GEOMETRIC AND COMBINATORIAL RIGIDITY OF PERIODIC FRAMEWORKS AS GRAPHS ON THE TORUS

ELISSA ROSS

A DISSERTATION SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

GRADUATE PROGRAM IN THE DEPARTMENT OF MATHEMATICS AND STATISTICS
YORK UNIVERSITY
TORONTO, ONTARIO
MAY 2011

GEOMETRIC AND COMBINATORIAL RIGIDITY OF PERIODIC FRAMEWORKS AS GRAPHS ON THE TORUS

by **Elissa Ross**

a dissertation submitted to the Faculty of Graduate Studies of York University in partial fulfilment of the requirements for the degree of

DOCTOR OF PHILOSOPHY
© 2011

GEOMETRIC AND COMBINATORIAL RIGIDITY OF PERIODIC FRAMEWORKS AS GRAPHS ON THE TORUS

by **Elissa Ross**

By virtue of submitting this document electronically, the author certifies that this is a true electronic equivalent of the copy of the dissertation approved by York University for the award of the degree. No alteration of the content has occurred and if there are any minor variations in formatting, they are as a result of the coversion to Adobe Acrobat format (or similar software application).

Examination Committee Members:

1. Walter Whiteley

2. Trueman MacHenry

3. Ada Chan

4. Mike Zabrocki

5. Gerald Audette

6. Robert Connelly

Abstract

A *periodic framework* is a simple infinite graph $\widetilde{G}$, together with a realization of its vertices $\widetilde{p}$ in $\mathbb{R}^d$ such that the resulting framework is invariant under a set L of d linearly independent translations. The translations generate a torus, which we view as a fundamental region for a tiling of d-space. The translations may remain fixed over time, generating the 'fixed torus' $\mathcal{T}_0^d$, or they may be permitted to vary, generating the 'flexible torus' $\mathcal{T}^d$. The periodic framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ can be described by a labeled multigraph $\langle G, m \rangle$ (a gain graph), together with a realization p of its vertices on to the torus. We call the pair $(\langle G, m \rangle, p)$ a *periodic orbit framework.*

We define what it means for such a framework to be rigid or flexible on $\mathcal{T}_0^d$ and $\mathcal{T}^d$, either continuously or infinitesimally. We define a number of standard rigidity theory definitions for this periodic setting, namely the rigidity matrix and the notion of generic rigidity (the rigidity of a graph $\langle G, m \rangle$ for almost all realizations of its vertices). We also introduce methods for treating groups of orbit graphs $\langle G, m \rangle$

together.

We find necessary conditions for the rigidity of frameworks on the fixed d-dimensional torus $\mathcal{T}_0^d$, and show that these conditions are also sufficient for generic rigidity in the case $d = 2$. In doing so, we define inductive constructions on periodic orbit frameworks: techniques for building up larger rigid graphs from smaller ones. We also outline an algorithm to test for the rigidity of a periodic orbit framework on $\mathcal{T}_0^2$.

We extend our characterization of frameworks on $\mathcal{T}_0^d$ to the partially flexible torus $\mathcal{T}_x^2$, where we again find necessary and sufficient conditions for generic rigidity. This result corresponds to a characterization of frameworks which are periodic in one direction only, that is, frameworks on a cylinder.

Finally we consider frameworks that are periodic, but also possess additional symmetry beyond translations. We find some necessary conditions for rigidity for some classes of such 'crystal structures', many of which offer surprising predictions of flexibility for frameworks that would be generically rigid without the symmetry.

In memory of Roxy "the pox" Ross.
September 1994 - May 7, 2011.

Acknowledgements

I first want to express my deepest gratitude to my supervisor, Walter Whiteley. He has been a constant source of knowledge, mentoring, and most importantly, inspiration. His willingness to share his enthusiasm for mathematics and learning is astounding, and I always feel challenged and motivated by our interactions. He has given so generously of his time and resources to my learning, and has been so patient with me. It is hard to believe just how much I have learned from him.

I feel fortunate to have had the privilege of learning from Asia Weiss and Trueman MacHenry, and to have encountered many corners of the geometric landscape under their guidance. They have both always been very supportive and generous with their time. In addition, I would also like to thank Mike Zabrocki, Ada Chan, Gerald Audette and Bob Connelly for reading the thesis and preparing such interesting questions in response.

I have benefitted from numerous conversations with Bob Connelly, Wendy Finbow-Singh, Bill Jackson, Tibor Jordán, Stephen Power, Bernd Schulze, and Brigitte Ser-

Contents

List of Tables

List of Figures

1 Introduction

1.1 Introduction to rigidity

Like many problems in the field of discrete geometry, the question of the *rigidity of a framework* admits a simple formulation. Given a set of physically rigid bars which are linked together by flexible joints, when is it possible to continuously deform the resulting framework into a non-congruent structure, without destroying the connectivity or the bars themselves? In other words, when is such a framework *flexible*, and therefore not *rigid*? We represent such a *bar-joint framework* by a graph $G = (V, E)$, together with a position of its vertices (*joints*) into some ambient space, say $p : V \to \mathbb{R}^d$.

The answer to this question is clearly of practical significance, with classical applications in engineering, as well as contemporary significance to computer aided design and molecular modelling. Indeed historical contributions have come from a range of sources, for example Euler's 1766 conjecture "A closed spatial figure allows no changes, as long as it is not ripped apart," [30], which was settled by the discovery

of a counterexample in 1978 by R. Connelly. Other contributions came from J.C. Maxwell (mid 1800's), who introduced the notion of a "stress" on a framework [30]. Yet another source of knowledge and questions came from engineers who were designing buildings and other structures, and who were developing or using "rules of thumb" to determine the rigidity of their plans. Many such methods were summarized and fleshed out in the 1911 book of engineer/mathematician Henneberg [36].

Since the 1970s, the study of rigidity has seen a dramatic rise in interest, with a formalization of the language and methods of the work of earlier researchers. In 1970, Laman proved his now celebrated result concerning the *generic* rigidity of graphs in two dimensions [46]. That is, he characterized the rigidity of *almost all* two-dimensional frameworks using only combinatorial methods. In fact, generic rigidity – the rigidity of almost all frameworks (G, p) with a given graph G – is a d-dimensional idea. In this way the study of the rigidity of a framework (G, p) can be seen to have two parts:

(i) the combinatorial properties of the graph G,

(ii) the geometric position p of the vertices of the graph in $\mathbb{R}^d$.

Generic rigidity is concerned with (i) only, in the sense that if a graph G is generically rigid, then the framework (G, p) is rigid for almost all positions p. This justifies

2

the use of the phrase "the graph is rigid". On the other hand, the geometric rigidity of graphs will be a study of (i) and (ii) together, and will include symmetry or other "geometrically special" frameworks. At this time, in dimensions 3 or more, generic rigidity is not well characterized. We always have tools to confirm the rigidity of particular (geometric) bar-joint frameworks (G, p) in $\mathbb{R}^d$ (namely the rigidity matrix), and we can generate certain classes of generically rigid graphs, but we lack general graph-theoretic results for dimensions $d > 2$. Furthermore, direct attempts to prove the rigidity or flexibility of a framework may be very hard without the use of additional tools from rigidity theory.

The study of rigidity has a rich history of questions generated by applications in structural engineering, mechanical engineering (in the study of linkages), chemistry, biology, materials science and computing, which then inspire and motivate a body of mathematical research. The study of *periodic rigidity* can be seen as exactly such a case, with the main inspirations coming from the study of zeolites and sphere packings.

Zeolites are a type of mineral with a crystalline structure characterized by a repetitive (periodic) porous pattern and a high internal surface area [62]. They are used as molecular sieves or catalysts in a variety of applications, from petrochemical manufacturing to carbon-sequestration. Their internal structure can be modelled using corner-sharing tetrahedra, with two at each corner (vertex), which
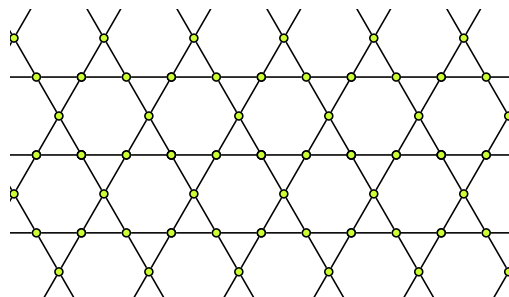
Figure 1.1: An example of an infinite two-dimensional mathematical zeolite. This particular configuration is frequently called the *Kagome Lattice*.

itself has inspired a flurry of zeolite-generating activity among mathematicians [70].

See Figure 1.1 for a two-dimensional example using corner-sharing triangles – the

Kagome lattice. Although there are many naturally occurring zeolites, it is also

possible to synthesize zeolites with some particular pore geometries and catalytic

chemistry. Since the activity of these materials in applications appears to depend

in part on their flexibility, it is desirable to have methods that would predict the

rigidity or flexibility of these hypothetical minerals prior to laboratory synthesis.

There have already been a number of articles which apply rigidity theory to the

study of zeolites, with M. Thorpe being one of the main instigators [42, 62]. In

parallel, rigidity theory has also been applied to other periodic frameworks in the

context of foam structures [21, 24].

Finally, on a separate track still, rigidity theory has been used to study sphere

packings, which can be modelled as *tensegrity frameworks* – frameworks in which

some bars are allowed to get longer, while others get shorter [4, 11, 12, 14, 25, 37]. In

4

the case of simulating sphere packings, one must define a container for the packing, which may simply be a finite box, or one may impose boundary conditions on the box to model an infinite periodic packing.

1.2 Introduction to periodic rigidity

A *d-periodic framework* is an infinite simple graph $\widetilde{G}$ together with a realization $\widetilde{p} : V \to \mathbb{R}^d$ of its vertices into Euclidean d-space such that the symmetry group of the framework contains d independent translations. We let the translations comprise the rows of a $d \times d$ matrix L (the *lattice*), and denote the d-periodic framework $\mathcal{F} = (\langle \widetilde{G}, L \rangle, \widetilde{p})$. Figure 1.2 depicts two 2-dimensional examples. When is such an infinite periodic framework rigid with respect to its periodicity? In other words, is there a motion of the framework that changes the distance between at least one pair of vertices while preserving the periodicity of the framework?
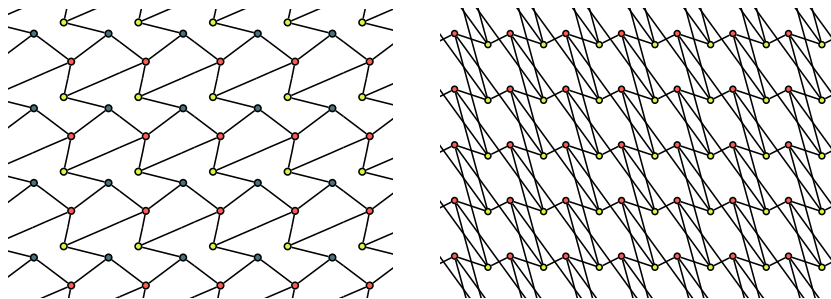
Figure 1.2: Two examples of infinite periodic frameworks.

Infinite periodic frameworks in 3-space are often used to model the molecular structure of crystalline materials, most notably zeolites [62], and understanding the rigidity properties of infinite periodic frameworks from a mathematical perspective may be of practical significance to the study of these compounds, as suggested above. Over the last decade, a number of studies of the rigidity of periodic frameworks have appeared, for example Fowler and Guest [28] and Guest and Hutchinson [69], both of which address two and three dimensional frameworks (with a view toward materials). Even more recently, work by Owen and Power [53], Power [54], Borcea and Streinu [7, 6] and Malestein and Theran [49] has formalized the mathematics involved in a general (d-dimensional) study of infinite periodic frameworks and provided substantial initial results.

In this thesis we describe one structure and a vocabulary for this investigation. We outline results from a natural "base case" for the study of general infinite periodic frameworks, namely frameworks on a torus of fixed dimensions. While at first the question of rigidity on a "fixed torus" may seem contrived, several materials scientists have confirmed that there may be some resonance with experiments on molecular compounds in which the time scales of lattice movement are several orders of magnitude slower than the molecular deformations within the lattice [76]. In other words, when we allow the lattice (torus) to deform, the velocities of the vertices that are "far away from the centre" will become arbitrarily large.

The central idea that underlies our program of research is to exploit the periodicity of the infinite graph to reduce the problem to a finite graph that captures the periodic structure. We accomplish this by considering quotient graphs on tori. For example, to study two-dimensional infinite periodic frameworks, we view the two-dimensional torus as a fundamental region for a tiling of the plane, and consider graphs realized on the torus as models of infinite periodic frameworks in the plane. Any motion of the elements of the framework on the torus can be viewed as a periodic motion of the plane graph. We can similarly consider graphs on the d-torus (equivalently the d-dimensional hypercube with pairs of opposite faces identified) and use this as a model of a d-dimensional periodic framework.

There are three qualities of infinite periodic frameworks that are of interest to the study of their rigidity:

(i) the combinatorial properties of the graph,

(ii) the geometric position of vertices of the graph on the torus, and in its cover in d-space,

(iii) the topological structure (up to homotopy) of the graph on the torus.

The usual study of rigidity of finite frameworks (as described in [30, 31, 84] for example) is an investigation of (i) and (ii), but the consideration of (iii) is unique to the study of periodic frameworks.

7

This thesis is concerned with the topic of *forced periodicity*. That is, we are interested motions of a periodic structure that must preserve the periodicity of the structure. An infinite periodic framework may have motions that break the periodic symmetry of the framework, but we will not address these motions here. The consideration of periodicity-breaking motions would be the study of *incidental periodicity*, frameworks which happen to be periodic, but do not necessarily preserve their periodicity through some motion of their joints. Of course, any flex of a framework (a continuous motion of its joints) in the forced periodicity setting will transfer to a flex of the framework in the incidental setting.

It should be further noted that for minerals such as zeolite, it may be possible that periodic motions are, in fact, energy-minimizing motions. Since the atomic structure of zeolite is periodic, there may already be such an energy minimizing behaviour at work. Therefore, the study of forced periodicity may not be only a special case of incidental periodicity, but a realistic model for the flexibility of zeolites.

An additional feature frequently observed in zeolites in symmetry. That is, there can be additional symmetry within the fundamental region of a periodic zeolite-type framework. From the perspective of rigidity theory, there has been a surge of recent activity in the study of symmetric finite frameworks, for example in the work of Schulze [63, 68]. In joint work with Schulze and Whiteley, we combine these

analyses for crystal-type periodic frameworks. The results are sometimes surprising predictions of flexibility beyond what the original graph without symmetry would have exhibited in the periodic setting alone. Again, this type of analysis may help in determining which theoretical zeolites to synthesize for further testing.

1.3   Contributions in context

In Chapter 3 we outline a basic vocabulary for the study of periodic frameworks using some ideas from topological graph theory. In particular, we demonstrate that a periodic framework can be represented using a *gain graph*, and define a rigidity matrix for these graphs realized on the torus. Related methods were used by Whiteley [82] and Guest and Hutchinson [69], however to our knowledge, this is the first time that gain graphs in particular have been used in this way. This idea, together with the main theorem of Chapter 4 were presented at an AMS meeting in Worcester [56]. A similar approach has subsequently been used by other authors in this field, for example Power's recent work [54], and Malestein and Theran [49] describe "coloured graphs" which are equivalent. The rigidity matrices we define are distinct from previous formulations in the literature, but can be shown to be equivalent.

In Chapter 4 we present a complete characterization of generic rigidity for frameworks on a two-dimensional fixed torus. We use *inductive techniques*, building up

isostatic frameworks from smaller isostatic frameworks by carefully adding vertices and edges to the underlying graph. This work was completed in early 2009 [56], prior to the appearance of the work of Borcea and Streinu [7] or Malestein and Theran [49]. The statements in the applicable cases are the same as in [49], as are the algorithms that are naturally obtained from these statements. However, our techniques are inductive, adding to a vocabulary of methods that may be applied to a broad class of problems concerning periodic frameworks.

Inductive techniques are both general and widely used. They easily adapt to d-dimensional frameworks, and have been used to generate special classes of three-dimensional rigid structures. Inductive techniques have also played a key role in the development of *global rigidity*, the study of graphs with unique realizations. Furthermore, inductive methods also appear in the study of special classes of frameworks, for example Schulze's work on symmetric frameworks [63], and Nixon, Owen and Power's recent exploration of frameworks supported on surfaces such as a torus embedded in $\mathbb{R}^3$ [52]. Inductive methods have also appeared as a tool for the study of pseudo-triangulations, a topic of current interest in computational geometry [59].

In summary, inductive methods are general, expandable and applicable. We hope that the techniques presented here will be of value to the study of periodic frameworks.

A second advantage of our consideration of the fixed torus is that it leads nat-

urally to an algorithm for detecting rigidity in this context. The same does not appear to be true for the methods of Malestein and Theran on the flexible torus.

We also present our own version of the rigidity of frameworks on a flexible torus, with a number of new results. In particular, we address frameworks on a torus with limited flexibility, namely continuous scaling in one or more directions. We use a novel method from algebraic geometry that has not appeared elsewhere in the periodic rigidity literature. It provides a ladder from the fixed torus case to a flexible torus with one degree of flexibility, and combines with the known necessary conditions to complete the characterization of necessary and sufficient conditions in this case. These conditions also extend to characterize the rigidity of frameworks on a cylinder of variable diameter.

1.3.1 Statement of authorship

With the exception of Chapter 7 which summarizes joint work with Bernd Schulze and Walter Whiteley, this thesis is the independent work of the author.

1.4 Outline of thesis

Chapter 2 provides the basic background definitions and results from tilings, graph theory, topological graph theory and rigidity theory for finite graphs. We there introduce our primary tool for representing periodic frameworks, namely the gain

graph.

Chapter 3 defines periodic (bar-joint) frameworks, and their representations as gain graphs on a torus (periodic orbit graphs). In particular, we consider frameworks on a torus of fixed dimensions, which we call the fixed torus $\mathcal{T}_0^d$. We define what it means for such a framework to be rigid or flexible, either continuously or infinitesimally. We introduce the rigidity matrix for frameworks on the fixed torus, and extract a number of necessary conditions from this tool. We introduce the notion of a generic realization of a periodic orbit graph on the torus, which means that we can speak of the generic rigidity properties of a particular orbit graph. We outline some fundamental results, namely the affine invariance of infinitesimal rigidity on a torus, and a method (the T-gain procedure) for grouping together graphs within the same homotopy class on the torus.

Chapter 4 elaborates on the necessary conditions found in Chapter 3 for rigidity of frameworks on the two-dimensional fixed torus $\mathcal{T}_0^2$, and examines to what extent these conditions are also sufficient. In particular, we prove necessary and sufficient conditions for rigidity of frameworks on the fixed two dimensional torus. In doing so, we define inductive constructions on periodic orbit graphs: techniques for building up larger rigid graphs from smaller ones. We conclude with a discussion of necessary conditions for rigidity of frameworks on the d-dimensional fixed torus $\mathcal{T}_0^d$.

In Chapter 5 we extend these results to frameworks on a partially flexible torus.

We look at the d-dimensional torus $\mathcal{T}_k^d$, which has $k < \binom{d+1}{2}$ degrees of freedom. We characterize the generic rigidity of frameworks on a two-dimensional torus which is allowed to scale in one direction only ($\mathcal{T}_x^2$). We observe that this result corresponds to a characterization of frameworks which are periodic in one direction only, that is, frameworks on a cylinder.

Chapter 6 explores a modification of the pebble game algorithm to check for generic rigidity of a periodic orbit graph on the fixed torus, as characterized in Chapter 4.

In Chapter 7 we sketch the results of recent joint work with Bernd Schulze and Walter Whiteley [58]. This work considers frameworks that are periodic, but also possess additional symmetry. We find some necessary conditions for rigidity for a number of classes of such 'crystal structures', some of which offer surprising predictions of flexibility for frameworks that would be infinitesimally rigid without the symmetry.

We conclude in Chapter 8 with an outline of likely extensions and topics for further investigation, some of which are in progress. In particular, we consider questions of static rigidity of periodic frameworks, periodic bar-body frameworks, and scaling the unit cell. We also collect some unanswered questions from earlier chapters.

Table 1.1 summarizes our contributions to the the study of periodic frameworks,

with the author's contributions in the highlighted cells. The arrows represent dependence on the finite $d \geq 3$ case.

Table 1.1: Bar-joint frameworks

Type	finite	periodic	periodic	periodic	symmetric periodic
Torus	–	fixed $\mathcal{T}_0^d$	scaling $\mathcal{T}_x^d$	flexible $\mathcal{T}^d$	fixed, scaling, flexible... $\mathcal{T}_k^d$
$d = 1, 2$	necessary, sufficient	necessary, sufficient (Chap. 4) algorithm (Chap. 6)	necessary, sufficient (Chap. 5)	necessary, sufficient [49]	necessary (Chap. 7)
$d \geq 3$	necessary, NOT sufficient	necessary (Chap. 4) $\leftarrow$	necessary (Chap. 5) $\leftarrow$	necessary $\leftarrow$ [7]	necessary (Chap. 7)

14

2 Background

In this chapter we outline the basic notations, definitions and results that we use throughout the thesis.

2.1 Periodic structures

Although the study of periodic frameworks will not explicitly refer to the theory of tilings, we introduce the basic notions here to develop intuition for periodic structures in general. For a detailed reference on tilings, see Grünbaum and Shephard [33], and for a discussion of tilings as instances of polytopes see Coxeter [19].

A *tiling* or *tessellation* of the plane is a collection of plane figures that may be joined together to cover the plane, without overlaps or gaps. We say that the plane figures *tile* together to create the tiling, which is itself an infinite plane figure. A tiling is called *regular* if it is composed of congruent polygons. For example, squares, equilateral triangles and hexagons each produce a tiling of the plane. In fact, these are the only regular tilings of the plane.

Suppose we have an infinite plane figure that has a symmetry group generated by two independent translations (the *lattice group*). The transformations of any single point by such a group generate a *lattice*, which consists of the vertices of a tessellation of the plane by equal parallelograms. The *fundamental region* of any such figure is a subset of the plane whose transforms (under the lattice group) completely cover the plane, without gaps or overlaps. That is, every point in the plane is equivalent (under the lattice group) to a point on the fundamental region, and no two points in the fundamental region are equivalent, unless they both lie on the boundary. The fundamental region of a figure whose symmetry group consists of two independent translations will be the parallelogram with the translations composing the sides. In general, if the symmetry group contains non-translational symmetries, such as a rotational symmetry, then the fundamental region may not be a parallelogram. However, with the exception of the special cases considered in Chapter 7, we assume throughout this work that the fundamental region is always generated by translational symmetries only.

