OPTIMAL CONTROL OF PRODUCTION RATE IN A FAILURE PRONE MANUFACTURING SYSTEM*

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ABSTRACT

We address the problem of controlling the production rate of a failure prone manufacturing system so as to minimize the discounted inventory cost, where certain cost rates are specified for both positive and negative inventories, and there is a constant demand rate for the commodity produced.

The underlying theoretical problem is the optimal control of a continuous time system with jump Markov disturbances, with an infinite horizon discounted cost criterion. We use two complementary approaches. First, proceeding informally, and using a combination of stochastic coupling, linear system arguments, stable and unstable eigenspaces, renewal theory, parametric optimization etc., we arrive at a conjecture for the optimal policy. Then we address the previously ignored mathematical difficulties associated with differential equations with discontinuous right hand sides, singularity of the optimal control problem, smoothness and validity of the dynamic programming equation etc., to give a rigorous proof of optimality of the conjectured policy. It is hoped that both approaches will find uses in other such problems also.

We obtain the complete solution and show that the optimal solution is simply characterized by a certain critical number, which we call the optimal inventory level. If the current inventory level exceeds the optimal, one should not produce at all, if less, one should produce at the maximum rate, while if exactly equal one should produce exactly enough to meet demand. We also give a simple explicit formula for the optimal inventory level.

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I. INTRODUCTION

We consider a manufacturing system producing a single commodity. There is a constant demand rate d for the commodity, and the goal of the manufacturing system is to try to meet this demand. The manufacturing system is however subject to occasional breakdowns and so there are two states, a "functional" state and a "breakdown" state, in which it can be. The transitions between these two states occur as a continuous time Markov chain, with q_1 the rate of transition from the functional to the breakdown state, and q_2 the rate of transition from the breakdown to the functional state. (Alternatively, the mean time between failures is q_1^{-1} and the mean repair time is q_2^{-1}). When the manufacturing system is in the breakdown state it cannot produce the commodity; while if it is in the functional state it can produce at any rate u upto a maximum production rate r. We assume that $r > d > 0$.

Let $x(t)$ be the inventory of the commodity at time t, i.e., $x(t) =$ (total production upto time t) – (total demand upto time t). $x(t)$ may be negative, which corresponds to a backlog. We suppose that positive inventories incur a holding cost of c^+ per unit commodity per unit time, while negative inventories incur a cost of c^-, with $c^+ > 0$, $c^- > 0$. Our goal is to control the production rate $u(t)$ at time t, so as to minimize the expected discounted cost,

$$E\int_0^\infty (c^+ x^+(t) + c^- x^-(t))e^{-\gamma t}\, dt \tag{1}$$

where $x^+ := \max(x,0)$, $x^- := \max(-x,0)$ and $\gamma > 0$ is the discount rate.

The problem that we address is this. When the manufacturing system is functioning, what is the optimal production rate u as a function of the inventory x? This problem has been motivated by the pioneering work of Kimemia [1] and Kimemia and Gershwin [2], where a more general problem is formulated (see Section XIII also).

We obtain the complete answer to this question. The optimal solution, $u(t) = \pi^{z^*}(x(t))$, is given by a critical number z^*. The optimal policy is,

$$\pi^{z^*}(x(t)) = r \qquad \text{if } x(t) < z^*$$
$$= d \qquad \text{if } x(t) = z^* \qquad\qquad (2)$$
$$= 0 \qquad \text{if } x(t) > z^*$$

Thus, whenever the manufacturing system is in the functional state, one should produce at the maximum rate r if the inventory x(t) is less than z^*, one should produce exactly enough to meet demand if the inventory x(t) is exactly equal to z^*, and one should not produce at all if the inventory x(t) exceeds z^*. Hence the production rate should always be controlled so as to drive the inventory level as rapidly as possible towards z^*, and once there should maintain it at the level z^*. For this reason we shall call z^* the optimal inventory level.

We also obtain the following simple formula for the optimal inventory level.

$$z^* = \max\left\{0, \ \frac{1}{\lambda_-} \log\left[\frac{c^+}{c^+ + c^-}\left(1 + \frac{\gamma d}{q_1 d - (\gamma + q_2 + d\lambda_-)(r-d)}\right)\right]\right\} \qquad (3)$$

where λ_- is the only negative eigenvalue of the matrix,

$$A_1 := \begin{bmatrix} \dfrac{\gamma + q_1}{r - d} & \dfrac{-q_1}{r - d} \\[3mm] \dfrac{q_2}{d} & -\left(\dfrac{q_2 + \gamma}{d}\right) \end{bmatrix} \tag{4}$$

The motivation for the problem studied here is that it is a basic problem for manufacturing systems. The optimal policy is trivial to compute and qualitatively simple to implement, and it is hoped that these two features will render it attractive enough for use as a guideline.

From a theoretical viewpoint also, both our solution procedure and method of proof possess several interesting features. First note that the system under consideration is the following.

5.i) $\dot{x}(t) = u(t) - d$

5.ii) $\{s(t); t \geq 0\}$ is a continuous time Markov chain with state space $\{1,2\}$ and generator $\begin{bmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{bmatrix}$

5.iii)   The constraint on $u(t)$ is,

$$u(t) = 0 \qquad \text{if } s(t) = 2$$
$$\in [0,r] \qquad \text{if } s(t) = 1$$

5.iv)   The goal is to minimize $E\displaystyle\int_0^\infty (c^+ x^+(t) + c^- x^-(t))e^{-\gamma t}\, dt$

Here $x(t)$ is the inventory at time t, $u(t)$ is the production rate at time t, and $s(t) = 1$ or 2 depending on whether the manufacturing system is in the functional state or the breakdown state respectively. Thus we have a continuous time system with jump Markov disturbances and an infinite horizon discounted cost criterion. For previous work on problems of this type, we refer the reader to Rishel [3] for the case of a finite horizon cost criterion; and Krassovskii and Lidskii [4] and Lidskii [5] for a case of an infinite horizon problem. In dealing with these types of systems there are two problematical issues.

The first set of problems arises because one encounters many mathematical difficulties, see Rishel [3], when studying the problem of optimal control for continuous time systems with jump Markov disturbances. Consider a feedback policy $u = \pi(x)$. Then the system (5.i) satisfies

$$\overset{\circ}{x} = \pi(x) - d \tag{6}$$

However, the types of functions π we wish to consider are essentially discontinuous functions. Standard existence and uniqueness conditions for the solution of the differential equation (6) are not satisfied. Rishel [3] has considered one notion of a solution; we use another method. There are also other difficulties. Let

$$V_i(x) := \text{minimum value of the cost (5.iv) when starting in the}$$
$$\text{state } s(0) = i, \ x(0) = x.$$

Then, informally, we have the Hamilton-Jacobi-Bellman dynamic programming equation:

$$\begin{bmatrix} \min_{u \in [0,r]} (u-d)\overset{\circ}{V}_1(x) \\ -d\overset{\circ}{V}_2(x) \end{bmatrix} = \begin{bmatrix} \gamma + q_1 & -q_1 \\ -q_2 & \gamma + q_2 \end{bmatrix} \begin{bmatrix} V_1(x) \\ V_2(x) \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} (c^+ x^+ + c^- x^-) \tag{7}$$

It is not _a priori_ clear that $V_i(\cdot)$ is a differentiable function. It is not also clear that there exists an optimal control law. Moreover, it turns out that $\overset{\circ}{V}_1(x)$ does vanish for some x, and for such a value of x, the left hand side of (7) is minimized by every u, and so the Hamilton-Jacobi-Bellman (HJB) dynamic programming equation does not prescribe the optimal u for such x. Thus, the optimal control problem is a singular one. It is this collection of mathematical problems that we shall address rigorously in the second half of this paper.

The second set of problems is this. Even ignoring all the mathematical difficulties mentioned above, how do we actually obtain the optimal solution? Why should we suspect that the optimal policy is of the critical number type? Why is the critical number z* always nonnegative? Given that we want to solve the HJB equation (7), what are the appropriate boundary conditions? After determining boundary conditions how does one determine an optimal choice for z*? It is this collection of issues dealing with the actual obtaining of the optimal policy that we address in the first half of the paper.

The two approaches complement each other and we hope that they will also be useful in dealing with other problems of the sort considered here.

The paper is organized as follows. In Sections II-VIII we ignore some technical questions and arrive at a conjecture for the optimal policy. Beginning with Section IX we address all the mathematical difficulties and rigorously prove the optimality of the conjectured policy.

II. OPTIMALITY OF CRITICAL NUMBER POLICY

Beginning with this section and continuing through Section VIII, we provide a sequence of informal arguments which will lead us to conjectures about the optimal policy and the optimal cost function.

In this section we give an argument to show that the optimal policy is characterized by a critical number.

Let us assume the existence of an optimal feedback policy $u(t) = \pi^*(x(t))$ and let $V_i(x)$ denote the optimal cost when starting in the state $(s(0) = i, x(0) = x)$. Fix $\{s(t,\omega); t \geq 0\}$, a realization of the continuous time Markov chain, with $s(0,\omega) = i$.

