On the Optimal Number of Active Receivers in Fading Broadcast Channels

Hengameh Keshavarz, *Student Member, IEEE,*
Liang-Liang Xie, *Member, IEEE,*
Ravi R. Mazumdar, *Fellow, IEEE*

***Abstract*—For broadcast channels, a power allocation scheme is proposed to maximize the number of active receivers, for each of which, a minimum rate $R_{\min} > 0$ can be achieved. Under the assumption of independent Rayleigh fading channels for different receivers, as the total number of receivers n goes to infinity, the maximum number of active receivers is shown to be arbitrarily close to $\ln(P \ln n)/R_{\min}$ with probability approaching 1, where P is the total transmit power.**

***Index Terms*—Broadcast channels, fading channels, minimum rate constraint, power allocation, scaling laws, user capacity.**

I. Introduction

In a broadcast system where the transmitter can allocate different portions of its total transmit power to different receivers according to their channel states, there is a basic trade-off between the total throughput and the minimum rate achievable for all the receivers. To increase the total throughput, it is always favorable to allocate more power to receivers with better channel states, while in order to increase the minimum rate, obviously, more power should be allocated to receivers with worse channel states.

In a dynamic environment, where the channel states are time-varying, opportunistic power allocation schemes can be exploited to increase the total throughput while maintaining an average rate constraint for each receiver. The basic idea is to adapt the power allocation to the variations of the channel states. The transmission rate for a receiver is increased when its channel state becomes better, thus higher rates can be achieved at the expense of less power. However, in delay-sensitive applications, it may not be admissible for a receiver to wait too long before its rate is increased. Basically, this raises an issue of the trade-off between ergodic capacity and outage capacity, for which, extensive studies have been given in [1], [2], [3] in the context of broadcast channels.

In this paper, we consider a power allocation scheme with a minimum rate constraint $R_{\min} > 0$. Since for a fixed $R_{\min}$, in a time-varying fading environment, it may not be always possible for all receivers to achieve this minimum rate simultaneously, we propose a scheme to maximize the number of active receivers, for each of which, such a minimum rate can be supported, while allocating no power to the other inactive receivers.

By adjusting the value of $R_{\min}$, different trade-offs between the total throughput and the delay can be achieved. Specifically, by increasing $R_{\min}$, the power is shared among fewer receivers with relatively better channel states, thus resulting in higher total throughput; However, this also results in delay for more inactive receivers, thus longer delay for each receiver on average. On the other hand, by choosing $R_{\min}$ small enough, it is possible to make it simultaneously achievable for all the receivers, thus resulting in no delay for any receiver; However, it may be too costly to save receivers at extremely bad channel states.

While the number of supportable active receivers depends on the specific channel states, we analyze the asymptotic behavior when the total number of receivers n is large. Under the assumption of independent Rayleigh fading channels for different receivers with unit noise variance, we show that the maximum number of active receivers m is very close to $\ln(P \ln n)/R_{\min}$ with probability approaching 1, where P is the total transmit power, and $R_{\min}$ is in the unit of nats. Actually this approximation is surprisingly close in the sense that both the lower and upper bounds of m differ only by $\epsilon > 0$, which can be made arbitrarily small.

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The authors are with the Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, ON N2L 3G1, Canada (email: hkeshava@uwaterloo.ca; llxie@ece.uwaterloo.ca; mazum@ece.uwaterloo.ca).

II. Main Results

Consider a broadcast system with one transmitter and n receivers with the following channel model in the time block $t = 1, 2, \ldots, T$:

$$Y_i(t) = g_i X(t) + Z_i(t), \quad i = 1, 2, \ldots, n, \tag{1}$$

where $X(t) \in \mathbb{C}$ is the signal sent by the transmitter, and $Y_i(t) \in \mathbb{C}$ is the signal received by receiver i. The noise $Z_i(t) \in \mathbb{C}$, $i = 1, \ldots, n$, $t = 1, \ldots, T$ are assumed to be i.i.d. complex Gaussian distributed according to $\mathcal{CN}(0, 1)$. The channel gains $g_i \in \mathbb{C}$, $i = 1, \ldots, n$ are assumed to be constant during this time block, and known to the transmitter and all the receivers.

Equivalently, the model (1) can be written as

$$Y_i'(t) = X(t) + Z_i(t)/g_i, \quad i = 1, 2, \ldots, n \tag{2}$$

where the noise $Z_i(t)/g_i$ is still complex Gaussian distributed, but with variance $1/|g_i|^2$.