The idea of a tiling can be generalized to higher dimensions, for example cubes will "tile" together to pack space. In addition, any group of d independent translations acting on $\mathbb{R}^d$ generate a lattice and a fundamental region in analogy with the plane. In the language of crystallography, that lattice is sometimes called the *period lattice*, and the fundamental region is frequently called the *unit cell*. The
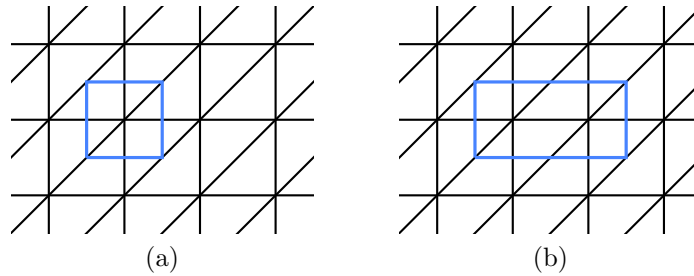
Figure 2.1: An example of a tiling of the plane by triangles (a) with the smallest parallelogram fundamental region. A larger fundamental region is shown in (b).

unit cell is always a box (a parallelopiped), while the fundamental region need not be, as noted above.

Remark 2.1.1. This definition of the fundamental region requires that it be the smallest possible cell that will tile together to create the overall infinite figure. However, in the remainder of this work, we do not necessarily assume that the fundamental region is the smallest. We are free to choose the fundamental region to contain as many copies of this smallest cell as we like (see Figure 2.1), although once selected, the fundamental region remains the same size throughout the analysis of a particular example. In practice, our examples will usually depict the smallest possible cell, but this is only for simplicity of presentation. We will remark further on this distinction later, and return to the idea of discrete scaling the fundamental region in Chapter 8. □

2.2 Graph theory

We now outline some basic graph theory terms and results. These definition are all standard, we refer to the books of Diestel [22] and West [78] for further details.

A *graph* is a pair $G = (V, E)$ of sets V and E, where V is a *vertex set* and E is an *edge set*, together with a relation that associates with each edge two vertices (not necessarily distinct) called its *endpoints*. The elements of V are the *vertices* of G and the elements of E are the *edges*. If $V = \{1, 2, \ldots, n\}$ is a finite set, we denote an edge e between vertices i and j by $\{i, j\} = \{j, i\}$. This notation may not uniquely determine an edge, however, if G contains two edges connecting a single pair of vertices. For example, there are two edges connecting the vertices 1 and 4 in Figure 2.2(b).

At times we will find it convenient to refer to the vertices and edges of a graph G by $V(G)$ and $E(G)$ respectively. Similarly, we will sometimes write the vertices of V as v_i, where the notation above might be confusing. The number of vertices of a graph G determines its *order*, which may be finite, infinite, countable etc.

The vertex i is *incident* to the edge e if $i \in e$. The *valence* or *degree* of a vertex v is the number of edges incident with v. We say that vertices i and j are *adjacent* if $\{i, j\} \in E$. A vertex of degree 1 is called a *pendent* vertex.

A *loop* is an edge whose endpoints are equal, and *multiple edges* are edges that
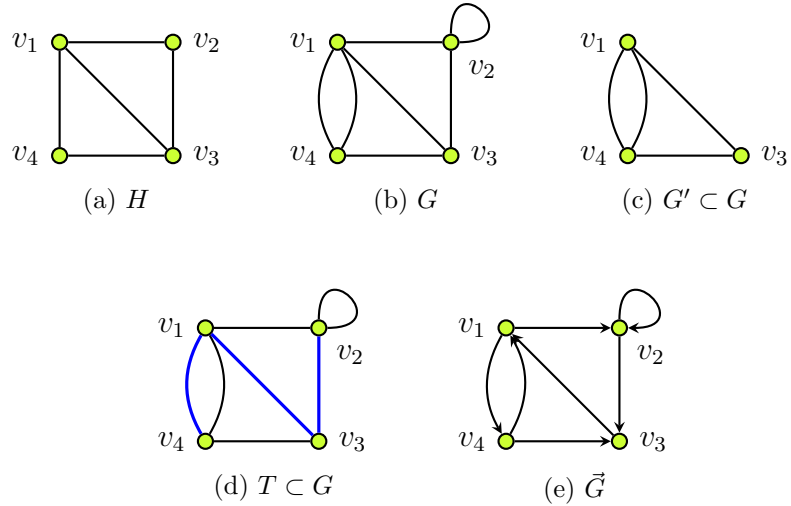
Figure 2.2: (a) A simple graph H; (b) a multigraph G on the same vertex set; (c) an induced subgraph $G' \subset G$; (d) a spanning tree of G (in blue); a directed graph $\vec{G}$.

have the same end points. A graph consisting of a single vertex with n loops will be called a *bouquet of n loops*. A graph without loops or multiple edges is called a *simple graph*. For a simple graph H, the notation $\{i, j\}$, $i, j \in V(H)$ will uniquely determine an edge. Sometimes graphs with loops and multiple edges are called *multigraphs* to distinguish them from simple graphs. The primary objects of study in this thesis will be infinite simple graphs, and finite multigraphs. In general, the word 'graph' should be taken to mean multigraph (as in the definition of 'graph' above), and we will note 'simple' where appropriate.

Let G and G' be two graphs. These graphs are *isomorphic* if there exists a bijection $\phi : V \to V'$, with $\{\phi(i), \phi(j)\} \in E'$ if and only if $\{i, j\} \in E$. We call ϕ a

19

(graph) isomorphism.

A *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$, and the assignment of endpoints to edges in G' is the same as in G. If G' contains all of the edges $\{i, j\} \in E$, with $i, j \in V'$, then G' is an *induced subgraph*. In this case we say that the vertex set V' *induces* or *spans* G', and we write $G' = G(V')$. Alternatively, we can define an induced subgraph to be any subgraph of G obtained by deleting vertices of G. If $1 < |V'| < |V|$, we say that G' is a *proper* subgraph of G. The graph shown in Figure 2.2(c) is a proper induced subgraph of the graph in (b).

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A graph G is called *connected* if any two of its vertices are linked by a path in G. The *components* of a graph G are its maximal connected subgraphs.

A *cycle* is a graph with an equal number of vertices and edges, whose vertices can be placed around a circle so that the two vertices are adjacent if and only if they appear consecutively along the circle. Note that a cycle is not necessarily a simple graph, since a loop on a single vertex forms a cycle, as do two edges connecting two vertices.

A *forest* is any graph without cycles, and a connected graph containing no cycles is a *tree*. If G is a graph with vertex set V, a *spanning subgraph of G* is any subgraph

20
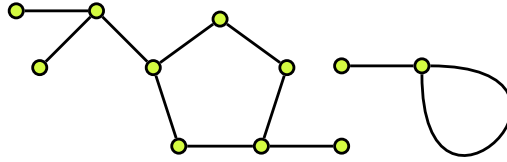
Figure 2.3: A map-graph

with vertex set V. A *spanning tree* is any spanning subgraph that is a tree.

A *map-graph* is a graph in which each connected component contains exactly one cycle (see Figure 2.3). If the map-graph is connected, then it is composed of a tree plus an edge.

A *walk* in a graph G is an alternating sequence of vertices and edges in G,

$$v_0 e_0 v_1 e_1 v_2 \ldots v_{k-1} e_{k-1} v_k,$$

such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If a walk begins and ends at the same vertex, it is a *closed walk*. If no vertex is visited twice on the walk (resp. closed walk), then it is clearly a path (resp. cycle).

Let $G = (V, E)$ be a graph. We write $G - e$ or $G - M$ to denote the subgraphs of G formed by deleting the edge e or the set of edges $M \subseteq E$. If $G - e$ or $G - M$ has more components than G, we call M an *edge cut*, or e is called a *cut-edge*. Similarly, we write $G - v$ or $G - S$ to denote the subgraphs of G formed by deleting the vertex v or the set of vertices $S \subseteq V$. If $G - v$ or $G - S$ has more connected components than G, then v and S are the *cut-vertex* or *vertex cut* respectively.

21

The *connectivity* of G is the minimum size of a vertex set S such that $G - S$ is disconnected or has more than one component. The graph G is *k-connected* if its connectivity is at least k. Similarly, the *edge-connectivity* of G is the minimum size of an edge cut of G. The graph G is *k-edge-connected* if every edge cut has at least k edges.

The *incidence matrix* of a graph G is the $|E| \times |V|$ matrix $M(G)$ in which the entry $m_{i,j}$ is 1 if v_i is an endpoint of e_j, and is 0 otherwise.

The *edge space* $\mathcal{E}(G)$ of a graph $G = (V, G)$ is the set of functions $E \to \mathbb{F}_2 = \{0, 1\}$. The elements of $\mathcal{E}(G)$ are naturally associated with the subsets of E, however the edge set thus defined has the structure of a vector space. The elements are the subsets of E, vector addition is the same as symmetric difference, $\emptyset \subseteq E$ is the zero element, and $F = -F$ for all $F \in \mathcal{E}(G)$. See [22] for further details.

The *cycle space* $\mathcal{C} = \mathcal{C}(G)$ of G is the subspace of $\mathcal{E}(G)$ spanned by the (edge sets of the) cycles of G. An *induced cycle* in G is a cycle which is also an induced subgraph. That is, it is a cycle without chords, a *chord* being an edge joining two vertices of a cycle, without being itself an edge of the cycle.

Proposition 2.2.1 ([22], Proposition 1.9.1). *The induced cycles in G generate its entire cycle space.*

Let T be a spanning tree of G. For each non-tree edge $e \notin E(T)$, there is a unique cycle C_e in $T + e$. These cycles are the *fundamental cycles* of G with respect

to T. The fundamental cycles of a graph span its cycle space [22].

A *directed graph* or *digraph* $\vec{G} = (V, E)$ consists of a vertex set V and an edge set E, together with a function which assigns an ordered pair of vertices to each edge. The first vertex of the ordered pair is called the *tail* or *origin* of the edge, and the second vertex of the ordered pair is called the *head* or *terminus* of the edge. For the directed edge $e = \{i, j\} \in E(\vec{G})$, we denote the origin by $o(e) = i$, and the terminus by $t(e) = j$. The *underlying graph* of a directed graph $\vec{G}$ is the graph G obtained by treating the edges of $\vec{G}$ as unordered pairs. The vertex set and edge set remain the same. A directed graph $\vec{G}$ is shown in Figure 2.2(e), with underlying graph G pictured in (b). The definitions of subgraph, isomorphism and connectivity for digraphs is the same as for undirected graphs.

A *path* in a directed graph $\vec{G}$ is a simple subgraph whose vertices can be linearly ordered so that the edges are ordered head to tail. That is, there is an edge with tail u and head v if and only if v immediately follows u in the vertex ordering. Similarly, a *cycle* in a directed graph uses the ordering of the vertices on a circle to order the edges head to tail.

2.3   Gain graphs

We will use the structure of *gain graphs* as a way of concisely describing our infinite periodic graphs. We will also view a gain graph as a set of instructions for how

to realize a graph on the torus, although this need not be a 2-cell embedding (an embedding without crossings), which distinguishes this treatment from other discussions of gain graph realizations [32]. Note that in some literature, namely Gross and Tucker's book [32], these graphs are called *voltage graphs*. Our discussion here is based on this presentation, but we use the word 'gain' to avoid the extra connotations given by the term 'voltage,' and to connect to the larger body of literature on the topic of gain graphs [89]. To allow for the generality required later in this thesis, we present the definition for general 'gains', and we will comment on the relevance for the periodic setting in the next chapter.

Let $G = (V, E)$ be a connected multigraph possibly having loops and multiple edges with vertices $V = \{v_0, v_1, \ldots, v_n\}$, $|V| = n < \infty$. Let the edges of G be assigned both plus and minus directions. Let m be a set function from the plus-directed edges into a group $\mathcal{A}$. The pair $\langle G, m \rangle$ is called a *gain graph*. G is called the *base graph* of $\langle G, m \rangle$, $m : E \to \mathcal{A}$ is called the *gain assignment*, and $\mathcal{A}$ is the *gain group*. Although G is technically a directed graph, we do not use the notation $\vec{G}$ here for reasons that will soon become clear (see Remark 2.3.1 below).

The vertices of $\langle G, m \rangle$ are the same as the vertices of G: $V\langle G, m \rangle = \langle V, m \rangle = V$. The edges of $\langle G, m \rangle$ are denoted $\langle E, m \rangle$ or $E\langle G, m \rangle$. An edge e in $\langle E, m \rangle$ is denoted

$$e = \{v_i, v_j; m_e\}, \ \text{ or } \ \{i, j; m_e\}, \tag{2.1}$$

24

where $\{v_i, v_j\} \in E$. This represents the directed edge from vertex v_i to vertex v_j, which is labeled with the gain m_e. This edge may equivalently be written in the reverse order, by using the group inverse m_e^{-1} of the gain assignment on e:

$$e = \{v_j, v_i; m_e^{-1}\}. \tag{2.2}$$

We borrow many of the notations from graph theory. In particular, we write $\langle G, m \rangle - e$ to denote the gain graph obtained from $\langle G, m \rangle$ be deleting the edge e from G, and its associated gain m_e from m. A *subgraph* of $\langle G, m \rangle$ is a gain graph $\langle G', m' \rangle$ where $G' \subset G$ is a subgraph of G, and m' is the restriction of m to the edges of G'.

A *path* of $\langle G, m \rangle$ is defined to be a path of the base graph G. In contrast to the definition of paths for directed graphs, we do not insist that the edges of a path of $\langle G, m \rangle$ be oriented head to tail. We record a path of $\langle G, m \rangle$ by

$$P = e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_k^{\alpha_k},$$

where $e_i \in E\langle G, m \rangle$, and α_i is either $+1$ or -1 depending on the orientation of the edge in the path. This allows us to define the *net gain on the path* to be

$$\prod_{i=1}^{k} m(e_i)^{\alpha_i},$$

where $m(e_i)^{+1} = m(e_i)$, and $m(e_i)^{-1}$ is the group inverse of $m(e_i)$. In other words, we multiply (using the group operation) the gains on the edges of the path according
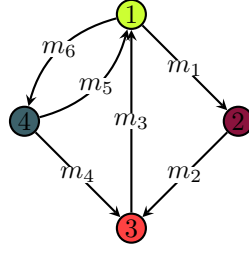
Figure 2.4: A gain graph $\langle G, m \rangle$, $m : E \to \mathcal{A}$.

to their orientation. We similarly define a *cycle* of $\langle G, m \rangle$ to be a cycle of the base graph G, and the *net gain on the cycle* is defined as for paths. If $\mathcal{A}$ is abelian, then of course the net gain on the path is simply the sum of the elements on the edges of the path, with the appropriate multiplier ($+1$ or -1) according to the orientation of the edges in the path:

$$\sum_{i=1}^{k} \alpha_i m(e_i).$$

For example, consider the graph in Figure 2.4. Suppose the edge e_i has gain m_i, as labeled. Then the cycle in the graph shown in Figure 2.4 given by

$$e_1^{+1} e_2^{+1} e_4^{-1} e_5^{+1} = \{1, 2; m_1\}\{2, 3; m_2\}\{3, 4; -m_4\}\{4, 1; m_5\}$$

has net gain $m_1 m_2 m_4^{-1} m_5$, or $m_1 + m_2 - m_4 + m_5$ is the net gain if $\mathcal{A}$ is abelian.

Remark 2.3.1. In contrast to our discussion of cycles in directed graphs, we here permit re-direction of the edges provided that they are accompanied by a relabelling of the gains on the edges as well (by the equivalence of (2.1) and (2.2)). In this way, we should think of cycles in the gain graph as corresponding one-to-one with

cycles in the base graph. That is, the gain graph $\langle G, m \rangle$ has the same cycle space as the base graph G, while a directed graph does not share the same cycle space as its underlying graph. For this reason we use the notation G for gain graphs, as opposed to the notation $\vec{G}$ as used for directed graphs. $\qquad\square$

Suppose $\langle G, m \rangle$ is a gain graph where $\mathcal{C}(G)$ is the cycle space of the (undirected) graph G. The *gain space* $\mathcal{M}_{\mathcal{C}}(G)$ is the vector space (over $\mathbb{Z}$) spanned by the net gains on the cycles of $\mathcal{C}(G)$.

2.3.1 Derived graphs corresponding to gain graphs

The key feature of gain graphs is that from a gain graph $\langle G, m \rangle$ we may define a related graph called the *derived graph* which we denote G^m. The derived graph G^m has vertex set V^m and E^m where V^m is the Cartesian product $V \times \mathcal{A}$, and E^m is the Cartesian product $E \times \mathcal{A}$. Vertices of V^m have the form (v_i, a), where $v_i \in V$, and $a \in \mathcal{A}$. Edges of E^m are denoted similarly. If e is the directed edge connecting vertex v_i to v_j in $\langle G, m \rangle$, and b is the gain assigned to the edge e, then the directed edge $\{e; a\}$ of G^m connects vertex (v_i, a) to (v_j, ab). In this way, the derived graph is a (directed) graph whose automorphism group contains $\mathcal{A}$.

If v is a vertex in the gain graph, then the set of vertices $\{(v, a) : a \in \mathcal{A}\}$ in the vertices V^m of the derived graph is called the *fiber* over v. Similarly, the set of edges $\{(e, a) : a \in \mathcal{A}\}$ is the *fiber* over the edge $e \in E$. There is a *natural projection*

27

from the derived graph to the base graph which is the graph map $\phi : G^m \to G$ that maps every vertex (resp. edge) in the fiber over v (resp. e) to the vertex v (resp. e) for all $v \in V$ (resp. $e \in E$).

For the majority of this thesis, we will be working exclusively with gain graphs whose gain group $\mathcal{A}$ is $\mathbb{Z}^d$. Since $\mathbb{Z}^d$ is an infinite group, this representation allows us to view gain graphs as a 'recipe' for an infinite periodic graph. The exception to this rule is Chapter 7, where we let $\mathcal{A} = \mathbb{Z}^d \rtimes \mathcal{S}$, where $\mathcal{S}$ is a symmetry group (or point group). Further details will be provided there. For the remainder of this section we work with $\mathcal{A} = \mathbb{Z}^d$, and move to additive notation for the group operation.

Example 2.3.2. Let $\langle G, m \rangle$ be the gain graph pictured in Figure 2.5a, with gain group $\mathbb{Z}^2$. The unlabeled, undirected edges have gain $(0,0)$. The corresponding derived graph is pictured in 2.5b.

$\square$

2.3.2 Local gain groups and the T-gain procedure

Let u be a vertex of the gain graph $\langle G, m \rangle$, and let W and W' be distinct closed walks that begin and end at u. The walk WW' is also a u-based closed walk. The set of all such walks forms a semigroup, with the product operation so defined. It
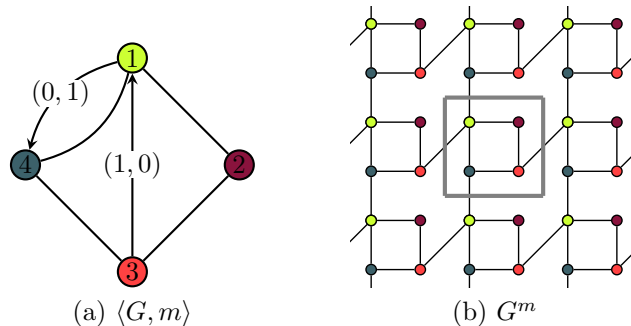
Figure 2.5: A gain graph $\langle G, m \rangle$, where $m : E \to \mathbb{Z}^2$, and its derived graph G^m. We use graphs with vertex labels as in (a) to depict gain graphs, and graphs without such vertex labels will record derived graphs, or graphs that are realized in some ambient space (see Section 2.5.1).

was observed by Alpert and Gross that the set of net gains occurring on u-based closed walks forms a subgroup of the gain group [32]. We call this group the *local gain group at u*. For a connected graph, it is clear that there is a unique local gain group that is independent of the choice of base vertex u. When the gain group, $\mathcal{A}$, is the integer lattice $\mathbb{Z}^d$, the gain space of $\langle G, m \rangle$ and the local gain group of $\langle G, m \rangle$ will be the same. In general, however, this is not the case, since the local gain group is a group, and the gain space is a vector space.

If our graph is a bouquet of loops, then the local gain group is simply the group generated by the gains of the loops. If our graph is not, however, a bouquet of loops, how do we find the local gain group? We have an algorithm called the *T-gain procedure* that will effectively transform our graph into a bouquet of loops. It appears in [32] and we outline it here. See Figure 2.6 for a worked example.
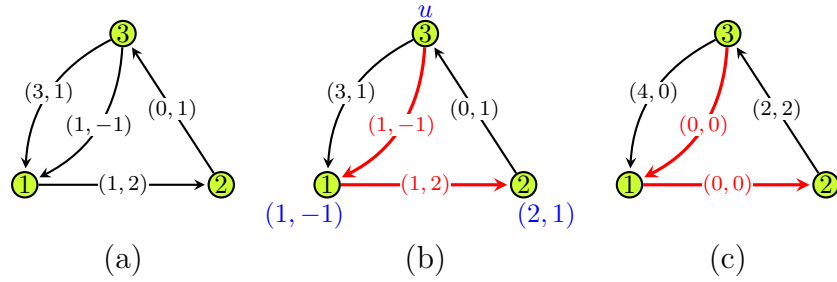
Figure 2.6: A gain graph $\langle G, m \rangle$ in (a), with identified tree T (in red), root u, and T-potentials in (b). The resulting T-gain graph $\langle G, m_T \rangle$ is shown in (c). The local gain graph is now seen to be generated by the elements $(4, 0)$ and $(2, 2)$, hence the local gain group is $2\mathbb{Z} \times 2\mathbb{Z}$.

T-gain Procedure

1. Select an arbitrary spanning tree T of G, and choose a vertex u to be the root vertex (of the local gain group). Such a spanning tree is known to exist, as we assumed G was connected.

2. For every vertex v in G, there is a unique path in the tree T from the root u to v. Denote the net gain along that path by $m(v, T)$, and we call this the T-*potential* of v. Compute the T-potential of every vertex v of G.

