Parameterized Constraint Satisfaction Problems: a Survey

Gregory Gutin*[1] and Anders Yeo[2]

1 **Royal Holloway, University of London, Egham, Surrey, UK; and University of Haifa, Mount Carmel, Haifa, Israel**
gutin@cs.rhul.ac.uk

2 **Singapore University of Technology and Design, Singapore; and University of Johannesburg, Auckland Park, South Africa**
anders.yeo.work@gmail.com

Abstract

We consider constraint satisfaction problems parameterized above or below guaranteed values. One example is MaxSat parameterized above $m/2$: given a CNF formula F with m clauses, decide whether there is a truth assignment that satisfies at least $m/2 + k$ clauses, where k is the parameter. Among other problems we deal with are MaxLin2-AA (given a system of linear equations over $\mathbb{F}_2$ in which each equation has a positive integral weight, decide whether there is an assignment to the variables that satisfies equations of total weight at least $W/2+k$, where W is the total weight of all equations), Max-r-Lin2-AA (the same as MaxLin2-AA, but each equation has at most r variables, where r is a constant) and Max-r-Sat-AA (given a CNF formula F with m clauses in which each clause has at most r literals, decide whether there is a truth assignment satisfying at least $\sum_{i=1}^{m}(1 - 2^{r_i}) + k$ clauses, where k is the parameter, r_i is the number of literals in clause i, and r is a constant). We also consider Max-r-CSP-AA, a natural generalization of both Max-r-Lin2-AA and Max-r-Sat-AA, order (or, permutation) constraint satisfaction problems parameterized above the average value and some other problems related to MaxSat. We discuss results, both polynomial kernels and parameterized algorithms, obtained for the problems mainly in the last few years as well as some open questions.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems

Keywords and phrases Constraint satisfaction problems, Fixed-parameter tractability

Digital Object Identifier 10.4230/DFU.Vol7.15301.179

1 Introduction

While the main body of papers in the area of parameterized algorithms and complexity deals with problems on graphs and hypergraphs, in this paper we will consider parameterized constraint satisfaction problems (CSPs). This article is an update of survey paper [27] on the topic. We provide basic terminology and notation on parameterized algorithms and complexity in Section 2.

To the best of our knowledge, the first study of parameterized CSPs was almost twenty years ago by Cai and Chen [7] on standard parameterization of MaxSat. In MaxSat, we are given a CNF formula F with m clauses and asked to determine the maximum number of clauses of F that can be satisfied simultaneously by a truth assignment. In the standard parametrization of MaxSat, denoted by k-MaxSat, we are to decide whether there is a

* G. Gutin's research was partially supported by Royal Society Wolfson Research Merit Award.

The Constraint Satisfaction Problem: Complexity and Approximability. *Dagstuhl Follow-Ups*, Volume 7, ISBN 978-3-95977-003-3.
Editors: Andrei Krokhin and Stanislav Živný; pp. 179–203
Dagstuhl Publishing
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Germany

truth assignment which satisfies at least k clauses of F, where k is the parameter. However, in the next paper on the topic Mahajan and Raman [41] already observed that the standard parameterization of MaxSat is not in the spirit of parameterized complexity. Indeed, it is well-known (and shown below, in Section 6) that there exists a truth assignment to the variables of F which satisfies at least $m/2$ clauses. Thus, for $k \leq m/2$ every instance of k-MaxSat is positive and thus only for $k > m/2$ the problem is of any interest. However, then the parameter k is quite large “in which range the fixed-parameter tractable algorithms are infeasible” [41].

Also it is easy to see that k-MaxSat has a kernel with a linear number of clauses. Indeed, consider an instance I of k-MaxSat. As we mentioned above, if $k \leq m/2$ then I is a positive instance. Otherwise, we have $k > m/2$ and $m \leq 2k-1$. Such a kernel should be viewed as *large* rather than *small* as the bound $2k-1$ might suggest at the first glance.

The bound $m/2$ is tight as we can satisfy only half clauses in the instances consisting of pairs $(x), (\bar{x})$ of clauses. This suggest the following parameterization of MaxSat *above tight bound* introduced by Mahajan and Raman [41]: decide whether there is a truth assignment which satisfies at least $m/2+k$ clauses of F, where k is the parameter.

To the best our knowledge, [41] was the first paper on problems parameterized above or below tight bounds. Since then a large number of papers have appeared on the topic, some on graph and hypergraph problems and others on CSPs. In this survey paper, we will overview results on CSPs parameterized above or below tight bounds, as well as some methods used to obtain these results. Since some graph problems can also be viewed as those on CSPs, we will mention some results initially proved for graphs. While not going into details of the proofs, we will discuss some ideas behind the proofs. We will also consider some open problems in the area.

In the remainder of this section we give an overview of the paper and its organization.

In the next section we provide basics on parameterized algorithms and complexity. In Section 3, we describe the Strictly-Above-Below-Expectation Method (SABEM) introduced in [26]. The method uses some tools from Probabilistic Method and Harmonic Analysis. A relatively simple example illustrates the method.

Another example for SABEM is given in Section 4, which is devoted to the Maximum r-CSPs parameterized above the average value, where r is a positive integral constant. In general, the Maximum r-CSP is given by a set V of n Boolean variables and a set of m Boolean formulas; each formula is assigned an integral positive weight and contains at most r variables from V. The aim is to find a truth assignment which maximizes the weight of satisfied formulas. Averaging over all truth assignments, we can find the average value A of the weight of satisfied formulas. It is easy to show that we can always find a truth assignment to the variables of V which satisfied formulas of total weight at least A. Thus, a natural parameterized problem is whether there exists a truth assignment that satisfies formulas of total weight at least $A+k$, where k is the parameter (k is a nonnegative integer). We denote such a problem by Max-r-CSP-AA.

In Subsection 4.1, we consider the Max-r-Lin2-AA problem, which is a special case of Max-r-CSP-AA when every formula is a linear equation over $\mathbb{F}_2$ with at most r variables. For Max-r-Lin2-AA, we have $A = W/2$, where W is the total weight of all equations. It is well-known that, in polynomial time, we can find an assignment to the variables that satisfies equations of total weight at least $W/2$, but, for any $\epsilon > 0$ it is NP-hard to decide whether there is an assignment satisfying equations of total weight at least $W(1+\epsilon)/2$ [31]. We will prove a result by Gutin, Kim, Szeider and Yeo [26] that Max-r-Lin2-AA has a kernel of quadratic size. We will mention some other results, in particular, a result of Crowston et al. [11] that Max-r-Lin2-AA has a kernel with at most $(2k-1)r$ variables.

In Subsection 4.2, we give a proof scheme of a result by Alon et al. [2] that Max-r-CSP-AA has a kernel of polynomial size. The main idea of the proof is to reduce Max-r-CSP-AA to Max-r-Lin2-AA and use results on Max-r-Lin2-AA and a lemma on bikernels given in the next section. The result of Alon et al. [2] solves an open question of Mahajan, Raman and Sikdar [42] not only for Max-r-Sat-AA but for the more general problem Max-r-CSP-AA. The problem Max-r-Sat-AA is a special case of Max-r-CSP-AA when every formula is a clause with at most r variables. For Max-r-Sat-AA, the reduction to Max-r-Lin2-AA can be complemented by a reduction from Max-r-Lin2-AA back to Max-r-Sat-AA, which yields a kernel of quadratic size.

(Note that while the size of the kernel for Max-r-CSP-AA is polynomial, any bound on the degree of the polynomial is unknown so far.)

Section 5 is devoted to two parameterizations of MaxLin2. The first is MaxLin2-AA, which is the same problem as Max-r-Lin2-AA, but the number of variables in an equation is not bounded. Thus, MaxLin2-AA is a generalization of Max-r-Lin2-AA. We present a scheme of a proof by Crowston et al. [11] that MaxLin2-AA is fixed-parameter tractable (FPT) and has a kernel with polynomial number of variables. This result solved an open question of Mahajan et al. [42] of whether MaxLin2-AA is FPT, but we still do not know whether MaxLin2-AA has a kernel of polynomial size and we present only partial results on the topic. The second parameterization of MaxLin2 is as follows. Let W be the total weight of all equations in MaxLin2. We are to decide whether there is an assignment satisfying equations of total weight at least $W - k$, where k is a nonnegative parameter. This problem was proved to be W[1]-hard by Crowston et al. [14]. Following [14] we will discuss special cases of this problem giving its classification into fixed-parameter tractable and W[1]-hard cases.

In Section 6 we consider several parameterizations of Max-Sat different from Max-r-Sat-AA. Subsection 6.1 is devoted to MaxSat-A($m/2$), where given a CNF formula F with m clauses, we are to decide whether there is a truth assignment with satisfies at least $m/2 + k$ clauses of F, where k is the parameter.

In Subsection 6.2 we consider Max-$r(n)$-Sat-AA, which is the same problem as Max-r-Sat-AA, but $r(n)$ now depends on n. We discuss bounds on $r(n)$, which make Max-$r(n)$-Sat-AA either fixed-parameter tractable or not fixed-parameter tractable under the assumption that the Exponential Time Hypothesis (ETH) holds (we introduce ETH in the next section).

Results on MaxSat parameterized above or below various other tight bounds are discussed in Subsection 6.3. We will consider the above-mentioned parameterization of MaxSat above $m/2$ and some “stronger” parameterizations of MaxSat introduced or inspired by Mahajan and Raman [41]. The stronger parameterizations are based on the notion of a t-satisfiable CNF formula (a formula in which each set of t clauses can be satisfied by a truth assignment) and asymptotically tight lower bounds on the maximum number of clauses of a t-satisfiable CNF formula satisfied by a truth assignment for $t = 2$ and 3. We will describe linear-variable kernels obtained for both $t = 2$ and 3. We will also consider the parameterization of 2-Sat below the upper bound m, the number of clauses. This problem was proved to be fixed-parameter tractable by Razgon and O’Sullivan [48]. Raman et al. [47] and Cygan et al. [17] designed faster parameterized algorithms. for the problem.

In Section 7 we discuss parameterizations above average for Ordering CSPs. An Ordering CSP of arity r is defined by a set $V = \{x_1, \ldots, x_n\}$ of variables and a set of constraints. Each constraint is a disjunction of clauses of the form $x_{i_1} < x_{i_2} < \cdots < x_{i_r}$. A linear ordering α of V satisfies such a constraint if one of the clauses in the disjunction agrees with α. The objective of the problem is to find an ordering of V which satisfies the maximum number of constraints. For the only nontrivial Ordering CSP of arity 2, 2-Linear Ordering, Guruswami, Manokaran and Raghavendra [22] proved that it is impossible to find, in polynomial time, an ordering that satisfies at least $|C|(1+\epsilon)/2$ constraints for every $\epsilon > 0$ provided the Unique Games Conjecture (UGC) of Khot [35] holds. (Note that $|C|/2$ is the expected number of constraints satisfied by a random uniformly-distributed ordering of V.) Similar approximation resistant results were proved for all Ordering CSPs of arity 3 by Charikar, Guruswami and Manokaran [8] and for Ordering CSPs of any arity by Guruswami et al. [21].

Thus it makes sense to consider Ordering CSPs parameterized above average. It was proved by Gutin, Kim, Szeider and Yeo [26] that 2-Linear Ordering parameterized above average is fixed-parameter tractable. Gutin, Iersel, Mnich and Yeo [23] showed that all Ordering CSPs of arity 3 parameterized above average are fixed-parameter tractable and conjectured the same results for every arity $r \geq 2$. Recently, Makarychev, Makarychev and Zhou [43] proved the conjecture. All the results can be proved using SABEM. This is already illustrated in Subsection 4.1 for 2-Linear Ordering parameterized above average. In Section 7, we provide a proof scheme for Betweenness parameterized above average by Gutin, Kim, Szeider and Yeo [25] who solved an open question of Benny Chor stated in Niedermeier's monograph [44]. This scheme was used also by Makarychev, Makarychev and Zhou [43], but their proof involves significantly more involved tools from Harmonic Analysis and we only provide some general remarks on their proof. We will also briefly discuss an interesting generalization of the result of Makarychev, Makarychev and Zhou [43].

