Compatibility of Shelah and Stupp's and Muchnik's iteration with fragments of monadic second order logic

Dietrich Kuske

Institut für Informatik, Universität Leipzig

Abstract. We investigate the relation between the theory of the iterations in the sense of Shelah-Stupp and of Muchnik, resp., and the theory of the base structure for several logics. These logics are obtained from the restriction of set quantification in monadic second order logic to certain subsets like, e.g., finite sets, chains, and finite unions of chains. We show that these theories of the Shelah-Stupp iteration can be reduced to corresponding theories of the base structure. This fails for Muchnik's iteration.

1 Introduction

Rabin's tree theorem states, via an automata-theoretic proof, the decidability of the monadic second order (short: MSO) theory of the complete binary tree. It allows to derive the decidability of seemingly very different theories (e.g., the MSO-theory of the real line where set quantification is restricted to closed sets [12]). Its importance is stressed by Seese's result that any class of graphs of bounded degree with a decidable MSO-theory has bounded tree-width (i.e., is "tree-like") [14].

In [16], Shelah reports a generalization of Rabin's tree theorem that was proved by Shelah and Stupp. The idea is to start with a structure $\mathfrak{A}$ and to consider the tree whose nodes are the finite words over the universe of $\mathfrak{A}$ together with the prefix order on these words. Then the immediate successors of any node in this tree can naturally be identified with the elements of the structure $\mathfrak{A}$ – hence they carry the relations of $\mathfrak{A}$. The resulting tree with additional relations is called *Shelah-Stupp-iteration*. The above mentioned result of Shelah and Stupp states that the MSO-theory of the Shelah-Stupp-iteration can be reduced to the MSO-theory of the base structure $\mathfrak{A}$. If $\mathfrak{A}$ is the two-elements set, then Rabin's tree theorem follows.

A further extension is attributed to Muchnik [15] who added a unary clone predicate to Shelah and Stupp's iteration resulting in the *Muchnik-iteration*. This clone predicate states that the last two letters of a word are the same. This allows, e.g., to define the unfolding of a rooted graph in its Muchnik-iteration [6]. Muchnik's theorem then gives a reduction of the MSO-theory of the Muchnik-iteration to the MSO-theory of the base structure. The proof was not published by Muchnik himself, but, using automata-theoretic methods,

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Walukiewicz showed that the reduction in Muchnik's theorem is even uniform (i.e., independent from the concrete base structure) [18]. Since, as mentioned above, the unfolding of a rooted graph can be defined in the Muchnik-iteration, the MSO-theory of this unfolding can be reduced to that of the graph [6]. This result forms the basis for Caucal's hierarchy [3] of infinite graphs with a decidable MSO-theory. Walukiewicz's automata-theoretic proof ideas have been shown to work for the Muchnik-iteration and stronger logics like Courcelle's counting MSO and guarded second-order logic by Blumensath & Kreutzer [2].

In [11], we asked for a first-order version of Muchnik's result – and failed. More precisely, we constructed structures with a decidable first-order theory whose Muchnik-iteration has an undecidable first-order theory. As it turns out, the only culprit is Muchnik's clone predicate since, on the positive side, we were able to uniformly reduce the first-order theory (and even the monadic chain theory where set variables range over chains, only) of the Shelah-Stupp-iteration to the first-order theory of the base structure.[1]

The aim of this paper is to clarify the role of weak monadic second order logic $\mathrm{MSO}^{\mathrm{w}}$ in the context of Shelah-Stupp- and Muchnik-iteration. We first define infinitary versions of these iterations that contain, in addition to the finite words, also ω-words. On the positive side, we prove a rather satisfactory relation between the theories of the infinitary Shelah-Stupp-iteration and the base structure. More precisely, the Shelah-Stupp result together with some techniques from [12] allows to uniformly reduce the $\mathrm{MSO}^{\mathrm{closed}}$-theory of the infinitary Shelah-Stupp-iteration (where set quantification is restricted to closed sets) to the MSO-theory of the base set. Our result from [11] ensures that Shelah-Stupp-iteration is FO-compatible in the sense of Courcelle (i.e., the FO-theory of the infinitary Shelah-Stupp-iteration can be reduced uniformly to the FO-theory of the base structure). Our new positive result states that Shelah-Stupp-iteration is also $\mathrm{MSO}^{\mathrm{w}}$-compatible. To obtain this result, one first observes that the finiteness of a set in the Shelah-Stupp-iteration is definable in $\mathrm{MSO}^{\mathrm{mch}}$ (where quantification is restricted to finite unions of chains), hence the $\mathrm{MSO}^{\mathrm{w}}$-theory of the Shelah-Stupp-iteration can be reduced to its $\mathrm{MSO}^{\mathrm{mch}}$-theory. For this logic, we then prove a result analogous to Rabin's basis theorem: Any consistent $\mathrm{MSO}^{\mathrm{mch}}$-property in the Shelah-Stupp-iteration of a finite union of chains (i.e., of a certain set of words over the base structure) has a witness that can be accepted by a small automaton. But an automaton over a fixed set of states can be identified with its transition matrix, i.e., with a fixed number of finite sets in the base structure. We then prove that $\mathrm{MSO}^{\mathrm{mch}}$-properties of the language of an automaton can effectively be translated into $\mathrm{MSO}^{\mathrm{w}}$-properties of the transition matrix.

On the negative side, we show that infinitary Muchnik-iteration is not $\mathrm{MSO}^{\mathrm{w}}$-compatible. Namely, there is a tree T_ω with decidable $\mathrm{MSO}^{\mathrm{w}}$-theory such that for any set M of natural numbers, there exists an $\mathrm{MSO}^{\mathrm{w}}$-equivalent tree $\mathfrak{A}_M$ such that M can be reduced to the $\mathrm{MSO}^{\mathrm{w}}$-theory of the infinitary Muchnik-iteration of $\mathfrak{A}_M$. This proof uses the fact that the existence of an infinite branch in a

[1] In the meantime, Alexis Bés [1] found a simpler proof of a stronger result based on the ideas of automatic structures and [17].

tree is not expressible in $\mathrm{MSO}^{\mathrm{w}}$, but it is a first-order (and therefore a $\mathrm{MSO}^{\mathrm{w}}$-) property of the infinitary Muchnik-iteration.