Let $N_i = 1/|g_i|^2$. Without loss of generality, assume that $N_1 \leq N_2 \leq \cdots \leq N_n$. It is well known [4, Sec.14.6] that the broadcast channel (2) is stochastically degraded, and the capacity region is given by

$$R_i < \ln\left(1 + \frac{P_i}{\sum_{j=1}^{i-1} P_j + N_i}\right), \quad i = 1, \ldots, n \tag{3}$$

where R_i is the achievable rate for receiver i, to which, the power $P_i \geq 0$ is allocated by the transmitter under the total transmit power constraint: $\sum_{i=1}^{n} P_i = P$.

Different rates can be achieved by different power allocations in (3). To increase the total throughput $\sum_{i=1}^{n} R_i$, it is always favorable to allocate more power to receivers with smaller N_i, as demonstrated by the following lemma.

Lemma 2.1: For any two power allocation schemes $\{P_i, i = 1, \ldots, n\}$ and $\{P_i', i = 1, \ldots, n\}$ in (3), where for some $1 \leq i_1 < i_2 \leq n$ and $\Delta > 0$, $P_{i_1}' = P_{i_1} + \Delta$, and $P_{i_2}' = P_{i_2} - \Delta$, and $P_i = P_i'$ for any $i \notin \{i_1, i_2\}$, the following inequality always holds:

$$\sum_{i=1}^{n} \ln\left(1 + \frac{P_i}{\sum_{j=1}^{i-1} P_j + N_i}\right) \leq \sum_{i=1}^{n} \ln\left(1 + \frac{P_i'}{\sum_{j=1}^{i-1} P_j' + N_i}\right) \tag{4}$$

where "=" holds if and only if $N_{i_1} = N_{i_1+1} = \cdots = N_{i_2}$.

The proof of Lemma 2.1, as well as the proofs of all the conclusions in this section, are presented in the Appendix.

Obviously, in order to maximize the total throughput, all power should be allocated to receiver 1, which has the maximum channel gain $|g_1|$, or the minimum equivalent noise variance N_1. However, as explained in the Introduction, in order to maintain a trade-off between throughput and delay, we consider the following power allocation scheme:

$$\begin{cases} \max\{m\} & (5) \\ \ln\left(1 + \dfrac{P_1}{N_1}\right) \geq R_{\min} & (6) \\ \ln\left(1 + \dfrac{P_i}{\sum_{j=1}^{i-1} P_j + N_i}\right) = R_{\min}; \quad 2 \leq i \leq m & (7) \\ \sum_{i=1}^{m} P_i = P & (8) \end{cases}$$

where $R_{\min} > 0$ (in nats) is a pre-set minimum rate constraint for all active receivers.

The reason for setting "=" instead of "$\geq$" in (7) is that once the minimum rate is satisfied, any redundant power should be given to receiver 1 in order to maximize the total throughput, as implied by Lemma 2.1.

A simple algorithm to solve the optimization problem (5)-(8) is as the following.

First, the maximum m can be determined by recursively defining P_i', $i = 1, 2, \ldots$, according to the following equations:

$$R_{\min} = \ln\left(1 + \frac{P_i'}{\sum_{j=1}^{i-1} P_j' + N_i}\right), \quad i = 1, 2, \ldots, \tag{9}$$

until some integer m such that $\sum_{i=1}^{m} P_i' \leq P$ but $\sum_{i=1}^{m+1} P_i' > P$, or $m = n$.

After the maximum m is determined, the optimal power allocation can be obtained by letting $P_i = 0$ for $i = m+1, \ldots, n$, and choosing P_i, $i = m, m-1, \ldots, 2$ recursively according to the following equations:

$$R_{\min} = \ln\left(1 + \frac{P_i}{P - \sum_{j=i}^{m} P_j + N_i}\right), \quad i = m, \ldots, 2,$$

and at last, setting $P_1 = P - \sum_{j=2}^{m} P_j$.

Obviously, with fixed P and $R_{\min}$, the maximum number of active receivers completely depends on the equivalent noise variance $N_i = 1/|g_i|^2$, $i = 1, \ldots, n$. When the channel gains g_i obey some statistical distribution, asymptotic behavior of the maximum m can be determined when the total number of receivers n becomes large.

For example, consider independent Rayleigh fading channels for different receivers, i.e., the gains g_i, $i = 1, \ldots, n$ are independent realizations of the complex Gaussian distribution $\mathcal{CN}(0, 1)$. We have the following theorem.