3. Let e be a plus-directed edge of G with initial vertex v and terminal vertex w. Define the T-*gain* of e, $m_T(e)$ to be

$$m_T(e) = m(v, T) + m(e) - m(w, T).$$

   Compute the T-gain of every edge in G. Note that the T-gain of every edge of the spanning tree will be zero.

30

4. Contract the graph along the spanning tree to obtain $|E| - (|V| - 1)$ loops at the root vertex u (there are $|V| - 1$ edges as part of the spanning tree). The gains on these loops will generate the local gain group. In other words, the gains on all of the edges of the graph that are not contained in T will generate the local gain group.

Since the net gain on any u-based closed walk is the same with respect to the T-gains as with respect to m, we have the following theorem:

Theorem 2.3.3 ([32])**.** *Let $\langle G, m \rangle$ be a gain graph, and let u be any vertex of G. Then the local gain group at u with respect to the T-gains, for any choice of spanning tree T, is identical to the local group of u with respect to m.*

In other words, the T-gain procedure supplies us with the net gains on a fundamental system of cycles.

What is important for the study of rigidity is that the gain graph with T-gains generates the same derived graph as the gain graph $\langle G, m \rangle$. Indeed:

Theorem 2.3.4 ([32])**.** *Let $\langle G, m \rangle$ be a gain graph, let u be any vertex of G, and let T be any spanning tree of G. Then the derived graph G^{m_T} corresponding to $\langle G, m_T \rangle$ is isomorphic to the derived graph G^m.*

Proof. This amounts to showing that there exists an appropriate relabeling of G^m. For each vertex v of G, relabel the vertices (v, z), $z \in \mathbb{Z}^d$ in the fiber over v according

31

to the rule $z \rightarrow z - c$, where c is the net gain on the unique path from the root vertex u to the vertex v. If e is an edge originating at v, then we also change the indices of edges $(e, i))$ in the fiber over e so that they agree with the relabeled indices of their initial points. This relabeling of vertices and edges defines an isomorphism $G^m \rightarrow G^{m_T}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We say that the graphs $\langle G, m \rangle$ and $\langle G, m_T \rangle$ are *T-gain equivalent.* Theorem 3.3.32 will demonstrate that T-gain equivalent graphs share the same generic rigidity properties.

2.3.3 The fundamental group of a graph

It is observed in Gross and Tucker [32] that we can extend the idea of local gain group to a notion of the fundamental group of a graph. Let G be a connected graph, and let W be a u-based closed walk, $u \in V(G)$. We call W a *reduced walk* if no directed edge of W is followed by its reverse. Any u-based closed walk W can be seen to be *equivalent* to a u-based closed walk W', in the sense that there is some sequence of walks $W = W_1, \ldots, W_k = W'$ such that each walk W_i differs from its predecessor W_{i-1} by the removal of a directed edge followed by its reverse.

The equivalence classes of u-based reduced walks form a group, which Gross and Tucker call the *fundamental group of the graph G based at u.* If G is connected, then we may simply refer to the *fundamental group of G,* since the isomorphism

type of its fundamental group is independent of the choice of u. They denote these groups by $\pi_1(G, u)$ and $\pi_1(G)$ respectively.

Since the net gain on any equivalent u-based closed walk is the same, any gain assignment on G induces a homomorphism $\pi_1(G, u) \to \mathcal{A}$, where $\mathcal{A}$ is the gain group. Every element of $\pi_1(G, u)$ is mapped onto the net gain on any u-based closed walk representing that element. On the other hand, every homomorphism $\pi_1(G, u) \to \mathcal{A}$ is induced by a gain assignment on G (where the gains are taken in $\mathcal{A}$). Given such a homomorphism, let T be a spanning tree of G, and let all edges of the tree be assigned the identity element of $\mathcal{A}$. For any non-tree directed edge, say $e = \{v, w\}$, let W be the u-based walk that traverses the unique path in the tree from u to v, then the edge e from v to w, and finally returns to u along the unique path through T from w to u. To the edge e, assign the image in the group $\mathcal{A}$ of the equivalence class in $\pi_1(G, u)$ of the walk W.

In this way, Gross and Tucker note that the standard topological theorems relating fundamental groups and covering spaces may be obtained for graphs. Furthermore, this justifies the use of the term "graph homotopic" to describe the T-gain procedure, or any other transformation which preserves the cycles of a gain graph $\langle G, m \rangle$ and their net gains.

2.4 The d-torus

Let $\widetilde{L}$ be the $d \times d$ matrix whose rows are the linearly independent vectors $\{t_1, \ldots, t_d\}$, $t_i \in \mathbb{R}^d$. Let $\widetilde{L}\mathbb{Z}^d$ denote the group generated by the rows of $\widetilde{L}$, viewed as translations of $\mathbb{R}^d$ (alternatively, we can think of this as the integer lattice, scaled by the rows of $\widetilde{L}$). We call $\widetilde{L}\mathbb{Z}$ the *fixed lattice*, and $\widetilde{L}$ is the *lattice matrix*. We call the quotient space $\mathbb{R}^d / \widetilde{L}\mathbb{Z}^d$ the *fixed d-torus* generated by $\widetilde{L}$, and denote it $\mathcal{T}_0^d$. It follows that $a \equiv b$ in $\mathcal{T}_0^d$ if and only if $a - b = \sum_{i=1}^{d} k_i t_i$, where $k_i \in \mathbb{Z}$.

There is an equivalence class of sets of translations (equivalently, lattice matrices) which all generate the 'same' torus, up to position at the origin. For any $d \times d$ matrix $\widetilde{L}$, there is a rotation matrix R such that $R\widetilde{L} = L_0$, where L_0 is a lower triangular matrix. R is a $d \times d$ rotation matrix which rotates the parallelotope generated by the rows of $\widetilde{L}$ (d-dimensional generalization of the parallelogram, see Coxeter [19]) such that $R\widetilde{L} = L_0$ is lower-triangular. For example, in 3-dimensions, this is the rotation that maps t_1 to a vector on the x-axis, and t_2 to a vector on the xy-plane. More generally, R is the product of $d - 1$ rotation matrices $R_1, \ldots, R_{d-1}$, where R_1 rotates t_1 onto the x-axis, and each subsequent rotation fixes the placement of the previously rotated generators, until R_{d-1} rotates t_{d-1} about the $(d-2)$-dimensional subspace of $\mathbb{R}^d$ which fixes the placements of $t_1, \ldots, t_{d-2}$.

We therefore assume, without loss of generality, that $\widetilde{L}$ is the lower triangular matrix L_0

$$
L_0 = \begin{pmatrix}
t_{11} & 0 & 0 & \ldots & 0 \\
t_{12} & t_{22} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & 0 \\
t_{d1} & t_{d2} & t_{d3} & \ldots & t_{dd}
\end{pmatrix},
$$

where $t_{\ell r} \in \mathbb{R}$ are the $\binom{d+1}{2}$ non-zero entries.

Let $L(t)$ be the matrix with the same form as L_0, but where the entries $t_{\ell r}$ are allowed to vary continuously with time. We call the quotient space $\mathbb{R}^d/L(t)\mathbb{Z}^d$ the *flexible d-torus*, and denote it by $\mathcal{T}^d$. We say that $L(t)\mathbb{Z}^d$ is the *flexible lattice*. There are $\binom{d+1}{2}$ variable entries $t_{\ell r}(t)$ in $L(t)$ (since it is lower triangular). We denote the initial position of $L(t)$ by L.

Let $L_k(t)$ be the matrix obtained from $L(t)$ by allowing some k-dimensional subset of the $\binom{d+1}{2}$ variables to remain continuous functions of time, and fixing the others. When we permit a mix of fixed and flexible elements, we denote the resulting *partially flexible d-torus* by $\mathcal{T}_k^d$, where $0 < k \leq \binom{d+1}{2}$ is the number of flexible elements. Again we denote the initial position of $L_k(t)$ by L_k. Note that $L_0 = L_0(t)$ is the fixed lattice. For example, $\mathcal{T}_1^2$ could be the 2-torus generated by the vectors $t_1 = (x(t), 0)$ and $t_2 = (y_1, y_2)$, where y_1, y_2 are fixed real numbers, but $x(t)$ is a continuous function of time. We will elaborate on these distinctions later

in the text.

To summarize the notation above:

- $\widetilde{L}$ is an arbitrary $d \times d$ matrix with non-zero determinant (linearly independent rows).

- L_0 is a lower triangular matrix with fixed entries.

- $L(t)$ is a lower triangular matrix with the $\binom{d+1}{2}$ lower triangular elements $t_{\ell r}$ continuous functions of time. $L = L(0)$.

- $L_k(t)$ is the lower triangular matrix obtained from $L(t)$ by fixing $d - k$ of the variable entries of $L(t)$. $L_k = L_k(0)$.

Remark 2.4.1. This representation of an abstract torus should not be confused with a realization of it. For example, we can realize the 2-torus $\mathcal{T}_0^2$ in $\mathbb{R}^3$ as the familiar donut. This realization will change the metric properties of $\mathcal{T}_0^2$, due to the curvature of the surface in $\mathbb{R}^3$. However, $\mathcal{T}_0^2$ can also be realized in $\mathbb{R}^4$ in the following way:

$$
\begin{aligned}
p: \quad & \mathbb{R}^2 \quad \longrightarrow \mathbb{R}^4 \\
& (x, y) \quad \longrightarrow \frac{1}{2\pi}(\cos 2\pi x, \sin 2\pi x, \cos 2\pi y, \sin 2\pi y)
\end{aligned}
$$

This is an isometric realization of $\mathcal{T}_0^2$ in $\mathbb{R}^4$, and it can be shown that this surface has zero Gaussian curvature everywhere, which explains why this realization is sometimes called the "flat" torus. See do Carmo [23] or [35] for details. □
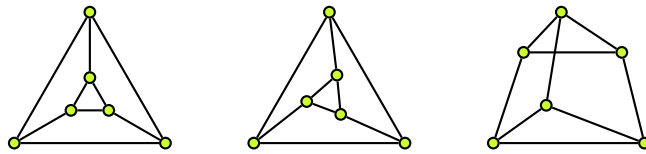
Figure 2.7: Three frameworks (G, p) with the same underlying graph G, and three different realizations p.

2.5 An introduction to the rigidity of finite frameworks

The development of the theory of rigidity for periodic frameworks in the remainder of this thesis is mostly self-contained. Nevertheless, as a preview and for context we collect here the basic ideas of rigidity theory for finite graphs. All of the definitions and results contained here are entirely standard, and can be found in the following general references on rigidity theory: [30, 31, 84, 85]. In addition, these concepts are also nicely summarized in the introduction of [63], which we borrow from here.

A *framework* in $\mathbb{R}^d$ is a pair (G, p) consisting of a finite simple graph $G = (V, E)$, together with an assignment $p = (p_1, \ldots, p_{|V|})$ of points $p_i \in \mathbb{R}^d$ to the vertices i of V, such that $p_i \neq p_j$ whenever $\{i, j\} \in E$. The graph G is called the *underlying graph*, and p is called the *configuration* or *realization* of G (See Figure 2.7). We will sometimes find it convenient to think of p as a point of $\mathbb{R}^{d|V|}$. We write $p(i) = p_i$.

We call the vertices of G in (G, p) *joints* of the framework, and edges $\{i, j\} \in E$ become *bars* of (G, p). The *length* of the bar $\{i, j\}$ is the Euclidean length $\|p(i) - p(j)\|$. Note that two vertices may have the same position in $\mathbb{R}^d$, provided

37

that they are not endpoints of an edge of G. This ensures that all edges of (G, p) have strictly positive length.

Remark 2.5.1. The frameworks considered in this section are finite, that is, the graph G has a finite number of vertices and edges. It is possible to consider infinite frameworks, $(\widetilde{G}, \widetilde{p})$, where $\widetilde{G}$ is an infinite graph, and the definitions of motions, infinitesimal motions, rigidity and infinitesimal rigidity (see below) are the same. However, many of the standard rigidity results do not apply directly to this setting.

$\square$

2.5.1 Rigidity

From [31]: A *motion* of the framework (G, p) is an indexed family of functions $P_i : [0, 1] \to \mathbb{R}^d$, $i = 1, \ldots, |V|$, such that

1. $P_i(0) = p_i$ for all i;

2. $P_i(t)$ is differentiable on the interval $[0, 1]$ for all i;

3. $\|P_i(t) - P_j(t)\| = \|p_i - p_j\|$, for all $t \in [0, 1]$ and $\{i, j\} \in E$.

A motion is called a *rigid motion* or *trivial motion* if it is an isometry of the whole framework. That is, $\{P_i\}$ is a rigid motion if the distances between all pairs of vertices of the framework are preserved by the motion: $\|P_i(t) - P_j(t)\| = \|p_i - p_j\|$,
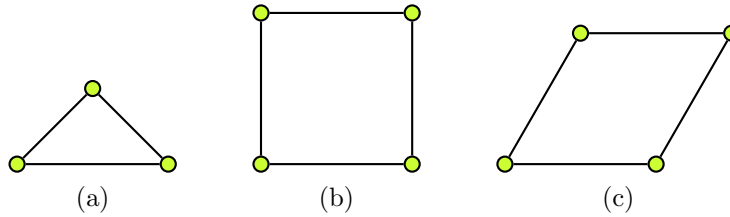
Figure 2.8: The triangle (a) is rigid, while the square in (b) is not: there is a continuous deformation of the joints of the square to the position shown in (c).

for all $t \in [0,1]$ and all $1 \le i \le j \le |V|$. Examples of rigid motions are translations and rotations of the framework.

The framework (G, p) is called *rigid* if all of its motions are rigid motions. That is, (G, p) admits no non-trivial *flex*: a motion of the framework which changes the distance between at least one pair of vertices

$$\|P_i(t) - P_j(t)\| \ne \|p_i - p_j\|$$

for some $t \in [0,1]$ and pair $\{i, j\} \notin E$. A flex is sometimes called a *deformation* of the framework [31]. If (G, p) is not rigid, then it is called *flexible* (See Figure 2.8). In fact, a result of Roth and Whiteley (Proposition 3.2, [60]) shows that we can replace "differentiable" with "continuous" in the definition of motion above, while preserving the same meaning of rigidity and flexibility. This justifies our use of the term "continuous rigidity" to distinguish the form of rigidity just described from the theory of infinitesimal rigidity which we turn toward now.

2.5.2 Infinitesimal rigidity

The problem of finding motions of frameworks is, in general, difficult. As a result the problem is frequently linearized to the theory of infinitesimal rigidity, which we shall now describe. Intuitively, if we differentiate the length $K = \|p_i - p_j\|$ of any bar of the framework (G, p), we find:

$$
\begin{aligned}
\frac{d}{dt}(K^2) &= \frac{d}{dt}\left((p_i - p_j)_1^2 + \cdots + (p_i - p_j)_d^2\right) \\
0 &= 2(p_i p_i' - p_j p_j')_1 + \cdots + 2(p_i p_i' - p_j p_j')_d \\
0 &= (p_i - p_j) \cdot (p_i' - p_j').
\end{aligned}
$$

Indeed we define an *infinitesimal motion* of (G, p) to be a function $u : V \to \mathbb{R}^d$ such that

$$
(p_i - p_j) \cdot (u_i - u_j) = 0 \text{ for all } \{i, j\} \in E, \tag{2.3}
$$

and we denote $u(i) = u_i$. We may think of an infinitesimal motion as an assignment of velocities to the vertices of the framework in such a way that the length of the bar is instantaneously preserved. Equivalently, the projections of the velocities at the endpoints of a bar onto the line of the bar must be equal in magnitude and direction (see Figure 2.9a).

An infinitesimal motion is called an *infinitesimal rigid motion* or a *trivial infinitesimal motion* if

$$
(p_i - p_j) \cdot (u_i - u_j) = 0
$$
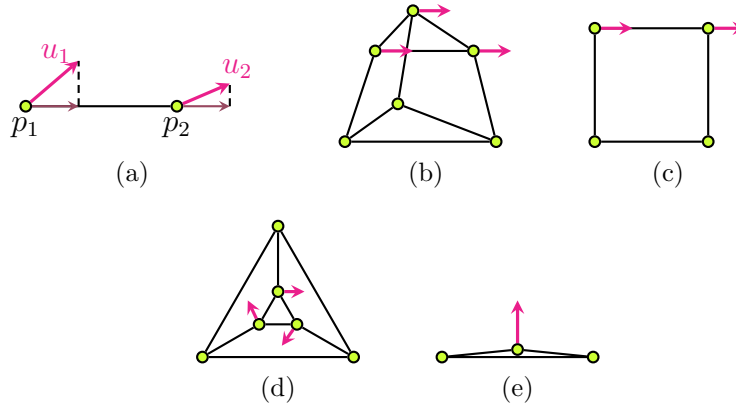
Figure 2.9: The infinitesimal velocities at the endpoints of a single edge must project onto that edge with equal magnitude and direction (a); The frameworks in (b) – (e) are infinitesimally flexible. However both (d) and (e) are rigid.

for *all* pairs of vertices $i, j \in V$. An infinitesimal motion is called an *infinitesimal flex* if it is non-trivial. For a framework whose joints affinely span $\mathbb{R}^d$, an infinitesimal flex has

$$(p_i - p_j) \cdot (u_i - u_j) \neq 0$$

for some pair of vertices $i, j \in V$ not connected by an edge. A framework (G, p) is *infinitesimally rigid* if all of its infinitesimal motions are trivial infinitesimal rigid motions. Otherwise, (G, p) is *infinitesimally flexible*. See Figure 2.9 for examples of infinitesimal motions. Note that the joints of the framework pictured in (e) do not affinely span $\mathbb{R}^2$, but the motion pictured is still a non-trivial infinitesimal motion.

The key result relating infinitesimal rigidity and rigidity is the following:

Theorem 2.5.2. *If the framework (G, p) is infinitesimally rigid, then it is rigid.*

A proof of this result can be found in [29] or [2]. Note that the converse is not necessarily true, as in Figure 2.9 (e). However, we will soon see that for the important class of *generic* frameworks (which we will shortly define), infinitesimal rigidity and rigidity are equivalent.

The essential advantage of linearizing the problem of rigidity is that we can apply the tools of linear algebra. In particular, the system of equations given by solving (2.3) for all edges $\{i, j\} \in E$ can be recorded in a matrix. The *rigidity matrix* of the framework (G, p) is the $|E| \times d|V|$ matrix $\mathbf{R}(G, p)$ given by

$$
\begin{array}{c}
\phantom{\text{edge } \{i,j\}} \quad\quad\quad\quad i \quad\quad\quad\quad\quad\quad\quad\quad j \\
\text{edge } \{i, j\} \left(\begin{array}{ccccccc} & & \vdots & & & & \\ 0 \cdots 0 & p_i - p_j & 0 \cdots 0 & p_j - p_i & 0 \cdots 0 \\ & & \vdots & & & & \end{array} \right).
\end{array}
$$

Note that each column of the rigidity matrix above actually represents d columns corresponding to the coordinates of the vertex $i \in V$.

We may identify an infinitesimal motion u of (G, p) with a point (column vector) of $\mathbb{R}^{d|V|}$. That is, infinitesimal motions are solutions of the matrix equation

$$
\mathbf{R}(G, p) \cdot u = 0.
$$

The space of infinitesimal motions of (G, p) are therefore the kernel of the rigidity matrix $\mathbf{R}(G, p)$. If the joints $p_1, \ldots, p_{|V|}$ span an affine subspace of $\mathbb{R}^d$ of dimension

42

at least $d-1$, then the space of infinitesimal rigid motions of (G,p) has a basis consisting of d infinitesimal translations and $\binom{d}{2}$ infinitesimal rotations [30, 84]. It follows that in such a situation, the kernel of $\mathbf{R}(G,p)$ always contains at least $\binom{d+1}{2}$ elements. Furthermore, since (G,p) is rigid if and only if all of its infinitesimal motions are trivial (rigid) motions, we have the following key result:

Theorem 2.5.3 ([2, 85]). *A framework (G,p) with $|V| > d$ is infinitesimally rigid in $\mathbb{R}^d$ if and only if*

$$\mathrm{rank}\mathbf{R}(G,p) = d|V| - \binom{d+1}{2}.$$

When $|V| \leq d$, the framework (G,p) is infinitesimally rigid if and only if G is the complete graph, and the joints $p_i \ldots, p_{|V|}$ do not lie on an affine space of dimension $|V| - 2$ or less.

The framework (G,p) is said to be *independent* if the rows of its rigidity matrix are linearly independent. Similarly, we say the set of edges $E' \subset E$ is independent if the rows corresponding to E' are linearly independent in (G,p). If the rows are not linearly independent, we say the framework (or set of edges) is *dependent*.

A framework that is both infinitesimally rigid and independent is called *isostatic*, or *minimally rigid*.

Theorem 2.5.4 ([85]). *For a framework (G,p) in $\mathbb{R}^d$, with $|V| > d$, the following are equivalent:*

43

1. (G, p) is isostatic;

2. (G, p) is infinitesimally rigid, and $|E| = d|V| - \binom{d+1}{2}$;

3. (G, p) is independent, and $|E| = d|V| - \binom{d+1}{2}$;

4. (G, p) is infinitesimally rigid, and the removal of any bar (but no vertices) leaves an infinitesimally flexible framework.

Remark 2.5.5. The notion of independence is closely related to the theory of *static rigidity*. We may interpret a row dependence of the rigidity matrix as a *stress* on the framework edges. A framework is *statically rigid* if every *equilibrium stress* has a resolution by the rows of a matrix. Static rigidity can be shown to be equivalent to *kinematic rigidity*, the theory of motions and infinitesimal motions outlined above. In this thesis, however, we do not address stresses in periodic frameworks, and so we will not comment further about static rigidity here. We sketch the basics of static rigidity in Section 8.2.4, in preparation for several topics of further work. □

2.5.3 Generic rigidity

We now turn to the idea of *generic rigidity*, which is concerned with the combinatorial characterization of rigidity of frameworks. In addition, we will soon see that for generic frameworks, rigidity and infinitesimal rigidity are the same (Theorem 2.5.7).

We note that there are a number of different ways to define generic frameworks, but they all have the same result: the rigidity of generic frameworks (G, p) is characterized by the properties of the underlying graph G, not the configuration p. Furthermore, since 'most' configurations are generic, we may say that a framework (G, p) is infinitesimally rigid for almost all configurations p. We now make these ideas precise.