Now we consider two different initial conditions $x_0(0)$ and $x_1(0)$ and also a convex combination $x_\alpha(0) = (1-\alpha)x_0(0) + \alpha x_1(0)$ where $0 \leq \alpha \leq 1$. If $\pi^*(\cdot)$ is used, then the trajectories starting with the initial conditions $x_0(0)$ and $x_1(0)$ satisfy,

$$\dot{x}_k(t,\omega) = \pi^*(x_k(t,\omega)) - d \qquad \text{if } s(t,\omega) = 1 \qquad (8)$$
$$= -d \qquad \text{if } s(t,\omega) = 2$$

$$x_k(0,\omega) = x_k(0)$$

for $k = 0,1$. Also, we have

$$V_1(x_k(0)) = E_\omega \int_0^\infty (c^+ x_k^+(t,\omega) + c^- x_k^-(t,\omega)) e^{-\gamma t} \, dt \qquad (9)$$

where E_ω signifies that the expectation is taken over ω.

Suppose now that for the initial state $x_\alpha(0)$, we use the control

$$u(t,\omega) = (1-\alpha)\pi^*(x_0(t,\omega)) + \alpha\pi^*(x_1(t,\omega)) \qquad (10)$$

Note that such a control is in fact implementable because by observing $\{s(t,\omega); t \geq 0\}$ one can in fact deduce what $\{x_k(t,\omega), t \geq 0\}$ would have been for $k = 0,1$. Such a control gives rise to the trajectory satisfying,

$$\dot{x}_\alpha(t,\omega) = (1-\alpha)\pi^*(x_0(t,\omega)) + \alpha\pi^*(x_1(t,\omega)) - d \qquad \text{if } s(t,\omega) = 1$$
$$= -d \qquad \text{if } s(t,\omega) = 2$$
$$\qquad (11)$$
$$x_\alpha(0,\omega) = (1-\alpha)x_0(0) + \alpha x_1(0)$$

It is easy to check from (8) and (11) that

$$x_\alpha(t,\omega) = (1-\alpha)x_0(t,\omega) + \alpha x_1(t,\omega) \qquad \text{for every } t \geq 0$$

From the convexity of the integrand in (5.iv) it follows that

$$E_\omega \int_0^\infty (c^+ x_\alpha^+(t,\omega) + c^- x_\alpha^-(t,\omega)) e^{-\gamma t} \, dt \leq (1-\alpha)V_1(x_0(0)) + \alpha V_1(x_1(0)) \qquad (12)$$

However, for the initial state $x_\alpha(0)$ the control (10) is not necessarily optimal, and so

$$V_1(x_\alpha(0)) \leq E_\omega \int_0^\infty (c^+ x_\alpha^+(t,\omega) + c^- x_\alpha^-(t,\omega)) e^{-\gamma t} \, dt \tag{13}$$

From (13) and (12) we deduce that $V_1(\cdot)$ is a _convex_ function. __Assuming__ that $V_1(\cdot)$ is continuously differentiable, we see that there is some z* for which

$$
\begin{aligned}
\overset{\bullet}{V}_1(x) &\leq 0 \qquad \text{for } x \leq z* \\
&\geq 0 \qquad \text{for } x \geq z*
\end{aligned}
\tag{14}
$$

From the left-hand side of the HJB equation (7), which we suspect $\{V_i(0);\ i = 1,2\}$ satisfy, we see that

$$
\begin{aligned}
u &= r \ \underline{\text{minimizes}} \ (u-d)\overset{\bullet}{V}_1(x) \qquad \text{if } x \leq z* \\
&= 0 \ \text{minimizes} \ (u-d)\overset{\bullet}{V}_1(x) \qquad \text{if } x \geq z*
\end{aligned}
$$

Hence, we suspect that the optimal policy $u(t) = \pi^*(x(t))$ is of the form

$$
\begin{aligned}
\pi^*(x) &= r \qquad \text{if } x < z* \\
&= 0 \qquad \text{if } x > z*
\end{aligned}
\tag{15}
$$

for some critical number z*.

What happens at x = z*? Any $u \in [0,r]$ minimizes $(u-d)\overset{\bullet}{V}_1(z*)$, but from the form of (15) it is clear that the inventory level is quickly driven back to z* if it deviates from z*. Hence, due to this "chattering" phenomenon, we suspect that

$$\pi^*(x) = d \qquad \text{if } x = z* \tag{16}$$

because such a choice keeps the inventory level exactly at z*, once it reaches z*.

(15) and (16) show that the optimal policy is characterized by

the critical number z*, which we have called the optimal inventory level.

III. NONNEGATIVITY OF OPTIMAL INVENTORY LEVEL

In this section we show that the optimal inventory level is nonnegative. Consider two policies $\pi^0(\cdot)$ and $\pi^z(\cdot)$, where

$$
\pi^0(x) = r \quad \text{if } x < 0
$$
$$
\quad\quad\;\; = d \quad \text{if } x = 0 \tag{17}
$$
$$
\quad\quad\;\; = 0 \quad \text{if } x > 0
$$

and

$$
\pi^z(x) = r \quad \text{if } x < z
$$
$$
\quad\quad\;\; = d \quad \text{if } x = z \tag{18}
$$
$$
\quad\quad\;\; = 0 \quad \text{if } x > z
$$

Let $z < 0$ be some strictly negative number. Denote by $V_i^0(x)$ and $V_i^z(x)$ the costs resulting from the policies $\pi^0(\cdot)$ and $\pi^z(\cdot)$ respectively, when starting in the state $(s(0) = i, x(0) = x)$.

If $\pi^z(\cdot)$ is optimal, then from (14) we see that $V_i^z(\cdot)$ should attain a minimum at $x = z$, i.e.

$$
V_i^z(z) \leq V_i^z(x) \quad \text{for all } x \tag{19}
$$

Moreover, if $\pi^z(\cdot)$ is optimal, we should also have

$$
V_i^z(x) \leq V_i^0(x) \quad \text{for all } x.
$$

In particular, from the above two inequalities we should have

$$
V_i^z(z) \leq V_i^0(0) \tag{20}
$$

Hence, to show that $\pi^z(\cdot)$ with $z < 0$ is not optimal, it will suffice to show that (20) is not true.

Indeed, let $\{s(t,\omega); t \geq 0\}$ with $s(0,\omega) = 1$ be a realization, and consider the two trajectories

$$\dot{x}^0(t,\omega) = r - d \qquad \text{if } x^0(t,\omega) < 0, \ s(t,\omega) = 1$$
$$= 0 \qquad \text{if } x^0(t,\omega) = 0, \ s(t,\omega) = 1$$
$$= -d \qquad \text{otherwise}$$
$$x^0(0,\omega) = 0$$

and

$$\dot{x}^z(t,\omega) = r - d \qquad \text{if } x^z(t,\omega) < z, \ s(t,\omega) = 1$$
$$= 0 \qquad \text{if } x^z(t,\omega) = 0, \ s(t,\omega) = 1$$
$$= -d \qquad \text{otherwise}$$
$$x^z(0,\omega) = z$$

which emanate from the initial states 0 and z, when the policies $\pi^0(\cdot)$ and $\pi^z(\cdot)$ are respectively used. It is easy to verify that

$$x^0(t,\omega) + z = x^z(t,\omega) \leq z < 0 \qquad \text{for all } t \geq 0$$

Hence,

$$c^+ x^{z+}(t,\omega) + c^- x^{z-}(t,\omega) = c^- x^{z-}(t,\omega)$$
$$= c^-[x^{0-}(t,\omega) + z^-]$$
$$= c^+ x^{0+}(t,\omega) + c^- x^{0-}(t,\omega) + c^-|z|$$

Hence

$$E_\omega \int_0^\infty (c^+ x^{0+}(t,\omega) + c^- x^{0-}(t,\omega)) e^{-\gamma t}\, dt < E_\omega \int_0^\infty (c^+ x^{z+}(t,\omega) + c^- x^{z-}(t,\omega)) e^{-\gamma t} dt$$

i.e.

$$V_1^0(0) < V_1^z(z)$$

showing that (20) is violated, and thus that $\pi^z(\cdot)$ cannot be an optimal policy.

Hence z^*, the optimal inventory level, has to be nonnegative.

IV. THE PIECEWISE LINEAR EQUATIONS FOR THE COST FUNCTION

Let $z \geq 0$ and denote by $V_i^z(x)$ the cost function for the policy $\pi^z(\cdot)$ defined in (18). The analog of (7) for the policy $\pi^z(\cdot)$ is

$$
\begin{pmatrix} (\pi^z(x) - d)\mathring{V}_1^z(x) \\ \\ -d\, \mathring{V}_2^z(x) \end{pmatrix} = \begin{bmatrix} \gamma + q_1 & -q_1 \\ \\ -q_2 & \gamma + q_2 \end{bmatrix} \begin{pmatrix} V_1^z(x) \\ \\ V_2^z(x) \end{pmatrix} - \begin{pmatrix} 1 \\ \\ 1 \end{pmatrix} (c^+ x^+ + c^- x^-) \tag{21}
$$

Denoting

$$
V^z(x) := \begin{pmatrix} V_1^z(x) \\ V_2^z(x) \end{pmatrix}
$$

$$
A_2 := \begin{bmatrix} -\left(\dfrac{\gamma + q_1}{d}\right) & \dfrac{q_1}{d} \\ \\ \dfrac{q_2}{d} & -\left(\dfrac{\gamma + q_2}{d}\right) \end{bmatrix}
$$

$$
b_1 := \begin{pmatrix} -\dfrac{1}{r - d} \\ \\ \dfrac{1}{d} \end{pmatrix}
$$

$$
b_2 := \begin{pmatrix} \dfrac{1}{d} \\ \\ \dfrac{1}{d} \end{pmatrix} \tag{22}
$$

and letting A_1 be as defined in (4), it is clear that (21) can be rewritten as,

$$
\frac{\partial}{\partial x} V^z(x) = A_1 V^z(x) - b_1 c^- x \qquad \text{for } x \leq 0
$$

$$
= A_1 V^z(x) + b_1 c^+ x \qquad \text{for } 0 \leq x < z \tag{23}
$$

$$
= A_2 V^z(x) + b_2 c^+ x \qquad \text{for } x > z
$$

Before we can utilize these piecewise linear equations to determine the cost

function corresponding to $\pi^z(\cdot)$, we need to determine appropriate boundary conditions for (23).