Theorem 2.1: Under the assumption of independent Rayleigh fading channels for different receivers with the gain $g_i \sim \mathcal{CN}(0, 1)$, the maximum number of active receivers m determined by (5)-(8) is bounded as: for any $\epsilon > 0$,

$$\mathbb{P}(\lfloor \nu(n) - \epsilon \rfloor \leq m \leq \nu(n) + \epsilon) \to 1, \quad \text{as } n \to \infty, \tag{10}$$

where, $\lfloor x \rfloor$ denotes the maximum integer no greater than x, n is the total number of receivers, and

$$\nu(n) = \ln(P \ln n)/R_{\min}. \tag{11}$$

Remark 2.1: The probability in (10) converges to 1 at the following rates:[1]

$$\mathbb{P}(m < \lfloor \nu(n) - \epsilon \rfloor) = o\left(\exp\left(-\frac{n^{1-\lambda}}{2+\sigma}\right)\right) \tag{12}$$

and

$$\mathbb{P}(m > \nu(n) + \epsilon) = o\left(n^{1 - \frac{1}{\lambda(1+\sigma)}}\right) \tag{13}$$

where $\lambda \triangleq e^{-\epsilon R_{\min}} < 1$, and $\sigma > 0$ can be arbitrarily small.

Remark 2.2: Theorem 2.1 states that the number of active receivers is close to $\nu(n)$ with high probability. Actually, for any $\epsilon < \frac{1}{2}$, there are at most two integers during the range $\lfloor \nu(n) - \epsilon \rfloor \leq m \leq \nu(n) + \epsilon$. An interesting observation of the equation (11) is that the number of active receivers will almost double by halving $R_{\min}$, with the total power P and the total number of receivers n fixed.

[1] As standard notation, $o(\cdot)$ and $O(\cdot)$ have the following interpretations: for any positive infinite sequences $f(n)$ and $g(n)$, $n = 1, 2, \ldots$, $f(n) = o(g(n))$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$; $f(n) = O(g(n))$ means $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$.

Basically, Theorem 2.1 states a double logarithmic scaling law. That is, the maximum number of active receivers scales double logarithmically with the total number of receivers. This is a rather slow scaling, and is basically determined by the tail of the Rayleigh distribution. Comparatively, (11) can also be written as

$$\nu(n) = (\ln P + \ln \ln n)/R_{\min}$$

which shows that the maximum number of active receivers scales logarithmically with the total transmit power, and as remarked before, is inversely proportional to the minimum rate constraint.

According to Theorem 2.1, there are about $\nu(n)$ active receivers, for each of which, a minimum rate $R_{\min}$ can be achieved. Hence, the total throughput scales at least as

$$\nu(n) R_{\min} = \ln(P \ln n). \tag{14}$$

It is interesting to compare (14) with the maximum achievable total throughput when all the power is allocated to the best receiver, which can be shown (see the Appendix) to be upper bounded with probability approaching 1 by

$$\ln(1 + \beta P \ln n) \tag{15}$$

where the constant $\beta > 1$ can be arbitrarily close to 1. Obviously, as n increases, the difference between (14) and (15) decreases to $\ln \beta$, which can be made arbitrarily small. The essential reason for such a negligible difference is that for large n, the gains of the best $\nu(n)$ receivers are very close to each other. It should also be pointed out that the smaller $\ln \beta$ is, the slower the probability converges to 1, as can be seen from the proof.

Besides Rayleigh fading, one can also consider Ricean and Nakagami fading models. The analytic techniques developed for the Rayleigh distribution as presented in the Appendix can be similarly applied. Especially, noting that the scaling behavior only depends on the tails of the distribution function, both Ricean and Nakagami fading channels obey the double logarithmic scaling law. The analytical details are more complicated than the Rayleigh case presented in this paper and space limitations prevent their presentation here. The details will appear elsewhere.

III. Simulations

Consider a system with channel bandwidth of $B = 100K$ Hz. Then a transmission rate of $100K$ bits per second is equivalent to $100K/2B = 0.5$ bits per real sample, or 1 bit per complex sample as in model (1).

Figure 1 shows the optimal number of active users versus the total number of users for both fixed ($P = 10^4$, or equivalently, SNR $=$ 40 dB for model (1)) and linearly increasing ($P = n$, or equivalently, SNR $= 10 \log_{10} n$ dB) transmit power. The value of $\nu(n)$ given by (11) is also indicated in Figure 1. As shown in Figure 1 and mentioned in Remark 2.2, the number of active users is almost doubled as $R_{\min}$ is halved.

For a further comparison, the optimal number of active users versus different $R_{\min}$ for fixed SNR $=$ 40dB and $n = 1000$ is shown in Figure 2, where the curve of $\nu(n)$ is also drawn.

It is worth explicitly pointing out that the curves drawn in Figures 1 and 2 are single realizations, not the Monte-Carlo averages. This is consistent with Theorem 2.1, which concludes on most single sample paths individually, instead of only on their mean. That such a regular pattern can exhibit for a single sample path is because each sample path already consists of a large number of independent users. For further illustration, Figure 3 shows that as n increases, the sample paths will become more concentrated.