Let K_n be the complete graph on n vertices (K_n is the graph with an edge connecting every distinct pair of vertices), and let K be the set of edges of K_n. For each $i = 1, \ldots, |V|$ we let $x_i = ((x_i)_1, \ldots, (x_i)_d)$ be a d-tuple of variables. Let $\mathbf{R}(K_n, x)$ be the matrix that is obtained from $\mathbf{R}(K_n, p)$ by replacing p with the indeterminants x. After Schulze [63], we call $\mathbf{R}(K_n, x)$ the *d-dimensional indeterminant rigidity matrix* of K_n.

Let $G = (V, E)$ and let $n = |V|$. We say that the configuration $p : V \to \mathbb{R}^d$ is *generic* if the determinant of any submatrix of $\mathbf{R}(K_n, p)$ is zero if and only if the determinant of the corresponding submatrix of $\mathbf{R}(K_n, x)$ is identically zero. The framework (G, p) is said to be *generic* if p is generic.

The subdeterminants of $\mathbf{R}(K_n, p)$ that are not identically zero correspond to algebraic curves in $\mathbb{R}^{d|V|}$. Each such curve is a closed set of measure zero, and hence the union of any finite number of such curves is also a closed set of measure zero. It follows that the set of generic configurations (the set of all configurations

avoiding these curves) is open and dense.

Theorem 2.5.6 ([30, 82]). *Let (G, p_0) be a framework in $\mathbb{R}^d$ (where p_0 is not necessarily generic). If (G, p_0) is infinitesimally rigid (independent, isostatic respectively), then (G, p) is infinitesimally rigid (independent, isostatic respectively) for all generic configurations p.*

In light of this result, we sometimes say that the graph G is *generically rigid* (*independent, isostatic*), meaning that any framework (G, p) where p is a generic configuration is infinitesimally rigid (independent, isostatic). For example, the frameworks pictured in Figure 2.7 share the same underlying graph G. This is generically rigid, but non-generic configurations may be infinitesimally flexible, as shown in Figure 2.9 (c) and (d).

There are other ways to define generic. For example, we might ask that the coordinates of p be algebraically independent over $\mathbb{Z}$, as in [13]. In other words, we demand that there is *no* polynomial $h(x) \in \mathbb{Z}[x]$ such that $h(p) = 0$. This is a strong condition which ensures that p is generic by the definition above. However, the set of all such embeddings is dense but not open in $\mathbb{R}^{d|V|}$.

For a particular graph G, we may also define a weaker form of generic, by replacing K_n by G in the definition given above. That is, we say that (G, p) is *G-generic* if the determinant of any submatrix of $\mathbf{R}(G, p)$ is zero if and only if the determinant of the corresponding submatrix of $\mathbf{R}(G, x)$ is identically zero [63]. We

will return to the two notions of generic and G-generic in Chapter 3.

An important result of Asimow and Roth [2] proves the equivalence of rigidity and infinitesimal rigidity for generic frameworks:

Theorem 2.5.7 ([2])**.** *If a framework (G, p) is generic, then (G, p) is rigid if and only if it is infinitesimally rigid.*

This justifies our use of the term *generically rigid* to mean a framework that is infinitesimally rigid and generic.

2.5.4 Fundamental results

We now mention some basic results of rigidity theory. Theorem 2.5.3 implies a simpler result known as Maxwell's Rule, after J.C. Maxwell who gave necessary conditions for a 2- or 3-dimensional framework to be isostatic.

Theorem 2.5.8 (Maxwell's Rule, 1864)**.** *Let (G, p) be a framework in $\mathbb{R}^d$ with $|V| \geq d$. If (G, p) is isostatic, then*

$$|E| = d|V| - \binom{d+1}{2}.$$

The advantage of this condition over the stronger form in Theorem 2.5.3 is that it is a counting condition only, and does not depend on the position of the vertices p.

We can extend Maxwell's condition to include a statement about subgraphs of the graph G. If a framework (G, p) is isostatic (and therefore independent), then any subgraph $G' \subset G$ with $G' = (V', E')$ must also be independent. It follows that

Theorem 2.5.9 ([31]). *Let (G, p) be an isostatic framework in $\mathbb{R}^d$ with $|V| \geq d$. Then*

1. *$|E| = d|V| - \binom{d+1}{2}$, and*

2. *for all subgraphs $G' \subseteq G$, $|E'| \leq d|V'| - \binom{d+1}{2}$.*

In dimensions 1 and 2, this result *characterizes* generic rigidity. In $\mathbb{R}$, this says that a graph G is isostatic on the line if and only it G is a tree, and is rigid on the line if G is connected [83].

In $\mathbb{R}^2$, Theorem 2.5.9 becomes Laman's Theorem:

Theorem 2.5.10 (Laman's Theorem, [46]). *The framework (G, p) with $|V| \geq 2$ is generically isostatic in $\mathbb{R}^2$ if and only if*

1. *$|E| = 2|V| - 3$, and*

2. *$|E'| \leq 2|V'| - 3$ for all subgraphs $G' \subseteq G$ with $|V'| \geq 2$.*

Proofs of Laman's theorem can be found in [30, 73], among others.
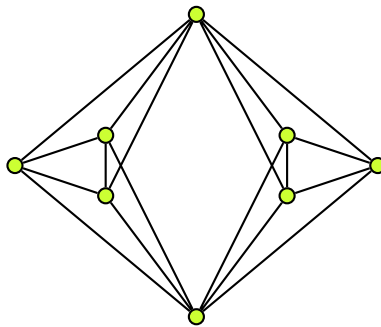
Figure 2.10: The "double bananas" are generically flexible, yet satisfy the necessary conditions of Theorem 2.5.9 in $\mathbb{R}^3$.

Theorem 2.5.9 is *not* sufficient for generic rigidity in $\mathbb{R}^3$, however. The "double banana" example shown in Figure 2.10 is a graph that satisfies the conditions of Theorem 2.5.9, but is generically flexible.

The final result that we mention here is known as Henneberg's Theorem. This is an inductive result, which says that every isostatic framework in $\mathbb{R}^2$ can be constructed from a smaller isostatic framework by a series of inductive constructions. In particular, it uses vertex additions and edge splits, which we will now define, for general d.

Let G be a graph, and let $U \subseteq V$ be a subset of vertices with $|U| = d$. A vertex d-addition (to G) is the addition of a vertex 0 to the vertex set V together with d new edges $\{0, u\}, u \in U$, to the edge set E, creating a new graph $G' = (V', E')$ (see Figure 2.11a for a 2-dimensional example).

Lemma 2.5.11 (Vertex Addition Lemma [31, 84])**.** *A vertex d-addition on a gener-*
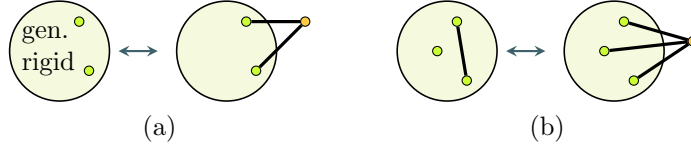
Figure 2.11: Vertex addition (a) and edge splitting (b) in $\mathbb{R}^2$. The large circular area represents a generically rigid graph, and both moves preserve generic rigidity.

ically isostatic graph (in $\mathbb{R}^d$) is generically isostatic. Conversely, deleting a d-valent vertex from a generically isostatic graph leaves a generically isostatic graph.

Let G be a graph, and let $U \subseteq V$ be a subset of vertices of size $d + 1$, and where one of the pairs of vertices of U, say $\{u_1, u_2\}$, is an edge of G. An *edge d-split (on $\{u_1, u_2\}$)* of G is the graph obtained from G by adding a vertex 0 to V, connecting 0 to each element of U, and deleting the edge $\{u_1, u_2\}$ from E. That is, an edge d-split is the graph $G' = (V', E')$, where $V' = V \cup \{0\}$, and $E' = E - \{u_1, u_2\} + \{\{0, u\}|u \in U\}$. See Figure 2.11(b)).

Lemma 2.5.12 (Edge Split Lemma [31, 84]). *An edge d-split of a generically isostatic graph in $\mathbb{R}^d$ is generically isostatic. Conversely, if one deletes a vertex v of degree $d + 1$ from a generically isostatic graph in $\mathbb{R}^d$, then one may add a single edge between two of the vertices formerly adjacent to v such that the resulting graph is generically isostatic.*

Henneberg used vertex additions and edge splits in two dimensions to give the following characterization of generically isostatic frameworks in $\mathbb{R}^2$:

50

Theorem 2.5.13 (Henneberg's Theorem (1911) [36])**.** *A framework* (G, p) *is generically isostatic in* $\mathbb{R}^2$ *if and only if it may be constructed from a single edge by a sequence of vertex additions and edge splits.*

It is interesting to note that Henneberg did not provide a correct proof of this result, and indeed claimed the same arguments worked in $\mathbb{R}^3$, which is false. A proof of Henneberg's theorem may be found in [84] or [31] for example.

This concludes our brief tour of the basics of rigidity theory. We will now turn our attention to infinite frameworks with periodic structure. Note that in some literature, continuous rigidity or simply rigidity as discussed in section 2.5.1 is called "finite rigidity". Since the subject of this thesis is (infinite) periodic frameworks, we reserve the use of the term *finite rigidity* to mean the rigidity or infinitesimal rigidity of finite frameworks. That is, it refers to the study of the rigidity of frameworks that are not periodic, and have a finite number of vertices and edges, which has been the content of the preceding sections. We will use the term *continuous rigidity* to describe the theory of motions described in Section 2.5.1, in contrast with *infinitesimal* rigidity.

3 Periodic frameworks and their rigidity

3.1 Introduction

In this chapter we introduce our basic object of study, the periodic framework.
We define its corresponding orbit framework on the torus, and define rigidity and
infinitesimal rigidity in this setting. Throughout this chapter and its sequel, we
consider frameworks on a fixed torus, while Chapter 5 will discuss frameworks on
a flexible torus.

The work of Borcea and Streinu is closely related to what is presented here. We
will note, where appropriate, the connections and terminology that appear in their
paper [7]. It should be emphasized however that the work of the present thesis was
completed independently, as reflected in a 2009 talk at the sectional AMS meeting
in Worcester [56]. Unless otherwise noted, the definitions and results in the present
work should be taken as the work of the author.

At a general level, the work of Borcea and Streinu treats periodic frameworks
as infinite simple graphs with periodic structure. In contrast, the original work

described here is concerned with finite frameworks on a torus, which *correspond* to infinite periodic frameworks. Both approaches share some common features: basic counting on orbit frameworks, a similar rigidity matrix, and basic results linking rigidity and infinitesimal rigidity. The two perspectives diverge on *genericity*. In [7], the edge directions are assumed to be generic, while in the present work, the edge directions are partially determined by the topology of the graph on the torus. In other words, we view this as part of the combinatorial information we seek to characterize. Only the positions of the vertices on the torus are assumed to be generic, as in finite rigidity.

We begin this chapter with a description of Borcea and Streinu's d-periodic frameworks, which we eventually show to be equivalent to our presentation of periodic orbit frameworks on a torus.

3.2 Periodic frameworks

3.2.1 d-periodic frameworks in $\mathbb{R}^d$

In [7], Borcea and Streinu set out notation for the study of infinite graphs with periodic structure. They say that the pair $(\widetilde{G}, \Gamma)$ is a *d-periodic graph* if $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ is a simple infinite graph with finite degree at every vertex, and $\Gamma \subset Aut(\widetilde{G})$ is a free abelian group of rank d, which acts without fixed points and has a finite number

of vertex orbits. In other words, Γ is isomorphic to $\mathbb{Z}^d$.

Let $(\widetilde{G}, \Gamma)$ be a d-periodic graph, with $\widetilde{G} = (\widetilde{V}, \widetilde{E})$. Borcea and Streinu define a *periodic placement* of $(\widetilde{G}, \Gamma)$ to be the pair $(\widetilde{p}, \pi)$ given by the functions

$$\widetilde{p} : \widetilde{V} \to \mathbb{R}^d \quad \text{and} \quad \pi : \Gamma \to Trans(\mathbb{R}^d),$$

where $\widetilde{p}$ assigns positions in $\mathbb{R}^d$ to each of the vertices of $\widetilde{G}$, and π is an injective homomorphism of Γ into the group of translations of $\mathbb{R}^d$, denoted $Trans(\mathbb{R}^d)$. The image $\pi(\gamma)$ has the form $\pi(\gamma)(x) = x + \gamma^*$, where $\gamma^* \in \mathbb{R}^d$ is a translation vector. The placement functions $\widetilde{p}$ and π must satisfy

$$p(\gamma v) = \pi(\gamma)(p(v)),$$

or equivalently,

$$\widetilde{p}(\gamma v) = \widetilde{p}(v) + \gamma^*. \tag{3.1}$$

Together, a d-periodic graph $(\widetilde{G}, \Gamma)$ and its periodic placement $(\widetilde{p}, \pi)$ define a d-*periodic framework*, which is denoted $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ [7].

3.2.2   Periodic orbit frameworks on $\mathcal{T}_0^d$

We now introduce our vocabulary for treating periodic frameworks, and we will later show these representations are equivalent. Let $\mathcal{T}_0^d$ be the fixed d-torus generated by a $d \times d$ matrix $\widetilde{L}$ (where $\widetilde{L}$ is not necessarily lower-triangular). A d-*periodic orbit framework* is a pair $(\langle G, m \rangle, p)$, where $\langle G, m \rangle$ is a gain graph with gain group

54

$\mathbb{Z}^d$, and p is an assignment of a unique geometric position on the fixed d-torus $\mathcal{T}_0^d$ to each vertex in V. That is, $p : V \longrightarrow \mathcal{T}_0^d \subset \mathbb{R}^d$, with $p(v_i) \neq p(v_j)$ for $i \neq j$. We denote the position of the vertex v_i by $p(v_i) = p_i$, and call p a *configuration* of $\langle G, m \rangle$. The geometric image of the edge $e = \{v_i, v_j; m_e\}$, is denoted $\{p_i, p_j + m_e\}$, and will be called a *bar* of the framework. The geometric vertices $p_1, \ldots p_m$ will be called the *joints*. We will also call a d-periodic orbit framework $(\langle G, m \rangle, p)$ simply a *framework on $\mathcal{T}_0^d$*. When we wish to talk only about the combinatorial structure of a periodic framework, we will refer to the gain graph $\langle G, m \rangle$ as a *d-periodic orbit graph*. Where it is clear from context we omit the 'd'.

The periodic framework $(\langle G, m \rangle, p)$ determines the *derived periodic framework* described by the pair $(\langle G^m, \widetilde{L} \rangle, p^m)$. The graph $G^m = (V^m, E^m)$ is determined as described in Section 2.3, with the vertices and edges indexed by the elements of the integer lattice: $V^m = V \times \mathbb{Z}^d$, and $E^m = E \times \mathbb{Z}^d$. The configuration $p^m : V^m \to \mathbb{R}^d$ is determined by the configuration p. The vertex $(v, z) \in V^m$ where $v \in V$, $z \in \mathbb{Z}^d$, has the following position:

$$p^m(v, z) = p(v) + z\widetilde{L},$$

where $\widetilde{L}$ is the lattice matrix whose rows are the generators of $\mathcal{T}_0^d$.

Similarly, from the derived periodic framework $(\langle G^m, \widetilde{L} \rangle, p^m)$ we can define the periodic framework $(\langle G, m \rangle, p)$. Let $G = (V, E)$ be the graph of vertices and edges consisting of all the elements of G^m whose indices are the zero vector. The gain

55

assignment m is determined by the edges E^m. If, for example, the edge $(e, 0)$ connects vertices $(v_1, 0)$ and (v_2, z) in G^m, then the directed edge $\{v_1, v_2\} \in E$ has gain z.

In contrast to the periodic orbit framework, the periodic framework has a countably infinite number of vertices and edges. The key relationship between these two different objects is the following:

Theorem 3.2.1. *A d-periodic framework $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ has a representation as the derived periodic framework $(\langle G^m, \widetilde{L} \rangle, p^m)$ corresponding to the periodic orbit framework $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d = \mathbb{R}^d / \widetilde{L} \mathbb{Z}^d$.*

The proof of this result consists of picking representatives from the vertex orbits of the periodic framework $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$, and using them to define the periodic orbit framework $(\langle G, m \rangle, p)$. We will describe this construction in detail, beginning with the following result about the graph $(\widetilde{G}, \Gamma)$.

The quotient multigraph $\widetilde{G}/\Gamma$ is finite since both $\widetilde{V}/\Gamma$ and $\widetilde{E}/\Gamma$ are finite [7]. Let $G = \widetilde{G}/\Gamma$, and let $q_\Gamma : \widetilde{G} \to G$ be the quotient map. Then q_Γ identifies each vertex orbit in $\widetilde{G}$ with a single vertex in G, and similarly for edges.

Theorem 3.2.2 (Theorem 2.2.2 in [32]). *Let $(\widetilde{G}, \Gamma)$ be a d-periodic graph, and let G be the resulting quotient graph by the action of Γ. Then there is an assignment m of gains in $\mathbb{Z}^d$ to the edges of G and a labeling of the vertices of $\widetilde{G}$ by the elements*

56

of $V_G \times \mathbb{Z}^d$, such that $\widetilde{G} = G^m$ and the action of Γ on $\widetilde{G}$ is the natural action of $\mathbb{Z}^d$ on G^m.

Proof. The following proof is an adaptation of the proof of Theorem 2.2.2 in [32] to the periodic setting.

For every element γ of Γ, let

$$\Phi_\gamma: \quad \widetilde{G} \quad \rightarrow \quad \widetilde{G} \quad \text{be the graph automorphism defined on}$$

$$\text{vertices:} \quad \alpha(v) \quad \rightarrow \quad \gamma(\alpha(v))$$

$$\text{edges:} \quad \alpha(e) \quad \rightarrow \quad \gamma(\alpha(e)),$$

where $\alpha \in \Gamma$. Φ_{id} is the identity automorphism.

Let $q_\Gamma : \widetilde{G} \to G$ be the quotient map described above, and let $\psi : \Gamma \to \mathbb{Z}^d$ be the isomorphism between Γ and $\mathbb{Z}^d$. Choose arbitrary orientations for the edges of both G and $\widetilde{G}$. For each vertex v of G, label one vertex of the orbit $q_\Gamma^{-1}(v)$ in $\widetilde{G}$ to be $(v, 0)$. For every element $\gamma \neq id$ of Γ, label the vertex $\Phi_\gamma(v, 0)$ as $(v, \psi(\gamma)) \in V \times \mathbb{Z}^d$.

Let e be a directed edge in G from the vertex v to the vertex w. Since the group Γ acts freely on $\widetilde{G}$, the edges in the orbit $q_\Gamma^{-1}(e)$ are in one-to-one correspondence with the elements of Γ, and therefore also $\mathbb{Z}^d$. That is, there is exactly one edge originating in each of the vertices of the vertex orbit of $q_\Gamma^{-1}(v)$, which are indexed by $\mathbb{Z}^d$. Hence the choice of the labeling of the edge (e, a) is unique for any $a \in \mathbb{Z}^d$.

Suppose the terminal vertex of the edge $(e, 0)$ is (w, b). Then assign gain b to

the edge e of the quotient graph G. It remains to be shown that this labeling

of edges in the orbit $q_\Gamma^{-1}(e)$, and the choice of the gain b for this edge, yields an

isomorphism $(\widetilde{G}, \Gamma) \to G^m$. That is, we must show that for each $\gamma \in \Gamma$, the edge

$(e, \psi(\gamma))$ terminates at $(w, \psi(\gamma) + b)$.

Note that $(e, \psi(\gamma)) = \Phi_\gamma((e, 0))$, so the terminal vertex of edge $(e, \psi(\gamma))$ is the

terminal vertex of $\Phi_\gamma((e, 0))$, which is

$$\Phi_\gamma((w, b)) = \Phi_\gamma(\Phi_b((w, 0))) = \Phi_{\gamma\psi^{-1}(b)}((w, 0)) = (w, \psi(\gamma) + b).$$

This labeling process is thus an isomorphism $(\widetilde{G}, \Gamma) \to G^m$, which identifies orbits

in $\widetilde{G}$ under Γ with fibres of G^m. In addition, this isomorphism has been defined so

that the action of Γ on $\widetilde{G}$ is consistent with the natural action of $\mathbb{Z}^d$ on G^m. $\qquad \square$

From Theorem 3.2.2, we know that the d-periodic graph $(\widetilde{G}, \Gamma)$ can be described

by the derived graph G^m corresponding to a d-periodic orbit graph $\langle G, m \rangle$, where

$G = (V, E)$. We now show that there is also a correspondence between the periodic

placement $(\widetilde{p}, \pi)$ of $(\widetilde{G}, \Gamma)$ and the map p^m on G^m.

Suppose that the generators of Γ are given by $\{\gamma_1, \ldots, \gamma_d\}$. Put

$$\widetilde{L} = \begin{pmatrix} \gamma_1^* \\ \vdots \\ \gamma_d^* \end{pmatrix}, \text{ where } \gamma_i^* \text{ is determined by (3.1).}$$

Then $\widetilde{L}$ is the matrix whose rows are the translations of $\mathbb{R}^d$ that are the images

under π of the generators of Γ (and again $\widetilde{L}$ is not necessarily lower-triangular).

For $\gamma \in \Gamma$, let $z \in \mathbb{Z}^d$ be the row vector of coefficients of γ written as a linear combination of $\{\gamma_1, \ldots, \gamma_d\}$. Then

$$\widetilde{p}(\gamma v) = \widetilde{p}(v) + \gamma^*$$

$$= \widetilde{p}(v) + z\widetilde{L}.$$

Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be the linear transformation satisfying

$$A(\gamma_i^*) = (\cdots, 0, 1, 0, \cdots),$$

where the non-zero entry occurs in the ith column of the row vector. Then define

$$A\widetilde{L} = \begin{pmatrix} A\gamma_1^* \\ \vdots \\ A\gamma_d^* \end{pmatrix} = I_{d\times d}.$$

This permits us to write

$$A\widetilde{p}(\gamma v) = A\widetilde{p}(v) + z \cdot A\widetilde{L} = A\widetilde{p}(v) + z.$$

For each $v \in V$, there is exactly one vertex in $q_\Gamma^{-1}(v)$ (the orbit of v in $\widetilde{G}$), whose image under $A\widetilde{p}$ is in $[0,1)^d$. Label this vertex by $(v, 0)$, and label the other vertices in $q_\Gamma^{-1}(v)$ according to Theorem 3.2.2. In addition, label the edges e of G by the same theorem, so that $\widetilde{G} = G^m$, the derived graph corresponding to $\langle G, m \rangle$.