V. BOUNDARY CONDITIONS

Since the vector $V^z(x)$ is two-dimensional, we need two boundary conditions for (23).

V.1 The First Boundary Condition

Let $\{s(t,\omega); t \geq 0\}$ be a realization. Under the policy $\pi^z(\cdot)$, the inventory is given by the differential equation,

$$\overset{\circ}{x}(t,\omega) = \pi^z(x(t,\omega)) - d \qquad \text{if } s(t,\omega) = 1$$
$$= -d \qquad \text{if } s(t,\omega) = 2$$

In any case $|\overset{\circ}{x}(t,\omega)| \leq r$ for all (t,ω), and so

$$|x(t,\omega)| \leq |x(0)| + rt \qquad \text{for all } (t,\omega).$$

Hence,

$$V_i^z(x(0)) = E_\omega[\int_0^\infty (c^+ x^+(t,\omega) + c^- x^-(t,\omega)) e^{-\gamma t} dt \,|\, s(0,\omega) = i, \; x(0,\omega) = x(0)]$$
$$\leq k_1 |x(0)| + k_2 \qquad \text{for some constants } k_1 \text{ and } k_2.$$

Hence, we see that

$$V^z(x) = 0(|x|) \qquad \text{as } x \to \pm \infty \tag{24}$$

Let us see how we can make (24) more usable. Solving (23) for $x < 0$ in terms of $V^z(0)$, we get

$$V^z(x) = e^{A_1 x}[V^z(0) - A_1^{-2} b_1 c^-] + [A_1^{-1} b_1 c^- x + A_1^{-2} b_1 c^-] \qquad \text{for } x < 0 \tag{25}$$

Now note the following easily verified fact.

A_1 has one strictly positive eigenvalue, say λ_+, and one strictly negative eigenvalue, say λ_-. $\tag{26}$

Let

$$w^+ = \begin{pmatrix} 1 \\ \\ w_2^+ \end{pmatrix} := \text{eigenvector of } A_1 \text{ corresponding to } \lambda_+.$$ (27)

To satisfy (24) as $x \to -\infty$, we clearly need

$$V^z(0) - A_1^{-2} b_1 c^- \in \langle w^+ \rangle$$ (28)

for otherwise, $V^z(x) = 0(e^{\lambda_- x})$ as $x \to -\infty$. Here, $\langle w^+ \rangle$ is the eigenspace generated by $\{w^+\}$.

(28) is one boundary condition for (23).

V.2 The Second Boundary Condition

To obtain the second boundary condition, let us see what $V_1^z(z)$ is. Consider a system starting in state $(s(0) = 1, x(0) = z)$ and let τ, a stopping time, be the first time at which $s(\tau+) = 2$. Clearly $x(t) = z$ for $0 \leq t < \tau$ when $\pi^z(\cdot)$ is used. Hence

$$V_1^z(z) = E_\tau [\int_0^\tau e^{-\gamma t} c^+ z \, dt + e^{-\gamma \tau} V_2^z(z)]$$

Noting that τ is exponentially distributed with mean q_1^{-1}, by evaluating the expectation in the above equation, we get

$$V_1^z(z) = \frac{1}{q_1 + \gamma} (q_1 V_2^z(z) + c^+ z)$$ (29)

or, equivalently,

$$[1,0] A_1 V^z(z) = \frac{c^+ z}{r - d}$$ (30)

(30) is the second boundary condition for (23).

By using the two boundary conditions (28) and (30) one can solve the piecewise linear differential equations (23) to obtain $V^z(\cdot)$, the cost function for any policy $\pi^z(\cdot)$ with $z \geq 0$.

The next question we have to face is; what is the optimal choice of z?

VI. OPTIMAL CHOICE OF z

Suppose $\pi^{z^*}(\cdot)$ with $z^* > 0$ is optimal. Then,

i) $V_1^{z^*}(z^*) \leq V_1^{z^*}(x)$ for all x, since by (19), the optimal cost

function attains a minimum at z^*.

ii) $V_1^{z^*}(x) \leq V_1^{z}(x)$ for all x,z, since $\pi^{z^*}(\cdot)$ is optimal and therefore

has lower cost than any other $\pi^{z}(\cdot)$.

From the above, we get

$$V_1^{z^*}(z^*) \leq V_1^{z^*}(z) \leq V_1^{z}(z) \qquad \text{for all z}$$

Hence $V_1^{z}(z)$ attains a minimum when $z = z^*$. Assuming now that $V_1^{z}(z)$ is a C^1

function of z, we see that

$$\frac{d}{dz}V_1^{z}(z)\bigg]_{z=z^*} = 0 \qquad\qquad (31)$$

We will call (31) the optimality condition and in the next section we will

see how it can be exploited to give the optimal solution.

VII. OPTIMAL INVENTORY LEVEL

We will now utilize the piecewise linear differential equations (23),

the two boundary conditions (28) and (30), and the optimality condition (31)

to obtain the optimal choice for z*.

Differentiating (29) and using (31), we get

$$\frac{d}{dz}V_2^{z}(z)\bigg]_{z=z^*} = -\frac{c^+}{q_1} \qquad\qquad (32)$$

However, by the chain rule,

$$\frac{d}{dz}V_2^z(z)\Big]_{z=z*} = \frac{\partial}{\partial x}V_2^{z*}(x)\Big]_{x=z*} + \frac{\partial}{\partial z}V_2^z(z*)\Big]_{z=z*}$$

Since $V_2^z(x)$ considered as a function of z is minimized at $z = z*$, by assuming continuous differentiability, we have,

$$\frac{\partial}{\partial z}V_2^z(z*)\Big]_{z=z*} = 0.$$

Hence,

$$\frac{d}{dz}V_2^z(z)\Big]_{z=z*} = \frac{\partial}{\partial x}V_2^{z*}(x)\Big]_{x=z*} = [0,1]\{A_2 V^{z*}(z*) + b_2 c^+ z*\} \tag{33}$$

where we have also used (21). From (32) and (33) we have

$$[0,1]\{A_2 V^{z*}(z*) + b_2 c^+ z*\} = -\frac{c^+}{q_1}$$

However, since $[0,1]A_2 = [0,1]A_1$, $[0,1]b_2 = \frac{1}{d}$, we have

$$[0,1]A_1 V^{z*}(z*) = -c^+\left(\frac{1}{q_1} + \frac{z*}{d}\right)$$

Combining the above equation with (30), we have

$$A_1 V^{z*}(z*) = \begin{pmatrix} \frac{c^+ z*}{z-d} \\ -\frac{c^+}{q_1} - \frac{c^+ z*}{d} \end{pmatrix} = -b_1 c^+ z* - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{c^+}{q_1} \tag{34}$$

Setting (34) temporarily aside, we turn to (28). Solving (23) for $V^z(0)$ in terms of $V^z(z)$, we get

$$V^z(0) = e^{-A_1 z}[V^z(z) + A_1^{-1}b_1 c^+ z + A_1^{-2}b_1 c^+] - A_1^{-2}b_1 c^+$$

and substituting this in (28), we have

$$e^{-A_1 z}[V^z(z) + A_1^{-1}b_1 c^+ z + A_1^{-2}b_1 c^+] - A_1^{-2}b_1(c^+ + c^-) \in \langle w^+\rangle$$

Since $\langle w^+\rangle$ is invariant under $e^{A_1 z}$, we have

$$[V^z(z) + A_1^{-1}b_1c^+z + A_1^{-2}b_1c^+] - e^{A_1z}A_1^{-2}b_1(c^+ + c^-) \in \langle w^+ \rangle \tag{35}$$

Combining (34) and (35), and noting that $\langle w^+ \rangle$ is invariant under A_1, we get

$$-\frac{c^+}{q_1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + A_1^{-1}b_1c^+ - e^{A_1z^*}A_1^{-1}b_1(c^+ + c^-) \in \langle w^+ \rangle \tag{36}$$

This equation now gives us a "formula" to choose z^*.

Recall now from Section VI, that in obtaining (36) we made the implicit assumption that $z^* > 0$. Therefore we now have to determine when there will be a positive solution z^* for (36).