We complete the paper with Section 8, where we briefly discuss two open problems.

2 Basics on Parameterized Algorithms and Complexity

A parameterized problem Π can be considered as a set of pairs (I, k) where I is the *problem instance* and k (usually a nonnegative integer) is the *parameter*. Π is called *fixed-parameter tractable (FPT)* if membership of (I, k) in Π can be decided by an algorithm of runtime $O(f(k)|I|^c)$, where $|I|$ is the size of I, $f(k)$ is an arbitrary function of the parameter k only, and c is a constant independent from k and I. Such an algorithm is called an *FPT* algorithm. Let Π and Π' be parameterized problems with parameters k and k', respectively. An *FPT-reduction R from Π to Π'* is a many-to-one transformation from Π to Π', such that (i) $(I, k) \in \Pi$ if and only if $(I', k') \in \Pi'$ with $k' \leq g(k)$ for a fixed computable function g, and (ii) R is of complexity $O(f(k)|I|^c)$.

If the nonparameterized version of Π (where k is just part of the input) is NP-hard, then the function $f(k)$ must be superpolynomial provided P≠NP. Often $f(k)$ is "moderately exponential," which makes the problem practically feasible for small values of k. Thus, it is important to parameterize a problem in such a way that the instances with small values of k are of real interest.

When the decision time is replaced by the much more powerful $|I|^{O(f(k))}$, we obtain the class XP, where each problem is polynomial-time solvable for any fixed value of k. There is a number of parameterized complexity classes between FPT and XP (for each integer $t \geq 1$, there is a class W[t]) and they form the following tower:

$$\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \cdots \subseteq \text{W}[\text{P}] \subseteq \text{XP}.$$

Here W[P] is the class of all parameterized problems (I, k) that can be decided in $f(k)|I|^{O(1)}$ time by a nondeterministic Turing machine that makes at most $f(k)\log|I|$ nondeterministic steps for some function f. For the definition of classes W[t], see, e.g., [16, 18] (we do not use these classes in the rest of the paper).

Π is in *para-NP* if membership of (I,k) in Π can be decided in nondeterministic time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $f(k)$ is an arbitrary function of the parameter k only, and c is a constant independent from k and I. Here, nondeterministic time means that we can use nondeterministic Turing machine. A parameterized problem Π' is *para-NP-complete* if it is in para-NP and for any parameterized problem Π in para-NP there is an FPT-reduction from Π to Π'.

Given a pair Π, Π' of parameterized problems, a *bikernelization from* Π *to* Π' is a polynomial-time algorithm that maps an instance (I,k) to an instance (I',k') (the *bikernel*) such that (i) $(I,k) \in \Pi$ if and only if $(I',k') \in \Pi'$, (ii) $k' \leq f(k)$, and (iii) $|I'| \leq g(k)$ for some functions f and g. The function $g(k)$ is called the *size* of the bikernel. A *kernelization* of a parameterized problem Π is simply a bikernelization from Π to itself. Then (I',k') is a *kernel*. The term bikernel was coined by Alon et al. [2]; in [4] a bikernel is called a generalized kernel.

It is well-known that a parameterized problem Π is fixed-parameter tractable if and only if it is decidable and admits a kernelization [16, 18]. This result can be extended as follows: A decidable parameterized problem Π is fixed-parameter tractable if and only if it admits a bikernelization from itself to a decidable parameterized problem Π' [2].

Due to applications, low degree polynomial size kernels are of main interest. Unfortunately, many fixed-parameter tractable problems do not have kernels of polynomial size unless the polynomial hierarchy collapses to the third level [4, 5, 19]. For further background and terminology on parameterized complexity we refer the reader to the monographs [16, 18].

The following lemma of Alon et al. [2] inspired by a lemma from [5] shows that polynomial bikernels imply polynomial kernels.

▶ **Lemma 1.** *Let* Π, Π' *be a pair of decidable parameterized problems such that the nonparameterized version of* Π' *is in NP, and the nonparameterized version of* Π *is NP-complete. If there is a bikernelization from* Π *to* Π' *producing a bikernel of polynomial size, then* Π *has a polynomial-size kernel.*

Recently many lower bound results for parameterized complexity were proved under the assumption that the Exponential Time Hypothesis (EHT) (see [16]) holds. ETH claims that 3-SAT cannot be solved in $O(2^{\delta n})$ time for some $\delta > 0$, where n is the number of variables in the CNF formula of 3-SAT.

Henceforth $[n]$ stands for the set $\{1, 2, \ldots, n\}$.

3 Strictly Above/Below Expectation Method

This section briefly describes basics of the method.

Let us start by outlining the very basic principles of the probabilistic method which will be implicitly used later. Given random variables $X_1, \ldots, X_n$, the fundamental property known as *linearity of expectation* states that $\mathbb{E}(X_1 + \ldots + X_n) = \mathbb{E}(X_1) + \ldots + \mathbb{E}(X_n)$. The *averaging argument* utilizes the fact that there is a point for which $X \geq \mathbb{E}(X)$ and a point for which $X \leq \mathbb{E}(X)$ in the probability space. Also a positive probability $\mathbb{P}(A) > 0$ for some event A means that there is at least one point in the probability space which belongs to A. For example, $\mathbb{P}(X \geq k) > 0$ tells us that there exists a point for which $X \geq k$.

A random variable is *discrete* if its distribution function has a finite or countable number of positive increases. A random variable X is *symmetric* if $-X$ has the same distribution function as X. If X is discrete, then X is symmetric if and only if $\mathbb{P}(X = a) = \mathbb{P}(X = -a)$ for each real a. Let X be a symmetric variable for which the first moment $\mathbb{E}(X)$ exists. Then $\mathbb{E}(X) = \mathbb{E}(-X) = -\mathbb{E}(X)$ and, thus, $\mathbb{E}(X) = 0$. The following easy to prove [26] result, is the simplest tool of the Strictly Above/Below Expectation method as it allows us sometimes to show that a certain random variable takes values (significantly) above/below its expectation.

► **Lemma 2.** *If X is a symmetric random variable and $\mathbb{E}(X^2)$ is finite, then*

$$\mathbb{P}(\ X \geq \sqrt{\mathbb{E}(X^2)}\ ) > 0.$$

We will illustrate the usefulness of the lemma using the 2-Linear Ordering Above Average problem. Let $D = (V, A)$ be a digraph on n vertices with no loops or parallel arcs in which every arc ij has a positive integral weight w_{ij}. Consider an ordering $\alpha : V \to [n]$ and the subdigraph $D_\alpha = (V, \{ij \in A : \ \alpha(i) < \alpha(j)\})$ of D. Note that D_α is acyclic.

2-Linear Ordering Above Average (2-Linear Ordering-AA)
Instance: A digraph $D = (V, A)$, each arc ij has an integral positive weight w_{ij}, and a positive integer κ.
Parameter: The integer κ.
Question: Is there a subdigraph D_α of D of weight at least $W/2 + \kappa$, where $W = \sum_{ij\in A} w_{ij}$?

Mahajan, Raman, and Sikdar [42] asked whether 2-Linear Ordering-AA is FPT for the special case when all arcs are of weight 1. Gutin et al. [26] solved the problem by obtaining a quadratic kernel for the problem. In fact, the problem can be solved using the following result of Alon [1]: there exists an ordering α such that D_α has weight at least $(\frac{1}{2} + \frac{1}{16|V|})W$. However, the proof in [1] uses a probabilistic approach for which a derandomization is not known yet and, thus, we cannot find the appropriate α deterministically. Moreover, the probabilistic approach in [1] is quite specialized. Thus, we will briefly describe a solution from [26]. Consider the following reduction rule:

► **Reduction Rule 1.** *Assume D has a directed 2-cycle iji; if $w_{ij} = w_{ji}$ delete the cycle, if $w_{ij} > w_{ji}$ delete the arc ji and replace w_{ij} by $w_{ij} - w_{ji}$, and if $w_{ji} > w_{ij}$ delete the arc ij and replace w_{ji} by $w_{ji} - w_{ij}$.*

It is easy to check that the answer to 2-Linear Ordering-AA for a digraph D is Yes if and only if the answer to 2-Linear Ordering-AA is Yes for a digraph obtained from D using the reduction rule as long as possible. Note that applying Rule 1 as long as possible results in an *oriented graph*, i.e., a digraph with no directed 2-cycle.

► **Theorem 3** ([26]). 2-Linear Ordering-AA *has a kernel with $O(\kappa^2)$ arcs.*

Proof. Consider a random ordering: $\alpha : V \to [n]$ and a random variable $X(\alpha)$ defined by $X(\alpha) = \frac{1}{2}\sum_{ij\in A} x_{ij}(\alpha)$, where $x_{ij}(\alpha) = w_{ij}$ if $\alpha(i) < \alpha(j)$ and $x_{ij}(\alpha) = -w_{ij}$, otherwise. It is easy to see that $X(\alpha) = \sum\{w_{ij} :\ ij \in A, \alpha(i) < \alpha(j)\} - W/2$. Thus, the answer to 2-Linear Ordering-AA is Yes if and only if there is an ordering $\alpha : V \to [n]$ such that $X(\alpha) \geq \kappa$.

By Rule 1, we may assume that the input of 2-Linear Ordering-AA is an oriented graph $D = (V, A)$. Let $\alpha : V \to [n]$ be a random ordering. Since $X(-\alpha) = -X(\alpha)$, where $-\alpha(i) = n + 1 - \alpha(i)$, X is a symmetric random variable and, thus, we can apply Lemma 2. It was proved in [26] that $\mathbb{E}(X^2) \geq |A|/12$. By this inequality and Lemma 2, we have

$\mathbb{P}(\ X \geq \sqrt{|A|/12}\ ) > 0$. Thus, if $\sqrt{|A|/12} \geq \kappa$, there is an ordering $\beta : V \to [n]$ such that $X(\beta) \geq k$ and so the answer to 2-Linear Ordering-AA is Yes. Otherwise, $\sqrt{|A|/12} \geq \kappa$ implying $|A| \leq 12\kappa^2$ and we are done. ◀

By deleting isolated vertices (if any), we can obtain a kernel with $O(\kappa^2)$ arcs and vertices. Kim and Williams [36] proved that 2-Linear Ordering has a kernel with a linear number of variables.

If a random variable X is not symmetric then the following lemma can be used instead of Lemma 2.

▶ **Lemma 4** (Alon et al. [2]). *Let X be a real random variable and suppose that its first, second and fourth moments satisfy $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = \sigma^2 > 0$ and $\mathbb{E}[X^4] \leq c\mathbb{E}[X^2]^2$, respectively, for some constant c. Then $\mathbb{P}(X > \frac{\sigma}{2\sqrt{c}}) > 0$.*

To check whether $\mathbb{E}[X^4] \leq c\mathbb{E}[X^2]^2$ we often can use the following well-known inequality whose proof can be found in [16] and [45].

▶ **Lemma 5** (Hypercontractive Inequality [6]). *Let $f = f(x_1, \ldots, x_n)$ be a polynomial of degree r in n variables $x_1, \ldots, x_n$ each with domain $\{-1, 1\}$. Define a random variable X by choosing a vector $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ uniformly at random and setting $X = f(\epsilon_1, \ldots, \epsilon_n)$. Then $\mathbb{E}[X^4] \leq 9^r\mathbb{E}[X^2]^2$.*

If $f = f(x_1, \ldots, x_n)$ is a polynomial in n variables $x_1, \ldots, x_n$ each with domain $\{-1, 1\}$, then it can be written as $f = \sum_{I\subseteq[n]} c_I \prod_{i\in S} x_i$, where $[n] = \{1, \ldots, n\}$ and c_I is a real for each $I \subseteq [n]$.