59

To determine the map $p^m : V^m \to \mathbb{R}^d$, for each $v \in V$, let $p^m(v, 0) = \widetilde{p}(v, 0)$.

For $a = (a_1, \ldots, a_d) \in \mathbb{Z}^d$, let $\gamma_a = a_1\gamma_1 + \cdots + a_d\gamma_d$. Now define

$$p^m(v, a) = \widetilde{p}(\gamma_a v).$$

Therefore, $Ap^m(v, a) = A\widetilde{p}(v) + a$, and applying the inverse linear transformation A^{-1},

$$p^m(v, a) = \widetilde{p}(v) + a\widetilde{L}.$$

These observations form the proof of Theorem 3.2.1.

Table 3.1 summarizes in chart form the different graphs and notations for periodic frameworks just described. Because every d-periodic framework $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ has a representation as the derived framework $(\langle G^m, \widetilde{L}\rangle, p^m)$ corresponding to the periodic orbit framework $(\langle G, m\rangle, p)$ on $\mathcal{T}_0^d$ (by Theorem 3.2.1), we adopt the following simplification of notation for d-periodic frameworks. Let $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ be an arbitrary periodic framework. Let $\widetilde{L}$ be the matrix described above,

$$\widetilde{L} = \begin{pmatrix} \gamma_1^* \\ \vdots \\ \gamma_d^* \end{pmatrix}, \text{ where } \gamma_i^* \text{ is determined by (3.1).}$$

Then (3.1) can be rewritten

$$\widetilde{p}(v_i, z) = \widetilde{p}(v_i, 0) + z\widetilde{L}.$$

60

Let R be the $d \times d$ rotation matrix which rotates the parallelotope generated by the rows of $\widetilde{L}$ such that $R\widetilde{L} = L_0$ is lower-triangular. Let $R\widetilde{p}$ be the rotated periodic placement of the vertices of $\widetilde{G}$. Then we denote this rotated d-periodic framework by $((\widetilde{G}, L_0), R\widetilde{p})$. The infinite framework $(\widetilde{G}, R\widetilde{p})$ is invariant with respect to the translations given by the rows of L_0. In this way, the d-periodic framework $((\widetilde{G}, L_0), R\widetilde{p})$ is actually an equivalence class of frameworks $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ up to rotation at the origin. Since every such framework has a representation as the derived graph of a periodic orbit framework $((G, m), p)$ on $\mathcal{T}_0^d = \mathbb{R}^d/L_0\mathbb{Z}^d$, we let the vertices of $\widetilde{G}$ be indexed in the same manner as the vertices of G^m. See Table 3.1. That is, we write

$$
\begin{aligned}
R\widetilde{p}(v_i, z) &= R(\widetilde{p}(v_i, 0)) + zR\widetilde{L} \\
&= R(\widetilde{p}(v_i, 0)) + zL_0.
\end{aligned}
$$

We have shown:

Proposition 3.2.3. *A d-periodic framework $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ is equivalent under rotation to a periodic framework $((\widetilde{G}, L_0), R\widetilde{p})$ which is represented as the derived periodic framework $((G^m, L_0), p^m)$ corresponding to the periodic orbit framework $((G, m), p)$ on $\mathcal{T}_0^d = \mathbb{R}^d/L_0\mathbb{Z}^d$, where L_0 is lower triangular.*

As a consequence of this result, we assume that the configuration $\widetilde{p}$ in all subsequent frameworks $((\widetilde{G}, L_0), \widetilde{p})$ is the rotated placement, and that $\mathcal{T}_0^d = \mathbb{R}^d/L_0\mathbb{Z}^d$,

where L_0 is lower triangular.

Table 3.1: Summary of notation for the different conceptions of periodic frameworks.

Graph	Vertices	Edges	Configuration
$(\widetilde{G}, \Gamma)$ $G = \widetilde{G}/\Gamma$	$\widetilde{V}, \;\; \|\widetilde{V}\| = \infty$ $V = \widetilde{V}/\Gamma, \;\; \|V\| < \infty$	$\widetilde{E}, \;\; \|\widetilde{E}\| = \infty$ $E = \widetilde{E}/\Gamma, \;\; \|E\| < \infty$ undirected edges	$(\widetilde{p}, \pi)$ $\widetilde{p} : \widetilde{V} \to \mathbb{R}^d$ $\pi : \Gamma \to Trans(\mathbb{R}^d)$
$\langle G, m \rangle$	$\langle V, m \rangle = V$	$m : E \to \mathbb{Z}^d$ $\langle E, m \rangle =$ $\{\{v, w; m_e\} : \{v, w\} \in E\}$ $\{v, w; m_e\} = \{w, v; -m_e\}$ $\|\langle E, m \rangle\| = \|E\|$ directed, labeled edges	$p : V \to \mathcal{T}_0^d$ $\mathcal{T}_0^d = \mathbb{R}^d / L_0 \mathbb{Z}^d$
$\langle G^m, L_0 \rangle$	$V^m =$ $\{(v, z) : v \in V, z \in \mathbb{Z}^d\}$ $\|V^m\| = \infty$	$E^m =$ $\{\{e, z\} : e \in E, z \in \mathbb{Z}^d\}$ $\{e, z\} = \{(v, z), (w, z + m_e)\}$ $\|E^m\| = \infty$	$p^m : V^m \to \mathbb{R}^d$ $p^m(v, z) = p(v) + z L_0$
$\langle \widetilde{G}, L_0 \rangle$	$\widetilde{V}, \;\; \|\widetilde{V}\| = \infty$	$\widetilde{E}, \;\; \|\widetilde{E}\| = \infty$	$R\widetilde{p} : \widetilde{V} \to \mathbb{R}^d$ $R\widetilde{p}(v, z) =$ $R\widetilde{p}(v, 0) + z L_0$

In the next section, we will explore to what extent the representation of d-periodic frameworks as d-periodic orbit graphs is unique. In addition, before we can define rigidity for periodic orbit frameworks, we first need to define length in this setting.

3.2.3 Equivalence relations among d-periodic orbit frameworks

We now define notions of length and congruence for frameworks on $\mathcal{T}_0^d$, which leads to an equivalence relation among all d-periodic orbit graphs. Let L_0 be the lower triangular matrix whose rows are the translations $\{t_1, \ldots, t_d\}$, where $\mathcal{T}_0^d = \mathbb{R}^d / L_0 \mathbb{Z}^d$.

Given an edge $e = \{v_i, v_j; m_e\} \in E\langle G, m \rangle$, we define the *length* of the edge e to be the Euclidean length of the vector $(p_i - (p_j + m_e L_0))$. That is,

$$\|e\| = \sqrt{(p_{i1} - p_{j1} - [m_e L_0]_1)^2 + \cdots + (p_{id} - p_{jd} - [m_e L_0]_d)^2}$$

$$= \|p_i - (p_j + m_e L_0)\|,$$

where $p_i = (p_{i1}, \ldots, p_{id})$, and $m_e L_0 = ([m_e L_0]_1, \ldots, [m_e L_0]_d)$.

More generally, for any pair of joints p_i, p_j and any element $m_{ij} \in \mathbb{Z}^d$, we write $\|\{p_i, p_j; m_{ij}\}\|$ to denote the Euclidean distance of the vector $(p_i - (p_j + m_{ij} L_0))$. Note that this need not be the same as $\|\{p_j, p_i; m_{ij}\}\|$. That is, the order of the vertices matters.

63

By definition, the edges of $(\langle G^m, L_0 \rangle, p^m)$, have the same lengths as the edges of $(\langle G, m \rangle, p)$. Let $e = \{v_i, v_j; m_e\}$ be an edge of $\langle G, m \rangle$. The edge $(e, z) \in E^m$ connects the vertex (v_i, z) to the vertex $(v_j, m_e + z)$. Hence

$$\|(e, z)\| = \|(p_i + zL_0) - (p_j + m_e L_0 + zL_0)\|$$

$$= \|p_i - (p_j + m_e L_0)\|$$

$$= \|e\|.$$

In other words, all edges in the fibre over e have length $\|e\|$.

Now let $L_0 = I_{d \times d}$ be the d-dimensional identity matrix, and consider $\mathcal{U}_0^d = [0, 1)^{d|V|}$ to be the unit torus generated by L_0. Let $p_i = (p_{i1}, \ldots, p_{id}) \in \mathbb{R}^d$. We write $\lfloor p_i \rfloor$ to denote the vector $(\lfloor p_{i1} \rfloor, \ldots, \lfloor p_{id} \rfloor)$, where $\lfloor x \rfloor, x \in \mathbb{R}$ is the *floor function*, defined to be the largest integer less than or equal to x. We say that the framework $(\langle G, n \rangle, q)$ is $\mathcal{U}_0^d$-*congruent* to $(\langle G, m \rangle, p)$ if there exists a vector $t \in \mathbb{R}^d$ such that

(a) $q_i = (p_i + t) - \lfloor p_i + t \rfloor$ for each vertex $v_i \in V$, and

(b) $n_e = m_e + (\lfloor p_j + t \rfloor - \lfloor p_i + t \rfloor)$.

We write $(\langle G, n \rangle, q) \cong (\langle G, m \rangle, p)$.

If $(\langle G, m \rangle, p)$ and $(\langle G, m \rangle, q)$ are two periodic frameworks with the same underlying gain graph $\langle G, m \rangle$, the description of congruence is more simple. In this case (b) is automatically satisfied, and (a) becomes simply $q_i = p_i + t$, for all $v_i \in V$.

64

More generally, if $\mathcal{T}_0^d$ is the fixed torus generated by the matrix L_0, then there is an affine transformation mapping L_0 to the $d \times d$ identity matrix. We say that the framework $(\langle G, n \rangle, q)$ is $\mathcal{T}_0^d$-*congruent* to $(\langle G, m \rangle, p)$ if their corresponding affine images on the unit torus are $\mathcal{U}_0^d$-congruent.

Proposition 3.2.4. $\mathcal{T}_0^d$-*congruence is an equivalence relation on the set of all periodic frameworks.*

Proof. We verify that $\mathcal{U}_0^d$-congruence satisfies the three conditions of an equivalence relation. The result for $\mathcal{T}_0^d$-congruence follows.

(i) *reflexivity.* $(\langle G, m \rangle, p)$ satisfies both (a) and (b) with $t = 0$.

(ii) *symmetry.* Suppose $(\langle G, n \rangle, q) \cong (\langle G, m \rangle, p)$. We show that $(\langle G, m \rangle, p) \cong (\langle G, n \rangle, q)$.

(a) If $(\langle G, n \rangle, q) \cong (\langle G, m \rangle, p)$, then there exists a vector $t \in \mathbb{R}^d$ such that $q_i = p_i + t - \lfloor p_i + t \rfloor$, for each vertex $v_i \in V$. Then

$$q_i - t = p_i - \lfloor p_i + t \rfloor \tag{3.2}$$

$$\Rightarrow \lfloor q_i - t \rfloor = \lfloor p_i - \lfloor p_i + t \rfloor \rfloor. \tag{3.3}$$

Recall that since $\lfloor p_i + t \rfloor \in \mathbb{Z}^d$,

$$\lfloor p_i - \lfloor p_i + t \rfloor \rfloor = \lfloor p_i \rfloor - \lfloor p_i + t \rfloor.$$

65

But $p_i \in [0, 1)^d$, hence $\lfloor p_i \rfloor = 0$, and (3.3) implies that

$$\lfloor q_i - t \rfloor = -\lfloor p_i + t \rfloor \tag{3.4}$$

Then (3.2) becomes

$$p_i = (q_i - t) - \lfloor q_i - t \rfloor.$$

(b) Applying (3.4) to the expression $n_e = m_e + (\lfloor p_j + t \rfloor - \lfloor p_i + t \rfloor)$, we see that

$m_e = n_e + (\lfloor q_j - t \rfloor - \lfloor q_i - t \rfloor)$, as desired. Hence $(\langle G, m \rangle, p) \cong (\langle G, n \rangle, q)$.

(iii) *transitivity.* Suppose $(\langle G, n \rangle, q) \cong (\langle G, m \rangle, p)$ and $(\langle G, m \rangle, p) \cong (\langle G, s \rangle, r)$.

We will show that $(\langle G, n \rangle, q) \cong (\langle G, s \rangle, r)$ too. Let t_1 and t_2 be the transla-

tions specified in the definition of congruence for $(\langle G, n \rangle, q) \cong (\langle G, m \rangle, p)$ and

$(\langle G, m \rangle, p) \cong (\langle G, s \rangle, r)$ respectively.

(a) For each vertex $v_i \in G$,

$$q_i = (p_i + t_1) - \lfloor p_i + t_1 \rfloor$$

$$p_i = (r_i + t_2) - \lfloor r_i + t_2 \rfloor.$$

Substituting,

$$q_i = ((r_i + t_2) - \lfloor r_i + t_2 \rfloor) + t_1 - \lfloor (r_i + t_2) - \lfloor r_i + t_2 \rfloor + t_1 \rfloor.$$

Since $\lfloor r_i + t_2 \rfloor \in \mathbb{Z}^d$, we may move it outside of the larger floor function,

and this term cancels. We obtain

$$q_i = r_i + t_3 - \lfloor r_i + t_3 \rfloor,$$

66

where $t_3 = t_1 + t_2$.

(b) For each edge $e \in G$,

$$n_e = m_e + (\lfloor p_j + t_1 \rfloor - \lfloor p_i + t_1 \rfloor)$$

$$m_e = s_e + (\lfloor r_j + t_1 \rfloor - \lfloor r_i + t_1 \rfloor).$$

Substituting,

$$n_e = s_e + (\lfloor p_j + t_1 \rfloor - \lfloor p_i + t_1 \rfloor + \lfloor r_j + t_1 \rfloor - \lfloor r_i + t_1 \rfloor).$$

Substituting the expressions $p_k = r_k + t_2 - \lfloor r_k + t_2 \rfloor$ for $k = i, j$, we obtain

$$n_e = s_e + (\lfloor r_j + t_3 \rfloor - \lfloor r_i + t_3 \rfloor),$$

as desired.

Hence $(\langle G, n \rangle, q) \cong (\langle G, s \rangle, r)$. $\qquad \square$

We say that the gain graphs $\langle G, m \rangle$ and $\langle G, n \rangle$ are *periodic equivalent* (write $\langle G, m \rangle \sim \langle G, n \rangle$) if there exist configurations p and q such that the periodic frameworks $(\langle G, m \rangle, p)$ and $(\langle G, n \rangle, q)$ are $\mathcal{T}_0^d$-congruent.

Proposition 3.2.5. *Periodic equivalence is an equivalence relation on the set of all d-periodic orbit graphs.*

Proof. This follows from the fact that $\mathcal{T}_0^d$-congruence is an equivalence relation on the set of all periodic frameworks. $\qquad \square$

For two periodic equivalent graphs $\langle G, m \rangle$ and $\langle G, n \rangle$, the net gain on any cycle is the same. For any vertex $v \in V$, let $\ell(v_i) = \lfloor p_i + t \rfloor$, where p is the configuration such that $(\langle G, m \rangle, p) \cong (\langle G, n \rangle, q)$ for some configuration q of $\langle G, n \rangle$. Consider a cycle C of edges in G. The net gain on C in $\langle G, m \rangle$ is

$$\sum_{e \in C} m_e.$$

In the graph $\langle G, n \rangle$, the same cycle has gain

$$\sum_{e \in C} n_e = \sum_{e \in C} (m_e + \ell(t(e)) - \ell(o(e)))$$

$$= \sum_{e \in C} m_e + \sum_{e \in C} \ell(t(e)) - \sum_{e \in C} \ell(o(e)) \tag{3.5}$$

where we denote the origin of the directed edge e by $o(e)$, and the terminus by $t(e)$. Since C is a cycle, each vertex appears exactly once as the origin of an edge, and exactly once as the terminus of another edge. Hence the last two sums in (3.5) cancel, and we obtain

$$\sum_{e \in C} m_e = \sum_{e \in C} n_e.$$

The following proposition follows from these observations:

Proposition 3.2.6. *If $\langle G, m \rangle$ and $\langle G, n \rangle$ are periodic equivalent, they have the same gain space:*

$$\mathcal{M}_{\mathcal{C}}\langle G, m \rangle = \mathcal{M}_{\mathcal{C}}\langle G, n \rangle.$$

3.3 Rigidity and infinitesimal rigidity on $\mathcal{T}_0^d$

3.3.1 Rigidity on $\mathcal{T}_0^d$

Let $(\langle G, m \rangle, p)$ be a periodic orbit framework with $m : E \to \mathbb{Z}^d$ and $p : V \to \mathcal{T}_0^d = \mathbb{R}^d / L_0 \mathbb{Z}^d$, where $V = \{v_1, v_2, \ldots, v_n\}$. A *motion* of the framework on $\mathcal{T}_0^d$ is an indexed family of functions $P_i : [0, 1] \to \mathbb{R}^d$, $i = 1, \ldots, |V|$ such that:

1. $P_i(0) = p(v_i)$ for all i;

2. $P_i(t)$ is continuous on $[0, 1]$, for all i;

3. For all edges $e = \{v_i, v_j; m_e\} \in E\langle G, m \rangle$,

$$\|P_i(t) - (P_j(t) + m_e L_0)\| = \|p(v_i) - (p(v_j) + m_e L_0)\|$$

for all $t \in [0, 1]$.

In other words, a motion P_i of a periodic orbit framework $(\langle G, m \rangle, p)$ preserves the distances between each pair of vertices connected by an edge. Let $M = \{-1, 0, 1\}$. Let M^d represent the set of all d-tuples with entries from the set M. If a motion P_i preserves all of the distances $\|\{p_i, p_j; m\}\|$, where $v_i, v_j \in V$, and $m \in M^d$, then we say that P_i is a *rigid motion* or *trivial motion*. Note that there will be some duplication among this set of distances, for example, $\|\{p_i, p_j; m_\alpha\}\| = \|\{p_j, p_i; -m_\alpha\}\|$, which we could eliminate with further restrictions on m.
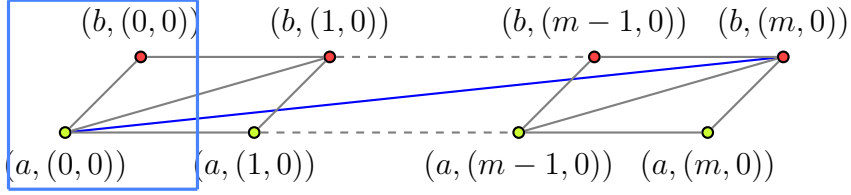
Figure 3.1: Part of the derived periodic framework $(\langle G^m, L_0 \rangle, p^m)$. The blue edge corresponds to the distance $\|\{a, b; (m, 0)\}\|$. This distance is fixed as a result of the fact that the distances between all vertices in any pair of adjacent copies of the fundamental region in $(\langle G^m, L_0 \rangle, p^m)$ are fixed.

Proposition 3.3.1. *Given any pair of vertices v_i, v_j, a rigid motion preserves the length of the segment $\|\{p_i, p_j; m\}\|$ for all $m \in \mathbb{Z}^d$.*

Proof. The distance between any two copies of a particular vertex, $\|\{p_i, p_i; m\}\|$ is trivially fixed by the motion. (In fact, since we are on the fixed torus, this distance is always fixed). Since the elements of each cell of the derived framework are fixed with respect to the elements of the adjacent cell, by a transitivity-type relation, everything is fixed. We offer Figure 3.1 by way of proof. $\qquad\square$

If the only motions of a framework $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$ are rigid motions, then we say that the framework $(\langle G, m \rangle, p)$ is *rigid on the fixed torus* $\mathcal{T}_0^d$. Alternative ways of defining "rigid" will be described later.

3.3.2 Infinitesimal rigidity of frameworks on $\mathcal{T}_0^d$

An *infinitesimal motion* of a periodic orbit framework $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$ is an assignment of velocities to each of the vertices, $u : V \to \mathbb{R}^d$, with $u(v_i) = u_i$ such that

70

(a) $\langle G, m \rangle$ (b) trivial   (c) non-trivial
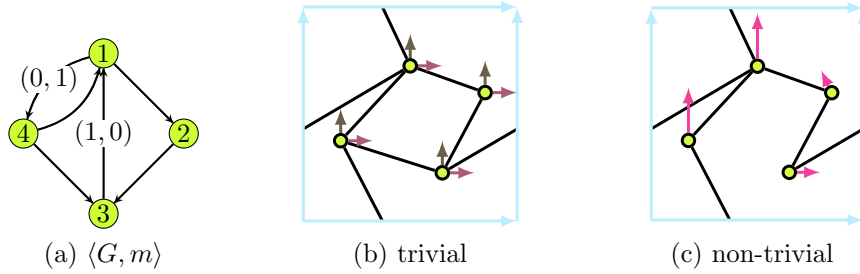
Figure 3.2: A periodic orbit framework (a). Two trivial infinitesimal motions (translations) for a framework on $\mathcal{T}_0^2$ are indicated in (b). Removing a single edge produces a non-trivial infinitesimal motion on the modified framework pictured in (c).

$$(u_i - u_j) \cdot (p_i - p_j - m_e L_0) = 0 \qquad (3.6)$$

for each edge $e = \{v_i, v_j; m_e\} \in E\langle G, m \rangle$. Such an infinitesimal motion preserves the lengths of any of the bars of the framework (see Figure 3.2).

A *trivial infinitesimal motion* of $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$ is an infinitesimal motion that preserves the distance between all pairs of vertices:

$$(u_i - u_j) \cdot (p_i - p_j - m_e L_0) = 0 \qquad (3.7)$$

for all triples $\{v_i, v_j; m_e\}$, $m \in \mathbb{Z}^d$. For any periodic orbit framework $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$, there will always be a d-dimensional space of trivial infinitesimal motions of the whole framework, namely the space of infinitesimal translations. See Figure 3.2b.