Let us first simplify (36) a bit. Note that

$$\det A_1 = \lambda_+\lambda_- = \frac{-\gamma(\gamma + q_1 + q_2)}{d(r-d)}$$

$$A_1^{-1}b_1 = \frac{-1}{\gamma}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{37}$$

Hence (36) simplifies to

$$e^{A_1z^*}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{c^+}{c^+ + c^-}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{c^+\gamma}{(c^+ + c^-)q_1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \langle w^+ \rangle \tag{38}$$

Now let

$$v^- = [v_1^-, v_2^-] := \text{a left eigenvector of } A_1 \text{ corresponding to } \lambda_- \tag{39}$$

Since left and right eigenvectors corresponding to different eigenvalues are orthogonal, i.e., $v^-w^+ = 0$, (38) is true, if and only if,

$$v^-e^{A_1z^*}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{c^+}{c^+ + c^-}v^-\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{c^+\gamma}{(c^+ + c^-)q_1}v^-\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

Since $v^-e^{A_1z^*} = e^{\lambda_-z^*}v^-$, this reduces to

$$e^{\lambda_- z^*} = \frac{c^+}{c^+ + c^-}\left[1 + \frac{\gamma}{q_1}\frac{v_2^-}{v_2^- + v_1^-}\right]$$

Since $v^- A_1 = \lambda_- v^-$, by equating the second components of both sides,

$$v_1^- = - v_2^- \frac{(r-d)}{q_1}\left[\lambda_- + \frac{(\gamma + q_2)}{d}\right]$$

and now substituting for $\dfrac{v_2^-}{v_2^- + v_1^-}$, we get

$$e^{\lambda_- z^*} = \frac{c^+}{c^+ + c^-}\left[1 + \frac{\gamma d}{q_1 d - (\gamma + q_2 + \lambda_- d)(r-d)}\right] \tag{40}$$

It is easy to check that

$$q_1 d - (\gamma + q_2 + \lambda_- d)(r-d) > 0 \tag{41}$$

and so the right hand side of (40) is strictly positive. Taking logarithms, we therefore get

$$z^* = \frac{1}{\lambda_-} \log\left[\frac{c^+}{c^+ + c^-}(1 + \frac{\gamma d}{q_1 d - (\gamma + q_2 + \lambda_- d)(r-d)})\right]$$

However, such a value of z^* may not be positive, and we already know from Section III, that if $z^* > 0$ is not optimal, then $z^* = 0$ is. Hence we arrive at our conjecture:

$$z^* = \max\left\{0, \frac{1}{\lambda_-} \log\left[\frac{c^+}{c^+ + c^-}(1 + \frac{\gamma d}{q_1 d - (\gamma + q_2 + \lambda_- d)(r-d)})\right]\right\} \tag{42}$$

Note: Due to (41), it follows that $\left[1 + \dfrac{\gamma d}{q_1 d - (\gamma + q_2 + \lambda_- d)(r-d)}\right] > 1.$

Hence, for _every_ $c^- > 0$, there exists $c^* > 0$ such that if $c^+ \geq c^*$, then $z^* = 0$. Thus the optimal inventory level may be zero even though $c^+ < +\infty$. This is somewhat counterintuitive and surprising.

VIII. OPTIMAL COST FUNCTION

In the previous section we have conjectured the optimal policy. In this section we shall conjecture the optimal cost function also.

Let us consider separately the two cases where z* as given by (42) is zero, and where it is positive.

VIII.1 Case 1: $z^* = 0$

When $z^* = 0$, the optimal cost function is $V^0(\cdot)$ which satisfies (23) with $z = 0$, and has the boundary conditions (28) and (30) with $z = 0$. Defining w^+ as in (27), (28) says that

$$V(0) = kw^+ + A_1^{-2}b_1c^- \qquad \text{for some constant k}$$

Then, from (30), we get

$$0 = [1,0]A_1 V(0) = k\lambda_+ + [1,0]A_1^{-1}b_1c^- = k\lambda_+ - \frac{c^-}{\gamma}$$

and so $k = \dfrac{c^-}{\gamma\lambda_+}$. Hence

$$V(0) = \frac{c^-}{\gamma\lambda_+} w^+ + A_1^{-2}b_1c^- \qquad (43)$$

Solving the differential equations (23) with the boundary condition (43), we get

$$V(x) = \frac{c^-}{\gamma\lambda_+} e^{\lambda_+ x} w^+ + A_1^{-1}b_1c^- x + A_1^{-2}b_1c^- \qquad \text{for } x \le 0$$

$$= e^{A_2 x}[\frac{c^-}{\gamma\lambda_+} w^+ + A_1^{-2}b_1c^- + A_2^{-2}b_2c^+] - [A_2^{-1}b_2c^+ x + A_2^{-2}b_2c^+]$$

$$\text{for } x \ge 0$$

(44)

VIII.2   Case 2: $z^* > 0$

From (34) we have

$$V(z^*) = - A_1^{-1} b_1 c^+ z^* - A_1^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{c^+}{q_1} \tag{45}$$

and so, solving the piecewise linear differential equations (23) with boundary condition (45), we get

$$V(x) = e^{A_1 x} \left\{ e^{-A_1 z^*} [A_1^{-2} b_1 c^+ - A_1^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{c^+}{q_1}] - A_1^{-2} b_1 (c^+ + c^-) \right\} + A_1^{-1} b_1 c^- x + A_1^{-2} b_1 c^-$$

$$\text{for } x \leq 0$$

$$= e^{A_1 (x - z^*)} [A_1^{-2} b_1 c^+ - A_1^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{c^+}{q_1}] - A_1^{-1} b_1 c^+ x - A_1^{-2} b_1 c^+$$

$$\text{for } 0 \leq x \leq z^* \tag{46}$$

$$= e^{A_2 (x - z^*)} \left\{ A_2^{-1} b_2 c^+ z^* - A_1^{-1} b_1 c^+ z^* - A_1^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{c^+}{q_1} + A_2^{-2} b_2 c^+ \right\}$$

$$- A_2^{-1} b_2 c^+ x - A_2^{-2} b_2 c^+ \quad \text{for } x \geq z^*$$

By simplification it can also be seen that

$$V(x) = e^{A_2 (x - z^*)} \left\{ -A_2^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{c^+}{q_1} + A_2^{-2} b_2 c^+ \right\} - A_2^{-1} b_2 c^+ x - A_2^{-2} b_2 c^+$$

$$\tag{47}$$

$$\text{for } x \geq z^*$$

Our conjectures for the optimal cost function in the two cases $z^* = 0$ and $z^* > 0$ are given by (44) and (46) respectively.

IX. SOLUTION OF HJB EQUATION

In the previous sections we have arrived at the conjecture that if z^* is as specified by (42), then $\pi^{z^*}(\cdot)$ defined as in (2) is the optimal policy and $V(\cdot)$ defined as in (44) and (46), for the two cases $z^* = 0$ and $z^* > 0$, is the optimal cost function.

Beginning with this section, we commence the rigorous proof of the validity of our conjectures.

In this section we will show that $V(\cdot)$ satisfies the Hamilton-Jacobi-Bellman dynamic programming equation (7).

Lemma 1

. $V(\cdot)$ is continuously differentiable.

Proof

Consider first the case $z^* = 0$ where $V(\cdot)$ is specified by (44). We only need to check the continuous differentiability at $x = 0$. Denote by $\dot{V}(a^+)$, $\lim_{h \downarrow 0} \frac{V(a+h) - V(a)}{h}$ and similarly for $\dot{V}(a^-)$. Now, from (44)

$$\dot{V}(0-) = \frac{c^-}{\gamma} w^+ + A_1^{-1} b_1 c^- = \frac{c^-}{\gamma} w^+ - \frac{c^-}{\gamma}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{c^-}{\gamma}\begin{pmatrix} 0 \\ w_2^+ - 1 \end{pmatrix}$$

$$\dot{V}(0+) = \frac{c^-}{\gamma \lambda_+} A_2 w^+ + A_2 A_1^{-2} b_1 c^- = \frac{c^-}{\gamma}\begin{pmatrix} 0 \\ w_2^+ - 1 \end{pmatrix} \tag{48}$$

where we have used (27), (37) and .

$$A_2 = \begin{pmatrix} -\frac{r-d}{d} & 0 \\ 0 & 1 \end{pmatrix} A_1 \tag{49}$$

Hence $V(\cdot)$ is continuously differentiable whenever $z^* = 0$. Now consider the case $z^* > 0$, where $V(\cdot)$ is specified by (46). We only need to check the continuous differentiability at $x = 0$ and $x = z^*$. Now, clearly, from (46),

$$\dot{V}(0-) = \dot{V}(0+) = e^{-A_1 z^*}[A_1^{-1} b_1 c^+ - \frac{c^+}{q_1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}] - A_1^{-1} b_1 c^+ \tag{50}$$

and so we proceed to consider $x = z^*$, for which,

$$\dot{V}(z^*-) = \dot{V}(z^*+) = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{c^+}{q_1} \tag{51}$$

as can be seen from (46) and (47). □

Lemma 2

$$V(0) - A_1^{-2} b_1 c^- \in \langle w^+ \rangle$$

Proof

From (44), this is clearly true for the case $z* = 0$. Considering $z* > 0$, we see from (46) that

$$V(0) = e^{-A_1 z*} [A_1^{-2} b_1 c^+ - A_1^{-1} \binom{0}{1} \frac{c^+}{q_1}] - A_1^{-2} b_1 c^+$$

and so noting that $\langle w^+ \rangle$ is invariant under $e^{A_1 z*} A_1$, we get

$$e^{A_1 z*} A_1^{-1} b_1 + \frac{c^+}{(c^+ + c^-) q_1} \binom{0}{1} - \frac{c^+}{(c^+ + c^-)} A_1^{-1} b_1 \in \langle w^+ \rangle$$

Noting that left and right eigenvectors corresponding to different eigenvalues are orthogonal, i.e. $v^- w^+ = 0$, where v^- and w^+ are given by (39) and (27), we only need to verify that,

$$v^- [e^{A_1 z*} A_1^{-1} b_1 + \frac{c^+}{(c^+ + c^-) q_1} \binom{0}{1} - \frac{c^+}{(c^+ + c^-)} A_1^{-1} b_1] = 0$$

Noting that $v^- e^{A_1 z*} = e^{\lambda_- z*} v^-$, by using (37) and simplifying, we see that we only have to show that

$$e^{\lambda_- z*} v^- \binom{1}{1} - \frac{c^+}{(c^+ - c^-) q_1} v^- \binom{0}{1} - \frac{c^+}{(c^+ + c^-)} v^- \binom{1}{1} = 0.$$

But noting the equivalence of this and (40) and (42) when $z* > 0$, the assertion follows. □

Now we are ready to consider the case $z* = 0$ and show that $V(\cdot)$ satisfies the HJB dynamic programming equation (7).

