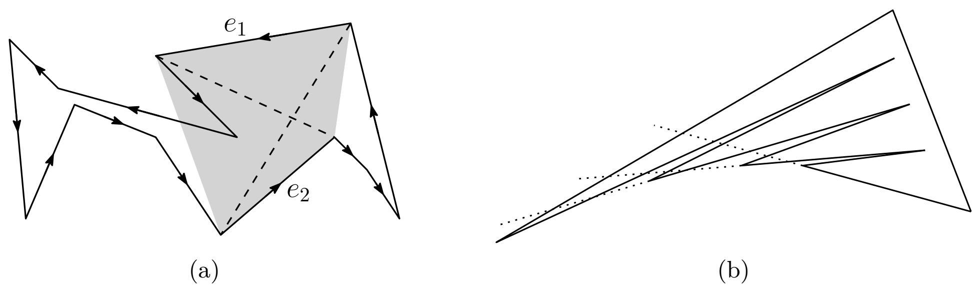

Figure 8 (a) Flipping two edges in convex position. (b) A simple polygon with no pair of edges in convex position that have the same orientation, no matter how we direct the polygon.

▶ **Observation 16.** *The longest cycle need not contain an edge between diametric points.*

To verify this observation consider an isosceles right triangle abc whose right angle is at b. Place one point at a, one point at c, and two or more points very close to b. Then, the longest cycle does not contain the diametric point pair $\{a, c\}$. This observation implies that a longest cycle may not be achieved by greedily choosing longest edges.

The following proposition implies that if the longest cycle is noncrossing, it contains some edge whose length is among the smallest three-quarters of all distances defined by its vertices.

▶ **Proposition 17.** *Let S be a simple polygon (a noncrossing cycle) on n points. Then S has an edge whose length is among the smallest $3n^2/8 + n/8$ distances of the $\binom{n}{2}$ point pairs.*

Proof. Let e and e' be two edges of S such that their distance along S (in terms of the number of edges) is at least 2. Since S is a simple polygon, e and e' do not cross. Thus, there is an endpoint p of e and an endpoint p' of e' such that $|pp'|$ is larger than the length of the shorter of e and e', and pp' is not an edge of S. The number of pairs of edges at distance 0 is n, and the number of pairs of edges at distance 1 is also n. Thus, the total number of pairs of edges at a distance at least 2 is $\binom{n}{2} - 2n$. Each such pair of edges yields a pair $\{p, p'\}$. Each $\{p, p'\}$ can be counted for 4 different pairs of edges that are obtained by combining the two edges incident to p and the two edges incident to p'. Therefore the total number of distinct pairs $\{p, p'\}$ is at least $\frac{1}{4}\left(\binom{n}{2} - 2n\right)$. Subtracting this from the total number $\binom{n}{2}$ of point pairs yields the claimed bound. ◀

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