2 Preliminaries

2.1 Logics

A *(relational) signature* σ consists of finitely many constant and relation symbols (together with the arity of the latter); a *purely relational signature* does not contain any constant symbols. Formulas use *individual* and *set variables*, usually denoted by small and capital, resp., letters from the end of the alphabet. *Atomic formulas* are $x_1 = x_2$, $R(x_1, \ldots, x_n)$, and $x_1 \in X$ where R is an n-ary relation symbol from σ, $x_1, x_2, \ldots, x_n$ are individual variables or constant symbols, and X is a set variable. *Formulas* are obtained from atomic formulas by conjunction, negation, and quantification $\exists Z$ for Z an individual or a set variable. A *sentence* is a formula without free variables. The satisfaction relation $\models$ between a σ-structure $\mathfrak{A}$ and formulas is defined as usual. For two σ-structures $\mathfrak{A}$ and $\mathfrak{B}$, we write $\mathfrak{A} \equiv_m^{\mathrm{MSO}} \mathfrak{B}$ if, for any sentence φ of quantifier depth at most m, we have $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$. If $\mathfrak{A}$ and $\mathfrak{B}$ agree on all first-order formulas (i.e., formulas without set quantification) of quantifier depth at most m, then we write $\mathfrak{A} \equiv_m^{\mathrm{FO}} \mathfrak{B}$.

Let $(V, \preceq)$ be a partially ordered set. A set $M \subseteq V$ is a *chain* if $(M, \preceq)$ is linearly ordered, it is a *multichain* if M is a finite union of chains. An element $x \in M$ is a *branching point* if $\{y \in M \mid x < y\}$ is nonempty and does not have a least element.

We will also consider different restrictions of the satisfaction relation $\models$ where set variables range over certain subsets, only. In particular, we will meet the following restrictions.

- Set quantification can be restricted to finite sets, i.e., we will discuss weak monadic second order logic. The resulting satisfaction relation is denoted $\models^{\mathrm{w}}$ and the equivalence of structures $\equiv_m^{\mathrm{w}}$.
- Set quantification can be restricted to chains (where we assume a designated binary relation symbol $\preceq$ in σ) which results in $\models^{\mathrm{ch}}$ and $\equiv_m^{\mathrm{ch}}$, cf. Thomas [17].
- $\models^{\mathrm{mch}}$ etc. refer to the restriction of set quantification to multichains.
- The superscript closed denotes that set variables range over closed sets, only (where we associate a natural topology to any σ-structure), cf. Rabin [12].

Let t be some transformation of σ-structures into τ-structures, e.g., transitive closure. A very strong relation between the $\mathcal{L}$-theory of $\mathfrak{A}$ and the $\mathcal{K}$-theory of $t(\mathfrak{A})$ is the existence of *a single* computable function red that reduces the $\mathcal{K}$-theory of $t(\mathfrak{A})$ to the $\mathcal{L}$-theory of $\mathfrak{A}$ *for any σ-structure* $\mathfrak{A}$. As shorthand for this fact, we say "The transformation t is $(\mathcal{K}, \mathcal{L})$-compatible" or, slightly less precise "The $\mathcal{K}$-theory of $t(\mathfrak{A})$ is *uniformly reducible* to the $\mathcal{L}$-theory of $\mathfrak{A}$." $(\mathcal{K}, \mathcal{K})$-compatible transformations are simply called $\mathcal{K}$-compatible.

Example 1. Any MSO-transduction is MSO-compatible [5] and finite set interpretations are $(\mathrm{MSO}^{\mathrm{w}}, \mathrm{FO})$-compatible [4]. Feferman & Vaught showed that any generalized product is FO-compatible [8]. Finally, any generalized sum is MSO-compatible by Shelah [16].

2.2 Shelah and Stupp's and Muchnik's iteration

Let A be a (not necessarily finite) alphabet. With A^* we denote the set of all finite words over A, A^ω is the set of infinite words, and $A^\infty = A^* \cup A^\omega$. The prefix relation on finite and infinite words is $\preceq$. The set of finite prefixes of a word $u \in A^\infty$ is denoted $\downarrow u = \{v \in A^* \mid v \preceq u\}$, if $C \subseteq A^\infty$, then $\downarrow C = \bigcup_{u \in C} \downarrow u$. For $L \subseteq A^\infty$ and $u \in A^*$ let $u^{-1}L = \{v \in A^\infty \mid uv \in L\}$ denote the left-quotient of L with respect to u.

Let σ be a relational signature and let $\mathfrak{A} = (A, (R^{\mathfrak{A}})_{R \in \sigma})$ be a structure over the signature σ. The *infinitary Shelah-Stupp-iteration* $\mathfrak{A}^\infty$ *of* $\mathfrak{A}$ is the structure

$$\mathfrak{A}^\infty = (A^\infty, \preceq, (\widehat{R})_{R \in \sigma}, \varepsilon)$$

where, for $R \in \sigma$,

$$\widehat{R} = \{(ua_1, \ldots, ua_n) \mid u \in A^*, (a_1, \ldots, a_n) \in R^{\mathfrak{A}}\} \ .$$

The *(finitary) Shelah-Stupp-iteration* $\mathfrak{A}^*$ is the restriction of $\mathfrak{A}^\infty$ to the set of finite words A^*.

Example 2. Suppose the structure $\mathfrak{A}$ has two elements a and b and two unary relations $R_1 = \{a\}$ and $R_2 = \{b\}$. Then $\widehat{R}_1 = \{a, b\}^* a$ and $\widehat{R}_2 = \{a, b\}^* b$. Hence the finitary Shelah-Stupp-iteration $\mathfrak{A}^*$ can be visualized as a complete binary tree with unary predicates telling whether the current node is the first or the second son of its father. In addition, the root ε is a constant of the Shelah-Stupp-iteration $\mathfrak{A}^*$. Furthermore, the infinitary Shelah-Stupp-iteration $\mathfrak{A}^\infty$ adds leaves to this tree at the end of any branch. Since this allows to define $(\mathbb{R}, \leq)$ in $\mathfrak{A}^\infty$, the unrestricted MSO-theory of $\mathfrak{A}^\infty$ is undecidable.

Example 3. **(cf. [10])** The Shelah-Stupp iteration allows to reduce the Cayley graph of a free product to the Cayley graphs of the factors. Let $M_i = (M_i, \circ_i, 1_i)$ be monoids finitely generated by Γ_i for $1 \leq i \leq n$ and let $G_i = (M_i, (E_i^a)_{a \in \Gamma_i}, \{1_i\})$ denote the rooted Cayley graph of M_i. Then the Cayley graph $G = (P, (E^a)_{a \in \bigcup \Gamma_i})$ of the free product $P = (P, \circ, 1)$ of these monoids can be defined in the Shelah-Stupp iteration of the disjoint union of the Cayley graphs G_i. For this to work, let $M = \bigcup_{1 \leq i \leq n} M_i$ be the disjoint union of the monoids M_i and consider the structure

$$\mathcal{A} = (M, (M_i)_{1 \leq i \leq n}, (E_i^a)_{\substack{1 \leq i \leq n \\ a \in \Gamma_i}}, U)$$

where $U = \{1_i \mid 1 \leq i \leq n\}$ is the set of units.