The following dual, in a sense, form of the Hypercontractive Inequality was proved by Gutin and Yeo [28]; for a weaker result, see [26].

▶ **Lemma 6.** *Let $f = f(x_1, \ldots, x_n)$ be a polynomial in n variables $x_1, \ldots, x_n$ each with domain $\{-1, 1\}$ such that $f = \sum_{I\subseteq[n]} c_I \prod_{i\in S} x_i$. Suppose that no variable x_i appears in more than ρ monomials of f. Define a random variable X by choosing a vector $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ uniformly at random and setting $X = f(\epsilon_1, \ldots, \epsilon_n)$. Then $\mathbb{E}[X^4] \leq (2\rho + 1)\mathbb{E}[X^2]^2$.*

The following lemma is easy to prove, cf. [26]. In fact, the equality there is a special case of Parseval's Identity in Harmonic Analysis, cf. [45].

▶ **Lemma 7.** *Let $f = f(x_1, \ldots, x_n)$ be a polynomial in n variables $x_1, \ldots, x_n$ each with domain $\{-1, 1\}$ such that $f = \sum_{I\subseteq[n]} c_I \prod_{i\in I} x_i$. Define a random variable X by choosing a vector $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ uniformly at random and setting $X = f(\epsilon_1, \ldots, \epsilon_n)$. Then $\mathbb{E}[X^2] = \sum_{i\in I} c_I^2$.*

We will give a relatively simple application of Lemmas 5 and 7 in Subsection 4.1. Another application is in Subsection 7.2.

4 Boolean Max-r-CSPs Above Average

Throughout this section, r is a positive integral constant. Recall that the problem Max-r-CSP-AA is given by a set V of n Boolean variables and a set of m Boolean formulas; each formula is assigned an integral positive weight and contains at most r variables from V. Averaging over all truth assignments, we can find the average value A of the weight of satisfied formulas. We wish to decide whether there exists a truth assignment that satisfies formulas of total weight at least $A + k$, where k is the parameter (k is a nonnegative integer).

Recall that the problem Max-r-Lin2-AA is a special case of Max-r-CSP-AA when every formula is a linear equation over $\mathbb{F}_2$ with at most r variables and that Max-Lin2-AA is the extension of Max-r-Lin2-AA when we do not bound the number of variables in an equation. We will see that for both Max-r-CSP-AA and Max-Lin2-AA, $A = W/2$, where W is the total weight of all equations.

Subsection 4.1 is devoted to parameterized complexity results on Max-r-Lin2-AA which are not only of interest by themselves, but also as tools useful for Max-r-CSP-AA. In particular, we will prove that Max-r-Lin2-AA has a kernel of quadratic size. Since some basic results on Max-r-Lin2-AA hold also for Max-Lin2-AA, in general, we will show them for Max-Lin2-AA.

In Subsection 4.2, we give a proof scheme of a result by Alon et al. [2] that Max-r-CSP-AA has a a kernel of polynomial size. The main idea of the proof is to reduce Max-r-CSP-AA to Max-r-Lin2-AA and use the above kernel result on Max-r-Lin2-AA and Lemma 1. This shows the existence of a polynomial-size kernel, but does not allow us to obtain a bound on the degree of the polynomial. We complete the section, by pointing out that for Max-r-Sat-AA, the reduction to Max-r-Lin2-AA can be complemented by a reduction from Max-r-Lin2-AA back to Max-r-Sat-AA and so we obtain a quadratic kernel for Max-r-Sat-AA.

4.1 Max-r-Lin-AA

In the Max-Lin2-AA problem, we are given a system S consisting of m linear equations in n variables over $\mathbb{F}_2$ in which each equation is assigned a positive integral weight. If we add the requirement that every equation has at most r variables then we get Max-r-Lin2-AA. Let us write the system S as $\sum_{i \in I} z_i = b_I$, $I \in \mathcal{F}$, and let w_I denote the weight of an equation $\sum_{i \in I} z_i = b_I$. Clearly, $m = |\mathcal{F}|$. Let $W = \sum_{I \in \mathcal{F}} w_I$ and let $\mathrm{sat}(S)$ be the maximum total weight of equations that can be satisfied simultaneously.

For each $i \in [n]$, set $z_i = 1$ with probability $1/2$ independently of the rest of the variables. Then each equation is satisfied with probability $1/2$ and the expected weight of satisfied equations is $W/2$ (as our probability distribution is uniform, $W/2$ is also the average weight of satisfied equations). Hence $W/2$ is a lower bound; to see its tightness consider a system of pairs of equations of the form $\sum_{i \in I} z_i = 0$, $\sum_{i \in I} z_i = 1$ of weight 1. The aim in both Max-Lin2-AA and Max-r-Lin2-AA is to decide whether for the given system S, $\mathrm{sat}(S) \geq W/2 + k$, where k is the parameter. It is well-known that, in polynomial time, we can find an assignment to the variables that satisfies equations of total weight at least $W/2$, but, for any $\epsilon > 0$ it is NP-hard to decide whether there is an assignment satisfying equations of total weight at least $W(1 + \epsilon)/2$ [31].

Henceforth, it will often be convenient for us to consider linear equations in their multiplicative form, i.e., instead of an equation $\sum_{i \in I} z_i = b_I$ with $z_i \in \{0, 1\}$, we will consider the equation $\prod_{i \in I} x_i = (-1)^{b_I}$ with $x_i \in \{-1, 1\}$. Clearly, an assignment $z^0 = (z_1^0, \ldots, z_n^0)$ satisfies $\sum_{i \in I} z_i = b_I$ if and only if the assignment $x^0 = (x_1^0, \ldots, x_n^0)$ satisfies $\prod_{i \in I} x_i = (-1)^{b_I}$, where $x_i^0 = (-1)^{z_i^0}$ for each $i \in [n]$.

Let $\varepsilon(x) = \sum_{I \in \mathcal{F}} w_I (-1)^{b_I} \prod_{i \in I} x_i$ (each $x_i \in \{-1, 1\}$) and note that $\varepsilon(x^0)$ is the difference between the total weight of satisfied and falsified equations when $x_i = x_i^0$ for each $i \in [n]$. We will call $\varepsilon(x)$ the *excess* and the maximum possible value of $\varepsilon(x)$ the *maximum excess*. The following claim is easy to check.

► **Lemma 8.** *Observe that the answer to* Max-Lin2-AA *and* Max-r-Lin2-AA *is* Yes *if and only if the maximum excess is at least* $2k$*.*

Let A be the matrix over $\mathbb{F}_2$ corresponding to the set of equations in S, such that $a_{ji} = 1$ if $i \in I_j$ and 0, otherwise.

Consider two reduction rules for Max-Lin2-AA introduced by Gutin et al. [26].

▶ **Reduction Rule 2.** *If we have, for a subset I of $[n]$, an equation $\prod_{i\in I} x_i = b'_I$ with weight w'_I, and an equation $\prod_{i\in I} x_i = b''_I$ with weight w''_I, then we replace this pair by one of these equations with weight $w'_I + w''_I$ if $b'_I = b''_I$ and, otherwise, by the equation whose weight is bigger, modifying its new weight to be the difference of the two old ones. If the resulting weight is 0, we delete the equation from the system.*

▶ **Reduction Rule 3.** *Let $t = \text{rank}A$ and suppose columns $a^{i_1}, \dots, a^{i_t}$ of A are linearly independent. Then delete all variables not in $\{x_{i_1}, \dots, x_{i_t}\}$ from the equations of S.*

▶ **Lemma 9** ([26]). *Let S' be obtained from S by Rule 2 or 3. Then the maximum excess of S' is equal to the maximum excess of S. Moreover, S' can be obtained from S in time polynomial in n and m.*

▶ **Definition 10.** If we cannot change a weighted system S using Rules 2 and 3, we call it *irreducible*.

Now we are read to prove the following result.

▶ **Theorem 11** ([26]). *The problem* Max-r-Lin2-AA *admits a kernel with at most $O(k^2)$ variables and equations.*

Proof. Let the system S be irreducible. Consider the excess

$$\varepsilon(x) = \sum_{I\in\mathcal{F}} w_I(-1)^{b_I} \prod_{i\in I} x_i. \tag{1}$$

Let us assign value -1 or 1 to each x_i with probability 1/2 independently of the other variables. Then $X = \varepsilon(x)$ becomes a random variable. By Lemma 7, we have $\mathbb{E}(X^2) = \sum_{I\in\mathcal{F}} w_I^2$. Therefore, by Lemmas 4 and 5,

$$\mathbb{P}[\ X \geq \sqrt{m}/(2\cdot 3^r)\ ] \geq \mathbb{P}\left[\ X \geq \sqrt{\sum_{I\in\mathcal{F}} w_I^2}/(2\cdot 3^r)\ \right] > 0.$$

Hence by Remark 8, if $\sqrt{m}/(2\cdot 3^r) \geq 2k$, then the answer to Max-r-Lin2-AA is Yes. Otherwise, $m = O(k^2)$ and, by Rule 3, we have $n \leq m = O(k^2)$. ◀

The bound on the number of variables can be improved and it was done by Crowston et al. [12] and Kim and Williams [36]. The best known improvement is by Crowston et al. [11]:

▶ **Theorem 12.** *The problem* Max-r-Lin2-AA *admits a kernel with at most $(2k-1)r$ variables.*

Theorem 12 implies the following:

▶ **Corollary 13.** *There is an algorithm of runtime $2^{O(k)} + m^{O(1)}$ for* Max-r-Lin2-AA.

Kim and Williams [36] proved that the last result is best possible, in a sense, if the Exponential Time Hypothesis (ETH) holds.

▶ **Theorem 14** ([36]). *If* Max-3-Lin2-AA *can be solved in $O(2^{\epsilon k} 2^{\epsilon m})$ time for every $\epsilon > 0$, then ETH does not hold.*

4.2 Max-r-CSP-AA and Max-r-Sat-AA

Consider first a detailed formulation of Max-r-CSP-AA. Let $V = \{v_1, \ldots, v_n\}$ be a set of variables, each taking values -1 (True) and 1 (False). We are given a set Φ of Boolean functions, each involving at most r variables, and a collection $\mathcal{F}$ of m Boolean functions, each $f \in \mathcal{F}$ being a member of Φ, each with a positive integral weight and each acting on some subset of V. We are to decide whether there is a truth assignment to the n variables such that the total weight of satisfied functions is at least $A + k$, where A is the average weight (over all truth assignments) of satisfied functions and k is the parameter.

Note that A is a tight lower bound for the problem, whenever the family Φ is closed under replacing each variable by its complement, since if we apply any Boolean function to all 2^r choices of literals whose underlying variables are any fixed set of r variables, then any truth assignment to the variables satisfies exactly the same number of these 2^r functions.

Note that if Φ consists of clauses, we get Max-r-Sat-AA. In Max-r-Sat-AA, $A = \sum_{j=1}^{m} w_j(1 - 2^{-r_j})$, where w_j and r_j are the weight and the number of variables of Clause j, respectively. Clearly, A is a tight lower bound for Max-r-Sat.

Following [3], for a Boolean function f of weight $w(f)$ and on $r(f) \le r$ Boolean variables $x_{i_1}, \ldots, x_{i_{r(f)}}$, we introduce a polynomial $h_f(x)$, $x = (x_1, \ldots, x_n)$ as follows. Let $S_f \subset \{-1, 1\}^{r(f)}$ denote the set of all satisfying assignments of f. Then

$$h_f(x) = w(f)2^{r-r(f)} \sum_{(v_1,\ldots,v_{r(f)})\in S_f} [\prod_{j=1}^{r(f)} (1 + x_{i_j} v_j) - 1].$$

Let $h(x) = \sum_{f\in\mathcal{F}} h_f(x)$. It is easy to see (cf. [2]) that the value of $h(x)$ at some x^0 is precisely $2^r(U - A)$, where U is the total weight of the functions satisfied by the truth assignment x^0. Thus, the answer to Max-r-CSP-AA is Yes if and only if there is a truth assignment x^0 such that $h(x^0) \ge k2^r$.