Lemma 3

Suppose z^* given by (42) is equal to 0, and $V(\cdot)$ is defined by (44). Then

52.i)
$$\begin{bmatrix} (\pi^{z^*}(x) - d)\dot{V}_1(x) \\ \\ -d\dot{V}_2(x) \end{bmatrix} = \begin{bmatrix} \gamma + q_1 & -q_1 \\ \\ -q_2 & \gamma + q_2 \end{bmatrix} \begin{bmatrix} V_1(x) \\ \\ V_2(x) \end{bmatrix} - \begin{bmatrix} 1 \\ \\ 1 \end{bmatrix} (c^+ x^+ + c^- x^-)$$

for all x

52.ii) $(\pi^{z^*}(x) - d)\dot{V}_1(x) = \displaystyle\min_{u \in [0,r]} (u - d)\dot{V}_1(x)$ for all x

Proof

It is easily checked that $V(\cdot)$ defined by (44) satisfies (23) for $x < 0$ and $x > 0$; and so (52.i) is valid for $x < 0$ and $x > 0$. Now considering $x = 0$ and using (48), we have

$$\begin{bmatrix} (\pi^{z^*}(0) - d)\dot{V}_1(0) \\ \\ -d\dot{V}_2(0) \end{bmatrix} = -\frac{c^- d}{\gamma} \begin{bmatrix} 0 \\ \\ w_2^+ - 1 \end{bmatrix}$$

while, by using (43), we have

$$\begin{bmatrix} \gamma + q_1 & -q_1 \\ \\ -q_2 & \gamma + q_2 \end{bmatrix} V(0) = -dA_2 V(0) = \frac{-dc^-}{\gamma \lambda_+} A_2 w^+ - dA_2 A_1^{-2} b_1 c^-$$

$$= \frac{-c^- d}{\gamma} \begin{bmatrix} 0 \\ \\ w_2^+ - 1 \end{bmatrix}$$

where we have also used (49) and (37). Hence (52.i) is also valid at $x = 0$. Now turning to (52.ii), since $\pi^{z^*}(\cdot)$ satisfies (2), we only need to show that

$$\overset{\circ}{V}_1(x) \leq 0 \qquad \text{for } x < 0 \qquad \text{and} \qquad \overset{\circ}{V}_1(x) \geq 0 \qquad \text{for } x > 0 \quad (53)$$

Consider $x < 0$ first. Then, from (44) it follows that

$$\overset{\circ\circ}{V}_1(x) = \frac{c^- \lambda_+}{\gamma} e^{\lambda_+ x} \geq 0 \qquad \text{for } x < 0$$

where, by $\overset{\circ\circ}{V}_1(0)$, we mean $\overset{\circ\circ}{V}_1(0-)$. Since (48) shows that $\overset{\circ}{V}_1(0) = 0$, it follows that (53) holds for $x < 0$. Now turning to $x > 0$, from (43) and (44) we see that

$$\overset{\circ}{V}(x) = e^{A_2 x}[A_2 V(0) + A_2^{-1} b_2 c^+] - A_2^{-1} b_2 c^+ \qquad \text{for } x \geq 0$$

From (43), (49) and (37), we have

$$A_2 V(0) = \frac{c^-}{\lambda_+ \gamma} A_2 w^+ + c^- A_2 A_1^{-2} b_1 = \frac{c^-}{\gamma}\begin{pmatrix} 0 \\ w_2^+ - 1 \end{pmatrix}$$

Recalling (22) we thus have,

$$\overset{\circ}{V}(x) = e^{A_2 x}\left\{\frac{c^-}{\gamma(w_2^+ - 1)}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{c^+}{d} A_2^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} - \frac{c^+}{d} A_2^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for } x \geq 0 \quad (54)$$

Now

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -q_1 \\ q_2 \end{pmatrix} \quad \text{are right eigenvectors of } A_2 \text{ corresponding to the}$$

$$\text{eigenvalues } \left(\frac{-\gamma}{d}\right) \text{ and } -\left(\frac{\gamma + q_1 + q_2}{d}\right) \text{ respectively} \quad (55)$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{q_1}{q_1 + q_2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{q_1 + q_2}\begin{pmatrix} -q_1 \\ q_2 \end{pmatrix} \quad (56)$$

Moreover, equating the first components of both sides of the equation $A_1 w^+ = \lambda_+ w^+$ and noting (4), we have

$$w_2^+ = \frac{1}{q_1}[\gamma + q_1 - \lambda_+(r - d)] \quad (57)$$

Using (55), (56) and (57) in (54), we get

$$\dot{V}_1(x) = \frac{c^+}{\gamma}\left[1 - e^{-\frac{\gamma}{d}x}\right] + \frac{c^-(\gamma - \lambda_+(r-d))}{\gamma(q_1+q_2)}\left[e^{-\frac{\gamma}{d}x} - e^{-\frac{(\gamma + q_1 + q_2)}{d}x}\right] \quad \text{for } x \geq 0$$

If $\gamma - \lambda_+(r - d) \geq 0$, then clearly $\dot{V}_1(x) > 0$ for $x > 0$ and (53) is valid.

So suppose that $\gamma - \lambda_+(r - d) < 0$ and note, by differentiating, that

$$\ddot{V}_1(x) = \left[\frac{c^+}{d} - \frac{c^-(\gamma - \lambda_+(r-d))}{\gamma(q_1 + q_2)} \cdot \frac{\gamma}{d}\right]e^{-\frac{\gamma}{d}x} + \frac{c^-(\gamma - \lambda_+(r-d))}{\gamma(q_1+q_2)} \frac{(\gamma + q_1 + q_2)}{d}$$

$$\text{for } x \geq 0$$

If $p \geq 0$ and $n \leq 0$ are constants, then $pe^{-\frac{\gamma}{d}x} + ne^{-\frac{(\gamma + q_1 + q_2)}{d}x} \geq 0$

for all $x \geq 0$ if and only if $p + n \geq 0$. Hence, to show that

$$\ddot{V}_1(x) \geq 0 \qquad \text{for all } x \geq 0$$

we only need to verify that

$$\frac{c^+}{d} + \frac{c^-(\gamma - \lambda_+(r-d))}{\gamma(q_1+q_2)}\left[\frac{\gamma + q_1 + q_2}{d} - \frac{\gamma}{d}\right] \geq 0 \qquad \text{for all } x \geq 0$$

or equivalently, that

$$c^+\gamma + c^-(\gamma - \lambda_+(r - d)) \geq 0$$

It is easy to check that $(\lambda_+ + \lambda_-)(r - d) = (\gamma + q_1) - (\gamma + q_2)\frac{(r-d)}{d}$, and so, substituting for λ_+, we only need to verify that

$$c^-\left[\frac{\gamma r}{d} + q_2 \frac{(r-d)}{d} - (\gamma + q_1 - \lambda_-(r - d))\right] + c^+\gamma \geq 0$$

But this is in turn equivalent to,

$$\frac{c^+}{c^+ + c^-} \left[1 + \frac{\gamma d}{q_1 d - (\gamma + q_2 + \lambda_- d)(r - d)} \right] \geq 1$$

which is in fact true, since $z*$ given by (42) satisfies $z* = 0$. Hence (53) is valid, proving (52.ii).

$\square$

Turning now to the case $z* > 0$ we show a similar result.

Lemma 4

Suppose $z*$ given by (42) is strictly positive, and $V(\cdot)$ is defined by (46), then (52.i) and (52.ii) are valid.