Then a word $w \in M^*$ belongs to the direct product P iff the following holds in the Shelah-Stupp iteration of $\mathcal{A}$:

$$\bigwedge_{1 \le i \le n} \forall x \lessdot y \preceq w : x \in \widehat{M_i} \to y \notin \widehat{M_i} \wedge y \notin \widehat{U}$$

where $\lessdot$ denotes the immediate successor relation of the partial order $\preceq$. For $a \in \Gamma_i$ and $v, w \in P$, we have $v \circ a = w$ (i.e., $(v, w) \in E^a$) iff the Shelah-Stupp iteration satisfies

$$\begin{aligned} &\left(\exists v' \in \widehat{U} : v \lessdot v' \wedge (v', w) \in \widehat{E_i^a}\right) \vee (v, w) \in \widehat{E_i^a} \\ \vee &\left(\exists w' \in \widehat{U} : w \lessdot w' \wedge (v, w') \in \widehat{E_i^a}\right) \ . \end{aligned}$$

Muchnik introduced the additional unary *clone predicate* $\mathrm{cl} = \{uaa \mid u \in A^*, a \in A\}$. The extension of the Shelah-Stupp-iterations by this clone predicate will be called *finitary and infinitary Muchnik-iteration* $(\mathfrak{A}^*, \mathrm{cl})$ and $(\mathfrak{A}^\infty, \mathrm{cl})$, resp. Courcelle and Walukiewicz [6] showed that the unfolding of a directed rooted graph G can be defined in the Muchnik iteration (G^*, cl) of G.

To simplify notation, we will occasionally omit the word "finitary" and just speak of the Shelah-Stupp- and Muchnik-iteration.

3 A basis theorem for MSO$^{\mathrm{mch}}$

Rabin's tree theorem [12] states the decidability of the monadic second order theory of the complete binary tree. As a *corollary* of his proof technique by tree automata, one obtains Rabin's basis theorem [13, Theorem 26]: Let φ be a formula with free variables $X_1, \ldots, X_\ell$ and let $L_1, \ldots, L_\ell \subseteq \{a, b\}^*$ be regular languages such that the binary tree satisfies $\varphi(L_1, \ldots, L_\ell)$. Then it satisfies $\psi(L_1, \ldots, L_\ell)$ where ψ is obtained from φ by restricting all quantifications to regular sets. To obtain this basis theorem, it suffices to show that validity of $\exists X_\ell : \varphi(L_1, \ldots, L_{\ell-1}, X_\ell)$ implies the existence of a regular set R_ℓ such that $\varphi(L_1, \ldots, L_{\ell-1}, R_\ell)$ holds true in the binary tree.

This is precisely what this section shows in our context of MSO$^{\mathrm{mch}}$ and the Shelah-Stupp-iteration $\mathfrak{A}^*$. Even more, we will not only show that the set R_ℓ can be chosen regular, but we will also bound the size of the automaton accepting it.

Throughout this section, σ denotes some purely relational signature.

3.1 Preliminaries

For $k, \ell \in \mathbb{N}$, let $\tau_{k,\ell}$ be the extension of the signature $(\sigma, \preceq)$ by k constants and ℓ unary relations. Using Hintikka-formulas (see [7] for the definition and properties of these formulas) one can show that for any of the signatures $\tau_{k,\ell}$ and $m \in \mathbb{N}$, there are only finitely many equivalence classes of $\equiv_m^{\mathrm{mch}}$. An upper bound $T(\ell, m)$ for the number of equivalence classes of $\equiv_m^{\mathrm{mch}}$ on formulas over the signature $\tau_{2,\ell}$ can be computed effectively.

Now let $\mathfrak{A} = (A, (R)_{R\in\sigma})$ be some σ-structure. For $u \in A^*$, let $\mathfrak{A}_u^*$ denote the $\tau_{1,0}$-structure $(uA^*, \sqsubseteq, (\overline{R})_{R\in\sigma}, u)$ where

- the relation $\sqsubseteq$ is the restriction of $\preceq$ to uA^* and
- $\overline{R}$ is the restriction of $\widehat{R}$ to uA^+.

For any $u, v \in A^*$, the mapping $f : \mathfrak{A}_u^* \to \mathfrak{A}_v^*$ with $f(ux) = vx$ is an isomorphism – this is the reason to consider $\overline{R}$ and not the restriction of $\widehat{R}$ to uA^*. Similarly, the $\tau_{2,0}$-structure $\mathfrak{A}_{u,v}^* = (uA^* \setminus vA^+, \sqsubseteq, (\overline{R})_{R\in\sigma}, u, v)$ is defined for $u, v \in A^*$ with $u \preceq v$. Here, again, $\overline{R}$ is the restriction of $\widehat{R}$ to $uA^+ \setminus vA^+$.

Frequently, we will consider the structure $\mathfrak{A}^*$ together with some additional unary predicates $L_1, \dots, L_\ell$. As for the plain structure $\mathfrak{A}^*$, we will also meet the restriction of $(\mathfrak{A}^*, L_1, \dots, L_\ell)$ to the set uA^*, i.e., the structure $(\mathfrak{A}_u^*, L_1 \cap uA^*, \dots, L_\ell \cap uA^*)$. To simplify notation, this will be denoted $(\mathfrak{A}_u^*, L_1, \dots, L_\ell)$; the structure $(\mathfrak{A}_{u,v}^*, L_1, \dots, L_\ell)$ is to be understood similarly.

Example 2 (continued). In the case of Example 2, $\mathfrak{A}_u^*$ is just the subtree rooted at the node u. On the other hand, $\mathfrak{A}_{u,v}^*$ is obtained from $\mathfrak{A}_u^*$ by deleting all descendants of v and marking the node v as a constant. Thus, we can think of $\mathfrak{A}_{u,v}^*$ as a tree with a marked leaf. These *special trees* are fundamental in the work of Gurevich & Shelah [9] and of Thomas [17].