Rotation is not a trivial motion for periodic orbit frameworks, because a rotation

71

of a graph on $\mathcal{T}_0^d$ will always change the distance between some pair of points. This is a consequence of the fact that we have fixed our representation of $\mathcal{T}_0^d$, and are considering motions of the periodic orbit framework relative to the fixed torus. This is in contrast to the approach of Borcea and Streinu, who do view rotations as trivial infinitesimal motions of the infinite framework $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ in $\mathbb{R}^d$. Recall that our frameworks on the torus are equivalence classes of the periodic frameworks $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$, where two such frameworks are equivalently represented by the orbit framework $(\langle G, m \rangle, p)$ if they are rotations of one another in $\mathbb{R}^d$.

Proposition 3.3.2. *If u is a trivial infinitesimal motion of $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$, then u is an infinitesimal translation.*

Proof. Let $u = (u_1, \ldots, u_{|V|})$ be an infinitesimal motion satisfying (3.7) for all values of m_e of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ (vectors of 0's with a single 1 in the i-th place). Elementary linear algebra demonstrates that the simultaneous solution of

$$(u_i - u_j) \cdot (p_i - p_j - m_e L_0) = 0$$

for all such values of m_e will yield the single solution, $u_1, = u_2 = \cdots = u_{|V|}$, which corresponds to an infinitesimal translation. $\square$

If the only infinitesimal motions of a framework $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$ are trivial (i.e. infinitesimal translations), then it is *infinitesimally rigid*. Otherwise, the framework

is *infinitesimally flexible*.

An infinitesimal motion u of $(\langle G, m\rangle, p)$ on $\mathcal{T}_0^d$ is called an *infinitesimal flex* if

$$(u_i - u_j) \cdot (p_i - p_j - zL_0) \neq 0 \tag{3.8}$$

for some triple $\{v_i, v_j; z\}$, where $v_i, v_j \in V$, and $z \in \mathbb{Z}^d$. Note that we no longer require that the vertices of the framework affinely span $\mathbb{R}^d$, in contrast to the analogous definition for finite frameworks. This is a a consequence that we need only find *some* triple $\{v_i, v_j; m_e\}$ for which the $(u_i - u_j) \cdot (p_i - p_j - zL_0) \neq 0$, and we are free to choose m_e from $\mathbb{Z}^d$.

Example 3.3.3. The framework on $\mathcal{T}_0^2$ shown in Figure 3.2b is infinitesimally rigid. The only infinitesimal motions of this framework are trivial, as indicated. Removing a single bar $(p_3, p_4; (0,0))$ from the orbit graph shown in (a) yields a framework with a non-trivial infinitesimal motion (a flex). Figure 3.2c depicts this motion, which was found by solving the rigidity matrix described below.

3.3.3 Infinitesimal rigidity of $(\langle \widetilde{G}, L_0\rangle, \widetilde{p})$ in $\mathbb{R}^d$

We now confirm that the representation of $(\langle \widetilde{G}, L_0\rangle, \widetilde{p})$ as an orbit framework on the torus provides us with the information we seek, namely the infinitesimal motions of $(\langle \widetilde{G}, L_0\rangle, \widetilde{p})$ in $\mathbb{R}^d$ that preserve its periodicity.

An infinitesimal periodic motion of $(\langle \widetilde{G}, L_0\rangle, \widetilde{p})$ in $\mathbb{R}^d$ is a function $\widetilde{u} : \widetilde{V} \to \mathbb{R}^d$

such that the infinitesimal velocity of every vertex of $\widetilde{G}$ in an equivalence class (under $\mathbb{Z}^d$) is identical. Recall that $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p})$ can be represented as $(\langle G^m, L_0 \rangle, p^m)$, and therefore the vertices of $\widetilde{G} = G^m$ are naturally indexed by the elements of $\mathbb{Z}^d$. Then an *infinitesimal periodic motion* of $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p})$ in $\mathbb{R}^d$ is a function $\widetilde{u} : \widetilde{V} \to \mathbb{R}^d$ such that the following two conditions are satisfied:

1. For every edge $e = \{(v_i, a), (v_j, b)\} \in \widetilde{E}$,

$$\big(\widetilde{p}(v_i, a) - \widetilde{p}(v_j, b)\big) \cdot \big(\widetilde{u}(v_i, a) - \widetilde{u}(v_j, b)\big) = 0,$$

2. $\widetilde{u}(v_i, z) = \widetilde{u}(v_i, 0)$, for all $z \in \mathbb{Z}^d$

The framework $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p})$ is *infinitesimally periodic rigid* in $\mathbb{R}^d$ if the only such motions assign the same infinitesimal velocity to all vertices of $\widetilde{V}$ (i.e. they are translations).

Remark 3.3.4. An infinitesimal motion of $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p})$ is a motion that is itself periodic in that $\widetilde{u}$ assigns the *same* infinitesimal velocity to every vertex in an equivalence class. In Chapter 5 we will relax this assumption to consider infinitesimal motions that preserve the periodicity of the framework, but that are not themselves periodic, since they also change the lattice. $\square$

Proposition 3.3.5. *Let $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p})$ be a d-periodic framework. Let $(\langle G, m \rangle, p)$ be its d-periodic orbit framework given by Proposition 3.2.1. Then the following are equivalent:*

(i) $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p})$ *is infinitesimally periodic rigid in* $\mathbb{R}^d$

(ii) $(\langle G, m \rangle, p)$ *is infinitesimally rigid on* $\mathcal{T}_0^d = \mathbb{R}^d / L_0 \mathbb{Z}^d$.

Proof. Let u be an infinitesimal motion of $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$. We extend u to an infinitesimal motion $\widetilde{u}$ of $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p}) = (\langle G^m, L_0 \rangle, p^m)$ by letting every vertex of $(\langle G^m, L_0 \rangle, p^m)$ in the fibre over $v \in V \langle G, m \rangle$ have the same infinitesimal velocity. More precisely, let

$$\widetilde{u}(v, z) = u(v), \ \forall z \in \mathbb{Z}^d.$$

Since an edge $(e, a) = \{(v_i, a), (v_j, b)\} \in \widetilde{E}$ if and only if $e = \{v_i, v_j; b - a\} \in E \langle G, m \rangle$, the fact that $\widetilde{u}$ is an infinitesimal periodic motion of $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p})$ is obvious.

On the other hand, given an infinitesimal motion $\widetilde{u}$ of $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p})$, let $u : V \to \mathbb{R}^d$ be given by

$$u(v_i) = \widetilde{u}(v_i, 0).$$

Again it is clear that u is an infinitesimal motion of $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$.

In both cases, the non-trivial motions assign the same velocities to all vertices of $(\langle G, m \rangle, p)$ or $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p})$ respectively, and therefore non-trivial infinitesimal motions of $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$ correspond to non-trivial infinitesimal periodic motions of $(\langle \widetilde{G}, L_0 \rangle, \widetilde{p})$ in $\mathbb{R}^d$. $\square$

Remark 3.3.6. Proposition 3.3.5 also holds when we replace "infinitesimally rigid"

with "rigid". Because our focus is infinitesimal rigidity, we omit the statement and proof of this version. □

Remark 3.3.7. The reader should be reminded that an infinite framework $(\widetilde{G}, \widetilde{p})$ may be infinitesimally periodic rigid without being infinitesimally rigid, since there may be non-trivial infinitesimal motions of the framework that do not preserve the periodicity. Hence it is important to distinguish between these forms of rigidity, and we emphasize that we are interested in forced periodicity, not incidental periodicity.

□

If $(\langle G, m \rangle, p)$ is infinitesimally rigid on $\mathcal{T}_0^d$, then $(\langle G, m \rangle, p)$ is rigid on $\mathcal{T}_0^d$. Or, in other words, if a framework is flexible, then it also has an infinitesimal flex. A periodic-adapted proof of this fact using the averaging technique is presented in Section 3.3.7, after the definition of the rigidity matrix. The converse is not true, as illustrated in the example pictured in Figure 3.3. However, geometrically this example is highly 'special'. It is known that for *generic* frameworks (defined in Section 3.3.8), infinitesimal rigidity and rigidity actually coincide. This is a periodic analogue of a well-known result due to Asimow and Roth [2] in the theory of rigidity for finite graphs (see Theorem 2.5.7), and will be discussed further in Section 3.3.9.

(a) $\langle G, m \rangle$ (b) $(\langle G, m \rangle, p)$
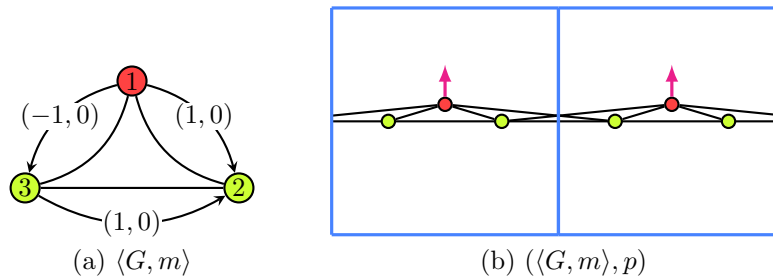
Figure 3.3: The framework $(\langle G, m \rangle, p)$ has a infinitesimal flex on $\mathcal{T}_0^2$ (b), but no finite flex. The position of the vertices of $(\langle G, m \rangle, p)$ has all three vertices on a line, however, the drawing has been exaggerated to indicate the connections between vertices in adjacent cells.

3.3.4 The fixed torus rigidity matrix

Rigidity matrices for periodic frameworks have been recorded by Guest and Hutchinson [69], Borcea and Streinu [7], and Malestein and Theran [49]. The matrix we present below is different from these other presentations, for two reasons. The first is that this is the matrix for the fixed torus, and the second is that we are considering equivalence classes of frameworks under rotation.

The *rigidity matrix*, $\mathbf{R}_0(\langle G, m \rangle, p)$, records equations for the space of possible infinitesimal motions of a d-periodic orbit framework. It is the $|E| \times d|V|$ matrix with one row of the matrix corresponding to each edge $e = \{i, j\}; m_e$ of $\langle G, m \rangle$ as

follows:

$$
\text{edge } \{i,j;m_e\} \begin{pmatrix} & & & \overset{i}{\vdots} & & & \overset{j}{} & \\ 0\cdots 0 & p_i-(p_j+m_eL_0) & 0\cdots 0 & (p_j+m_eL_0)-p_i & 0\cdots 0 \\ & & & \vdots & & & \end{pmatrix},
$$

where each entry is actually a d-dimensional vector, and the non-zero entries occur in the columns corresponding to vertices v_i and v_j respectively. By construction, the kernel of this matrix will be the space of infinitesimal motions of $(\langle G,m\rangle,p)$ on $\mathcal{T}_0^d$. By an abuse of notation we may write

$$
\mathbf{R}_0(\langle G,m\rangle,p)\cdot u^T = 0
$$

where $u=(u_1,u_2,\ldots,u_{|V|})$, and $u_i \in \mathbb{R}^d$. That is, u is an infinitesimal motion of the joints of $(\langle G,m\rangle,p)$ on $\mathcal{T}_0^d$.

Example 3.3.8. Consider the periodic orbit graph $\langle G,m\rangle$ shown in Figure 3.2a. Let L_0 be the matrix generating the torus $\mathcal{T}_0^2$. The rigidity matrix $\mathbf{R}_0(\langle G,m\rangle,p)$ will have have six rows, and eight columns (two columns corresponding to the two coordinates of each vertex).
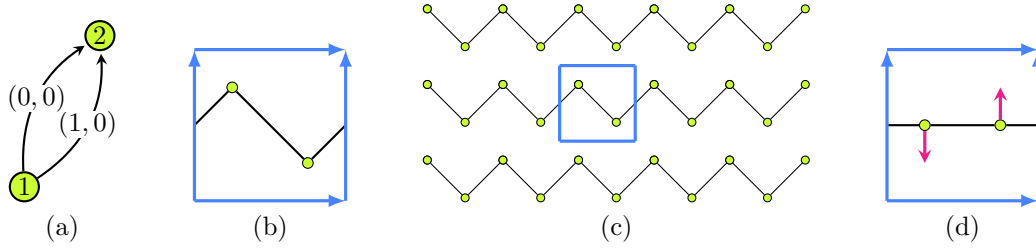
Figure 3.4: The zig zag framework has a gain graph with two vertices (a). Realized as a framework on the 2-dimensional torus (b). The derived framework is shown in (c). A non-generic position of the vertices on $\mathcal{T}_0^2$ (d). The framework pictured in (d) is not infinitesimally rigid, but the framework $(\langle G, m \rangle, p)$ shown in (b) is infinitesimally rigid on $\mathcal{T}_0^2$, and the corresponding derived framework $(\langle G^m, L_0 \rangle, p^m)$ (c) is infinitesimally rigid in $\mathbb{R}^2$.

$$
\begin{array}{c}
\begin{array}{cccc}
\quad p_1 & \qquad p_2 & \qquad p_3 & \qquad p_4
\end{array} \\
\begin{array}{c}
\{1,2;(0,0)\} \\
\{2,3;(0,0)\} \\
\{3,4;(0,0)\} \\
\{1,4;(0,0)\} \\
\{1,3;(-1,0)\} \\
\{1,4;(0,1)\}
\end{array}
\left(
\begin{array}{cccc}
p_1 - p_2 & p_2 - p_1 & 0 & 0 \\
0 & p_2 - p_3 & p_3 - p_2 & 0 \\
0 & 0 & p_3 - p_4 & p_4 - p_3 \\
p_1 - p_4 & 0 & 0 & p_4 - p_1 \\
p_1 - p_3 + (1,0)L_0 & 0 & p_3 - p_1 - (1,0)L_0 & 0 \\
p_1 - p_4 - (0,1)L_0 & 0 & 0 & p_4 - p_1 + (0,1)L_0
\end{array}
\right)
\end{array}
$$

$\square$

As stated, a framework on $\mathcal{T}_0^d$ is infinitesimally rigid if and only if the only infinitesimal motions of the framework are infinitesimal translations. In addition, any periodic framework $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$ has a d-dimensional space of trivial motions.

It follows that the rigidity matrix always has at least d trivial solutions, and hence

Theorem 3.3.9. *A periodic orbit framework* $(\langle G, m \rangle, p)$ *is infinitesimally rigid on the fixed torus* $\mathcal{T}_0^d$ *if and only if the rigidity matrix* $\mathbf{R}_0(\langle G, m \rangle, p)$ *has rank* $d|V| - d$.

The rigidity matrix of the framework in Example 3.3.8 above has rank 6, which is exactly $2|V| - 2$, and hence $(\langle G, m \rangle, p)$ is infinitesimally rigid on $\mathcal{T}_0^2$.

Example 3.3.10 (the *zig-zag* framework). Consider the graph $G = (V, E)$ where $V = \{v_1, v_2\}$ and E consists of two copies of the edge connecting the two vertices v_1 and v_2 (Figure 3.4a). If the gains on the two edges are the same, then the framework is not infinitesimally rigid, since both rows of the rigidity matrix will be identical. Let m be a gain assignment on G with $m_1 \neq m_2$. The periodic orbit framework $(\langle G, m \rangle, p)$ is infinitesimally rigid on $\mathcal{T}_0^2$ if and only if:

1. $p_1 \neq p_2$

2. both edges have distinct directions (that is, the vectors $p_1 - p_2 - m_1$ and $p_1 - p_2 - m_2$ are independent). See Figure 3.4d.

Figures b and c depict $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^2$ and $(\langle G^m, L_0 \rangle, p^m)$ in $\mathbb{R}^2$ respectively. $\quad\square$

There are a number of simple observations which we record here for future reference.

Corollary 3.3.11. *A periodic orbit framework* $(\langle G, m \rangle, p)$ *where* G *has* $|E| < d|V| - d$ *is not infinitesimally rigid on* $\mathcal{T}_0^d$.

A collection of edges $E' \subset E$ of the periodic orbit framework $(\langle G, m \rangle, p)$ is called *independent* if the corresponding rows of the rigidity matrix are linearly independent. For each set of multiple edges $e_{i_1} = e_{i_2} = \cdots = e_{i_t}$ in E, we can have at most d independent copies. If a framework $(\langle G, m \rangle, p)$ has edges corresponding to dependent rows in the rigidity matrix, we say that the edges are *dependent*. We may also refer to a framework $(\langle G, m \rangle, p)$ as being independent or dependent, and for clarity we will at times write *dependent on* $\mathcal{T}_0^d$ to differentiate this setting from the finite case (frameworks which are not on a torus).

Corollary 3.3.12. *Any periodic orbit framework* $(\langle G, m \rangle, p)$ *where* G *has* $|E| > d|V| - d$ *is dependent on* $\mathcal{T}_0^d$.

We sometimes call such a framework *over-counted*. A periodic orbit framework $(\langle G, m \rangle, p)$ whose underlying gain graph satisfies $|E| = d|V| - d$ and is infinitesimally rigid on $\mathcal{T}_0^d$ will be called *minimally rigid*. In other words, a minimally rigid framework on $\mathcal{T}_0^d$ is one that is both infinitesimally rigid and independent. In fact, such a framework is maximally independent – adding any new edge will introduce a dependence among the edges. If a periodic orbit framework is minimally rigid, then the removal of any edge will result in a framework that is not infinitesimally

rigid.

We observe a periodic analogue of the extension of Maxwell's rule (Theorem 2.5.9) we mentioned in Chapter 2.

Corollary 3.3.13. *Let $(\langle G, m \rangle, p)$ be a minimally rigid periodic orbit framework. Then*

1. *$|E| = d|V| - d$, and*

2. *for all subgraphs $G' \subseteq G$, $|E'| \leq d|V'| - d$.*

Corollary 3.3.14. *Any loop edge in the d-periodic orbit framework $(\langle G, m \rangle, p)$ is dependent on $\mathcal{T}_0^d$.*

The following useful result is a direct consequence of the fact that the row rank of a matrix is equal to its column rank.

Corollary 3.3.15. *A d-periodic framework $(\langle G, m \rangle, p)$ whose underlying gain graph satisfies $|E| = d|V| - d$ is independent on $\mathcal{T}_0^d$ if and only if it is infinitesimally rigid on $\mathcal{T}_0^d$. Moreover, the vector space of non-trivial infinitesimal motions of $(\langle G, m \rangle, p)$ is isomorphic to the vector space of row dependencies of $\mathbf{R}_0(\langle G, m \rangle, p)$.*

We also now confirm that if $(\langle G, m \rangle, p)$ is infinitesimally rigid, then all frameworks that are $\mathcal{T}_0^d$-congruent to $(\langle G, m \rangle, p)$ are also infinitesimally rigid.

Proposition 3.3.16. *Let $(\langle G, m \rangle, p)$ and $(\langle G, n \rangle, q)$ be $\mathcal{T}_0^d$-congruent. Then*

$$\mathrm{rank}\mathbf{R}_0(\langle G, m \rangle, p) = \mathrm{rank}\mathbf{R}_0(\langle G, n \rangle, q).$$

Proof. This is a straightforward application of the definition of $\mathcal{T}_0^d$-congruence. $\square$

The rows of $\mathbf{R}_0(\langle G, m \rangle, p)$ corresponding to edges with zero gains can be viewed as rows in the rigidity matrix of a finite framework, as described in the background (Chapter 2). Since at most $d|V| - \binom{d+1}{2}$ rows can be independent in the finite matrix, we have the following proposition:

Proposition 3.3.17. *Let $\langle G, m \rangle$ be a d-periodic orbit graph with all edges having zero gains, $m = 0$. If $|E| > d|V| - \binom{d+1}{2}$, then the edges of $(\langle G, m \rangle, p)$ are dependent for any configuration p.*

The goal of Chapter 4 is to characterize minimal periodic rigidity for periodic orbit graphs on $\mathcal{T}_0^2$. Because loop edges are always dependent by Corollary 3.3.14, we restrict our attention to frameworks $(\langle G, m \rangle, p)$ that do not have loop edges. On the flexible torus, however, loops may be independent, but the consideration of this case is left to Chapter 5.

Remark 3.3.18. The derived periodic framework corresponding to the periodic orbit framework in Example 3.3.3 would not be considered minimally rigid as an infinite framework in the sense of being both independent and rigid. That is, disregarding

83

the periodic qualities of the graph and recording an infinite dimensional rigidity matrix, it is not true that row rank equals column rank, and hence Corollary 3.3.15 is no longer true. Further details on this problem can be found in Guest and Hutchinson, [69]. □

Remark 3.3.19. We can define a *d-periodic rigidity matroid* $\mathcal{R}_0(\langle G, m \rangle, p)$ on the edges of the d-periodic orbit framework: A set of edges is independent in the rigidity matroid $\mathcal{R}_0(\langle G, m \rangle, p)$ if the corresponding rows are independent in the rigidity matrix $\mathbf{R}_0(\langle G, m \rangle, p)$. □

3.3.5 Stresses and independence

A row dependence among the rows of the rigidity matrix can be thought of as a *stress* on the edges of the periodic orbit matrix, or equivalently a *periodic stress* on the edges of a periodic framework. This topic has been considered by Guest and Hutchinson [69]. The minimally rigid graphs are therefore the graphs that do not have any infinitesimal motions, or any stresses among their edges. In finite rigidity, this state is called *isostatic*, but we avoid this terminology here for the reasons outlined in [69]. Borcea and Streinu also define stresses for d-periodic frameworks with a flexible lattice in [7]. We return to these ideas in Chapter 8, as background for several areas of further work.

3.3.6   The unit torus and affine transformations

In this section we show that frameworks on the unit torus can be used to model all d-periodic orbit frameworks, since Theorem 3.3.20 will demonstrate that infinitesimal rigidity of periodic orbit frameworks is affinely invariant.

An *affine transformation* is a map

$$A: \quad \mathbb{R}^d \quad \longrightarrow \quad \mathbb{R}^d$$

$$x \quad \longmapsto \quad xB + t$$

where B is an invertible $d \times d$ matrix, and $t \in \mathbb{R}^d$.

The next result was shown independently in [7] (see also Chapter 5 of this thesis).

Theorem 3.3.20. *Let* $(\langle G, m \rangle, p)$ *be a d-periodic orbit framework on* $\mathcal{T}_0^d$. *Let* L_0 *be the $d \times d$ lattice matrix whose rows are the generators of* $\mathcal{T}_0^d$. *Let A be an affine transformation of* $\mathbb{R}^d$, *with* $A(x) = xB + t$, *and where* $A(p) = (A(p_1), \ldots, A(p_{|V|})) \in \mathbb{R}^{d|V|}$. *Then the edges of* $(\langle G, m \rangle, A(p))$ *are independent on* $\mathbb{R}^d / L_0 B \mathbb{Z}^d$ *if and only if the edges of* $(\langle G, m \rangle, p)$ *are independent on* $\mathcal{T}_0^d = \mathbb{R}^d / L_0 \mathbb{Z}^d$.