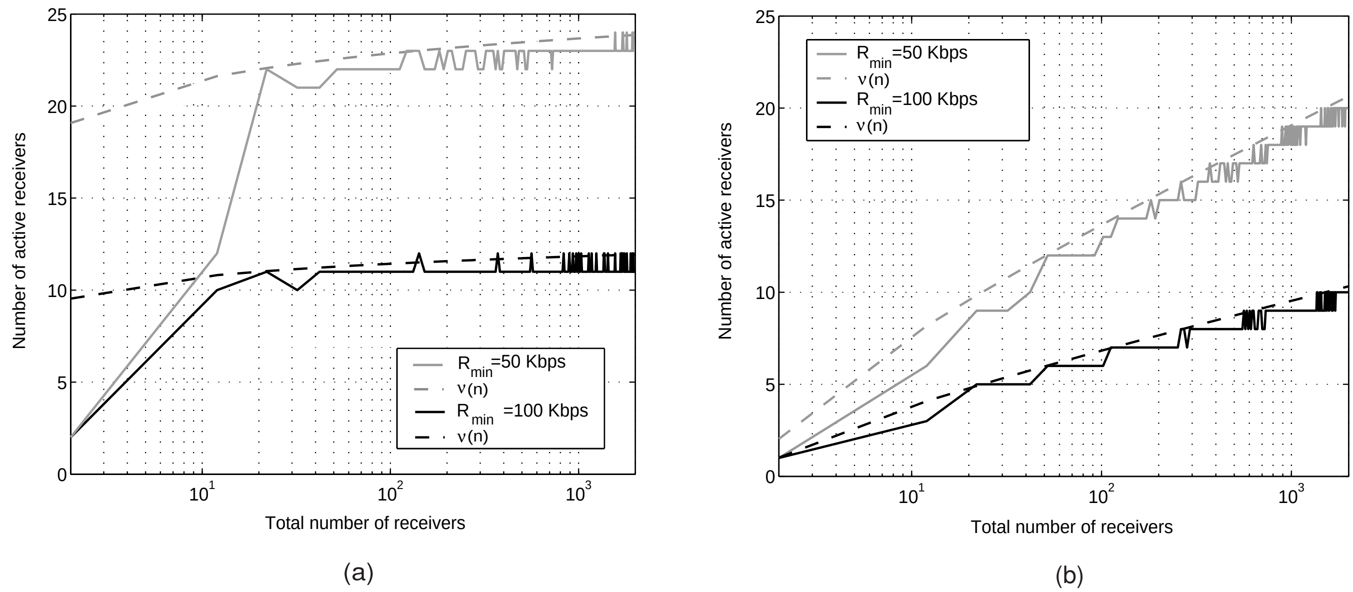

Fig. 1. The optimal number of active receivers versus the total number of users for model (1) and $R_{\min} = 50, 100\,\text{Kbp}$, (a) Fixed total transmit power: $P = 10^4$, or equivalently, SNR $= 40\,\text{dB}$, (b) Linearly increasing transmit power: $P = n$, or equivalently, SNR $= 10\log_{10} n\,\text{dB}$.

APPENDIX

Proof of Lemma 2.1: Obviously, by induction, we only need to prove the case when $i_2 = i_1 + 1$, for which, (4) is equivalent to

$$\sum_{i=i_1}^{i_1+1} \ln\left(1 + \frac{P_i}{\sum_{j=1}^{i-1} P_j + N_i}\right) \le \sum_{i=i_1}^{i_1+1} \ln\left(1 + \frac{P_i'}{\sum_{j=1}^{i-1} P_j' + N_i}\right)$$

which is equivalent to

$$\left(1 + \frac{P_{i_1}}{\sum_{j=1}^{i_1-1} P_j + N_{i_1}}\right)\left(1 + \frac{P_{i_1+1}}{\sum_{j=1}^{i_1-1} P_j + P_{i_1} + N_{i_1+1}}\right) \le$$
$$\left(1 + \frac{P_{i_1}+\Delta}{\sum_{j=1}^{i_1-1} P_j + N_{i_1}}\right)\left(1 + \frac{P_{i_1+1}-\Delta}{\sum_{j=1}^{i_1-1} P_j + P_{i_1} + \Delta + N_{i_1+1}}\right)$$

which is equivalent to

$$\frac{\sum_{j=1}^{i_1} P_j + N_{i_1}}{\sum_{j=1}^{i_1} P_j + N_{i_1+1}} \le \frac{\sum_{j=1}^{i_1} P_j + N_{i_1} + \Delta}{\sum_{j=1}^{i_1} P_j + N_{i_1+1} + \Delta}$$

which holds obviously for any $\Delta > 0$ and $N_{i_1} \le N_{i_1+1}$, where "=" holds if and only if $N_{i_1} = N_{i_1+1}$.