Algebraic simplification of $h(x)$ will lead us the following (Fourier expansion of $h(x)$, cf. [45]):

$$h(x) = \sum_{S\in\mathcal{F}} c_S \prod_{i\in S} x_i, \qquad (2)$$

where $\mathcal{F} = \{\emptyset \ne S \subseteq [n] : \ c_S \ne 0, |S| \le r\}$. Thus, $|\mathcal{F}| \le n^r$. The sum $\sum_{S\in\mathcal{F}} c_S \prod_{i\in S} x_i$ can be viewed as the excess of an instance of Max-r-Lin2-AA and, thus, we can reduce Max-r-CSP-AA into Max-r-Lin2-AA in polynomial time (since r is fixed, the algebraic simplification can be done in polynomial time and it does not matter whether the parameter of Max-r-Lin2-AA is k or $k' = k2^r$). By Theorem 11, Max-r-Lin2-AA has a kernel with $O(k^2)$ variables and equations. This kernel is a bikernel from Max-r-CSP-AA to Max-r-Lin2-AA. Thus, by Lemma 1, we obtain the following theorem of Alon et al. [2].

▶ **Theorem 15.** Max-r-CSP-AA *admits a polynomial-size kernel.*

Applying a reduction from Max-r-Lin2-AA to Max-r-Sat-AA in which each monomial in (2) is replaced by 2^{r-1} clauses, Alon et al. [2] obtained the following:

▶ **Theorem 16.** Max-r-Sat-AA *admits a kernel with $O(k^2)$ clauses and variables.*

Using also Theorem 12, it is easy to improve this theorem with respect to the number of variables in the kernel. Note that this result was first obtained by Kim and Williams [36].

▶ **Theorem 17.** Max-r-Sat-AA *admits a kernel with $O(k)$ variables.*

4.3 Max-r-CSP-AA with global cardinality constraint

Recall the formulation of Max-r-CSP-AA: $V = \{v_1, \ldots, v_n\}$ is a set of variables, each taking values from $\{-1, 1\}$. We are given m Boolean formulas, each with an integral positive weight. We wish to decide if we can satisfy clauses with a total weight of k more than the average weight (if every variables is assigned -1 or 1 with equal probability).

Chen and Zhou [9] consider the unweighted case of Max-r-CSP-AA but allow for a global cardinality constraint. That is, we can restrict the number of 1's (or -1's) to be a given fraction of the total number of variables. Consider the sum $\sum_{i=1}^{n} v_i$ and note that this is an integer between $-n$ and n. We now consider the case when $\sum_{i=1}^{n} v_i = \alpha n$ and $-p_0 \leq \alpha \leq p_0$ for some fixed integer $p_0 < 1$. Note that α may depend on n, but has to be bounded by constants $-p_0$ and p_0 (p_0 does not depend on n and will be considered a constant in the complexity). We now formally describe the problem.

Let $0 \leq p_0 < 1$ be a constant and for every n, let α_n satisfy $-p_0 \leq \alpha_n \leq p_0$. We consider all instances that satisfy the following:

$$\sum_{i=1}^{n} v_i = \alpha_n n \,.$$

For example if $\alpha_n = 0$ then we require to be equally many 1's and -1's in the assignments to $v_1, v_2, \ldots, v_n$ (this is called a *bisection constraint*). If $\alpha_n = 1/2$ then we require exactly one quarter of all variables $v_1, v_2, \ldots, v_n$ to be assigned -1 and three quarters to be assigned 1.

One application of the global cardinality constraint can be found in the MaxBisection problem where we are given a graph G and want to partition the vertices into two equal-size sets such that we have the maximum possible number of edges between the two sets. Let $v_1, v_2, \ldots, v_n$ be the vertices of the graph G and, with abuse of notation, also the variables in our instance of Max-r-CSP-AA. If $v_i v_j$ is an edge in G then add the constraint $v_i \neq v_j$. Adding the global cardinality constraint $\sum_{i=1}^{n} v_i = 0$ now gives us an instance of Max-r-CSP-AA (with global cardinality constraint) which has a solution k above the average if and only if MaxBisection has a solution with k more edges in the cut than an average cut (given that both partite sets are equally large).

If we do not require both partite sets to be equally large we get the problem MaxCut, in which every edge has probability $1/2$ of belonging to a random cut. However for the MaxBisection problem this probability is slightly higher. Consider en edge $v_i v_j$ in the graph G. Without loss of generality, let v_i be assigned 1. Of the remaining $n-1$ vertices $n/2-1$ will be assigned 1 and $n/2$ will be assigned -1. Therefore the probability that $v_i v_j$ is in the cut will be as follows.

$$\frac{n/2}{n-1} = \frac{1}{2} + \frac{1}{2(n-1)} \,.$$

Therefore in the MaxBisection-AA we are looking to decide if there is a solution with at least $m\left(\frac{1}{2} + \frac{1}{2(n-1)}\right) + k$ edges in the cut, where k is the parameter.

MaxBisection-AA was shown to be FPT and have a $O(k^2)$ kernel by Chen and Zhou [9]. This significantly improves a result by Gutin and Yeo [29] who showed a similar result when looking for a solution with at least $m/2 + k$ edges, where k is the parameter.

In fact, in [9] it is proved that each unweighted Max-r-CSP-AA with a global cardinality constraint is FPT and has a kernel of size $O(k^2)$.

▶ **Theorem 18** ([9]). *For every p_0 with $0 < p_0 \leq 1$ there exists a kernel of size $O(k^2)$ for the unweighted* Max-r-CSP-AA *with global cardinality constraint (given by p_0).*

If $m \in n^{O(r)}$ (which is the case if we do not repeat constraints), then in Theorem 18 the polynomial algorithm that produces the kernel has runtime $n^{O(r)}$ (where r was the number of variables in each constraint). If $m \notin n^{O(r)}$, it is not difficult to obtain a runtime of $m^{O(r)}$.

The size of the kernel in Theorem 18 depends on p_0 and r, but in the theorem these are constants. Theorem 18 furthermore implies the following result (where p_0 and r again are considered constants).

▶ **Theorem 19** ([9]). *For every $0 < p_0 \leq 1$ and unweighted instance* Max-r-CSP-AA *with global cardinality constraint (given by p_0) and $m \in n^{O(1)}$ there exists an algorithm with runtime $n^{O(1)} + 2^{O(k^2)}$ that decides if there is a solution satisfying k constraints more than the average.*

The proofs of the above results are deep and beyond the scope of this survey. They use a version of the hypercontractive inequality where the probability space is given by all assignments satisfying the global cardinality constraint. Therefore the variables are not independent, which complicates matters compared to previous proofs of the ordinary hypercontractive inequality. The proof of this new hypercontractive inequality relies on the analysis of the eigenvalues of several $n^{O(r)} \times n^{O(r)}$ set-symmetric matrices.

5 Parameterizations of MaxLin2

In the Max-Lin2 problem, we are given a system S of m linear equations in n variables over $\mathbb{F}_2$ in which each equation is assigned a positive integral weight. Our aim is to find am assignment to the variables that maximizes the total weight of satisfied equations. In this section, we will consider the following two parameterizations of Max-Lin2 :

- MaxLin2-AA is the same problem as Max-r-Lin2-AA, but the number of variables in an equation is not bounded. Thus, MaxLin2-AA is a generalization of Max-r-Lin2-AA. In Subsection 5.1 we present a scheme of a recent proof by Crowston et al. [11] that MaxLin2-AA is FPT and has a kernel with polynomial number of variables. This result finally solved an open question of Mahajan et al. [42]. Still, we do not know whether MaxLin2-AA has a kernel of polynomial size and we are able to give only partial results on the topic.
- Let W be the total weight of all equations in S. In Subsection 5.2 we consider the following parameterized version of MaxLin2: decide whether there is an assignment satisfying equations of total weight at least $W - k$, where k is a nonnegative parameter.

5.1 MaxLin2-AA

Let S be an irreducible system of Max-Lin2-AA (recall Definition 10). Consider the following algorithm introduced in [12]. We assume that, in the beginning, no equation or variable in S is marked.

Algorithm $\mathcal{H}$

While the system S is nonempty do the following:

1. Choose an equation $\prod_{i \in I} x_i = b$ and mark a variable x_l such that $l \in I$.
2. Mark this equation and delete it from the system.
3. Replace every equation $\prod_{i \in I'} x_i = b'$ in the system containing x_l by $\prod_{i \in I \Delta I'} x_i = bb'$, where $I \Delta I'$ is the symmetric difference of I and I' (the weight of the equation is unchanged).
4. Apply Reduction Rule 2 to the system.

The *maximum* $\mathcal{H}$*-excess* of S is the maximum possible total weight of equations marked by $\mathcal{H}$ for S taken over all possible choices in Step 1 of $\mathcal{H}$. The following lemma indicates the potential power of $\mathcal{H}$.

▶ **Lemma 20** ([12]). *Let S be an irreducible system. Then the maximum excess of S equals its maximum $\mathcal{H}$-excess.*

This lemma gives no indication on how to choose equations in Step 1 of Algorithm $\mathcal{H}$. As the problem Max-Lin2-AA is NP-hard, we cannot hope to obtain an polynomial-time procedure for optimal choice of equations in Step 1 and, thus, have to settle for a good heuristic. For the heuristic we need the following notion first used in [12]. Let K and M be sets of vectors in $\mathbb{F}_2^n$ such that $K \subseteq M$. We say K is M*-sum-free* if no sum of two or more distinct vectors in K is equal to a vector in M. Observe that K is M-sum-free if and only if K is linearly independent and no sum of vectors in K is equal to a vector in $M \backslash K$.

The following lemma was proved implicitly in [12]; we provide a short proof of this result.

▶ **Lemma 21.** *Let S be an irreducible system of* Max-Lin2-AA *and let A be the matrix corresponding to S. Let M be the set of rows of A (viewed as vectors in $\mathbb{F}_2^n$) and let K be an M-sum-free set of k vectors. Let $w_{\min}$ be the minimum weight of an equation in S. Then, in time in $(nm)^{O(1)}$, we can find an assignment to the variables of S that achieves excess of at least $w_{\min} \cdot k$.*

Proof. Let $\{e_{j_1}, \ldots, e_{j_k}\}$ be the set of equations corresponding to the vectors in K. Run Algorithm $\mathcal{H}$, choosing at Step 1 an equation of S from $\{e_{j_1}, \ldots, e_{j_k}\}$ each time, and let S' be the resulting system. Algorithm $\mathcal{H}$ will run for k iterations of the while loop as no equation from $\{e_{j_1}, \ldots, e_{j_k}\}$ will be deleted before it has been marked.

Indeed, suppose that this is not true. Then for some e_{j_l} and some other equation e in S, after applying Algorithm $\mathcal{H}$ for at most $l-1$ iterations e_{j_l} and e contain the same variables. Thus, there are vectors $v_j \in K$ and $v \in M$ and a pair of nonintersecting subsets K' and K'' of $K \setminus \{v, v_j\}$ such that $v_j + \sum_{u \in K'} u = v + \sum_{u \in K''} u$. Thus, $v = v_j + \sum_{u \in K' \cup K''} u$, a contradiction to the definition of K. ◀

The main result of this subsection is the following theorem whose proof is based on Theorems 23 and 25.