In the following, fix some $\ell \in \mathbb{N}$. We then define the operations of product and infinite product of $\tau_{k,\ell}$-structures: If $\mathfrak{A} = (A, \preceq^{\mathfrak{A}}, (R^{\mathfrak{A}})_{R\in\sigma}, a_1, a_2, L_1^{\mathfrak{A}}, \dots, L_\ell^{\mathfrak{A}})$ is a $\tau_{2,\ell}$-structure and $\mathfrak{B} = (B, \preceq^{\mathfrak{B}}, (R^{\mathfrak{B}})_{R\in\sigma}, b_1, \dots, b_k, L_1^{\mathfrak{B}}, \dots, L_\ell^{\mathfrak{B}})$ a disjoint $\tau_{k,\ell}$-structure with $k \geq 1$, then their *product* $\mathfrak{A} \cdot \mathfrak{B}$ is a $\tau_{k,\ell}$-structure. It is obtained from the structure

$$(A \cup B, \preceq^{\mathfrak{A}} \cup \preceq^{\mathfrak{B}}, (R^{\mathfrak{A}} \cup R^{\mathfrak{B}})_{R\in\sigma}, L_1^{\mathfrak{A}} \cup L_2^{\mathfrak{B}}, \dots, L_\ell^{\mathfrak{A}} \cup L_\ell^{\mathfrak{B}})$$

by identifying a_2 and b_1, taking the transitive closure of the partial orders, and extending the resulting structure by the list of constants $a_1, b_2, b_3, \dots, b_k$. Now let $\mathfrak{A}_n$ be disjoint $\tau_{2,\ell}$-structures with constants u_n and v_n for $n \in \mathbb{N}$. Then the *infinite product* $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$ is a $\tau_{1,\ell}$-structure. It is obtained from the disjoint union of the structures $\mathfrak{A}_n$ by identifying v_n and u_{n+1} for any $n \in \mathbb{N}$. The only constant of this infinite product is u_0. If $\mathfrak{A} \cong \mathfrak{A}_n$ for all $n \in \mathbb{N}$, then we write simply $\mathfrak{A}^\omega$ for the infinite product of the structures $\mathfrak{A}_n$.

Standard applications of Ehrenfeucht-Fraïssé-games (see [7]) yield:

Proposition 1. *Let $j, \ell, m \in \mathbb{N}$, $\mathfrak{A}_n, \mathfrak{A}'_n$ be $\tau_{2,\ell}$-structures for $n \in \mathbb{N}$ and let $\mathfrak{B}, \mathfrak{B}'$ be some $\tau_{j+1,\ell}$-structures such that $\mathfrak{A}_n \equiv_m^{\mathrm{mch}} \mathfrak{A}'_n$ for $n \in \mathbb{N}$ and $\mathfrak{B} \equiv_m^{\mathrm{mch}} \mathfrak{B}'$. Then*

$$\mathfrak{A}_0 \cdot \mathfrak{B} \equiv_m^{\mathrm{mch}} \mathfrak{A}'_0 \cdot \mathfrak{B}' \quad \textit{and} \quad \prod_{n\in\mathbb{N}} \mathfrak{A}_n \equiv_m^{\mathrm{mch}} \prod_{n\in\mathbb{N}} \mathfrak{A}'_n \ .$$

Remark 1. We sketch a typical use of the above proposition in this section. Let $x \in A^*$ be some sufficiently long word. Since $\equiv_m^{\mathrm{mch}}$ has only finitely many equivalence classes, there exist words u, v, w with $x = uvw$ and $v \neq \varepsilon$ such that $(\mathfrak{A}^*_u, \{x\}) \equiv_m^{\mathrm{mch}} (\mathfrak{A}^*_{uv}, \{x\})$. Hence we obtain

$$(\mathfrak{A}^*, \{x\}) = (\mathfrak{A}^*_{\varepsilon,u}, \emptyset) \cdot (\mathfrak{A}^*_u, \{uvw\}) \equiv_m^{\mathrm{mch}} (\mathfrak{A}^*_{\varepsilon,u}, \emptyset) \cdot (\mathfrak{A}^*_{uv}, \{uvw\}) \cong (\mathfrak{A}^*, \{uw\}) \ .$$

(This proves that every consistent property of a single element of $\mathfrak{A}^*$ is witnessed by some "short" word.)

The last isomorphism does not hold for the Muchnik-iteration since the clone predicate allows to express that the last letter of u and the first letter of v are connected by some edge in the graph $\mathfrak{A}$.

Convention We speak of *automata* when we actually mean complete deterministic finite automata $\mathcal{M} = (Q, B, \iota, \delta, F)$. Its language is denoted $L(\mathcal{M})$. We will also write $p.w$ for $\delta(p, w)$. The *transition matrix* of $\mathcal{M}$ is the tuple $T = (T_{p,q})_{p,q \in Q}$ with $T_{p,q} = \{b \in B \mid \delta(p, b) = q\}$.

As explained above, we will use automata to describe subsets of the Shelah-Stupp iteration $\mathfrak{A}^*$, i.e., the alphabet B will always be a finite subset of the universe of $\mathfrak{A}$. These regular subsets have the following nice property whose proof is obvious.

Lemma 1. *Let $\mathfrak{A}$ be a σ-structure with universe A and let $\mathcal{M} = (Q, B, \iota, \delta, F)$ be an automaton with alphabet $B \subseteq A$. Then, for any $u, v \in B^*$ with $\delta(\iota, u) = \delta(\iota, v)$, the mapping $f_{u,v} : uA^* \to vA^* : ux \mapsto vx$ is an isomorphism from $(\mathfrak{A}^*_u, L(\mathcal{M}))$ onto $(\mathfrak{A}^*_v, L(\mathcal{M}))$.*

Consequently, the number of isomorphism classes of structures $(\mathfrak{A}^*_v, L(\mathcal{M}))$ is finite. This fails in the Muchnik-iteration even for $L(\mathcal{M}) = \emptyset$: With $\mathfrak{A} = (\mathbb{N}, \mathrm{succ})$ and $m, n \in \mathbb{N}$, we have $(\mathfrak{A}^*_m, \mathrm{cl}) \cong (\mathfrak{A}^*_n, \mathrm{cl})$ iff $m = n$ since the structure $(\mathbb{N}, \mathrm{succ}, m)$ can be defined in $(\mathfrak{A}^*_m, \mathrm{cl})$.

3.2 Quantification

While multichains in the Shelah-Stupp-iteration can be rather complicated, this section shows that, up to logical equivalence, we can restrict attention to "simple" multichains. Here, "simple" means that they are regular and, even more, can be accepted by a "small" automaton.

For the rest of this section, let $\mathfrak{A} = (A, (R)_{R \in \sigma})$ be some fixed σ-structure and $\ell, m \in \mathbb{N}$. For $1 \leq i \leq \ell$, let $\mathcal{M}_i = (Q_i, B_i, \iota_i, F_i)$ be automata with $B_i \subseteq A$ such that $L(\mathcal{M}_i) \subseteq A^*$ is a multichain in the Shelah-Stupp iteration $\mathfrak{A}^*$. Write $\overline{L}$ for the tuple of multichains $(L(\mathcal{M}_1), \ldots, L(\mathcal{M}_\ell))$.