□

Proof of Theorem 2.1: Consider the broadcast channel (1), with the independent gains $g_i \sim \mathcal{CN}(0,1)$, for $i = 1, \ldots, n$. For the equivalent model (2), the noise variance $N_i = 1/|g_i|^2$ is of the following distribution function:

$$\begin{aligned} F(y) &= \mathbb{P}(N_i < y) = \mathbb{P}(1/|g_i|^2 < y) = \mathbb{P}(|g_i|^2 > 1/y) \\ &= \int_{1/y}^{\infty} e^{-x}dx = e^{-\frac{1}{y}}, \text{ for } y > 0. \end{aligned}$$

For any fixed $N_0 > 0$, we can characterize the number of "good" channels with the equivalent noise variance N_i less than N_0 as the following.

Let $p_0 = F(N_0) = e^{-\frac{1}{N_0}}$. Then with probability p_0, a channel is good. Consider a Bernoulli sequence:

$$x_i = \begin{cases} 1, & \text{with probability } p_0 \\ 0, & \text{with probability } 1 - p_0 \end{cases}$$

for $i = 1, 2, \ldots, n$. Then the number of good channels has the same distribution as $X = \sum_{i=1}^{n} x_i$, which satisfies the binomial distribution $B(n, p_0)$.

For any integer $m \ge 1$, obviously,

$$\mathbb{P}(X \le m-1) = \sum_{j=0}^{m-1} \binom{n}{j} p_0^j (1-p_0)^{n-j}$$

which, however, is not easy to analyze. But if $m - 1 \le np_0$ (this condition can be verified later for $m \le \nu(n) - \epsilon$), we can use the Chernoff inequality [5, page 70]:

$$\mathbb{P}(X \le m-1) \le \exp\left(-\frac{1}{2p_0}\frac{(np_0 - m + 1)^2}{n}\right).$$

Hence,

$$\mathbb{P}(X \ge m) \ge 1 - \exp\left(-\frac{1}{2p_0}\frac{(np_0 - m + 1)^2}{n}\right). \qquad (16)$$

Now, consider the following power allocations for the m best receivers:

$$P_i = \frac{c}{\alpha^{m-i}}, \text{ for } i = 1, \ldots, m,$$

where $\alpha = e^{R_{\min}} > 1$, and $c = (1 - 1/\alpha)P$. It is easy to check that the total power constraint is satisfied:

$$\sum_{i=1}^{m} \frac{c}{\alpha^{m-i}} = c\frac{1 - (1/\alpha)^m}{1 - 1/\alpha} \le c\frac{1}{1 - 1/\alpha} = P.$$

If $\max_{1 \le i \le m} N_i \le P/\alpha^m$, then we have the following uniform lower bound for the SINRs at all these m receivers: for $i = 1$,

$$\frac{P_1}{N_1} \ge \frac{c/\alpha^{m-1}}{P/\alpha^m} = \alpha - 1,$$

and for any $i = 2, \ldots, m$,

$$\begin{aligned} \frac{P_i}{\sum_{j=1}^{i-1} P_j + N_i} &\ge \frac{c/\alpha^{m-i}}{\sum_{j=1}^{i-1} c/\alpha^{m-j} + P/\alpha^m} \\ &= \frac{1/\alpha^{m-i}}{\frac{(1/\alpha)^{m-i+1} - (1/\alpha)^m}{1-1/\alpha} + \frac{(1/\alpha)^m}{1-1/\alpha}} = \alpha - 1. \end{aligned}$$

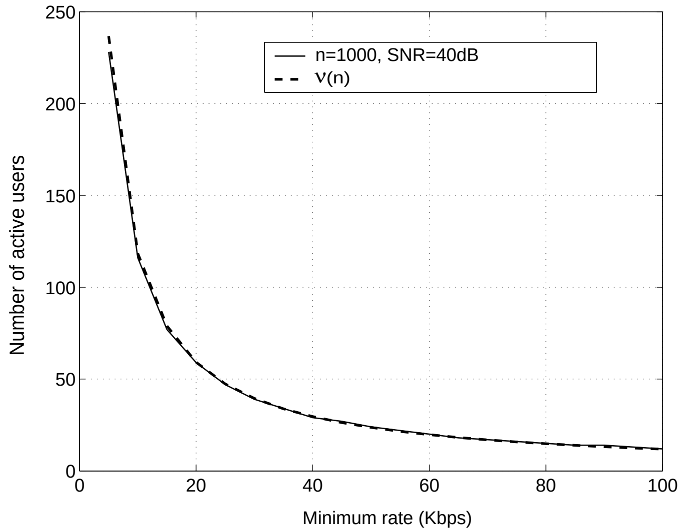

Fig. 2. The optimal number of active users versus the minimum rate for n=1000 and SNR$=40\,$dB.