▶ **Theorem 22** ([11]). *The problem* MaxLin2-AA *has a kernel with at most $O(k^2 \log k)$ variables.*

▶ **Theorem 23** ([12]). *Let S be an irreducible system of* MaxLin2-AA *and let $k \geq 2$. If $k \leq m \leq 2^{n/(k-1)} - 2$, then the maximum excess of S is at least k. Moreover, we can find an assignment with excess of at least k in time $m^{O(1)}$.*

This theorem can easily be proved using Lemma 21 and the following lemma.

▶ **Lemma 24** ([12]). *Let M be a set in $\mathbb{F}_2^n$ such that M contains a basis of $\mathbb{F}_2^n$, the zero vector is in M and $|M| < 2^n$. If k is a positive integer and $k+1 \leq |M| \leq 2^{n/k}$ then, in time $|M|^{O(1)}$, we can find an M-sum-free subset K of M with at least $k+1$ vectors.*

▶ **Theorem 25** ([11]). *There exists an $n^{2k}(nm)^{O(1)}$-time algorithm for* MaxLin2-AA *that returns an assignment of excess of at least $2k$ if one exists, and returns* No *otherwise.*

The proof of this theorem in [11] is based on constructing a special depth-bounded search tree.

Now we will present the proof of Theorem 22 from [11].

Proof of Theorem 22. Let $\mathcal{L}$ be an instance of MaxLin2-AA and let S be the system of $\mathcal{L}$ with m equations and n variables. We may assume that S is irreducible. Let the parameter k be an arbitrary positive integer.

If $m < 2k$ then $n < 2k = O(k^2 \log k)$. If $2k \le m \le 2^{n/(2k-1)} - 2$ then, by Theorem 23 and Remark 8, the answer to $\mathcal{L}$ is Yes and the corresponding assignment can be found in polynomial time. If $m \ge n^{2k} - 1$ then, by Theorem 25, we can solve $\mathcal{L}$ in polynomial time.

Finally we consider the case $2^{n/(2k-1)} - 2 \le m \le n^{2k} - 2$. Hence, $n^{2k} \ge 2^{n/(2k-1)}$. Therefore, $4k^2 \ge 2k + n/\log n \ge \sqrt{n}$ and $n \le (2k)^4$. Hence, $n \le 4k^2 \log n \le 4k^2 \log(16k^4) = O(k^2 \log k)$.

Since S is irreducible, $m < 2^n$ and thus we have obtained the desired kernel. ◀

Now let us consider some cases where we can prove that MaxLin2-AA has a polynomial-size kernel. Consider first the case when each equation in S has odd number of variables. Then we have the following theorem proved by Gutin et al. [26] using the Strictly Above/Below Expectation Method (in particular, Lemmas 2 and 7).

▶ **Theorem 26.** *The following special case of* MaxLin2-AA *admits a kernel with at most* $4k^2$ *variables and equations: there exists a subset* U *of variables such that each equation in* $Ax = b$ *has odd number of variables from* U.

Let us turn to results on MaxLin2-AA that do not require any parity conditions. One such result is Theorem 11. Gutin et al. [26] also proved the following 'dual' theorem.

▶ **Theorem 27.** *Let* $\rho \ge 1$ *be a fixed integer. Then* MaxLin2-AA *restricted to instances where no variable appears in more than* ρ *equations, admits a kernel with* $O(k^2)$ *variables and equations.*

The proof is similar to that of Theorem 11, but Lemma 6 (in fact, its weaker version obtained in [26]) is used instead of Lemma 5.

5.2 MaxLin2-B

In 2011, Arash Rafiey asked to determine the parameterized complexity of the following parameterized problem denoted MaxLin2-B. Let W be the total weight of all equations in S. Let us we consider the following parameterized version of MaxLin2: decide whether there is an assignment satisfying equations of total weight at least $W - k$, where k is a nonnegative parameter.

Crowston, Gutin, Jones and Yeo [14] proved that MaxLin2-B is W[1]-hard. This hardness result prompts us to investigate the complexity of MaxLin2-B in more detail by considering special cases of this problem. Let Max-($\le r$,$\le s$)-Lin2 (Max-($= r$,$= s$)-Lin2, respectively) denote the problem MaxLin2 restricted to instances, which have at most (exactly, respectively) r variables in each equation and at most (exactly) s appearances of any variable in all equations. In the special case when each equation has weight 1 and there are no two equations with the same left-hand side, MaxLin2-B will be denoted by MaxLin2-B[m]. Crowston et al. [14] proved that MaxLin2-B remains hard even after significant restrictions are imposed on it, namely, even Max-($= 3$,$= 3$)-Lin2-B[m] is W[1]-hard.

No further improvement of this result is possible unless FPT=W[1] as Crowston et al. [14] proved that Max-(≤ 2,*)-Lin2-B is fixed-parameter tractable, where symbol * indicates that no restriction is imposed on the number of appearances of a variable in the equations. Moreover, they showed that the nonparameterized problem Max-(*,≤ 2)-Lin2 is polynomial time solvable, where symbol * indicates that no restriction is imposed on the number of variables in any equation.

6 Parameterizations of MaxSat

In the well-known problem MaxSat, we are given a CNF formula F with m clauses and asked to determine the maximum number of clauses of F that can be satisfied by a truth assignment. In this section, we overview results on various parameterizations of MaxSat apart from Max-r-Sat-AA, where r is a constant (see Theorems 16 and 17 for this parameterized problem).

6.1 MaxSat above $m/2$

Let us assign True to each variable of F with probability $1/2$ and observe that the probability of a clause to be satisfied is at least $1/2$ and thus, by linearity of expectation, the expected number of satisfied clauses in F is at least $m/2$. Thus, by the averaging argument, there exists a truth assignment to the variables of F which satisfies at least $m/2$ clauses of F.

Let us denote by $\mathrm{sat}(F)$ the maximum number of clauses of F that can be satisfied by a truth assignment. The lower bound $\mathrm{sat}(F) \geq m/2$ is tight as we have $\mathrm{sat}(H) = m/2$ if $H = (x_1) \wedge (\bar{x}_1) \wedge \cdots \wedge (x_{m/2}) \wedge (\bar{x}_{m/2})$. Consider the following parameterization of MaxSat above tight lower bound introduced by Mahajan and Raman [41].

MaxSat-A($m/2$)
Instance: A CNF formula F with m clauses (clauses may appear several times in F) and a nonnegative integer k.
Parameter: k.
Question: $\mathrm{sat}(F) \geq m/2 + k$?

Mahajan and Raman [41] proved that MaxSat-A($m/2$) admits a kernel with at most $6k+3$ variables and $10k$ clauses. Crowston et al. [15] improved this result, by obtaining a kernel with at most $4k$ variables and $(2\sqrt{5}+4)k$ clauses. The improved result is a simple corollary of a new lower bound on $\mathrm{sat}(F)$ obtained in [15], which is significantly stronger than the simple bound $\mathrm{sat}(F) \geq m/2$. We give the new lower bound below, in Theorem 34.

For a variable x in F, let $m(x)$ denote the number of pairs of unit of clauses $(x), (\bar{x})$ that have to be deleted from F such that F has no pair $(x), (\bar{x})$ any longer. Let $\mathrm{var}(F)$ be the set of all variables in F and let $\ddot{m} = \sum_{x \in \mathrm{var}(F)} m(x)$. The following is a stronger lower bound on $\mathrm{sat}(F)$ than $m/2$.

▶ **Theorem 28.** *For a CNF formula F, we have $\mathrm{sat}(F) \geq \ddot{m}/2 + \hat{\phi}(m - \ddot{m})$, where $\hat{\phi} = (\sqrt{5}-1)/2 \approx 0.618$.*

6.2 Max-$r(n)$-Sat-AA

Max-$r(n)$-Sat-AA is a generalization of Max-r-Sat-AA, where $r(n)$ is no longer just a constant, it depends on n. The following results for Max-$r(n)$-Sat-AA were obtained by Crowston et al. [13].

▶ **Theorem 29.** Max-$r(n)$-Sat-AA *is para-NP-complete for $r(n) = \lceil \log n \rceil$.*

Assuming ETH, one can prove the following stronger result.

▶ **Theorem 30.** *Assuming ETH,* Max-$r(n)$-Sat-AA *is not FPT for any $r(n) \geq \log\log n + \phi(n)$, where $\phi(n)$ is any unbounded strictly increasing function of n.*

The following theorem shows that Theorem 30 provides a bound on $r(n)$ which is not far from optimal.

▶ **Theorem 31** ([13]). Max-$r(n)$-Sat-AA *is FPT for* $r(n) \leq \log\log n - \log\log\log n - \phi(n)$, *for any unbounded strictly increasing function* $\phi(n)$.

6.3 Parameterizations for MaxSat with t-Satisfiable CNF Formulas

A CNF formula F is *t-satisfiable* if for any t clauses in F, there is a truth assignment which satisfies all of them. It is easy to check that F is 2-satisfiable if and only if $\ddot{m} = 0$ and clearly Theorem 28 is equivalent to the assertion that if F is 2-satisfiable then $\mathrm{sat}(F) \geq \hat{\phi}m$. The proof of this assertion by Lieberherr and Specker [38] is quite long; Yannakakis [51] gave the following short probabilistic proof. For $x \in \mathrm{var}(F)$, let the probability of x being assigned True be $\hat{\phi}$ if (x) is in F, $1 - \hat{\phi}$ if $(\bar{x})$ is in F, and $1/2$, otherwise, independently of the other variables. Let us bound the probability $p(C)$ of a clause C to be satisfied. If C contains only one literal, then, by the assignment above, $p(C) = \hat{\phi}$. If C contains two literals, then, without loss of generality, $C = (x \vee y)$. Observe that the probability of x assigned False is at most $\hat{\phi}$ (it is $\hat{\phi}$ if $(\bar{x})$ is in F). Thus, $p(C) \geq 1 - \hat{\phi}^2$. If C containes more than two literals then it is easy to see that $p(C) \geq 1 - \hat{\phi}^2$. It remains to observe that $1 - \hat{\phi}^2 = \hat{\phi}$. Now to obtain the bound $\mathrm{sat}(F) \geq \hat{\phi}m$ apply linearity of expectation and the averaging argument.

Note that $\hat{\phi}m$ is an *asymptotically* tight lower bound: for each $\epsilon > 0$ there are 2-satisfiable CNF formulae F with $\mathrm{sat}(F) < m(\hat{\phi}+\epsilon)$ [38]. Thus, the following problem stated by Mahajan and Raman [41] is natural.

Max-2S-Sat-A($\hat{\phi}m$)
Instance: A 2-satisfiable CNF formula F with m clauses (clauses may appear several times in F) and a nonnegative integer k.
Parameter: k.
Question: $\mathrm{sat}(F) \geq \hat{\phi}m + k$?

Mahajan and Raman [41] conjectured that Max-2S-Sat-A($\hat{\phi}m$) is FPT. Crowston et al. [15] solved this conjecture in the affirmative; moreover, they obtained a kernel with at most $(7 + 3\sqrt{5})k$ variables. This result is an easy corollary from a lower bound on $\mathrm{sat}(F)$ given in Theorem 34, which, for 2-satisfiable CNF formulas, is stronger than the one in Theorem 28. The main idea of [15] is to obtain a lower bound on $\mathrm{sat}(F)$ that includes the number of variables as a factor. It is clear that for general CNF formula F such a bound is impossible. For consider a formula containing a single clause c containing a large number of variables. We can arbitrarily increase the number of variables in the formula, and the maximum number of satisfiable clauses will always be 1. We therefore need a reduction rule that cuts out 'excess' variables. Our reduction rule is based on the notion of an expanding formula given below. Lemma 32 and Theorem 33 show the usefulness of this notion.

A CNF formula F is called *expanding* if for each $X \subseteq \mathrm{var}(F)$, the number of clauses containing at least one variable from X is at least $|X|$ [20, 50]. The following lemma and its parts were proved by many authors, see, e.g., Fleischner et al. [20], Lokshtanov [40] and Szeider [50].