Proposition 2. *Let $C \subseteq A^*$ be a chain. Then there exist $u, v \in A^*$, $E \subseteq {\downarrow}u \setminus \{u\}$, and $F \subseteq {\downarrow}v \setminus \{v\}$ such that $\iota_i.u = \iota_i.uv$ for all $1 \leq i \leq \ell$ and $(\mathfrak{A}^*, \overline{L}, C) \equiv_m^{\mathrm{mch}} (\mathfrak{A}^*, \overline{L}, D)$ with $D = E \cup uv^*F$.*

Proof. One shows the existence of $u_1 \prec u_2 \in A^*$ such that $C \cup \{u_1, u_2\}$ is a chain, $\iota_i.u_1 = \iota_i.u_2$ for all $1 \leq i \leq \ell$, $(\mathfrak{A}^*, \overline{L}) \cong (\mathfrak{A}^*_{\varepsilon,u_1}, \overline{L}) \cdot (\mathfrak{A}^*_{u_1,u_2}, \overline{L})^\omega$, and $(\mathfrak{A}^*, \overline{L}, C) \equiv^{\text{mch}}_m (\mathfrak{A}^*_{\varepsilon,u_1}, \overline{L}, C) \cdot (\mathfrak{A}^*_{u_1,u_2}, \overline{L}, C)^\omega$. This uses arguments similar to those in Remark 1 and Ramsey's theorem. The result follows with $u = u_1$, $uv = u_2$, $E = C \cap {\downarrow}u \setminus \{u\}$, and $F = u^{-1}(C \cap {\downarrow}u_2 \setminus \{u_2\})$.

The above proposition shows that every consistent property of a chain is witnessed by some regular chain D. Using the pigeonhole principle and arguments as in Remark 1, one can bound the lengths of u and v to obtain

Proposition 3. *Let $C \subseteq A^*$ be a chain. Then there exists an automaton $\mathcal{N}$ with at most $2\prod_{1\leq i\leq \ell} |Q_i| \cdot T(\ell+1, m)$ states such that $L(\mathcal{N})$ is a chain and $(\mathfrak{A}^*, \overline{L}, C) \equiv^{\text{mch}}_m (\mathfrak{A}^*, \overline{L}, L(\mathcal{N}))$.*

It is our aim to prove a similar result for arbitrary multichains in place of the chain C in the proposition above. Certainly, in order to get a small automaton for a multichain, the branching points of this set have to be short words. Again using arguments as in Remark 1, one obtains

Lemma 2. *Let $M \subseteq A^*$ be a multichain. Then there exists a multichain $N \subseteq A^*$ such that*

- $(\mathfrak{A}^*, \overline{L}, M) \equiv^{\text{mch}}_m (\mathfrak{A}^*, \overline{L}, N)$ *and*
- *any branching point of N has length at most $k = \prod_{1\leq i\leq \ell}(|Q_i|+1)\cdot T(\ell+1, m)$.*

Lemma 3. *Let M be a multichain such that all branching points of M have length $\leq s-1$. Then there exists an automaton $\mathcal{N}$ with at most $(2\prod_{1\leq i\leq \ell} |Q_i| \cdot T(\ell+1, m))^{s+1}$ many states such that $L(\mathcal{N})$ is a multichain and $(\mathfrak{A}^*, \overline{L}, M) \equiv^{\text{mch}}_m (\mathfrak{A}^*, \overline{L}, L(\mathcal{N}))$.*

Proof. Let $n = \prod_{1\leq i\leq \ell} |Q_i|$ and $\overline{L} = (L(\mathcal{M}_1), \ldots, L(\mathcal{M}_\ell))$.

The lemma is shown by induction on s. If $s = 0$, then M is a chain, i.e., the result follows from Prop. 3.

Now let M be a multichain such that any branching point has length at most $s > 0$. By the induction hypothesis, for every $a \in A$, there exists an automaton $\mathcal{N}_a$ with at most $(2nT(\ell+1, m))^{s+1}$ many states such that $L(\mathcal{N}_a)$ is a multichain and

$$(\mathfrak{A}^*_a, \overline{L}, M) \equiv^{\text{mch}}_m (\mathfrak{A}^*, a^{-1}L(\mathcal{M}_1), \ldots, a^{-1}L(\mathcal{M}^a_\ell), L(\mathcal{N}_a)) \ .$$

Let θ be the equivalence relation on A with $(a, b) \in \theta$ if and only if

1. $\delta_i(\iota_i, a) = \delta_i(\iota_i, b)$ for all $1 \leq i \leq \ell$ and
2. $(\mathfrak{A}^*_a, \overline{L}, M) \equiv^{\text{mch}}_m (\mathfrak{A}^*_b, \overline{L}, M)$.

Let $H \subseteq A$ contain precisely one element h from any θ-equivalence class. Then the set $\bigcup\{aL(\mathcal{N}_h) \mid a\,\theta\, h \in H \text{ and } a^{-1}M \neq \emptyset\} \cup (\{\varepsilon\} \cap M)$ is a multichain and can be accepted by some automaton $\mathcal{N}$ with the right number of states.

Then $(\mathfrak{A}^*, \overline{L}, L(\mathcal{N}))$ is obtained from $(\mathfrak{A}^*, \overline{L}, M)$ by replacing any subtree $(\mathfrak{A}^*_a, \overline{L}, M)$ with the equivalent structure $(\mathfrak{A}^*, a^{-1}L(\mathcal{M}_1), \ldots, a^{-1}L(\mathcal{M}^a_\ell), L(\mathcal{N}_h))$ for $a\,\theta h \in H$. Hence, by Prop. 1, $(\mathfrak{A}^*, \overline{L}, M) \equiv^{\text{mch}}_m (\mathfrak{A}^*, \overline{L}, L(\mathcal{N}))$.

Putting these two lemmas together, we obtain that, indeed, every consistent property of a multichain M is witnessed by some multichain that can be accepted by some "small" automaton:

Proposition 4. *Let $M \subseteq A^*$ be some multichain. Then there exists an automaton $\mathcal{N}$ with at most $(2nT(\ell{+}1,m))^{s+1}$ many states (where $s = n{\cdot}T(\ell{+}1,m)$, $n = \prod_{1\le i\le \ell} |Q_i|$) such that $L(\mathcal{N})$ is a multichain and $(\mathfrak{A}^*, \overline{L}, M) \equiv_m^{\mathrm{mch}} (\mathfrak{A}^*, \overline{L}, L(\mathcal{N}))$.*

Now a result analogous to Rabin's basis theorem follows immediately

Theorem 1. *Let $\mathfrak{A}$ be a σ-structure, let φ be an $\mathrm{MSO}^{\mathrm{mch}}$-formula in the language of the Shelah-Stupp-iteration $\mathfrak{A}^*$ with free variables $X_1, \ldots, X_\ell$ and let $L_1, \ldots, L_\ell \subseteq A^*$ be regular languages such that $(\mathfrak{A}^*, L_1, \ldots, L_\ell) \models^{\mathrm{mch}} \varphi$. Then $(\mathfrak{A}^*, L_1, \ldots, L_\ell) \models^{\mathrm{reg-mch}} \varphi$ where $\models^{\mathrm{reg-mch}}$ denotes that set quantification is restricted to regular multichains.*

Recall that Rabin's basis theorem follows from his tree theorem whose proof, in turn, uses the effective complementation of Rabin tree automata. While the above theorem is an analogue of Rabin's basis theorem, the proof is more direct and does in particular not rest on any complementation of automata.