Proof. Suppose some edges of $(\langle G, m \rangle, p)$ are dependent on $\mathcal{T}_0^d = \mathbb{R}^d / L_0 \mathbb{Z}^d$. Then there exist scalars ω_e, for each $e = \{i, j; m_e\} \in E$, not all of which are zero, such that for each $v_i \in V$, the column sum corresponding to the vertex v_i in $\omega \cdot \mathbf{R}_0(\langle G, m \rangle, p)$

is zero. Since the gains on the oriented edges in G depend on the direction of the edge, for a particular vertex we consider the edges directed into and out from the vertex separately. That is, for a vertex $v_i \in V$, let E_+ denote the set of edges directed *out* from the vertex v_i, and let E_- denote the set of edges directed *into* the vertex v_i. Then for a vertex v_i we have (note the sign on the gain in the two summands):

$$
\begin{aligned}
0 &= \sum_{e_\alpha \in E_+} \omega_{e_\alpha}(p_i - (p_j + m_{e_\alpha} L_0)) + \sum_{e_\beta \in E_-} \omega_{e_\beta}(p_i - (p_k - m_{e_\beta} L_0)) \\
&= \sum_{e_\alpha \in E_+} \omega_{e_\alpha}(p_i - (p_j + m_{e_\alpha} L_0))B + \sum_{e_\beta \in E_-} \omega_{e_\beta}(p_i - (p_k - m_{e_\beta} L_0))B \\
&= \sum_{e_\alpha \in E_+} \omega_{e_\alpha}(p_i B + t - (p_j B + t + m_{e_\alpha} L_0 B)) + \\
&\qquad\qquad \sum_{e_\beta \in E_-} \omega_{e_\beta}(p_i B + t - (p_k B + t - m_{e_\beta} L_0 B)) \\
&= \sum_{e_\alpha \in E_+} \omega_{e_\alpha}(A(p_i) - (A(p_j) + m_{e_\alpha} L_0 B)) + \\
&\qquad\qquad \sum_{e_\beta \in E_-} \omega_{e_\beta}(A(p_i) - (A(p_k) - m_{e_\beta} L_0 B)).
\end{aligned}
$$

Hence the corresponding rows of $\mathbf{R}_0(\langle G, m \rangle, A(p))$ are dependent, and therefore the edges of $(\langle G, m \rangle, A(p))$ are dependent on $\mathbb{R}/L_0 B \mathbb{Z}^d$. Because the transformation A is invertible, the reverse direction follows. $\qquad\square$

Corollary 3.3.21. *Let $\mathcal{F} = (\langle G, m \rangle, p)$ be a d-periodic orbit framework on $\mathcal{T}_0^d$, where L_0 is the $d \times d$ matrix of generators of $\mathcal{T}_0^d$. Let $\mathcal{F}'$ be the image of $\mathcal{F}$ under*

86
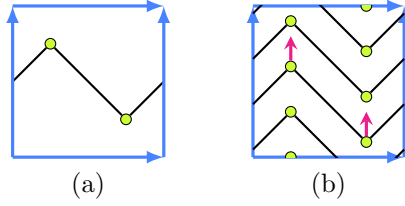
Figure 3.5: The framework pictured in (a) is infinitesimally rigid on the fixed torus $\mathcal{T}_0^2$. The affine transformation of the framework shown in (b) is *not* infinitesimally rigid, as indicated.

the unique affine transformation of $\mathbb{R}^d$ which maps L_0 to the d-dimensional identity matrix $I_{d \times d}$. Then $\mathcal{F}$ is infinitesimally rigid on $\mathcal{T}_0^d$ if and only if $\mathcal{F}'$ is infinitesimally rigid on the d-dimensional unit torus, $\mathcal{U}_0^d$.

Remark 3.3.22. It is essential that the affine transformation of Corollary 3.3.21 act on both the points of the framework, and the generators of the torus (the rows of L_0). In other words, it is *not* true that a framework $(\langle G, m \rangle, p)$ is infinitesimally rigid on $\mathcal{T}_0^d$ if and only if an affine image of the framework $(\langle G, m \rangle, p)$ is infinitesimally rigid on $\mathcal{T}_0^d$. The framework pictured in Figure 3.5 is an example. □

3.3.7 Infinitesimal rigidity implies rigidity (averaging)

Averaging is a technique for constructing a flex on a framework, and it provides one proof of the fact that infinitesimal rigidity implies rigidity. This technique and its implications for rigidity are described in a chapter of an unpublished book on rigidity theory [10]. For this reason we reproduce the full theory here, with the

appropriate modifications to handle the periodic case.

Proposition 3.3.23. *Let $\langle G, m \rangle$ be a periodic orbit graph, and suppose p and q are two distinct realizations of the periodic orbit graph on the fixed torus, $\mathcal{T}_0^d$. Then*

(a) $(\langle G, m \rangle, p)$ and $(\langle G, m \rangle, q)$ have equal edge lengths, i.e. $\|p_i - (p_j + m_e L_0)\| = \|q_i - (q_j + m_e L_0)\|$ for $e = \{i, j; m_e\} \in E \iff p - q$ is an infinitesimal flex of $(\langle G, m \rangle, \frac{p+q}{2})$

(b) $p - q$ is a trivial infinitesimal flex of $(\langle G, m \rangle, \frac{p+q}{2}) \iff (\langle G, m \rangle, p)$ and $(\langle G, m \rangle, q)$ are $\mathcal{T}_0^d$-congruent.

Note that for finite (i.e. not periodic) frameworks as in [10], the second condition appears as:

(b) if $p - q$ is a trivial infinitesimal flex of $\frac{p+q}{2}$, then p and q are congruent.

Proof. For any $i, j \in 1, \ldots |V|$, consider

$$\left(\frac{p_i + q_i}{2} - \frac{p_j + q_j}{2} - m_e L_0 \right) \cdot \left((p_i - q_i) - (p_j - q_j) \right)$$

$$= 1/2 \left((p_i - p_j) + (q_i - q_j) - 2m_e L_0 \right) \cdot \left((p_i - p_j) - (q_i - q_j) \right)$$

$$= 1/2 \left((p_i - p_j - m_e L_0) + (q_i - q_j - m_e L_0) \right) \cdot \left((p_i - p_j - m_e L_0) - (q_i - q_j - m_e L_0) \right)$$

$$= 1/2 \left((p_i - p_j - m_e L_0)^2 - (q_i - q_j - m_e L_0)^2 \right).$$

When $\{i, j, m_e\} \in E$, the final row of this equality is zero since the edge lengths are the same, which proves *(a)*.

To show *(b)*, note that if $p - q$ is a *trivial* infinitesimal flex of $(\langle G, m \rangle, \frac{p+q}{2})$, then it must be a translation (since $(\langle G, m \rangle, \frac{p+q}{2})$ is a periodic orbit framework on $\mathcal{T}_0^d$, and the only trivial motions of a periodic orbit framework are translations). It follows that p is a translate of q, and since the framework is periodic, $(\langle G, m \rangle, p)$ is $\mathcal{T}_0^d$-congruent to $(\langle G, m \rangle, q)$. On the other hand, if $(\langle G, m \rangle, p) \sim (\langle G, m \rangle, q)$, then the last line of the expression above, $1/2\Big((p_i - p_j - m_e L_0)^2 - (q_i - q_j - m_e L_0)^2\Big)$, is zero for any triple $\{v_i, v_j; m_e\}$. Hence the first line is also zero for any such triple, and the claim holds. $\qquad\square$

We will now show that infinitesimal rigidity implies rigidity on $\mathcal{T}_0^d$. We use a slightly different definition of rigidity than that given in Section 3.3.1. These definitions can be shown to be equivalent. We say that a framework $(\langle G, m \rangle, p)$ is *rigid on $\mathcal{T}_0^d$* if there is an $\epsilon > 0$ such that if $(\langle G, m \rangle, p)$ and $(\langle G, m \rangle, q)$ have equal edge lengths, and $|p - q| < \epsilon$, then $(\langle G, m \rangle, p)$ is $\mathcal{T}_0^d$-congruent to $(\langle G, m \rangle, q)$ (i.e. $(\langle G, m \rangle, p) \sim (\langle G, m \rangle, q)$). We will also make use of the following proposition.

Proposition 3.3.24. *The set $I = \{q \mid (\langle G, m \rangle, q)$ is infinitesimally rigid on $\mathcal{T}_0^d\}$ is open in $\mathbb{R}^{d|V|}$, if $I \neq \emptyset$.*

Proof. Recall that an orbit framework $(\langle G, m \rangle, q)$ is infinitesimally rigid on $\mathcal{T}_0^d$

if and only if the corresponding rigidity matrix $\mathbf{R}_0(\langle G, m\rangle, q)$ has rank $d|V| - d$. There are a finite number of polynomials defining the "bad positions" where rank$\mathbf{R}_0(\langle G, m\rangle, q) < d|V| - d$, and hence the positions q avoiding this set form an open set in $\mathbb{R}^{d|V|}$. $\qquad\square$

Theorem 3.3.25. *Let $(\langle G, m\rangle, p)$ be any infinitesimally rigid periodic framework on $\mathcal{T}_0^d$. Then $(\langle G, m\rangle, p)$ is rigid on $\mathcal{T}_0^d$.*

Proof. Assume $(\langle G, m\rangle, p)$ is infinitesimally rigid on $\mathcal{T}_0^d$. By proposition 3.3.24, the set $I = \{q \mid (\langle G, m\rangle, q) \text{ is infinitesimally rigid on } \mathcal{T}_0^d\}$ is open in $\mathbb{R}^{d|V|}$. Let q be close enough to p so that $\frac{p+q}{2} \in I$. If $(\langle G, m\rangle, p)$ and $(\langle G, m\rangle, q)$ have corresponding edges with equal lengths, then $p - q$ is an infinitesimal flex of $\frac{p+q}{2}$ by 3.3.23*(a)*. But since $\frac{p+q}{2} \in I$, $p - q$ must be a *trivial* infinitesimal flex, and hence $(\langle G, m\rangle, p) \sim (\langle G, m\rangle, q)$ by 3.3.23*(b)*. By the definition of rigidity, $(\langle G, m\rangle, p)$ is rigid. $\qquad\square$

3.3.8 Generic frameworks

Let V be a finite set of vertices, and let p be a realization of these vertices on to the d-dimensional unit torus $\mathcal{U}_0^d = [0, 1)^d$. Let $k \in \mathbb{Z}_+$ be given, and let K be the set of all edges between pairs of vertices of V with gains $m_e = (m_{e,1}, m_{e,2}, \ldots, m_{e,d})$ where $|m_{e,i}| \leq k$ for $i = 1, \ldots d$. Then K is the set of all edges with bounded gains.

Consider a set of edges $E \subset K$ such that, for some realization p, the rows of $\mathbf{R}_0$ corresponding to E are independent. The determinants of the $|E| \times |E|$ submatrices

90

of these rows will either be identically zero or will define an algebraic variety in $\mathbb{R}^{d|V|}$ (by setting these determinants equal to zero, and taking the p_i's as variables). The collection of all such varieties, corresponding to all such subsets E will define a closed set of measure zero (this set is a finite union of closed sets of measure zero). Let this set be denoted $\mathcal{X}_k$. The complement of $\mathcal{X}_k$ in $\mathbb{R}^{d|V|}$ is an open dense set in $\mathbb{R}^{d|V|}$, and hence its restriction to the subspace of realizations p of the vertices V on the unit torus, $[0,1)^{d|V|}$ is also open and dense.

Any realization p of the vertex set V where $p \notin \mathcal{X}_k$ will be called *k-generic* (recall that k was the upper bound on the gain assignments). More generally, we may consider graphs that are k-generic for any k. By the Baire Category Theorem, the countable intersection

$$\bigcap_{k \in \mathbb{Z}} \left(\mathbb{R}^{d|V|} - \mathcal{X}_k \right)$$

is dense in $\mathbb{R}^{d|V|}$, as the intersection of open dense sets in the Baire space $\mathbb{R}^{d|V|}$ [50]. We refer to a realization in this set as simply *generic*, and it is this definition that we use throughout the remainder of this paper.

It is also possible to define generic positions p by demanding that the coordinates of the realization p are algebraically independent over $\mathbb{Q}$. This is the definition used by R. Connelly in [13] and others. The set of all such realizations is dense, but not open. The relationships between these different conceptions of generic are indicated in Figure 3.6. Note that in the case of generic and algebraically
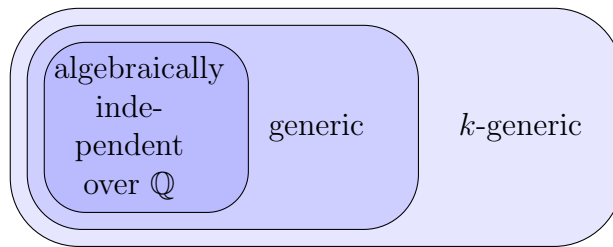
Figure 3.6: Three different types of generic.

independent positions, we need to avoid a countably infinite number of polynomial conditions. In contrast, k-generic frameworks need only avoid a finite number of polynomial conditions. In Chapter 6 we discuss yet another conception of generic, in which we need to consider a smaller finite number of polynomial conditions. This is essential for the combinatorial algorithms we describe there. However for the inductive techniques of Chapter 4 we need the stronger (more restrictive) definition of generic described above (although we do not require algebraic independence).

Corollary 3.3.26. (to Theorem 3.3.20) *Let A be an affine transformation of $\mathbb{R}^d$ which maps L_0 to the identity matrix $I_{d \times d}$, and let $\langle G, m \rangle$ be a periodic orbit graph. $A(p)$ is a generic realization of $\langle G, m \rangle$ on the unit torus, $\mathcal{U}_0^d = [0, 1)^d$ if and only if p is a generic realization of $\langle G, m \rangle$ on $\mathcal{T}_0^d$.*

As a consequence of this result, from this point forward we assume that all frameworks are realized on the unit torus. That is, $p : V \longrightarrow [0, 1)^d$, and $L_0 = I_{d \times d}$, the identity matrix. We continue to write $\mathcal{T}_0^d$, but drop the matrix "L_0" from expressions involving gains, since $m_e L_0 = m_e I_{d \times d} = m_e$.

The following result states that for a given d-periodic orbit graph, all generic realizations share the same rigidity properties.

Lemma 3.3.27 (Special Position Lemma). *Let $\langle G, m \rangle$ be a d-periodic orbit graph, and suppose that for some realization p_0 of $\langle G, m \rangle$ on $\mathcal{T}_0^d$ the framework $(\langle G, m \rangle, p_0)$ is infinitesimally rigid. Then for all generic realizations p of $\langle G, m \rangle$ on $\mathcal{T}_0^d$, the framework $(\langle G, m \rangle, p)$ is infinitesimally rigid.*

Proof. Since the framework $(\langle G, m \rangle, p_0)$ is infinitesimally rigid on $\mathcal{T}_0^d$, the rigidity matrix for $(\langle G, m \rangle, p_0)$ has maximum rank, $\text{rank}\mathbf{R}_0(\langle G, m \rangle, p_0) = d|V| - d$ (Theorem 3.3.9). By definition of generic, any framework $(\langle G, m \rangle, p)$ with p generic will have

$$\text{rank}\mathbf{R}_0(\langle G, m \rangle, p) \geq \text{rank}\mathbf{R}_0(\langle G, m \rangle, p_0)$$

It follows that $\text{rank}\mathbf{R}_0(\langle G, m \rangle, p) = d|V| - d$, and the framework $(\langle G, m \rangle, p)$ is infinitesimally rigid. $\square$

The following modification of the Special Position Lemma states that the coordinates of p_0 need not be on the unit torus, but can in fact be taken anywhere in $\mathbb{R}^{d|V|}$.

Corollary 3.3.28 (Modified Special Position Lemma). *Let $\langle G, m \rangle$ be a d-periodic orbit graph, and suppose that for some realization $p_0 : V \to \mathbb{R}^{d|V|}$ the rigidity matrix $\mathbf{R}_0(\langle G, m \rangle, p_0)$ has rank $d|V| - d$. Then for all generic realizations p of $\langle G, m \rangle$ on $\mathcal{T}_0^d = [0, 1)^d$, the framework $(\langle G, m \rangle, p)$ is infinitesimally rigid.*

Proof. Recall that the set of generic realizations of a vertex set V is dense in $\mathbb{R}^{d|V|}$, and that the set of generic realizations on the torus is simply the restriction of this larger set to $[0,1)^{d|V|}$. By the arguments of the proof of Lemma 3.3.27, if $\mathbf{R}_0(\langle G, m \rangle, p_0) = d|V| - d$ for *some* realization $p_0 \in \mathbb{R}^{d|V|}$, then $\mathbf{R}_0(\langle G, m \rangle, p) = d|V| - d$ for all generic realizations in $\mathbb{R}^{d|V|}$, which includes all generic realizations on $[0,1)^{d|V|}$. $\qquad\square$

This result should be understood to mean that we can pick any representatives of a vertex, provided that the edge representatives are the same, in the sense that the corresponding rows of the rigidity matrix are unchanged. In light of these results, we may say that a periodic orbit graph $\langle G, m \rangle$ is *generically rigid* on $\mathcal{T}_0^2$, meaning that the periodic orbit framework $(\langle G, m \rangle, p)$ is rigid for all generic realizations p of the vertices of G.

3.3.9 Infinitesimal flex as a sufficient condition for a finite flex in a generic framework

We now formulate a definition of flexibility for periodic frameworks, in the style of Asimow and Roth [2]. We will use this definition to prove a fundamental fact about the rigidity of periodic frameworks, namely that infinitesimal rigidity and rigidity are equivalent for generic frameworks.

In what follows, we assume that no vertex lies on the boundary of $\mathcal{T}_0^d$. This

is possible, since we only have a finite number of vertices, and we can therefore translate the entire framework by ϵ to move any vertices off the boundary. Let $\langle G, m \rangle$ be a d-periodic orbit graph, with $G = (V, E)$. Order the edges $\langle E, m \rangle$ in some way. Define

$$f_{\langle G,m \rangle} : (0,1)^{d|V|} \to \mathbb{R}^{|E|}$$

$$f_{\langle G,m \rangle}(p_1, \ldots, p_{|V|}) = (\ldots, \|p_i - (p_j + m_e)\|, \ldots),$$

where $e = \{v_i, v_j; m_e\} \in \langle E, m \rangle$, $p_i \in \mathbb{R}^d$, and $\| \cdot \|$ represents distance on the flat d-torus $\mathcal{T}_0^d$, as described in Section 3.2.3. The function $f_{\langle G,m \rangle}$ is called the *edge function* of $\langle G, m \rangle$, and is a list of edge lengths of the framework.

We now define a special d-periodic orbit graph, which we think of as a kind of complete graph on the torus. Let $\langle K, m_K \rangle$ be the gain graph with $|V|$ vertices, and let E be the set of all edges such that each pair of vertices is connected by 3^d edges. Let m_K be the map which assigns to each set of edges between two vertices, all of the possible gains $(m_1, \ldots, m_d)$, where $m_i \in \{-1, 0, 1\}$. By our previous definition of rigidity, the graph $\langle K, m_K \rangle$ is generically rigid on $\mathcal{T}_0^d$. Let $\langle K, m_K^* \rangle$ be periodic equivalent to $\langle K, m_K \rangle$. It follows that $f_{\langle K, m_K \rangle}(p) = f_{\langle K, m_K^* \rangle}(q)$ for $p, q \in \mathbb{R}^{d|V|}$ if and only if the frameworks $(\langle K, m_K \rangle, p)$ and $(\langle K, m_K^* \rangle, q)$ are $\mathcal{T}_0^d$-congruent.

For ease of notation in the remainder of the section, we let $f_G = f_{\langle G,m \rangle}$, and $f_K = f_{\langle K, m_K \rangle}$. Let $X(p) = f_G^{-1}(f_G(p))$, the set of all configurations of the vertices of

$\langle G, m \rangle$ with the same edge lengths as $(\langle G, m \rangle, p)$. Let $M(p) = f_K^{-1}(f_K(p))$, the set of all configurations of the vertices of $\langle G, m \rangle$ that come from congruent frameworks on $\mathcal{T}_0^d$. Note that $X(p)$ and $M(p)$ are subsets of $\mathbb{R}^{d|V|}$.

We say that a periodic framework $(\langle G, m \rangle, p)$ is *rigid* on $\mathcal{T}_0^d$ if there exists a neighbourhood U of p in $\mathbb{R}^{d|V|}$ such that

$$X(p) \cap U = M(p) \cap U.$$

The framework $(\langle G, m \rangle, p)$ is *flexible* on $\mathcal{T}_0^d$ if there exists a continuous family of functions $p_t : V \to \mathbb{R}^n$, where $t \in [0, 1]$ such that $p_0 = p$ and $p_t \in X(p) - M(p)$ for some $t \in (0, 1]$.

In other words, $(\langle G, m \rangle, p)$ is rigid on $\mathcal{T}_0^d$ if all configurations of a given d-periodic orbit graph with specified edges lengths in a neighbourhood of the original position must come from congruent frameworks. On the other hand, $(\langle G, m \rangle, p)$ is flexible if we may continuously move the vertices of $(\langle G, m \rangle, p)$ to positions that are not $\mathcal{T}_0^d$-congruent, while preserving the edge lengths of the frameworks.

Because there are a finite number of vertices and edges in this description, a number of important results from the usual study of rigidity transfer almost directly. For example, from the work of Asimow and Roth [2] and Roth and Whiteley [60] we have the following equivalent notions of flexibility for such a framework:

Theorem 3.3.29. *[60] Let $(\langle G, m \rangle, p)$ be a periodic framework on $\mathcal{T}_0^d$. Then the*

96

following are equivalent:

(i) $(\langle G, m \rangle, p)$ *is not rigid on* $\mathcal{T}_0^d$;

(ii) *there exists a family of real analytic paths* $p_t : V \to \mathbb{R}^d$, $t \in [0, 1]$, *with* $p_0 = p$ *and* $p_t \in X(p) - M(p)$ *for all* $t \in (0, 1]$;

(iii) *there exists a family of continuous paths* $p_t : V \to \mathbb{R}^d$, $t \in [0, 1]$, *with* $p_0 = p$ *and* $p_t \in X(p) - M(p)$ *for all* $t \in (0, 1]$;

(iv) $(\langle G, m \rangle, p)$ *is flexible on* $\mathcal{T}_0^d$.