Proof

It is easily checked that $V(\cdot)$ satisfies (23) for $x < 0$, $0 < x < z*$ and $x > z*$, and so in all three of these cases (52.i) is satisfied. At $x = 0$, (50) and (46) again show that $\overset{\circ}{V}(0) = A_1 V(0)$ and so (52.i) is also valid at $x = 0$. Turning now to $x = z*$, we note from (51) that

$$\begin{pmatrix} (\pi^{z*}(z*) - d)\overset{\circ}{V}_1(z*) \\ -d\overset{\circ}{V}_2(z*) \end{pmatrix} = \frac{c^+ d}{q_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -d\overset{\circ}{V}(z*)$$

Also,

$$\begin{bmatrix} \gamma + q_1 & -q_1 \\ -q_2 & \gamma + q_2 \end{bmatrix} V(z*) - \begin{pmatrix} 1 \\ 1 \end{pmatrix} c^+ z* = -d\{A_2 V(z*) + b_2 c^+ z*\}$$

$$= -d \lim_{x \to z*} \{A_2 V(x) + b_2 c^+ x\} = -d \lim_{x \to z*} \overset{\circ}{V}(x) = -d\overset{\circ}{V}(z*)$$

where we have used the continuous differentiability of $V(\cdot)$. Hence (52.i) is valid for all x. Now turning to (52.ii), since $\pi^{z*}(\cdot)$ is of the form shown in (2), we need to show that

$$\dot{V}_1(x) \le 0 \quad \text{for } x < z* \quad \text{and} \quad \dot{V}_1(x) \ge 0 \quad \text{for } x > z* \tag{58}$$

Consider $0 \le x < z*$ first. From (46) we obtain

$$\ddot{V}(x) = e^{A_1(x - z*)} \left[b_1 c^+ - A_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{c^+}{q_1} \right] \quad \text{for } 0 \le x \le z*$$

where, by $\ddot{V}(0)$ and $\ddot{V}(z*)$ we mean $\ddot{V}(0+)$ and $\ddot{V}(z*-)$ respectively. Now

$$b_1 c^+ - A_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{c^+}{q_1} = \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

where $\theta := \frac{c^+}{d} + \frac{c^+}{d} \frac{(\gamma + q_2)}{q_1} > 0$. Let $t := -(x - z*) \ge 0$ and denoting by $\mathscr{L}^{-1}$ the inverse Laplace transform, we have

$$\ddot{V}_1(x) = [1,0] e^{-A_1 t} \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \mathscr{L}^{-1} \left\{ [1,0](sI + A_1)^{-1} \begin{pmatrix} 0 \\ \theta \end{pmatrix} \right\}$$

$$= \mathscr{L}^{-1} \left\{ \frac{\theta q_1}{(s - \lambda_+)(s - \lambda_-)(r - d)} \right\} = \frac{q_1 \theta (e^{\lambda_+ t} - e^{\lambda_- t})}{(r - d)(\lambda_+ - \lambda_-)} \ge 0$$

with strict inequality for $x \ne z*$. Moreover, from (51), $\dot{V}_1(z*) = 0$, and so the validity of (58) for $0 < x < z*$ is established. By continuity of $\dot{V}_1(x)$, we see now that $\dot{V}_1(0) < 0$, thereby establishing (58) for $x = 0$ also. Now we consider $x < 0$. Since (23) is satisfied for $x < 0$, we see that

$$V(x) = e^{A_1 x}[V(0) - A_1^{-2} b_1 c^-] + A_1^{-1} b_1 c^- x + A_1^{-2} b_1 c^- \quad \text{for } x \le 0 \tag{59}$$

Hence

$$\dot{V}(x) = A_1 e^{A_1 x}[V(0) - A_1^{-2} b_1 c^-] + A_1^{-1} b_1 c^- \quad \text{for } x \le 0$$

From Lemma 2, we know that $V(0) - A_1^{-2} b_1 c^- = k w^+$ for some constant k, and so

$$\dot{V}(x) = k \lambda_+ e^{\lambda_+ x} w^+ + A_1^{-1} b_1 c^- \quad \text{for } x \le 0$$

Noting (27) and (37), we have

$$\dot{V}_1(x) = k\lambda_+ e^{\lambda_+ x} - \frac{c^-}{\gamma} = e^{\lambda_+ x} \dot{V}_1(0) - \frac{c^-}{\gamma}(1 - e^{\lambda_+ x}) \qquad \text{for } x \leq 0$$

Since $\lambda_+ x \leq 0$ for $x \leq 0$ and since $\dot{V}_1(0) < 0$ as previously shown, it follows that $\dot{V}_1(x) < 0$ for $x \leq 0$ and so (58) is valid for $x \leq 0$ in addition to $0 \leq x < z^*$. Now we consider $x > z^*$. From (47) we have,

$$\ddot{V}(x) = e^{A_2(x - z^*)} \left\{ -A_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{c^+}{q_1} + b_2 c^+ \right\} \qquad \text{for } x \geq z^*$$

$$= \mu e^{A_2(x - z^*)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{for } x \geq z^*$$

where $\mu := \frac{c^+}{d}(1 + \frac{\gamma + q_2}{q_1}) > 0$, and by $\ddot{V}(z^*)$ we mean $\ddot{V}(z^*+)$. Now using (55) and (56) we get

$$\ddot{V}_1(x) = \frac{\theta q_1}{q_1 + q_2} \left\{ e^{-\frac{\gamma}{d}(x - z^*)} - e^{-\frac{(\gamma + q_1 + q_2)}{d}(x - z^*)} \right\} \geq 0 \qquad \text{for } x \geq z^*$$

with strict inequality except at $x = z^*$. Since $\dot{V}_1(z^*) = 0$, the validity of (58) is also established for $x > z^*$. □

Theorem 5

Let

60.i) z^* be defined as in (42)

60.ii) $\pi^{z^*}(\cdot)$ be defined as in (2)

60.iii) $V(\cdot)$ be defined as in (44) or (46) depending on whether $z^* = 0$ or $z^* > 0$.

Then

60.iv) $V(\cdot)$ is continuously differentiable

60.v) $\pi^{z^*}(\cdot)$ and $V(\cdot)$ satisfy (52.i) and (52.ii) and hence the Hamilton-Jacobi-Bellman dynamic programming equation.

60.vi) $V(x) = 0(|x|)$ as $x \to \pm\infty$.

Proof

We have already established (60.iv) and (60.v) in Lemmas 1, 3 and 4. To show (60.vi) note that (44) and (46), (26), (27) together with Lemma 2 show that $V(x) = 0(|x|)$ as $x \to -\infty$. Moreover, since both eigenvalues of A_2 are strictly negative, it follows from (44) and (46) that $V(x) = 0(x)$ as $x \to +\infty$ also. □

X. ADMISSIBLE POLICIES

It is now time for us to address more general issues. We begin by defining the class of admissible policies.

Definition: A measurable function $\pi : R \to [0,r]$ will be called an __admissible__ policy if, for every $(\tau,\xi) \in R^2$ with $\tau \geq 0$, there exists a function $y_\pi(t;\tau,\xi)$ which satisfies

(61.i) $y_\pi(t;\tau,\xi)$ is absolutely continuous in t

(61.ii) $y_\pi(t;\tau,\xi) = \xi + \int_\tau^t (\pi(y_\pi(s;\tau,\xi)) - d)ds$ for $t \geq \tau$

(61.iii) $y_\pi(t;\tau,\xi)$ is continuous in (t,τ,ξ).

(61.iv) $y_\pi(\cdot)$ is the unique function satisfying (i and ii) above.

Given such an admissible we now describe the manner in which we interpret the differential equation (5.i). Let $\{s(t,\omega); t \geq 0\}$ be a realization of (5.ii) with, say, $s(0,\omega) = 1$ and suppose x_0 is the initial inventory level. Define $\tau_0(\omega) := \inf\{t > 0 : s(t,\omega) = z\}$ and $\tau_{i+1}(\omega) = \inf\{t > \tau_i(\omega); s(t+,\omega) \neq s(t-,\omega)\}$. Then we construct the process $\{x_\pi(t,\omega)\}$ by,

$$x_\pi(t,\omega) := y_\pi(t;0,x_0) \quad \text{for } 0 \leq t \leq \tau_0(\omega)$$
$$:= x_\pi(\tau_i(\omega),\omega) - d(t - \tau_i(\omega)) \quad \text{for } \tau_i(\omega) \leq t \leq \tau_{i+1}(\omega)$$
$$\text{and } i = 0,2,4,\ldots$$
$$:= y_\pi(t;\tau_i(\omega), x_\pi(\tau_i(\omega),\omega)) \quad \text{for } \tau_i(\omega) \leq t \leq \tau_{i+1}(\omega)$$
$$\text{and } i = 1,3,5,\ldots .$$

Note that an immediate consequence is

$$x_\pi(t,\omega) = x_0 + \int_0^t (u_\pi(s,\omega) - d)ds \qquad \text{for all } t \geq 0 \tag{62}$$

where

$$u_\pi(t,\omega) = 0 \qquad \text{if } s(t,\omega) = 2$$
$$= \pi(x_\pi(t,\omega)) \qquad \text{if } s(t,\omega) = 1$$

Thus, the differential equation (5.i) is interpreted in integral form in (62).

One can use the theory of semigroups of nonlinear contractions in Banach spaces, see Barbu [6], to obtain sufficient conditions for a policy π to be admissible. We now use this to establish the admissibility of policies of the $\pi^z(\cdot)$ type.

Theorem 6

$\pi^z(\cdot)$ defined by (18) is admissible.