Then obviously, the minimum rate constraint is satisfied for all these m receivers, since

$$\ln\left(1+(\alpha-1)\right)=\ln\alpha=R_{\min}.$$

Next, we show that for any $\epsilon>0$, if $m\leq\nu(n)-\epsilon$, $\max_{1\leq i\leq m}N_i\leq P/\alpha^m$ holds with probability approaching one as n tends to infinity. Let $N_0=P/\alpha^m$. Then,

$$\begin{aligned} p_0=F(N_0)=\exp\left(-\frac{\alpha^m}{P}\right) &\geq \exp\left(-\frac{\alpha^{\nu(n)-\epsilon}}{P}\right) \\ &= \exp\left(-\alpha^{-\epsilon}\ln n\right)=n^{-\lambda}, \end{aligned}$$

where $\lambda=\alpha^{-\epsilon}<1$. Then it is obvious that as $n\to\infty$,

$$\frac{1}{2p_0}\frac{(np_0-m+1)^2}{n}\sim\frac{n^2p_0^2}{2np_0}=\frac{np_0}{2}\geq\frac{n^{1-\lambda}}{2}\to\infty. \qquad (17)$$

Hence, by (16), the probability of $\max_{1\leq i\leq m}N_i\leq\alpha^mP$ approaches 1 as $n\to\infty$.

Therefore, we proved that as $n\to\infty$, with probability approaching 1, there are at least $m=\lfloor\nu(n)-\epsilon\rfloor$ good channels with $N_i\leq P/\alpha^m$, for which the minimum rate constraint is satisfied.

Next, we prove the upper bound, i.e., $m\leq\nu(n)+\epsilon$ holds with probability approaching 1.

First, we show that for any $\delta>0$, for sufficiently large m, the best receiver should have the equivalent noise variance $N_1\leq P_\delta/\alpha^m$, with $P_\delta:=P+\delta$. Otherwise, if $\min_{1\leq i\leq n}N_i>P_\delta/\alpha^m$, then by the minimum rate constraint, i.e.,

$$\frac{P_i}{\sum_{j=1}^{i-1}P_j+N_i}\geq\alpha-1,\quad\text{for } i=1,2,\ldots,m,$$

we have

$$P_1\geq(\alpha-1)N_1>(\alpha-1)P_\delta/\alpha^m,$$

and inductively, for $i=2,\ldots,m$,

$$\begin{aligned} P_i &\geq (\alpha-1)\left(\textstyle\sum_{j=1}^{i-1}P_j+N_i\right) \\ &> (\alpha-1)\left(\textstyle\sum_{j=1}^{i-1}(\alpha-1)P_\delta/\alpha^{m-j+1}+P_\delta/\alpha^m\right) \\ &= (\alpha-1)P_\delta/\alpha^{m-i+1}, \end{aligned}$$

which violates the total power constraint since

$$\sum_{i=1}^{m}P_i>\sum_{i=1}^{m}(\alpha-1)P_\delta/\alpha^{m-i+1}=(1-1/\alpha^m)P_\delta>P$$