4 Shelah-Stupp-iteration is ($\mathrm{MSO}^{\mathrm{mch}}$, $\mathrm{MSO}^{\mathrm{w}}$)-compatible

The results of the previous section, as explained at the beginning, imply that quantification in an $\mathrm{MSO}^{\mathrm{mch}}$-sentence can be restricted to regular sets that are accepted by "small" automata. In this section, we will use this insight to reduce the $\mathrm{MSO}^{\mathrm{mch}}$-theory of the Shelah-Stupp-iteration to the $\mathrm{MSO}^{\mathrm{w}}$-theory of the base structure.

Fix some σ-structure $\mathfrak{A}$ with universe A, some finite set of states Q, some initial state ι, and some set of final states $F \subseteq Q$. Then, for any automaton $\mathcal{M} = (Q, B, \iota, \delta, F)$ with $B \subseteq A$, the language $L(\mathcal{M})$ is a set in the Shelah-Stupp-iteration $\mathfrak{A}^*$ while its transition matrix is a tuple of finite sets in the base structure $\mathfrak{A}$. *The idea of our reduction is that $\mathrm{MSO}^{\mathrm{mch}}$-properties of the set $L(\mathcal{M})$ in the Shelah-Stupp-iteration $\mathfrak{A}^*$ can (effectively) be translated into $\mathrm{MSO}^{\mathrm{w}}$-properties of the transition matrix T in the base structure $\mathfrak{A}$.*

In precisely this spirit, the following lemma expresses simple properties of the automaton $\mathcal{M}$ and of the language $L(\mathcal{M})$ in terms of FO-properties of $(\mathfrak{A}, T) = (\mathfrak{A}, (T_{p,q})_{p,q\in Q})$.

Lemma 4. *Let $F \subseteq Q$ be finite sets and $\iota \in Q$. There exist formulas $\mathrm{reach}_{(Q,p,q)}$ for $p, q \in Q$ and $\mathrm{mchain}_{(Q,\iota,F)}$ of FO with free variables $T_{p,q}$ for $p, q \in Q$ such that for any σ-structure $\mathfrak{A}$ and any automaton $\mathcal{M} = (Q, B, \iota, \delta, F)$ with transition matrix T:*

(1) $(\mathfrak{A}, T) \models^{\mathrm{w}} \mathrm{reach}_{(Q,p,q)}$ iff there exists a word $w \in A^*$ with $\delta(p, w) = q$.
(2) $(\mathfrak{A}, T) \models^{\mathrm{w}} \mathrm{mchain}_{(Q,\iota,F)}$ iff $L(\mathcal{M})$ is a multichain.

Proof. The proof is based on the observation that (1) one only needs to search for a path of length at most $|Q|$ and (2) that $L(\mathcal{M})$ is a multichain iff no branching point belongs to some cycle.

So far, we showed that simple properties of $L(\mathcal{M})$ are actually FO- (and therefore $\mathrm{MSO}^{\mathrm{w}}$-) properties of the transition matrix of $\mathcal{M}$. We now push this idea further and consider arbitrary $\mathrm{MSO}^{\mathrm{mch}}$-properties of a tuple of languages $L(\mathcal{M}_1), \ldots, L(\mathcal{M}_\ell)$.

Theorem 2. *There is an algorithm with the following specification*

input:
- $\ell \in \mathbb{N}$,
- *finite sets* $F_i \subseteq Q_i$ *and states* $\iota_i \in Q_i$ *for* $1 \le i \le \ell$,
- *and a formula* α *with free variables among* $L_1, \ldots, L_\ell$ *in the language of the Shelah-Stupp-iteration* $\mathfrak{A}^*$.

output: *A formula* $\alpha_{(\overline{Q},\overline{\iota},\overline{F})}$ *in the language of* $\mathfrak{A}$ *with free variables among* $T^i_{p,q}$ *for* $p, q \in Q_i$ *and* $1 \le i \le \ell$ *with the following property:*
If $\mathfrak{A}$ *is a* σ*-structure and* $\mathcal{M}_i = (Q_i, B_i, \iota_i, T^i, F_i)$ *are automata with* $B_i \subseteq A$ *for* $1 \le i \le \ell$*, then*

$$(\mathfrak{A}^*, L(\mathcal{M}_1), L(\mathcal{M}_2), \ldots, L(\mathcal{M}_\ell)) \models^{\mathrm{mch}} \alpha$$
$$\iff (\mathfrak{A}, T^1, T^2, \ldots, T^\ell) \models^{\mathrm{w}} \alpha_{(\overline{Q},\overline{\iota},\overline{F})} \; .$$

Proof. The proof proceeds by induction on the construction of the formula α, we only sketch the most interesting part $\alpha = \exists X\, \beta$. Set $n = \prod_{1 \le i \le \ell} |Q_i|$, $s = nT(\ell+1, m)$, and $k = (2nT(\ell+1,m))^{s+1}$. Let $\mathfrak{A}$ be a σ-structure and let $\mathcal{M}_i = (Q_i, B_i, \iota_i, \delta_i, F_i)$ be automata with $B_i \subseteq A$ and transition matrix T^i. Then, by Prop. 4, $(\mathfrak{A}^*, L(\mathcal{M}_1), \ldots, L(\mathcal{M}_\ell)) \models^{\mathrm{mch}} \alpha$ iff there exists an automaton $\mathcal{N}$ with k states such that

$$(\mathfrak{A}^*, L(\mathcal{M}_1), L(\mathcal{M}_2), \ldots, L(\mathcal{M}_\ell), L(\mathcal{N})) \models^{\mathrm{mch}} \beta \; .$$

Using the induction hypothesis on β and $\beta_{(\overline{Q},\overline{\iota},\overline{F})}$, this is the case if and only if there exist finite sets $T^{\ell+1}_{i,j}, B \subseteq A$ for $i, j \in [k] = \{1, 2, \ldots, k\}$ such that

- $T^{\ell+1}$ forms the transition matrix of some automaton with alphabet B
- for some $F \subseteq [k]$, the automaton $\mathcal{M}_{\ell+1} = ([k], B, 1, T^{\ell+1}, F)$
  - accepts a multichain M (i.e., $(\mathfrak{A}, T^{\ell+1}) \models^{\mathrm{w}} \mathrm{mchain}_{([k],1,F)}$) and
  - this multichain satisfies β (i.e., $\mathfrak{A}, T^1, \ldots, T^{\ell+1} \models^{\mathrm{w}} \beta_{((\overline{Q},[k]),(\overline{\iota},1),(\overline{F},F))}$).