Proof

Let A, a multivalued operator, or equivalently a subset of R^2, be defined by

$$A(x) := \{r - d\} \qquad \text{if } x < z$$
$$:= [-d, r - d] \qquad \text{if } x = z$$
$$:= \{-d\} \qquad \text{if } x > z$$

Then $x_1 \leq x_2$ and $y_1 \in A(x_1)$, $y_2 \in A(x_2)$ implies that $(x_1 - x_2)(y_1 - y_2) \leq 0$ and so A is a dissipative operator, see Definition 3.1, page 71 of Barbu [6]. Moreover

$$\bigcup_{x \in R} \bigcup_{y \in A(x)} \{x - y\} = R$$

and so A is m-dissipative, see [6, page 71]. Also, for every $x \in R$,

$(\pi^z(x) - d) \in A(x)$ and $|\pi^z(x) - d| \leq |y|$ for every $y \in A(x)$

By Corollary 1.1 and Theorem 1.6 of [6, page 118] we see that (61.i, ii, and iv) are satisfied. Moreover, by Proposition 1.2 [6, page 110], $y_{\pi^z}(t;\tau,\xi)$ as a function of t, for each ξ, is a semigroup of nonlinear contractions, and so from Definition 1.1 of [6, page 98] we see that $|y_{\pi^z}(t;\tau,\xi_1)$ $- y_{\pi^z}(t;\tau,\xi_2)| \leq |\xi_1 - \xi_2|$. Since by uniqueness $y_{\pi^z}(t;\tau,\xi) = y_{\pi^z}(t-\tau; 0,\xi)$ for all $t \geq \tau$, it follows from $|y_{\pi^z}(t;\tau_1,\xi_1) - y_{\pi^z}(s;\tau_2,\xi_2)| \leq |y_{\pi^z}(t-\tau_1; 0,\xi_1)$ $- y_{\pi^z}(t-\tau_2; 0,\xi_1)| + |y_{\pi^z}(t-\tau_2; 0,\xi_1)| - y_{\pi^z}(t-\tau_2; 0,\xi_2)|$ that $y_{\pi^z}(\cdot)$ is a continuous function, and so (61.iii) is also satisfied.   □

XI. INTEGRAL EQUATION FOR COST FUNCTION

In this section we will show that the cost function corresponding to a policy π satisfies a certain integral equation.

Let $\{\tau_i\}$ be the successive jump times of $\{s(t)\}$. If $\{x_\pi(t); t \geq 0\}$ is the trajectory resulting from a policy π, define

$$V_{i,\pi}^n(\xi) := E[\int_0^{\tau_n} c(x_\pi(t))e^{-\gamma t}dt \Big| s(0) = i, x_\pi(0) = \xi]$$

as the cost of using π upto the n-th jump of $\{s(t)\}$. Here

$$c(x) := c^+x^+ + c^-x^-$$

Clearly

$$V_{i,\pi}(\xi) = \lim_{n \to \infty} V_{i,\pi}^n(\xi) \tag{63}$$

is the corresponding expected cost of using π indefinitely.

Define

$$x_\pi^1(t,\xi) := y_\pi(t;0,\xi)$$

$$x_\pi^2(t,\xi) := \xi - td$$

Clearly $x_\pi^i(t,\xi)$ represents the inventory level at time t if initially the inventory level is ξ, $s(0) = i$, and there are no jumps of $\{s(t)\}$ in $[0,t)$. By a renewal argument it follows that

$$V_{i,\pi}^0(\xi) = 0$$

and

$$V_{i,\pi}^{n+1}(\xi) = \int_0^\infty q_i e^{-q_i\sigma} [\int_0^\sigma e^{-\gamma t} c(x_\pi^i(t,\xi))dt + e^{-\gamma\sigma} V_{j(i),\pi}^n(x_\pi^i(\sigma,\xi))]d\sigma \qquad (64)$$

where

$$j(i) \in \{1,2\}, \quad j(i) \neq i \quad \text{for } i = 1,2.$$

For $\epsilon > 0$, let $\mathcal{F}_\epsilon$ be the Banach space of all measurable functions mapping R into R, with norm defined by $||f||_\epsilon := \sup_x |e^{-\epsilon|x|} f(x)|$, and let $\mathcal{F} := \bigcap_{\epsilon > 0} \mathcal{F}_\epsilon$. On $\mathcal{F}_\epsilon^2 = \mathcal{F}_\epsilon \times \mathcal{F}_\epsilon$, define $||(f_1,f_2)||_\epsilon := \max_i ||f_i||_\epsilon$, and note that $\mathcal{F}^2 = \bigcap_{\epsilon > 0} \mathcal{F}_\epsilon^2$. For $(f_1,f_2) \in \mathcal{F}^2$, define $T_\pi(f_1,f_2) = (T_{1,\pi}f_2, T_{2,\pi}f_1)$ by

$$(T_{i,\pi}f_{j(i)})(\xi) := \int_0^\infty q_i e^{-q_i\sigma} [\int_0^\sigma e^{-\gamma t} c(x_\pi^i(t,\xi))dt + e^{-\gamma\sigma} f_{j(i)}(x_\pi^i(\sigma,\xi))]d\sigma$$

$$\qquad (65)$$

Lemma 7

i) If $(f_1,f_2) \in \mathcal{F}^2$, then $T_\pi(f_1,f_2) \in \mathcal{F}^2$.

ii) T_π is a contraction with respect to the norm $||\cdot||_\epsilon$ for every $\epsilon > 0$ sufficiently small.

Proof

It suffices to show that $T_{i,\pi}f_{j(i)} \in \mathcal{F}$ and that $T_{i,\pi}$ is a contraction for $i = 1,2$. Note first that by (62),

$$|x_\pi^i(t,\xi)| \leq |\xi| + k_1 t$$

In the above and what follows, all the k_i's are constants chosen appropriately. Since $c(x) \le k_2|x|$, it follows that

$$\int_0^\infty q_i e^{-q_i\sigma}\Big[\int_0^\sigma e^{-\gamma t}c(x_\pi^i(t,\xi))dt\Big]d\sigma \le k_3|\xi| + k_4$$

Also, for $0 < \epsilon < \dfrac{(\gamma + q_i)}{k_1}$

$$\int_0^\infty q_i e^{-(q_i+\gamma)\sigma}|f_{j(i)}(x_\pi^i(\sigma,\xi))|d\sigma \le \int_0^\infty q_i e^{-(q_i+\gamma)\sigma}||f_{j(i)}||_\epsilon \, e^{\epsilon|x_\pi^i(\sigma,\xi)|}d\sigma$$

$$\le \int_0^\infty q_i e^{-(q_i+\gamma)\sigma}||f_{j(i)}||_\epsilon \, e^{\epsilon(|\xi| + k_1\sigma)}d\sigma$$

$$\le k_5 e^{\epsilon|\xi|} + k_6$$

Hence $|T_{i,\pi}f_{j(i)}(\xi)| \le k_5 e^{\epsilon|\xi|} + k_3|\xi| + k_7$ and so, $T_{i,\pi}f_{j(i)} \in \mathcal{F}_\epsilon$ for all $\epsilon > 0$ sufficiently small, i.e. $T_{i,\pi}f_{j(i)} \in \mathcal{F}$. To show that $T_{i,\pi}$ is a contraction, consider $0 < \epsilon < \dfrac{\gamma}{k_1}$, then

$$(T_{i,\pi}f - T_{i,\pi}g)(\xi) \le \int_0^\infty q_i e^{-(q_i+\gamma)\sigma}|f(x_\pi^i(\sigma,\xi)) - g(x_\pi^i(\sigma,\xi))|d\sigma$$

$$\le \int_0^\infty q_i e^{-(q_i+\gamma)\sigma}e^{\epsilon|x_\pi^i(\sigma,\xi)|}||f - g||_\epsilon \, d\sigma$$

$$\le \int_0^\infty q_i e^{-(q_i+\gamma)\sigma}e^{\epsilon(|\xi| + k_1\sigma)}||f - g||_\epsilon \, d\sigma$$

$$= e^{\epsilon|\xi|}||f - g||_\epsilon \int_0^\infty q_i e^{-q_i\sigma}e^{(\epsilon k_1 - \gamma)\sigma}d\sigma$$

$$\le \beta e^{\epsilon|\xi|}||f - g||_\epsilon$$

where $0 < \beta < 1$. Hence $|T_{i,\pi}f - T_{i,\pi}g||_\epsilon \le \beta||f - g||_\epsilon$. □

From (64) it follows that if $V_\pi^n := (V_{1,\pi}^n, V_{2,\pi}^n)$, then $V_\pi^{n+1} = T_\pi V_\pi^n$ for $n = 0,1,2,\ldots$ with $V_\pi^0 := 0$.

Theorem 8

Let $V_{i,\pi}(\xi)$ denote the cost of using π starting in the state $(s(0) = 1, x(0) = \xi)$. Let $V_\pi(\xi) := (V_{1,\pi}(\xi), V_{2,\pi}(\xi))$. Then

66.i) V_π is the unique solution in $\mathscr{F}^2$ of the integral equation.

$$V_{i,\pi}(\xi) = \int_0^\infty q_i e^{-q_i \sigma} [\int_0^\infty e^{-\gamma t} c(x_\pi^i(t,\xi)) dt + e^{-\gamma \sigma} V_{j(i),\pi}(x_\pi^i(\sigma,\xi))] d\sigma$$

for every $\xi \in R$.