▶ **Lemma 32.** *Let F be a CNF formula and let V and C be its sets of variables and clauses. There exists a subset $C^* \subseteq C$ that can be found in polynomial time, such that the formula F' with clauses $C \setminus C^*$ and variables $V \setminus V^*$, where $V^* = \mathrm{var}(C^*)$, is expanding. Moreover,* $\mathrm{sat}(F) = \mathrm{sat}(F') + |C^*|$.

The following result was shown by Crowston et al. [15]. The proof is nontrivial and consists of a deterministic algorithm for finding the corresponding truth assignment and a detailed combinatorial analysis of the algorithm.

▶ **Theorem 33.** *Let F be an expending 2-satisfiable CNF formula with n variables and m clauses. Then* $\text{sat}(F) \geq \hat{\phi}m + n(2 - 3\hat{\phi})/2$.

Lemma 32 and Theorem 33 imply the following:

▶ **Theorem 34.** *Let F be a 2-satisfiable CNF formula and let V and C be its sets of variables and clauses. There exists a subset $C^* \subseteq C$ that can be found in polynomial time, such that the formula F' with clauses $C \setminus C^*$ and variables $V \setminus V^*$, where $V^* = \text{var}(C^*)$, is expanding. Moreover, we have*

$$\text{sat}(F) \geq \hat{\phi}m + (1 - \hat{\phi})m^* + (n - n^*)(2 - 3\hat{\phi})/2,$$

where $m = |C|$, $m^* = |C^*|$, $n = |V|$ and $n^* = |V^*|$.

Let us turn now to 3-satisfiable CNF formulas. If F is 3-satisfiable then it is not hard to check that the forbidden sets of clauses are pairs of the form $\{x\}, \{\bar{x}\}$ and triplets of the form $\{x\}, \{y\}, \{\bar{x}, \bar{y}\}$ or $\{x\}, \{\bar{x}, y\}, \{\bar{x}, \bar{y}\}$, as well as any triplets that can be derived from these by switching positive literals with negative literals.

Lieberherr and Specker [39] and, later, Yannakakis [51] proved the following: if F is 3-satisfiable then $\text{sat}(F) \geq \frac{2}{3}w(\mathcal{C}(F))$. This bound is also asymptotically tight. Yannakakis [51] gave a probabilistic proof which is similar to his proof for 2-satisfiable formulas, but requires consideration of several cases and, thus, not as short as for 2-satisfiable formulas. For details of his proof, see, e.g., Gutin, Jones and Yeo [24] and Jukna [33] (Theorem 20.6). Yannakakis's approach was extended by Gutin, Jones and Yeo [24] to prove the following theorem using a quite complicated probabilistic distribution for a random truth assignment.

▶ **Theorem 35.** *Let F be an expanding 3-satisfiable CNF formula with n variables and m clauses. Then $\text{sat}(F) \geq \frac{2}{3}m + \rho n$, where $\rho (> 0.0019)$ is a constant.*

This theorem and Lemma 32 imply the following:

▶ **Theorem 36.** *Let F be a 3-satisfiable CNF formula and let V and C be its sets of variables and clauses. There exists a subset $C^* \subseteq C$ that can be found in polynomial time, such that the formula F' with clauses $C \setminus C^*$ and variables $V \setminus V^*$, where $V^* = \text{var}(C^*)$, is expanding. Moreover, we have*

$$\text{sat}(F) \geq \frac{2}{3}m + \frac{1}{3}m^* + \rho(n - n^*),$$

where $\rho (> 0.0019)$ is a constant, $m = |C|$, $m^* = |C^*|$, $n = |V|$ and $n^* = |V^*|$.

Using this theorem it is easy to obtain a linear-in-number-of-variables kernel for the following natural analog of Max-2S-Sat-A($\hat{\phi}m$), see [24] for details.

Max-3S-Sat-A($\frac{2}{3}m$)
Instance: A 3-satisfiable CNF formula F with m clauses and a nonnegative integer k.
Parameter: k.
Question: $\text{sat}(F) \geq \frac{2}{3}m + k$?

Now let us consider the following important parameterization of r-Sat below the tight upper bound m:

r-SAT-B(m)
Instance: An r-CNF formula F with m clauses (every clause has at most r literals) and a nonnegative integer k.
Parameter: k. *Question:* $\text{sat}(F) \geq m - k$?

Since MAX-r-SAT is NP-hard for each fixed $r \geq 3$, r-SAT-B(m) is not FPT unless P=NP. However, the situation changes for $r = 2$: Razgon and O'Sullivan [48] proved that 2-SAT-B(m) is FPT. The algorithm in [48] is of complexity $O(15^k km^3)$ and, thus, MAX-2-SAT-B(m) admits a kernel with at most $15^k k$ clauses. Raman et al. [47] and Cygan et al. [17] designed algorithms for 2-SAT-B(m) of runtime $9^k(km)^{O(1)}$ and $4^k(km)^{O(1)}$, respectively. Kratsch and Wahlström [37] proved that 2-SAT-B(m) admits a randomized kernel with a polynomial number of variables. The existence of a deterministic polynomial kernel 2-SAT-B(m) is an open problem.

7 Ordering CSPs

In this section we will discuss recent results in the area of *Ordering Constraint Satisfaction Problems (Ordering CSPs)* parameterized above average. Ordering CSPs include several well-known problems such as BETWEENNESS, CIRCULAR ORDERING and ACYCLIC SUBDIGRAPH (which is equivalent to 2-LINEAR ORDERING). These three problems have applications in circuit design and computational biology [10, 46], in qualitative spatial reasoning [32], and in economics [49], respectively. Our main interest are Ordering CSPs parameterized above average, Ordering CSPs-AA.

2-LINEAR ORDERING-AA was already considered in Subsection 3. In the next subsection, we give some basic definitions and results on Ordering CSPs. In Subsection 7.2, we give a proof scheme that BETWEENNESS-AA is fixed-parameter tractable. The proof uses SABEM supplemented by additional approaches. In Subsection 7.3, we discuss how to combine fixed-parameter tractability of 2-LINEAR ORDERING-AA and BETWEENNESS-AA to show that 3-LINEAR ORDERING-AA is fixed-parameter tractable. Finally, in Subsection 7.4, we briefly discuss the recent paper of Makarychev, Makarychev and Zhou [43], where it was proved that any Ordering CSP-AA is fixed-parameter tractable. Moreover, the authors of [43] extended their result to a linear programming generalization of Ordering CSPs-AA.

7.1 Basic Definitions and Results

Let us define Ordering CSPs of arity 3. The reader can easily generalize it to any arity $r \geq 2$ and we will do it below for LINEAR ORDERING of arity r. Let V be a set of n variables and let

$$\Pi \subseteq \mathcal{S}_3 = \{(123), (132), (213), (231), (312), (321)\}$$

be arbitrary. A *constraint set over* V is a multiset $\mathcal{C}$ of *constraints*, which are permutations of three distinct elements of V. A bijection $\alpha : V \to [n]$ is called an *ordering* of V. For an ordering $\alpha : V \to [n]$, a constraint $(v_1, v_2, v_3) \in \mathcal{C}$ is Π-*satisfied by* α if there is a permutation $\pi \in \Pi$ such that $\alpha(v_{\pi(1)}) < \alpha(v_{\pi(2)}) < \alpha(v_{\pi(3)})$. Thus, given Π the problem Π-CSP, is the problem of deciding if there exists an ordering of V that Π-satisfies all the constraints. Every such problem is called an Ordering CSP of arity 3. We will consider the maximization version of these problems, denoted by MAX-Π-CSP, parameterized above the average number of constraints satisfied by a random ordering of V (which can be shown to be a tight bound).

Table 1 Ordering CSPs of arity 3 (after symmetry considerations).

$\Pi \subseteq \mathcal{S}_3$	Name	Complexity
$\Pi_0 = \{(123)\}$	Linear Ordering-3	polynomial
$\Pi_1 = \{(123), (132)\}$		polynomial
$\Pi_2 = \{(123), (213), (231)\}$		polynomial
$\Pi_3 = \{(132), (231), (312), (321)\}$		polynomial
$\Pi_4 = \{(123), (231)\}$		NP-comp.
$\Pi_5 = \{(123), (321)\}$	Betweenness	NP-comp.
$\Pi_6 = \{(123), (132), (231)\}$		NP-comp.
$\Pi_7 = \{(123), (231), (312)\}$	Circular Ordering	NP-comp.
$\Pi_8 = \mathcal{S}_3 \setminus \{(123), (231)\}$		NP-comp.
$\Pi_9 = \mathcal{S}_3 \setminus \{(123), (321)\}$	Non-Betweenness	NP-comp.
$\Pi_{10} = \mathcal{S}_3 \setminus \{(123)\}$		NP-comp.

Guttmann and Maucher [30] showed that there are in fact only 13 distinct Π-CSP's of arity 3 up to symmetry, of which 11 are nontrivial. They are listed in Table 1 together with their complexity. Note that if $\Pi = \{(123), (321)\}$ then we obtain the Betweenness problem and if $\Pi = \{(123)\}$ then we obtain 3-Linear Ordering.

Gutin et al. [23] proved that all 11 nontrivial Max-Π-CSP problems are NP-hard (even though four of the Π-CSP are polynomial).

Now observe that given a variable set V and a constraint multiset $\mathcal{C}$ over V, for a random ordering α of V, the probability of a constraint in $\mathcal{C}$ being Π-satisfied by α equals $\frac{|\Pi|}{6}$. Hence, the expected number of satisfied constraints from $\mathcal{C}$ is $\frac{|\Pi|}{6}|\mathcal{C}|$, and thus there is an ordering α of V satisfying at least $\frac{|\Pi|}{6}|\mathcal{C}|$ constraints (and this bound is tight). A derandomization argument leads to $\frac{|\Pi_i|}{6}$-approximation algorithms for the problems Max-Π_i-CSP [8]. No better constant factor approximation is possible assuming the Unique Games Conjecture [8].

We will study the parameterization of Max-Π_i-CSP above tight lower bound:

Π-Above Average (Π-AA)

Input:	A finite set V of variables, a multiset $\mathcal{C}$ of ordered triples of distinct variables from V and an integer $\kappa \geq 0$.
Parameter:	κ.
Question:	Is there an ordering α of V such that at least $\frac{|\Pi|}{6}|\mathcal{C}| + \kappa$ constraints of $\mathcal{C}$ are Π-satisfied by α?

In [23] it is shown that all 11 nontrivial Π-CSP-AA problems admit kernels with $O(\kappa^2)$ variables. This is shown by first reducing them to 3-Linear Ordering-AA (or 2-Linear Ordering-AA), and then finding a kernel for this problem, which is transformed back to the original problem. The first transformation is easy due to the following:

▶ **Proposition 37** ([23]). *Let* Π *be a subset of* $\mathcal{S}_3$ *such that* $\Pi \notin \{\emptyset, \mathcal{S}_3\}$. *There is a polynomial time transformation* f *from* Π-AA *to* 3-Linear Ordering-AA *such that an instance* $(V, \mathcal{C}, k)$ *of* Π-AA *is a* Yes*-instance if and only if* $(V, \mathcal{C}_0, k) = f(V, \mathcal{C}, k)$ *is a* Yes*-instance of* 3-Linear Ordering-AA.

Proof. From an instance $(V, \mathcal{C}, k)$ of Π-AA, construct an instance $(V, \mathcal{C}_0, k)$ of 3-Linear

Ordering-AA as follows. For each triple $(v_1, v_2, v_3) \in \mathcal{C}$, add $|\Pi|$ triples $(v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)})$, $\pi \in \Pi$, to $\mathcal{C}_0$.