Since all these properties can be expressed in $\mathrm{MSO}^{\mathrm{w}}$, the construction of $\alpha_{(\overline{Q},\overline{\iota},\overline{F})}$ is complete.

As an immediate consequence, we get a uniform version of Shelah and Stupp's theorem for the logics $\mathrm{MSO}^{\mathrm{w}}$ and $\mathrm{MSO}^{\mathrm{mch}}$:

Theorem 3. *Finitary Shelah-Stupp-iteration is* $(\mathrm{MSO}^{\mathrm{mch}}, \mathrm{MSO}^{\mathrm{w}})$*-compatible.*

Remark 2. $(\mathrm{MSO}^{\mathrm{ch}}, \mathrm{FO})$-compatibility of Shelah-Stupp-iteration [11] can alternatively be shown along the same lines: One allows incomplete automata and proves an analogue of Prop. 3 for the logic $\mathrm{MSO}^{\mathrm{ch}}$. Then Theorem 3 can be shown for the pair of logics $(\mathrm{MSO}^{\mathrm{ch}}, \mathrm{FO})$.

5 Infinitary Muchnik-iteration is not $(\mathrm{FO}, \mathrm{MSO}^{\mathrm{w}})$-compatible

Our argument goes as follows: From a set $M \subseteq \mathbb{N}$, we construct a tree $\mathfrak{A}_M$. The $\mathrm{MSO}^{\mathrm{w}}$-theory of this tree will be independent from M and M will be FO-definable in the infinitary Muchnik-iteration $(\mathfrak{A}_M^\infty, \mathrm{cl})$. Assuming $(\mathrm{FO}, \mathrm{MSO}^{\mathrm{w}})$-compatibility of the infinitary Muchnik-iteration, the set M will be reduced uniformly to the $\mathrm{MSO}^{\mathrm{w}}$-theory of $\mathfrak{A}_M$. For $M \neq N$, this yields a contradiction.

A *tree* is a structure $(V, \preceq, r)$ where $\preceq$ is a partial order on V such that, for any $v \in V$, $(\downarrow v, \preceq)$ is a finite linear order and $r \preceq v$ for all $v \in V$.

We will consider the set $T_\omega = \{(a_1, m_1)(a_2, m_2) \dots (a_k, m_k) \in (\mathbb{N} \times \mathbb{N})^* \mid m_1 > m_2 > m_3 \cdots > m_k\}$ of sequences in $\mathbb{N}^2$ whose second components decrease. This set, together with the prefix relation $\preceq$, forms a tree $(T_\omega, \preceq, \varepsilon)$ with root ε that we also denote T_ω. Nodes of the form $w(a, 0)$ are leaves of T_ω. Any inner node of T_ω has infinitely many children (among them, there are infinitely many leaves). Furthermore, all the branches of T_ω are finite. Even more, if x is a node different from the root, then the branches passing through x have bounded length.

We will also consider the set $T_\infty = a^* T_\omega$ where a is an arbitrary symbol. Together with the prefix relation, this yields another tree $(T_\infty, \preceq, \varepsilon)$ that we denote T_∞. Differently from T_ω, it has an infinite branch, namely the set of all nodes a^n for $n \in \mathbb{N}$.

For two trees S and T and a node v of S, let $S \cdot_v T$ denote the tree obtained from the disjoint union of S and T by identifying v with the root of T (i.e., the node v gets additional children, namely the children of the root in T).

It is important for our later arguments that this operation transforms trees equivalent wrt. $\equiv_m^{\mathrm{w}}$ into equivalent structures. More precisely

Proposition 5. *Let S, T, and T' be trees and $k \in \mathbb{N}$ such that $T \equiv_k^{\mathrm{w}} T'$. Then $S \cdot_v T \equiv_k^{\mathrm{w}} S \cdot_v T'$ for any node v of S.*

With $a^{\leq n} = \{\varepsilon, a, a^2, \dots, a^n\}$, the set $a^{\leq n} T_\omega$ together with the prefix relation and the root, is considered as a tree that we denote $a^{\leq n} T_\omega$.

Proposition 6. *For any $k \in \mathbb{N}$, we have $T_\omega \equiv_k^{\mathrm{w}} T_\infty$.*

Proof. The statement is shown by induction on k where the base case $k = 0$ is trivial. To show $T_\omega \equiv_{k+1}^{\mathrm{w}} T_\infty$, it suffices to prove for any formula $\varphi(X)$ of quantifier-depth at most k

$$T_\omega \models^{\mathrm{w}} \exists X\, \varphi(X) \iff T_\infty \models^{\mathrm{w}} \exists X\, \varphi(X) \ .$$

Assuming $T_\infty \models^{\mathrm{w}} \exists X\, \varphi$, there exist $n \in \mathbb{N}$ and $M \subseteq a^{\leq n} T_\omega$ finite with $(T_\infty, M) \models^{\mathrm{w}} \varphi$. Hence we have

$$\begin{aligned}(T_\infty, M) &\cong (a^{\leq n} T_\omega, M) \cdot_{a^n} (T_\infty, \emptyset)\\ &\equiv_k^{\mathrm{w}} (a^{\leq n} T_\omega, M) \cdot_{a^n} (T_\omega, \emptyset) \text{ by Prop. 5 and the induction hypothesis}\end{aligned}$$

$$\cong (a^{\leq n}T_\omega, M) \ .$$

Hence $(a^{\leq n}T_\omega, M) \models^{\mathrm{w}} \varphi$ and therefore $a^{\leq n}T_\omega \models^{\mathrm{w}} \exists X\, \varphi$. Using $T_\omega \equiv^{\mathrm{w}}_{k+1} a^{\leq n}T_\omega$ (see complete paper for the proof), we obtain $T_\omega \models^{\mathrm{w}} \exists X\, \varphi$.

Conversely, one can argue similarly again using $T_\omega \equiv^{\mathrm{w}}_{k+1} a^{\leq n}T_\omega$.

Remark 3. This proves that the existence of an infinite path cannot be expressed in weak monadic second order logic since T_∞ has such a path and T_ω does not.