66.ii)   For every $f \in \mathscr{F}^2$, $\lim_{n \to \infty} T_\pi^{(n)} f = V$.

Proof

From (63) and (64), we see that $V_\pi = \lim_{n \to \infty} T_\pi^{(n)} 0$, where $T_\pi^{(n)}$ denotes the n-fold iterate of T_π, and 0 is the identically zero function. However, since T_π is a contraction, $\lim_{n \to \infty} T_\pi^{(n)} f$ is a unique fixed point of T_π, for every $f \in \mathscr{F}$. Hence V_π is the unique solution, in, $\mathscr{F}$ of $V_\pi = T_\pi V_\pi$.   □

It is important to note that the boundary condition for the integral equation is really the condition that the solution be in $\mathscr{F}_2$. This is a condition on the asymptotic growth rate, and serves to differentiate the infinite horizon problem treated here from the finite horizon problem in [3].

XII. OPTIMALITY of π^{z^*}

We are now ready to prove the optimality of the suggested policy. We shall actually show that optimality is a consequence of π^{z^*} and $V(\cdot)$ satisfying the HJB equation (60.iv,v,vi) for the infinite time problem and so our proof is quite general.

Theorem 9

Let z^* and $\pi^{z^*}(\cdot)$ be as in (3) and (2). If $z^* = 0$, define $V(\cdot)$ by (44), while if $z^* > 0$, define $V(\cdot)$ by (46). Then

i) If $V_\pi(\cdot)$ represents the cost function corresponding to an admissible policy π, then

$$V_{i,\pi^{z^*}}(\xi) \leq V_{i,\pi}(\xi) \qquad \text{for } i = 1,2; \xi \in R \text{ and all admissible } \pi$$

ii) $V_{\pi^{z*}}(\xi) = V(\xi)$ for every $\xi \in R$

Proof

We will show that

$$T_\pi V_{\pi^{z*}} \geq V_{\pi^{z*}} \qquad \text{for every admissible } \pi \tag{67}$$

i.e. $T_{i,\pi} V_{j(i),\pi^{z*}}(\xi) \geq V_{i,\pi^{z*}}(\xi) \qquad$ for every $\xi \in R$ and $i = 1,2.$ (68)

Since T_π is monotone, (67) implies that $T_\pi^{(n)} V_{\pi^{z*}} \geq V_{\pi^{z*}}$. Taking the limit in n and using (66.ii), we obtain $V_\pi \geq V_{\pi^{z*}}$, that is (i) for i = 1,2. So our goal is to show (67), along with equality when $\pi = \pi^{z*}$. Considering (52.i, ii) we have

$$\min_{u \in [0,r]} (u - d)\dot{V}_1(x) = (\gamma + q_1)V_1(x) - q_1 V_2(x) - c(x) \tag{69}$$

$$-d\dot{V}_2(x) = -q_2 V_1(x) + (\gamma + q_2)V_2(x) - c(x) \tag{70}$$

For any π therefore, (69) implies that

$$c(x) \geq (\gamma + q_1)V_1(x) - q_1 V_2(x) - (\pi(x) - d)\dot{V}_1(x) \tag{71}$$

and so for any admissible π,

$$c(x_\pi^1(t,\xi)) \geq (\gamma + q_1)V_1(x_\pi^1(t,\xi)) - q_1 V_2(x_\pi^1(t,\xi)) - (\pi(x_\pi^1(t,\xi)) - d)\dot{V}_1(x_\pi^1(t,\xi))$$
$$\tag{72}$$

Now noting that $x_\pi^1(t,\xi)$ is absolutely continuous in t, with derivative $(\pi(x_\pi^1(t,\xi)) - d)$, and $\dot{V}_1(\cdot)$ is continuous, we can apply Corollary 7 of [7] showing that the chain rule is valid, and so obtain

$$\frac{d}{dt}V_1(x_\pi^1(t,\xi)) = \dot{V}_1(x_\pi^1(t,\xi))(\pi(x_\pi^1(t,\xi)) - d) \qquad \text{a.e.} \tag{73}$$

Hence, from (72) and (73), we have

$$\int_0^\sigma e^{-\gamma t} c(x_\pi^1(t,\xi))dt \geq \int_0^\sigma e^{-\gamma t}[(\gamma + q_1)V_1(x_\pi^1(t,\xi)) - q_1 V_2(x_\pi^1(t,\xi))]dt$$

$$\tag{74}$$

$$- \int_0^\sigma e^{-\gamma t}\frac{d}{dt}V_1(x_\pi^1(t,\xi))dt \qquad \text{for } \sigma \geq 0$$

Integrating the last term in (74) by parts, see Hewitt and Stromberg [9, p.287],
we have

$$\int_0^\sigma e^{-\gamma t}c(x_\pi^1(t,\xi))dt \geq \int_0^\sigma e^{-\gamma t}q_1(V_1(x_\pi^1(t,\xi)) - V_2(x_\pi^1(t,\xi)))dt$$

$$+ V_1(\xi) - e^{-\gamma\sigma}V_1(x_\pi^1(\sigma,\xi)) \qquad \text{for } \sigma \geq 0$$

Hence,

$$\int_0^\infty q_1 e^{-q_1\sigma}\left[\int_0^\sigma e^{-\gamma t}c(x_\pi^1(t,\xi))dt\right]d\sigma \geq V_1(\xi) - \int_0^\infty q_1 e^{-(\gamma + q_1)\sigma}V_1(x_\pi^1(\sigma,\xi))d\sigma$$

$$+ \int_0^\infty q_1 e^{-q_1\sigma}\left[\int_0^\sigma e^{-\gamma t}q_1(V_1(x_\pi^1(t,\xi)) - V_2(x_\pi^1(t,\xi)))dt\right]d\sigma$$

$$= V_1(\xi) - \int_0^\infty q_1 e^{-(\gamma + q_1)\sigma}V_1(x_\pi^1(\sigma,\xi))d\sigma$$

$$+ \int_0^\infty q_1 e^{-\gamma t}(V_1(x_\pi^1(t,\xi)) - V_2(x_\pi^1(t,\xi)))\left[\int_t^\infty q_1 e^{-q_1\sigma}d\sigma\right]dt$$

$$= V_1(\xi) - \int_0^\infty q_1 e^{-(\gamma + q_1)t}V_2(x_\pi^1(t,\xi))dt$$

Hence

$$\int_0^\infty q_1 e^{-q_1\sigma}\left[\int_0^\sigma e^{-\gamma t}c(x_\pi^1(t,\xi))dt + e^{-\gamma\sigma}V_2(x_\pi^1(\sigma,\xi))\right]d\sigma \geq V_1(\xi)$$

i.e.

$$(T_{1,\pi}V_2)(\xi) \geq V_1(\xi)$$

noting that from (60.iv), $V(x) = 0(x)$ and so $V \in \mathcal{F}^2 = \text{Domain }(T_\pi)$. Using
(70), similarly we deduce that $(T_{2,\pi}V_1)(\xi) = V_2(\xi)$. Thus we have shown
$T_\pi V \geq V$. On the other hand since $\pi^{z^*}(\cdot)$ attains equality in (71), we have

$T_{\pi^{z^*}} V = V$. Thus $V_{\pi^{z^*}} = V \leq T_\pi V = T_\pi V_{\pi^{z^*}}$ with equality when $\pi = \pi^{z^*}$. $\square$

XIII. CONCLUDING REMARKS

There are two directions in which more work is needed. The first is to realize the full program for flexible manufacturing systems outlined in Kimemia and Gershwin [2]. Consider a flexible manufacturing system making p parts on m machines. Part j requires a_{ij} units of time on machine i. Thus if a subset $s_k \subseteq \{1,2,\ldots,m\}$ of machines is functioning, while the rest have failed, then a vector $u = (u_1, u_2, \ldots, u_p)^T$ of production rates is feasible if and only if $u \in U_k$ where $U_k := \left\{ u : \sum_{j=1}^{p} a_{ij} u_j \leq 1 \text{ if } i \in s_k, \text{ or } = 0 \right.$ $\left. \text{if } i \notin s_k \right\}$. Suppose now that each machine is subject to occasional failure and let $\{s(t); t \geq 0\}$ be a Markov chain with state space $\{s_1, s_2, \ldots, s_{2^m}\}$. Given a demand rate vector $d = (d_1, d_2, \ldots, d_p)^T$, we have the problem

$$\dot{x} = u(t) - d$$

$\{s(t); t \geq 0\}$ is a Markov chain with state space $\{s_1, \ldots, s_{2^m}\}$

$u(t) \in U_k$ if $s(t) = s_k$

Minimize $E \int_0^\infty e^{-\gamma t} c(x(t)) dt$

where $c(\cdot)$ is some convex cost function. Due to the multidimensional nature of $x(t)$, this problem is much more difficult than the one solved here. [2] has proposed an approximation, but the optimal solution needs more study.

The other direction in which more research is needed is theoretical, and is the problem of optimal control of continuous time systems with jump Markov disturbances. As in Rishel [3], we also have proved optimality only within the class of Markov policies. For discrete time systems, see Blackwell [10] for example, much more progress has been made on optimal control, and one usually considers a much more general class of policies within which optimality is proven. The question of existence of optimal controls also needs more study. Finally, more work needs to be done on the average cost problem for systems with jump Markov disturbances, see Tsitsiklis [8].

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