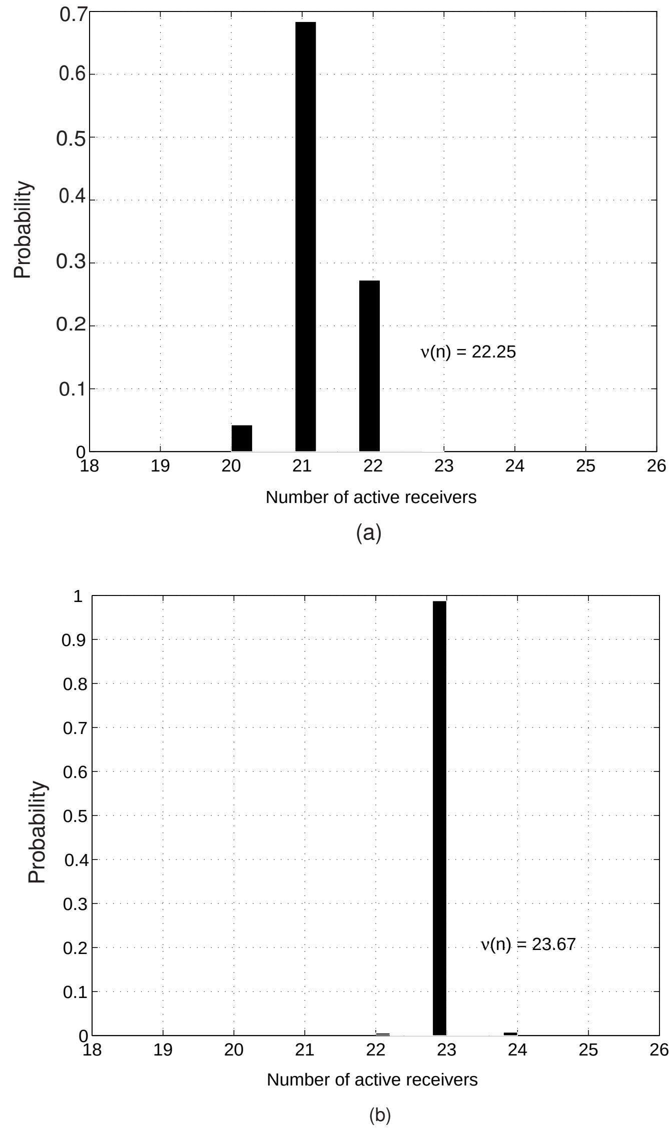

Fig. 3. The histogram of the number of active receivers for $R_{\min}=50\,$Kbps, SNR$=40\,$dB, and (a) $n=30$, and (b) $n=1000$.

for sufficiently large m.

Therefore, to show that

$$\mathbb{P}(m\leq\nu(n)+\epsilon)\to1,$$

i.e.,

$$\mathbb{P}(m>\nu(n)+\epsilon)\to0,$$

we only need to show that

$$\mathbb{P}(N_1\leq P_\delta/\alpha^{\nu(n)+\epsilon})\to0.$$

Let $c_1=P_\delta$ and $p_1=F(c_1/\alpha^{\nu(n)+\epsilon})$. Then, $(1-p_1)^n$ is the probability that all the receivers have equivalent noise variance greater than $P_\delta/\alpha^{\nu(n)+\epsilon}$. Hence,

$$\mathbb{P}(N_1\leq P_\delta/\alpha^{\nu(n)+\epsilon})=1-(1-p_1)^n, \qquad (18)$$

which tends to 0 if and only if

$$\left(1-\exp\left(-\frac{\alpha^{\nu(n)+\epsilon}}{c_1}\right)\right)^n\to1. \qquad (19)$$

Since

$$\left(1-\exp\left(-\frac{\alpha^{\nu(n)+\epsilon}}{c_1}\right)\right)^{\exp\left(\frac{\alpha^{\nu(n)+\epsilon}}{c_1}\right)} \to e^{-1},$$

(19) holds if

$$n\cdot\exp\left(-\frac{\alpha^{\nu(n)+\epsilon}}{c_1}\right) = n\cdot\exp\left(-\frac{P\alpha^{\epsilon}\ln n}{P+\delta}\right) \to 0, \tag{20}$$

which holds by choosing $\delta < (\alpha^{\epsilon}-1)P$.

$\square$

Proof of Remark 2.1: Following the proof of Theorem 2.1, especially noting (16), to prove (12), we only need to show that for $m = \lfloor \nu(n)-\epsilon \rfloor$,

$$\frac{1}{2p_0}\frac{(np_0-m+1)^2}{n} \geq \frac{n^{1-\lambda}}{2+\sigma}, \text{ for sufficiently large } n,$$

which actually follows from (17) with the following modification

$$\frac{1}{2p_0}\frac{(np_0-m+1)^2}{n} \geq \frac{n^2p_0^2}{(2+\sigma)np_0}, \text{ for sufficiently large } n.$$

To prove (13), noting (18), we have

$$\begin{aligned}
\mathbb{P}(m>\nu(n)+\epsilon) &\leq \mathbb{P}(N_1 \leq P_\delta/\alpha^{\nu(n)+\epsilon}) \\
&= 1-\left(1-\exp\left(-\frac{\alpha^{\nu(n)+\epsilon}}{c_1}\right)\right)^n \\
&= O\left(n\cdot\exp\left(-\frac{\alpha^{\nu(n)+\epsilon}}{c_1}\right)\right) \\
&= O\left(n\cdot\exp\left(-\frac{P\ln n}{\lambda(P+\delta)}\right)\right) \\
&= o\left(n^{1-\frac{1}{\lambda(1+\sigma)}}\right),
\end{aligned}$$

where, $\sigma > 0$ can be arbitrarily small, since $\delta > 0$ can be arbitrarily small.