Using an idea from [6], the existence of an infinite path is a first-order property of the infinitary Muchnik-iteration. The following lemma pushes this idea a bit further:

Lemma 5. *Let $T = (T, \leq, r)$ be a tree and let $U \subseteq T$ be the union of all infinite branches of T. Then the* $\mathrm{MSO}^{\mathrm{w}}$*-theory of $(T, \leq, r, U)$ is uniformly reducible to the* $\mathrm{MSO}^{\mathrm{w}}$*-theory of the infinitary Muchnik-iteration* (T^∞, cl) *of the tree* $(T, \leq, r)$ *without the extra predicate.*

For $M \subseteq \mathbb{N}$, let $A_M = \{b^m \mid m \in M\}T_\infty \cup \{b^m \mid m \notin M\}T_\omega$ and $\mathfrak{A}_M = (A_M, \preceq, \varepsilon)$. Then $\mathfrak{A}_M$ is obtained from the linear order $(\mathbb{N}, \leq) \cong (b^*, \preceq)$ by attaching the tree T_∞ to elements from M and the tree T_ω to the remaining numbers.

Theorem 4. *For $M \subseteq \mathbb{N}$, we have $\mathfrak{A}_M \equiv^{\mathrm{w}}_k T_\omega$ for all $k \in \mathbb{N}$, and M can be reduced to the* FO*-theory of the infinitary Muchnik-iteration* $(\mathfrak{A}_M^\infty, \mathrm{cl})$.

Proof. Using Ehrenfeucht-Fraïssé-games and Prop. 6, one obtains

$$\mathfrak{A}_M \equiv^{\mathrm{w}}_k (b^*T_\omega, \preceq, \varepsilon) \cong T_\infty \equiv^{\mathrm{w}}_k T_\omega \ .$$

For the second statement, it suffices, by Lemma 5, to reduce M to the first-order theory of $(A_M, \preceq, \varepsilon, U)$ where $U = b^* \cup \{b^m \mid m \in M\}a^*$ is the set of nodes of the tree $\mathfrak{A}_M$ that belong to some infinite branch.

If a transformation t is $(\mathrm{FO}, \mathrm{MSO}^{\mathrm{w}})$-compatible, then for any structure $\mathfrak{A}$, the FO-theory of $t(\mathfrak{A})$ can be reduced to the the $\mathrm{MSO}^{\mathrm{w}}$-theory of $\mathfrak{A}$. Contrary to this, the above theorem states that the FO-theory of the infinitary Muchnik-iteration can be arbitrarily more complicated than the $\mathrm{MSO}^{\mathrm{w}}$-theory of the base structure. Hence we obtain

Corollary 1. *Infinitary Muchnik-iteration is not* $(\mathrm{FO}, \mathrm{MSO}^{\mathrm{w}})$*-compatible.*

6 Summary

Table 1 summarizes our knowledge about the compatibility of Muchnik's and Shelah & Stupp's iteration. It consists of four subtables dealing with finitary and infinitary Muchnik-iteration and with finitary and infinitary Shelah-Stupp-iteration. The sign + in cell $(\mathcal{K}, \mathcal{L})$ of a subtable denotes that the respective iteration is $(\mathcal{K}, \mathcal{L})$-compatible, – denotes the opposite. Minus-signs without further marking hold since the base structure can be defined in any of its iterations. Capital letters denote references: (A) is [16], (B) [18], (C) [11, Prop. 3.4], (D) [11, Thm. 4.10], (E) Theorem 3, (F) Theorem 4, and (G) since the base structure is definable in its iteration and finiteness of a set is no MSO-property. Small letters denote that the result follows from Theorem 5 below and some further "simple" arguments from the result marked by the corresponding capital letter.

Theorem 5. *Let* $(\mathcal{K}, \mathcal{L})$ *be any of the pairs* $(\mathrm{MSO}^{\mathrm{closed}}, \mathrm{MSO})$*,* $(\mathrm{MSO}^{\mathrm{ch}}, \mathrm{MSO}^{\mathrm{ch}})$*, or* $(\mathrm{MSO}^{\mathrm{mch}}, \mathrm{MSO}^{\mathrm{mch}})$*. There exists a computable function* red *such that, for any* σ*-structure* $\mathfrak{A}$*,* red *reduces the* $\mathcal{K}$*-theory of* $(\mathfrak{A}^{\infty}, \mathrm{cl})$ *to the* $\mathcal{L}$*-theory of* $(\mathfrak{A}^{*}, \mathrm{cl})$*.*

The same holds for the Shelah-Stupp-iterations.

The two questions marks in Table 1 express that it is not clear whether finitary Muchnik-iteration is $\mathrm{MSO}^{\mathrm{w}}$-compatible or not.

Note the main difference between Muchnik- and Shelah-Stupp-iteration: the latter is $\mathcal{K}$-compatible for all relevant logics while only MSO behaves that nicely with respect to (infinitary) Muchnik-iteration

A referee proposed to also consider the variant of MSO where set quantification is restricted to countable sets. As to whether Muchnik iteration is compatible with this logic is not clear at the moment.

	Muchnik				inf. Muchnik		
	MSO	$\mathrm{MSO}^{\mathrm{w}}$	FO		MSO	$\mathrm{MSO}^{\mathrm{w}}$	FO
MSO	+ (B)	–	–	$\mathrm{MSO}^{\mathrm{closed}}$	+ (b)	–	–
$\mathrm{MSO}^{\mathrm{w}}$	– (g)	?	–	$\mathrm{MSO}^{\mathrm{w}}$	– (g)	– (f)	–
FO	+ (b)	?	– (C)	FO	+ (b)	– (F)	– (c)
	Shelah-Stupp				inf. Shelah-Stupp		
	MSO	$\mathrm{MSO}^{\mathrm{w}}$	FO		MSO	$\mathrm{MSO}^{\mathrm{w}}$	FO
MSO	+ (A)	–	–	$\mathrm{MSO}^{\mathrm{closed}}$	+ (a)	–	–
$\mathrm{MSO}^{\mathrm{mch}}$	– (g)	+ (E)	–	$\mathrm{MSO}^{\mathrm{mch}}$	– (g)	+ (e)	–
$\mathrm{MSO}^{\mathrm{w}}$	– (G)	+ (e)	–	$\mathrm{MSO}^{\mathrm{w}}$	– (g)	+ (e)	–
$\mathrm{MSO}^{\mathrm{ch}}$	+ (a)	+ (e)	+ (D)	$\mathrm{MSO}^{\mathrm{ch}}$	+ (a)	+ (e)	+ (d)
FO	+ (a)	+ (e)	+ (d)	FO	+ (a)	+ (e)	+ (d)

Table 1. summary

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