Observe that a triple $(v_1, v_2, v_3) \in \mathcal{C}$ is Π-satisfied if and only if exactly one of the triples $(v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)})$, $\pi \in \Pi$, is satisfied by 3-Linear Ordering. Thus, $\frac{|\Pi|}{6}|\mathcal{C}| + k$ constraints from $\mathcal{C}$ are Π-satisfied if and only if the same number of constraints from $\mathcal{C}_0$ are satisfied by 3-Linear Ordering. It remains to observe that $\frac{|\Pi|}{6}|\mathcal{C}| + k = \frac{1}{6}|\mathcal{C}_0| + k$ as $|\mathcal{C}_0| = |\Pi| \cdot |\mathcal{C}|$. ◀

r-Linear Ordering ($r \geq 2$) can be defined as follows. An instance of such a problem consists of a set of variables V and a multiset of constraints, which are ordered r-tuples of distinct variables of V (note that the same set of r variables may appear in several different constraints). The objective is to find an ordering α of V that maximizes the number of constraints whose order in α follows that of the constraint (we say that these constraints are satisfied). It is well-known that 2-Linear Ordering is NP-hard (it follows immediately from the fact proved by Karp [34] that the feedback arc set problem is NP-hard). It is easy to extend this hardness result to all r-Linear Ordering problems (for each fixed $r \geq 2$). Note that in r-Linear Ordering Above Average (r-Linear Ordering-AA), given a multiset $\mathcal{C}$ of constraints over V we are to decide whether there is an ordering of V that satisfies at least $|\mathcal{C}|/r! + \kappa$ constraints.

7.2 Betweenness-AA

Let $V = \{v_1, \ldots, v_n\}$ be a set of variables and let $\mathcal{C}$ be a multiset of m *betweenness* constraints of the form $(v_i, \{v_j, v_k\})$. For an ordering $\alpha: V \to [n]$, a constraint $(v_i, \{v_j, v_k\})$ *is satisfied* if either $\alpha(v_j) < \alpha(v_i) < \alpha(v_k)$ or $\alpha(v_k) < \alpha(v_i) < \alpha(v_j)$. In the Betweenness problem, we are asked to find an ordering α satisfying the maximum number of constraints in $\mathcal{C}$. Betweenness is NP-hard as even the problem of deciding whether all betweenness constraints in $\mathcal{C}$ can be satisfied by an ordering α is NP-complete [46].

Let $\alpha: V \to [n]$ be a random ordering and observe that the probability of a constraint in $\mathcal{C}$ to be satisfied is 1/3. Thus, the expected number of satisfied constraints is $m/3$. A triple of betweenness constraints of the form $(v, \{u, w\}), (u, \{v, w\}), (w, \{v, u\})$ is called a *complete triple.* Instances of Betweenness consisting of complete triples demonstrate that $m/3$ is a tight lower bound on the maximum number of constraints satisfied by an ordering α. Thus, the following parameterization is of interest:

Betweenness Above Average (Betweenness-AA)
Instance: A multiset $\mathcal{C}$ of m betweenness constraints over variables V and an integer $\kappa \geq 0$.
Parameter: The integer κ.
Question: Is there an ordering $\alpha: V \to [n]$ that satisfies at least $m/3 + \kappa$ constraints from $\mathcal{C}$?

In order to simplify instances of Betweenness-AA we introduce the following reduction rule.

▶ **Reduction Rule 4.** *If $\mathcal{C}$ has a complete triple, delete it from $\mathcal{C}$. Delete from V all variables that appear only in the deleted triple.*

Benny Chor's question (see [44, p. 43]) to determine the parameterized complexity of Betweenness-AA was solved by Gutin, Kim, Mnich and Yeo [25] who proved that Betweenness-AA admits a kernel with $O(\kappa^2)$ variables and constraints (in fact, [25] considers only the case when $\mathcal{C}$ is a set, not a multiset, but the proof for the general case is the same [23]). Below we briefly describe the proof in [25].

Suppose we define a random variable $X(\alpha)$ just as we did for 2-Linear Ordering. However such a variable is not symmetric and therefore we would need to use Lemma 7 on $X(\alpha)$. The problem is that α is a permutation and in Lemma 7 we are looking at polynomials, $f = f(x_1, x_2 \dots, x_n)$, over variables $x_1, \dots, x_n$ each with domain $\{-1, 1\}$. In order to get around this problem the authors of [25] considered a different random variable $g(Z)$, which they defined as follows.

Let $Z = (z_1, z_2, \dots, z_{2n})$ be a set of $2n$ variables with domain $\{-1, 1\}$. These $2n$ variables correspond to n variables $z_1^*, z_2^*, \dots, z_n^*$ such that z_{2i-1} and z_{2i} form the binary representation of z_i^*. That is, z_i^* is 0, 1, 2 or 3 depending on the value of $(z_{2i-1}, z_{2i}) \in \{(-1,-1), (-1,1), (1,-1), (1,1)\}$. An ordering: $\alpha : V \to [n]$ complies with Z if for every $\alpha(i) < \alpha(j)$ we have $z_i^* \le z_j^*$. We now define the value of $g(Z)$ as the average number of constraints satisfied over all orderings which comply with Z. Let $f(Z) = g(Z) - m/3$, and by Lemma 38 we can now use Lemma 7 on $f(Z)$ as it is a polynomial over variables whose domain is $\{-1, 1\}$. We consider variables z_i as independent uniformly distributed random variables and then $f(Z)$ is also a random variable. In [25] it is shown that the following holds if Reduction Rule 4 has been exhaustively applied.

▶ **Lemma 38.** *The random variable $f(Z)$ can be expressed as a polynomial of degree 6. We have $\mathbb{E}[f(Z)] = 0$. Finally, if $f(Z) \ge \kappa$ for some $Z \in \{-1, 1\}^{2n}$ then the corresponding instance of* Betweenness-AA *is a* Yes*-instance.*

▶ **Lemma 39** ([23]). *For an irreducible (by Reduction Rule 4) instance we have $\mathbb{E}[f(Z)^2] \ge \frac{11}{768} m$.*

▶ **Theorem 40** ([23]). Betweenness-AA *has a kernel of size $O(\kappa^2)$.*

Proof. Let $(V, \mathcal{C})$ be an instance of Betweenness-AA. We can obtain an irreducible instance $(V', \mathcal{C}')$ such that $(V, \mathcal{C})$ is a Yes-instance if and only if $(V', \mathcal{C}')$ is a Yes-instance in polynomial time. Let $m' = |\mathcal{C}'|$ and let $f(Z)$ be the random variable defined above. Then $f(Z)$ is expressible as a polynomial of degree 6 by Lemma 38; hence it follows from Lemma 5 that $\mathbb{E}[f(Z)^4] \le 2^{36} \mathbb{E}[f(Z)^2]^2$. Consequently, $f(Z)$ satisfies the conditions of Lemma 4, from which we conclude that $\mathbb{P}\left(f(Z) > \frac{1}{4 \cdot 2^{18}} \sqrt{\frac{11}{768} m'}\right) > 0$, by Lemma 39. Therefore, by Lemma 38, if $\frac{1}{4 \cdot 2^{18}} \sqrt{\frac{11}{768} m'} \ge \kappa$ then $(V', \mathcal{C}')$ is a Yes-instance for Betweenness-AA. Otherwise, we have $m' = O(\kappa^2)$. This concludes the proof of the theorem. ◀

By deleting variables not appearing in any constraint, we obtain a kernel with $O(\kappa^2)$ constraints and variables.

7.3 3-Linear Ordering-AA

In this subsection, we will give a short overview of the proof in [23] that 3-Linear Ordering has a kernel with at most $O(\kappa^2)$ variables and constraints.

Unfortunately, approaches which we used for the 2-Linear Ordering-AA problem and the Betweenness-AA problem do not work for this problem. In fact, if we wanted to remove subsets of constraints where only the average number of constraints can be satisfied such that after these removals we are guaranteed to have more than the average number of constraints satisfied, then, in general case, an *infinite* number of reduction rules would be needed [23].

However, we can reduce an instance of 3-Linear Ordering-AA to instances of Betweenness-AA *and* 2-Linear Ordering-AA as follows. With an instance $(V, \mathcal{C})$ of 3-Linear Ordering-AA, we associate an instance $(V, \mathcal{B})$ of Betweenness-AA and two instances (V, A') and (V, A'') of 2-Linear Ordering-AA such that if $C_p = (u, v, w) \in \mathcal{C}$, then add $B_p = (v, \{u, w\})$ to $\mathcal{B}$, $a'_p = (u, v)$ to A', and $a''_p = (v, w)$ to A''.

Let α be an ordering of V and let $\mathsf{dev}(V, \mathcal{C}, \alpha)$ denote the number of constraints satisfied by α minus the average number of satisfied constraints in $(V, \mathcal{C})$, where $(V, \mathcal{C})$ is an instance of 3-Linear Ordering-AA, Betweenness-AA or 2-Linear Ordering-AA.

▶ **Lemma 41** ([23]). *Let (V, C, κ) be an instance of* 3-Linear Ordering-AA *and let α be an ordering of V. Then*

$$\mathsf{dev}(V, \mathcal{C}, \alpha) = \frac{1}{2} \left[\mathsf{dev}(V, A', \alpha) + \mathsf{dev}(V, A'', \alpha) + \mathsf{dev}(V, \mathcal{B}, \alpha) \right].$$

Therefore, we want to find an ordering satisfying as many constraints as possible from both of our new type of instances (note that we need to use the same ordering for all the problems).

Suppose we have a No-instance of 3-Linear Ordering-AA. As above, we replace it by three instances of Betweenness-AA and 2-Linear Ordering-AA. Now we apply the reduction rules for Betweenness-AA and 2-Linear Ordering-AA introduced above as well as the proof techniques described in the previous sections in order to show that the total number of variables and constraints left in any of our instances is bounded by $O(\kappa^2)$. We then transform these reduced instances back into an instance of 3-Linear Ordering-AA as follows. If $\{v, \{u, w\}\}$ is a Betweenness constraint then we add the 3-Linear Ordering-AA constraints (u, v, w) and (w, v, u) and if (u, v) is an 2-Linear Ordering-AA constraint then we add the 3-Linear Ordering-AA constraints (u, v, w), (u, w, v) and (w, u, v) (for any $w \in V$). As a result, we obtain a kernel of 3-Linear Ordering-AA with at most $O(\kappa^2)$ variables and constraints.

This result has been partially improved by Kim and Williams [36] who showed that 3-Linear Ordering-AA has a kernel with at most $O(\kappa)$ variables.

7.4 Ordering CSPs AA

Recall that an Ordering CSP of arity r is defined by a set $V = \{x_1, \ldots, x_n\}$ of variables and set of constraints. Each constraint is a disjunction of clauses of the form $x_{i_1} < x_{i_2} < \cdots < x_{i_r}$. A linear ordering α of V satisfies such a constraint if one of the clauses in the disjunction agrees with α.

Gutin, Iersel, Mnich and Yeo [23] conjectured that all Ordering CSPs parameterized above average are fixed-parameter tractable. One of the difficulties in proving this conjecture is that, as we mentioned in the previous subsection, we may need an infinite number of reduction rules. The approach of the previous section will not work as it is designed for the case when some Ordering CSPs of the same arity have already proved to be fixed-parameter tractable. Recently, Makarychev, Makarychev and Zhou [43] proved the conjecture. Their proof uses SABEM together the idea to define variables x_i not on a discrete domain, but on the continuous interval $[-1, 1]$. Such a domain allows to order all the variables with ties being almost impossible.