$\square$

Proof of the upper bound (15): First, it follows from (18)-(20) that for any $0 < \delta < (\alpha^{\epsilon}-1)P$

$$\mathbb{P}(N_1 \leq P_\delta/\alpha^{\nu(n)+\epsilon}) \to 0.$$

Hence,

$$\mathbb{P}(N_1 > P_\delta/\alpha^{\nu(n)+\epsilon}) \to 1.$$

Since

$$P_\delta/\alpha^{\nu(n)+\epsilon} = (P+\delta)/\alpha^{\nu(n)+\epsilon} = (\alpha^{\epsilon}P-\eta)/\alpha^{\nu(n)+\epsilon}$$

where $\eta = (\alpha^{\epsilon}-1)P-\delta > 0$ can be arbitrarily small, the maximum achievable total throughput is upper bounded with probability approaching 1 as

$$\begin{aligned}
\ln\left(1+\frac{P}{N_1}\right) &< \ln\left(1+\frac{P}{P_\delta/\alpha^{\nu(n)+\epsilon}}\right) \\
&= \ln\left(1+\frac{P\alpha^{\nu(n)+\epsilon}}{\alpha^{\epsilon}P-\eta}\right) \\
&= \ln\left(1+\beta\alpha^{\nu(n)}\right) \\
&= \ln\left(1+\beta P\ln n\right)
\end{aligned}$$

where $\beta = \frac{\alpha^{\epsilon}P}{\alpha^{\epsilon}P-\eta} > 1$ can be arbitrarily close to 1.

$\square$

References

[1] L. Li and A. J. Goldsmith, "Capacity and optimal resource allocation for fading broadcast channels—Part I: Ergodic Capacity," *IEEE Trans. Information Theory*, vol. 47, no. 3, pp. 1083-1102, March 2001.

[2] L. Li and A. J. Goldsmith, "Capacity and optimal resource allocation for fading broadcast channels—Part II: Outage Capacity," *IEEE Trans. Information Theory*, vol. 47, no. 3, pp. 1103-1127, March 2001.

[3] N. Jindal and A. J. Goldsmith, "Capacity and optimal power allocation for fading broadcast channels with minimum rates," *IEEE Trans. Information Theory*, vol. 49, no. 11, pp. 2895-2909, November 2003.

[4] T. Cover and J. Thomas, *Elements of Information Theory*. New York: Wiley and Sons, 1991.

[5] R. Motwani and P. Raghavan, *Randomized Algorithms*. Cambridge University Press, 1995.

Biography of Hengameh Keshavarz

Hengameh Keshavarz (S'04) received the B.Sc. and the M.Sc. degrees, both in electrical engineering, from Ferdowsi University of Mashad, Mashad, Iran, in 1997 and 2001, respectively. From 2001 to 2003, she was a lecturer and laboratory instructor in the Department of Electrical Engineering, University of S & B, Zahedan, Iran. During this period, she taught some undergraduate courses. She is currently a research assistant and pursuing her Ph.D. in the Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, Canada. Her research interests include multi-user information theory, wireless networks, estimation theory, and array signal processing with applications in wireless communications.

Biography of Liang-Liang Xie

Liang-Liang Xie (M'03) received the B.S. degree in mathematics from Shandong University, Jinan, China, and the Ph.D. degree in control theory from the Chinese Academy of Sciences, Beijing, China, in 1995 and 1999, respectively.

He held postdoctoral positions at the Automatic Control Group, Linköping University, Sweden during 1999-2000 and the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, during 2000-2002. He was a faculty member of the Institute of Systems Science, Chinese Academy of Sciences before joining the Department of Electrical and Computer Engineering, University of Waterloo, Canada in 2005, where he is currently an Associate Professor. His research interests include network information theory, wireless networks, adaptive control and system identification.

Biography of Ravi Mazumdar

Ravi Mazumdar (M'83, SM'94, F'05) was born in Bangalore, India. He obtained the B.Tech. in Electrical Engineering from the Indian Institute of Technology, Bombay, India in 1977, the M.Sc. DIC in Control Systems from Imperial College, London, U.K. in 1978 and the Ph.D. in Systems Science from the University of California, Los Angeles, USA in 1983.

He is currently a University Research Chair Professor of Electrical and Computer Engineering at the University of Waterloo, Waterloo, Canada and an Adjunct Professor of ECE at Purdue University. Prior to this he was Professor of ECE at Purdue University, W. Lafayette, USA. He has held visiting positions and sabbatical leaves at UCLA, the University of Twente (Netherlands), the Indian Institute of Science (Bangalore); and the Ecole Nationale Superieure desTelecommunications (Paris).

He is a Fellow of the IEEE and the Royal Statistical Society. He is a member of the working groups WG6.3 and 7.1 of the IFIP and a member of SIAM and the IMS. His work has won the IEEE INFOCOM 2006 Best Paper Award and was runner-up for the Best Paper at INFOCOM 1998.

His research interests are in applied probability, optimization, stochastic analysis and game theory with applications to networks, traffic engineering, filtering theory, and mathematical finance.