Makarychev, Makarychev and Zhou [43] also use the Efron-Stein decomposition instead of the (standard) Fourier Analysis on $[-1, +1]^n$ since "we have no control over the Fourier coefficients of the functions we need to analyze." For terminology and results on the Efron-Stein decomposition, see [43] and for a more detailed account [45]. Here we will only give some very basic definitions. Let (Ω, μ) be a probability space and consider the product probability space (Ω^n, μ^n). Let $f : \Omega^n \to \mathbb{R}$ be a function (random variable). Informally, the Efron-Stein decomposition of f is $f = \sum_{S \subseteq [n]} f_S$, where f_S depends only on variables x_i, $i \in S$. The functions f_S have some very useful properties such as $\mathbb{E}[f_S f_T] = 0$ if $S \neq T$ (this implies that the variance of f equals the sum of the variances of f_S in the decomposition). To use the Efron-Stein decomposition for SABEM, Makarychev, Makarychev and Zhou [43] obtained the following Hypercontractive Inequality for functions defined on arbitrary product probability spaces:

▶ **Theorem 42.** *Consider $f \in L_2(\Omega^n, \mu^n)$. Let $f = \sum_{S \subseteq [n]} f_S$ be the Efron-Stein decomposition of f and let $d = \max\{t : f_S \neq 0$ and $|S| = d\}$. Assume that for every S_1, S_2, S_3, S_4,*

$$\mathbb{E}[f_{S_1} f_{S_2} f_{S_3} f_{S_4}] \leq C(\mathbb{E}[f^2_{S_1}]\mathbb{E}[f^2_{S_1}]\mathbb{E}[f^2_{S_1}]\mathbb{E}[f^2_{S_1}])^{1/2}\,. \tag{3}$$

Then

$$\mathbb{E}[f(X_1, \ldots, X_n)^4] \leq 81^d C \mathbb{E}[f(X_1, \ldots, X_n)^2]^2\,. \tag{4}$$

Note that Condition (3) is usually much easier to verify than Inequality (4 [43].

The idea to use a continuous domain allows one to define various systems of linear inequalities rather than just those of the form $x_{i_1} < x_{i_2} < \cdots < x_{i_r}$. For example, we can require that x_4 is to the left of the average of x_1, x_2 and x_3, which corresponds to the system $3x_4 - x_1 - x_2 - x_3 < 0$. Makarychev, Makarychev and Zhou [43] define the following generalization of Ordering CSPs. An (r, b)-LP CSP is defined by a set $V = \{x_1, \ldots, x_n\}$ of variables taking values in $[-1, 1]$ and set of constraints. Each constraint is a disjunction of clauses of the form $Ax < c$, where A is a matrix with integral entries in the range $[-b, b]$ and there are at most r non-zero columns in A, and c is a vector with integral entries in the range $[-b, b]$. The aim is to assign distinct real values to variables x_i so as to maximize the number of satisfied constraints. Makarychev, Makarychev and Zhou [43] proved that every (r, b)-LP CSP above average is also fixed-parameter tractable.

8 Two Open Problems

Many results described in the previous sections were obtained in order to solve open problems. Two problems stated a while ago remain unsolved. The first is whether r-SAT-B(m), usually called ALMOST 2-SAT, admits a (deterministic) polynomial kernel. It seems it widely believed to be the case, but no proof is obtained. The second is whether MAX-2-LIN-AA admits a polynomial kernel (in the number of constraints). For the second problem, the possible answer is unclear.

References

1 N. Alon, Voting paradoxes and digraphs realizations, Advances in Applied Math. 29:126–135, 2002.
2 N. Alon, G. Gutin, E. J. Kim, S. Szeider, and A. Yeo, Solving MAX-r-SAT above a tight lower bound. Algorithmica 61(3):638-655, 2011.
3 N. Alon, G. Gutin and M. Krivelevich. Algorithms with large domination ratio. J. Algorithms 50:118–131, 2004.
4 H. L. Bodlaender, R. G. Downey, M. R. Fellows, and D. Hermelin, On problems without polynomial kernels. J. Comput. Syst. Sci. 75(8):423–434, 2009.
5 H. L. Bodlaender, S. Thomassé, and A. Yeo, Kernel bounds for disjoint cycles and disjoint paths. Theor. Comput. Sci. 412(35): 4570–4578, 2011.

6 A. Bonami, Étude des coefficients de Fourier des fonctions de $L^p(G)$. Ann. Inst. Fourier, 20(2):335–402, 1970.

7 L. Cai and J. Chen, On fixed-parameter tractability and approximation of NP optimization problems, J. Comput. Syst. Sci. 54:465–474, 1997.

8 M. Charikar, V. Guruswami, and R. Manokaran, Every permutation CSP of arity 3 is approximation resistant. *Proc. Computational Complexity 2009*, 62–73.

9 X. Chen and Y. Zhou, Parameterized Algorithms for Constraint Satisfaction Problems Above Average with Global Cardinality Constraints. Manuscript

10 B. Chor and M. Sudan. A geometric approach to betweenness. SIAM J. Discrete Math., ll(4):511-523, 1998.

11 R. Crowston, M. Fellows, G. Gutin, M. Jones, F. Rosamond, S. Thomassé and A. Yeo, Satisfying more than half of a system of linear equations over GF(2): A multivariate approach. J. Comput. Syst. Sci. 80(4): 687-696 (2014).

12 R. Crowston, G. Gutin, M. Jones, E. J. Kim, and I. Ruzsa. Systems of linear equations over $\mathbb{F}_2$ and problems parameterized above average. *Proc. SWAT 2010*, Lect. Notes Comput. Sci. 6139: 164–175, 2010.

13 R . Crowston, G. Gutin, M. Jones, V. Raman, and S. Saurabh, Parameterized Complexity of MaxSat Above Average. Theor. Comput. Sci. 511:77-84, 2013.

14 R. Crowston, G. Gutin, M. Jones and A. Yeo, Parameterized Complexity of Satisfying Almost All Linear Equations over $\mathbb{F}_2$, Theory Comput. Syst. 52(4):719–728, 2013.

15 R. Crowston, G. Gutin, M. Jones, and A. Yeo, A new lower bound on the maximum number of satisfied clauses in Max-SAT and its algorithmic applications. Algorithmica 64(1): 56-68 (2012).

16 M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. Parameterized Algorithms. Springer-Verlag, 2015.

17 M. Cygan, M. Pilipczuk, M. Pilipczuk, and J.O. Wojtaszczyk, On Multiway Cut parameterized above lower bounds. ACM Trans. Comput. Theory 5(1):1–11, 2013.

18 R. G. Downey and M. R. Fellows, Fundamentals of Parameterized Complexity. Springer, 2013.

19 H. Fernau, F.V. Fomin, D. Lokshtanov, D. Raible, S. Saurabh, and Y. Villanger, Kernel(s) for problems with no kernel: On out-trees with many leaves. *Proc. STACS 2009*, 421–432.

20 H. Fleischner, O. Kullmann and S. Szeider, Polynomial-time recognition of minimal unsatisfiable formulas with fixed clause-variable difference. *Theoret. Comput. Sci.*, 289(1):503–516, 2002.

21 V. Guruswami, J. Håstad, R. Manokaran, P. Raghavendra, and M. Charikar, Beating the random ordering is hard: Every ordering CSP is approximation resistant. SIAM J. Comput. 40(3): 878–914 (2011).

22 V. Guruswami, R. Manokaran, and P. Raghavendra, Beating the random ordering is hard: Inapproximability of maximum acyclic subgraph. *Proc. FOCS 2008*, 573–582.

23 G. Gutin, L. van Iersel, M. Mnich, and A. Yeo, Every ternary permutation constraint satisfaction problem parameterized above average has a kernel with a quadratic number of variables, J. Comput. Syst. Sci., in press, `doi:10.1016/j.jcss.2011.01.004`.

24 G. Gutin, M. Jones, D. Scheder, and A. Yeo, A New Bound for 3-Satisfiable MaxSat and its Algorithmic Application. Inf. Comput. 231:117-124, 2013.

25 G. Gutin, E. J. Kim, M. Mnich, and A. Yeo. Betweenness parameterized above tight lower bound. J. Comput. Syst. Sci., 76:872–878, 2010.

26 G. Gutin, E. J. Kim, S. Szeider, and A. Yeo. A probabilistic approach to problems parameterized above tight lower bound. J. Comput. Syst. Sci. 77: 422–429, 2011.

27 G. Gutin and A. Yeo, Constraint satisfaction problems parameterized above or below tight bounds: a survey. In Fellows Festschrift, Lect. Notes Comput. Sci. 7370 (2012), 257–286.

28 G. Gutin and A. Yeo, Hypercontractive inequality for pseudo-boolean functions of bounded Fourier width. Discrete Appl. Math. 160 (2012), 2323–2328.
29 G. Gutin and A. Yeo, Note on maximal bisection above tight lower bound. Information Proc. Letters. 110 (2010) 966–969.
30 W. Guttmann and M. Maucher. Variations on an ordering theme with constraints. *Proc. 4th IFIP International Conference on Theoretical Computer Science-TCS 2006*, pp. 77–90, Springer.
31 J. Håstad, Some optimal inapproximability results. *J. ACM* 48: 798–859, 2001.
32 A. Isli and A. G. Cohn. A new approach to cyclic ordering of 2D orientations using ternary relation algebras. Artif. Intelligence, 122(1-2):137–187, 2000.
33 S. Jukna, Extremal Combinatorics With Applications in Computer Science, Springer-Verlag, 2001.
34 R. M. Karp, Reducibility among combinatorial problems, *Proc. Complexity of Computer Computations*, Plenum Press, 1972.
35 S. Khot, On the power of unique 2-prover 1-round games. *Proc. STOC 2002*, 767–775.
36 E. J. Kim and R. Williams, Improved parameterized algorithms for constraint satisfaction. *Proc. IPEC 2011*, Lect. Notes Comput. Sci. 7112 (2011) 118–131.
37 S. Kratsch and M. Wahlström, Representative Sets and Irrelevant Vertices: New Tools for Kernelization. In 54th Annual Symposium on Foundations of Computer Science (FOCS), 450–459, 2012.
38 K. J. Lieberherr and E. Specker, Complexity of partial satisfaction. J. ACM, 28(2):411-421, 1981.
39 K. J. Lieberherr and E. Specker, Complexity of partial satisfaction, II. Tech. Report 293, Dept. of EECS, Princeton Univ., 1982.
40 D. Lokshtanov, New Methods in Parameterized Algorithms and Complexity, PhD thesis, Bergen, 2009.
41 M. Mahajan and V. Raman. Parameterizing above guaranteed values: MaxSat and Max-Cut. J. Algorithms, 31(2):335–354, 1999. Preliminary version in Electr. Colloq. Comput. Complex. (ECCC), TR-97-033, 1997.
42 M. Mahajan, V. Raman, and S. Sikdar. Parameterizing above or below guaranteed values. J. Computer System Sciences, 75(2):137–153, 2009. Preliminary version in *Proc. IWPEC 2006*, Lect. Notes Comput. Sci. 4169: 38–49, 2006.
43 K. Makarychev, Y. Makarychev and Y. Zhou, Satisfiability of Ordering CSPs Above Average Is Fixed-Parameter Tractable. In FOCS 2015, 975–993.
44 R. Niedermeier. Invitation to Fixed-Parameter Algorithms. Oxford University Press, 2006.
45 R. O'Donnell, Analysis of Boolean Functions, Cambridge UP, 2014.
46 J. Opatrný, Total ordering problem. SIAM J. Comput., 8: 111–114, 1979.
47 V. Raman, M. S. Ramanujan and S. Saurabh, Paths, Flowers and Vertex Cover. *Proc. ESA 2011*, Lect. Notes Comput. Sci. 6942: 382–393, 2011.
48 I. Razgon and B. O'Sullivan. Almost 2-SAT is fixed-parameter tractable. J. Comput. Syst. Sci. 75(8):435–450, 2009.
49 G. Reinelt, The linear ordering problem: Algorithms and applications, Heldermann Verlag, 1985.
50 S. Szeider, Minimal unsatisfiable formulas with bounded clause-variable difference are fixed-parameter tractable. J. Comput. Syst. Sci., 69(4):656–674, 2004.
51 M. Yannakakis, On the approximation of maximum satisfiability. J. Algorithms, 17:475–502, 1994.