This establishes the equivalence of non-rigidity and flexibility.

We also have an important connection between infinitesimal rigidity and rigidity for generic frameworks, which is a consequence of the main theorem of [2]:

Theorem 3.3.30. *A periodic orbit framework* $\langle G, m \rangle$ *is generically infinitesimally rigid if and only if it is generically rigid, and it is generically infinitesimally flexible, if and and only if it is generically flexible.*

We sketch the proof of this result, but note that the key ideas from [2] transfer almost directly. We first observe that the rigidity matrix $\mathbf{R}_0(\langle G, m \rangle, p)$ is the Jacobian of the edge function $f_G(p)$ (up to a constant). We denote the Jacobian of f_G by df_G. We need the notion of a *regular point*. The point $p \in (0, 1)^{d|V|}$ is a

regular point of the periodic orbit graph $\langle G, m \rangle$ if there is a neighborhood N_p of p

in $(0,1)^{d|V|}$ such that

$$\mathrm{rank}(df_G(p)) \geq \mathrm{rank}(df_G(q)),$$

for all $q \in N_p$. We also need the following lemma, the proof of which follows directly

from Proposition 2 in [2].

Lemma 3.3.31. *Let $(\langle G, m \rangle, p)$ be a periodic orbit framework. If p is a regular*

point of $\langle G, m \rangle$, then there exists a neighborhood N_p of p in $(0,1)^{d|V|}$ such that

$f_G^{-1} f_G(p) \cap N_p$ is a smooth manifold of dimension $d - \mathrm{rank}(df_G(p))$.

Note that the fixed torus $\mathcal{T}_0^d$ has a d-dimensional manifold of isometries. We

are now ready to prove Theorem 3.3.30.

Proof of Theorem 3.3.30. We will show that $\mathrm{rank}(df_G(p)) = \mathrm{rank}(df_K(p))$ if and

only if $(\langle G, m \rangle, p)$ is rigid. The point $p \in (0,1)^{d|V|}$ is a regular point of both $\langle G, m \rangle$

and $\langle K, m_K \rangle$. Therefore, there exist neighborhoods N_p and N_p' of p in $(0,1)^{d|V|}$ such

that $f_G^{-1} f_G(p) \cap N_p$ is a manifold of dimension $d - \mathrm{rank}(df_G(p))$, and $f_K^{-1} f_K(p) \cap N_p'$ is

a manifold of dimension $d - \mathrm{rank}(df_K(p))$. Now since $f_K^{-1} f_K(p) \cap N_p'$ is a submanifold

of $f_G^{-1} f_G(p) \cap N_p$ for some sufficiently small neighborhood (for example $N_p \cap N_p'$),

it must be the case that

$$\mathrm{rank}(df_K(p)) \geq \mathrm{rank}(df_G(p)).$$

But note that $\mathrm{rank}(df_K(p)) = \mathrm{rank}(df_G(p))$ if and only if there exists a neighborhood U of p such that $f_K^{-1}f_K(p) \cap U = f_G^{-1}f_G(p) \cap U$. But this is true if and only if $(\langle G, m \rangle, p)$ is rigid on $\mathcal{T}_0^d$ (by the definition earlier in this section).

On the other hand, if $\mathrm{rank}(df_K(p)) > \mathrm{rank}(df_G(p))$, then every neighborhood N_p of p in $(0,1)^{d|V|}$ contains elements of $f_G^{-1}f_G(p) \backslash f_K^{-1}f_K(p)$, which implies the existence of a non-trivial flex of $(\langle G, m \rangle, p)$, by Theorem 3.3.29. $\qquad\square$

3.3.10 T-gain procedure preserves infinitesimal rigidity on $\mathcal{T}_0^d$

In section 2.3.2 we described the T-gain procedure for identifying the local gain group of a graph. We noted that the original gain assignment m and the T-gain assignment m_T can be seen as simply two different ways to describe the same infinite periodic graph. Most importantly, we now confirm that the rigidity matrices corresponding to these two periodic orbit graphs have the same rank. In fact this is a *geometric* statement, with a generic corollary.

Theorem 3.3.32. *For any framework* $(\langle G, m \rangle, p)$,

$$\mathrm{rank}\mathbf{R}_0(\langle G, m \rangle, p) = \mathrm{rank}\mathbf{R}_0(\langle G, m_T \rangle, p'),$$

where $p' : V \to \mathbb{R}^d$ *is given by* $p_i' = p_i + m_T(v_i)$.

The essence of the following argument is that the T-gain procedure changes the representatives of the vertices used in the rigidity matrix, which, together with the

new gains, leaves the rows of the matrix unchanged.

Proof. Let T be a spanning tree in $\langle G, m \rangle$. Each vertex v_i of G is labeled with a T-potential, which we denote $m(v_i, T) = m_T(i)$. The edge $e = \{v_i, v_j; m_e\}$ has T-gain

$$m_T(e) = m_T(i) + m_e - m_T(j).$$

We know that the derived graphs G^m and G^{m_T} are isomorphic by Theorem 2.3.4. For each vertex $v \in V$, we relabel the indices of the vertices in the fibre over v according to the rule

$$z \longrightarrow z - m_T(v).$$

In other words, the vertex (v_i, z) in G^m, where $z \in \mathbb{Z}^d$ is mapped to the vertex $(v_i, z - m_T(i))$ in G^{m_T}.

Suppose that a set of rows is dependent in $\mathbf{R}_0(\langle G, m \rangle, p)$. Then there exists a vector of scalars, say $\omega = [\ \omega_1 \ \ \cdots \ \ \omega_{|E|}\ ]$ such that

$$\omega \cdot \mathbf{R}_0(\langle G, m \rangle, p) = 0.$$

As in the proof of affine invariance, for a particular vertex we consider the edges directed into and out from the vertex separately. That is, for a vertex $v_i \in V$, let E_+ denote the set of edges directed *out* from the vertex v_i, and let E_- denote the set of edges directed *into* the vertex v_i. For each vertex $v_i \in V$ the column sum of

100

$\mathbf{R}_0(\langle G, m \rangle, p)$ becomes

$$\sum_{e_\alpha \in E_+} \omega_{e_\alpha}(p_i - (p_j + m_{e_\alpha})) + \sum_{e_\beta \in E_-} \omega_{e_\beta}(p_i - (p_k - m_{e_\beta})) = 0. \qquad (3.9)$$

Adding and subtracting $m_T(i)$ and $m_T(j)$ to the first summand of (3.9), we

obtain

$$\sum_{e_\alpha \in E_+} \omega_e \Big(p_i - p_j + m_T(i) - m_T(j) - [m_T(i) + m_e - m_T(j)] \Big),$$

which is equivalent to

$$\sum_{e_\alpha \in E_+} \omega_e \Big(p_i + m_T(i) - (p_j + m_T(j)) - m_T(e) \Big).$$

Similarly, the second summand of (3.9) becomes

$$\sum_{e_\beta \in E_-} \omega_e \Big(p_i + m_T(i) - (p_j + m_T(j)) + m_T(e) \Big).$$

Putting them together, (3.9) becomes the column sum of the column of $\mathbf{R}_0(\langle G, m_T \rangle, p')$

corresponding to the vertex v_i. Hence this set of rows is dependent in $\mathbf{R}_0(\langle G, m_T \rangle, p)$.

The argument reverses for the converse. $\qquad\square$

Corollary 3.3.33. *The periodic orbit graph $\langle G, m \rangle$ is generically rigid on $\mathcal{T}_0^d$ if*

and only if $\langle G, m_T \rangle$ is generically rigid on $\mathcal{T}_0^d$.

Proof. Let p be a generic position of $\langle G, m \rangle$ on $\mathcal{T}_0^d$. Let $p_i' = p_i + m_T(v_i)$. While

$p : V \longrightarrow \mathcal{U}_0^d$, $p' : V \longrightarrow \mathbb{R}^{d|V|}$. By the Modified Special Position Lemma (3.3.28),

the rank of the matrix $\mathbf{R}_0(\langle G, m_T \rangle, p)$ is generically the same as the rank of the

matrix $\mathbf{R}_0(\langle G, m_T \rangle, p')$, which, by Theorem 3.3.32, is the same as the rank of the matrix $\mathbf{R}_0(\langle G, m \rangle, p)$. □

3.3.11 A sample prior result

The following theorem says that given a graph G with certain combinatorial proper-ties, we can always find an appropriate gain assignment m and geometric realization p to yield a minimally rigid framework on $\mathcal{T}_0^d$.

Theorem 3.3.34 (Whiteley, [82])**.** *For a multigraph G, the following are equivalent:*

(i) G satisfies $|E| = d|V| - d$, and every subgraph $G' \subseteq G$ satisfies $|E'| \le d|V'| - d$,

(ii) G is the union of d edge-disjoint spanning trees,

(iii) For some gain assignment m and some realization p, the framework $(\langle G, m \rangle, p)$ is minimally rigid on $\mathcal{T}_0^d$.

For completeness, and as a preview of Chapter 4 we outline the proof.

Proof. The equivalence of *(i)* and *(ii)* is due to Nash-Williams [51] and Tutte [77]. We sketch the proof of the equivalence of *(ii)* and *(iii)*.

(ii) $\rightarrow$ *(iii)* Suppose G is the union of d edge-disjoint spanning trees. Let all vertices be assigned the same position p, and let m be the gain assignment which assigns the k-th basis vector to all edges in the k-th tree, $k = 1, \ldots, d$. That is,

$m(e) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the k-th entry is non-zero. The non-zero entries in the resulting rigidity matrix correspond only to the gains m. This is the matrix of a d-frame (a generalization of the rigidity matrix), and the rows of each spanning tree are independent in this matrix, by the arguments of [82].

(iii) → *(ii)* is a consequence of Corollary 3.3.13. □

This proof from [82] constructs some gain assignments which are sufficient for infinitesimal rigidity (in fact, it produces an infinite space of such gains!). In a nutshell it says that given any graph satisfying the necessary conditions of Corollary 3.3.13, we can define a gain assignment, with basis vector gains, that will be infinitesimally rigid on $\mathcal{T}_0^2$. It is true, however, that these are not the only infinitesimally rigid frameworks. The question of interest then becomes:

Question 3.3.35. When is a periodic orbit graph $\langle G, m \rangle$ generically rigid on $\mathcal{T}_0^d$?

The goal of Chapter 4 will be devoted to broadening the scope of Theorem 3.3.34 for periodic orbit frameworks on the two-dimensional fixed torus, and to characterize more precisely the interactions between combinatorics, geometry and topology in defining rigid frameworks.

As previously noted, the approach of Borcea and Streinu [7] *does not* consider the gains to be part of the combinatorial information of a periodic framework. Instead they work with the notion of generic edge directions, which involves both the gain and the position of the vertices. We will consider the gains of a periodic

orbit framework to be part of the combinatorial information of the graph, and will characterize the rigidity of periodic orbit frameworks for *all* gains.

Malestein and Theran [49] do consider gain graphs. In their language, our gain graphs are "coloured graphs".

4 Infinitesimal rigidity on the fixed torus $\mathcal{T}_0^d$

4.1 Introduction

In this chapter we find sufficient conditions for infinitesimal rigidity on $\mathcal{T}_0^2$. These results can be framed in relation to two well known results in finite rigidity, namely Henneberg's Theorem (Theorem 2.5.13) and Laman's Theorem (Theorem 2.5.10). In Section 4.3 it is shown that every minimally rigid periodic orbit framework on $\mathcal{T}_0^2$ can be constructed from smaller graphs through a series of inductive constructions. This is a periodic version of Henneberg's theorem about finite graphs. Section 4.4 describes a characterization of the rigidity of a two-dimensional periodic framework through a consideration of the gain assignment on the corresponding periodic orbit framework. This can be viewed as a periodic analogue of Laman's theorem about finite graphs. In this section we develop the idea of a *constructive* gain assignment, which is a gain assignment for which every fully-counted subgraph contains a cycle with non-zero net gain. In Section 4.5 we indicate what extensions to d-dimensions are possible.

105

4.2 Statement of main result

The goal of the rest of this chapter is to prove the following theorem, which builds

on Theorem 3.3.34, characterizing infinitesimal rigidity on $\mathcal{T}_0^2$.

Theorem 4.2.1. *For a multigraph* $G = (V, E)$, *the following are equivalent:*

(i) *G is the union of 2 edge-disjoint spanning trees*

(ii) *G satisfies $|E| = 2|V| - 2$ and every subgraph $G' \subset G$ satisfies $|E'| \leq 2|V| - 2$*

(iii) *If $\langle G, m \rangle$ is generically minimally rigid on $\mathcal{T}_0^2$, then it can be constructed from*

a single vertex by a sequence of periodic vertex-additions and edge-splits

(iv) *for all constructive gain assignments m, $\langle G, m \rangle$ is generically minimally rigid*

on $\mathcal{T}_0^2$

(v) *for some gain assignment m and some realization p, the framework $(\langle G, m \rangle, p)$*

is minimally rigid on $\mathcal{T}_0^2$

This clearly builds on Theorem 3.3.34, but is extended in two key ways:

- (iii) is a periodic version of Henneberg's Theorem, and is the subject of Section

4.3

- (iv) is a periodic version of Laman's Theorem, and is the subject of Section

4.4.

Proof. The theorem is proved as follows:

$$
\begin{array}{ccccc}
\text{(i)} & \leftrightarrow & \text{(ii)} & \rightarrow & \text{(iv)} \\
 & & & \nwarrow \quad \downarrow & \\
 & & \text{(v)} & \leftrightarrow & \text{(iii)}
\end{array}
$$

The equivalence of (i), (ii) and (v) is the content of Theorem 3.3.34. We will show:

(iii) $\iff$ (v) is the periodic Henneberg theorem, Theorem 4.3.8.

(iv) $\implies$ (v) is immediate.

(ii) $\implies$ (iv) is the content of Theorem 4.4.5. $\qquad\qquad\qquad\qquad$ □

4.3 Generating minimally rigid frameworks on $\mathcal{T}_0^2$

We now describe methods for generating infinitesimally rigid frameworks on the fixed two dimensional torus $\mathcal{T}_0^2$.

4.3.1 Inductive constructions

Let $(\langle G, m \rangle, p)$ be an infinitesimally rigid periodic orbit framework. It is possible to construct other infinitesimally rigid frameworks from $(\langle G, m \rangle, p)$ using *periodic vertex additions* and *edge splits*. We present here a periodic adapted version of these *inductive constructions*, which we will also call *periodic Henneberg moves* after their finite counterparts which were developed by Henneberg [36]. Details about the finite versions of these moves can be found in [74] and [82]. The following arguments are

based on the rigidity matrix. In particular, we will show that the periodic inductive constructions preserve the independence of the rows of $\mathbf{R}_0$.

4.3.1.1 Vertex addition
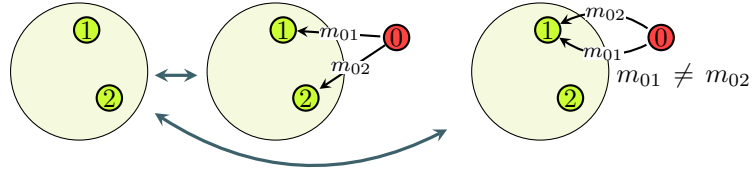
Figure 4.1: Periodic vertex addition. The large circular region represents a generically rigid periodic orbit graph.

Given a periodic orbit graph $\langle G, m \rangle$, a *periodic vertex addition* is the addition of a single new vertex v_0 to V, and the edges $\{v_0, v_{i_1}; m_{01}\}$ and $\{v_0, v_{i_2}; m_{02}\}$ to E, such that $m_{01} \neq m_{02}$ whenever $v_{i_1} = v_{i_2}$ (see Figure 4.1). Provided that $v_{i_1} \neq v_{i_2}$, by definition, m_{01} and m_{02} may always taken to be $(0, 0)$.

Proposition 4.3.1 (Periodic Vertex Addition)**.** *Let $\langle G, m \rangle$ be a periodic orbit graph, and let $\langle G', m' \rangle$ be the graph created by performing a vertex addition on $\langle G, m \rangle$, adding the vertex v_0 to G. For a generic choice of $p : V \to \mathcal{T}_0^2$, and with p_0 chosen generically with respect to p, the rows of $\mathbf{R}_0(\langle G, m \rangle, p)$ are independent if and only if the rows of $\mathbf{R}_0(\langle G', m' \rangle, p')$ are independent, where $p' = p \cup p_0$.*

Proof. Suppose that the vertex v_0 is connected to the vertices v_{i_1} and v_{i_2} by the edges $\{v_0, v_{i_1}; m_{01}\}$ and $\{v_0, v_{i_2}; m_{02}\}$, where v_{i_1} and v_{i_2} may or may not be the

108

same vertex. The rigidity matrix of $\langle G', m' \rangle$ is

$$
\mathbf{R}_0(\langle G', m' \rangle, p') =
\begin{array}{c}
\\
\\
\\
e_1 \\
\vdots \\
e_{|E|} \\
\\
\{v_0, v_{i_1}; m_{01}\} \\
\{v_0, v_{i_2}; m_{02}\}
\end{array}
\begin{array}{c}
\begin{array}{cccc}
v_0 & | & v_1 & \cdots & v_{|V|}
\end{array} \\
\left(
\begin{array}{c|ccc}
0 & | & & \\
\vdots & | & \multicolumn{3}{c}{\mathbf{R}_0(\langle G, m \rangle, p)} \\
0 & | & \\
\hline
p_0 - p_{i_1} - m_{01} & | & \cdots & \\
p_0 - p_{i_2} - m_{02} & | & \cdots &
\end{array}
\right)
\end{array}.
$$

Suppose, toward a contradiction, that the rows of $\mathbf{R}_0(\langle G', m' \rangle, p')$ are dependent. Then the columns of $\mathbf{R}_0(\langle G', m' \rangle, p')$ corresponding to v_0 provide the relationship:

$$
\omega_{01}(p_0 - p_{i_1} - m_{01}) + \omega_{02}(p_0 - p_{i_2} - m_{02}) = 0
$$

for some $\omega_{01}, \omega_{02} \in \mathbb{R}$. However, $(p_0 - p_{i_1} - m_{01})$ and $(p_0 - p_{i_2} - m_{02})$ are linearly independent (as vectors in $\mathbb{R}^2$) if and only if the points p_0, $p_{i_1} + m_{01}$ and $p_{i_2} + m_{02}$ are *not* collinear.

If $v_{i_1} \neq v_{i_2}$, or if $v_{i_1} = v_{i_2}$ but $m_{01} \neq m_{02}$, then these points are not collinear, since we chose p_0 generically with respect to p. Hence $\omega_{01} = \omega_{02} = 0$, which leaves a dependence among the rows of $\mathbf{R}_0(\langle G, m \rangle, p)$ and contradicts our assumption that the rows of $\mathbf{R}_0(\langle G, m \rangle, p)$ were independent. The argument reverses for the converse. (Assume the rows of $\mathbf{R}(\langle G', m' \rangle, p')$ are dependent and proceed from there.) $\square$

Corollary 4.3.2. *Let $\langle G, m \rangle$ be a periodic orbit graph, and let $\langle G', m' \rangle$ be a vertex addition of $\langle G, m \rangle$. Then $\langle G, m \rangle$ is generically minimally rigid if and only if $\langle G', m' \rangle$ is generically minimally rigid.*

Proof. If $\langle G', m' \rangle$ is generically minimally rigid, then $\mathbf{R}_0 \langle G', m' \rangle$ has rank

$$\mathrm{rank}\mathbf{R}_0 \langle G', m' \rangle = 2|V'| - 2 = 2|V| - 2 + 2.$$

Since two rows were added to $\mathbf{R}_0 \langle G, m \rangle$ to obtain $\mathbf{R}_0 \langle G', m' \rangle$, it must be the case that $\mathrm{rank}\mathbf{R}_0 \langle G, m \rangle = 2|V| - 2$. The argument reverses for the converse. $\square$

Note that Proposition 4.3.1 also has a geometric meaning. In fact, the proof of that result was geometric in nature, in the sense that we chose p so that the points p_0, $p_{i_1} + m_{01}$ and $p_{i_2} + m_{02}$ were not collinear in $\mathbb{R}^2$ (in fact we chose p to be generic, and the non-collinearity followed). This observation is summarized in the following corollary:

Corollary 4.3.3. *Let $(\langle G, m \rangle, p)$ be a periodic orbit framework, and let $\langle G', m' \rangle$ be the graph created by performing a vertex addition on $\langle G, m \rangle$, adding the vertex v_0 to G. If p_0 is not collinear with $p_{i_1} + m_{01}$ and $p_{i_2} + m_{02}$ in $\mathbb{R}^2$, then the rows of $\mathbf{R}_0(\langle G, m \rangle, p)$ are independent if and only if the rows of $\mathbf{R}_0(\langle G', m' \rangle, p')$ are independent, where $p' = p \cup p_0$.*

4.3.1.2 Edge splitting

Let $\langle G, m \rangle$ be a periodic orbit graph, and let $e = \{v_{i_1}, v_{i_2}; m_e\}$ be an edge of $\langle G, m \rangle$.

A *periodic edge split* $\langle G', m' \rangle$ of $\langle G, m \rangle$ is a graph with vertex set $V \cup \{v_0\}$ and edge set consisting of all of the edges of E except e, together with the edges

$$\{\{v_0, v_{i_1}; (0,0)\}, \{v_0, v_{i_2}; m_e\}, \{v_0, v_{i_3}; m_{03}\}\}$$

where $v_{i_1} \neq v_{i_3}$, and $m_{03} \neq m_e$ if $v_{i_2} = v_{i_3}$ (see Figure 4.2).
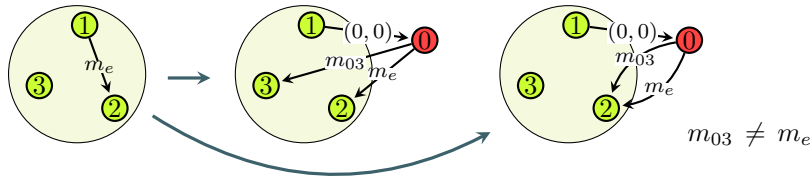
Figure 4.2: Periodic edge split.

Periodic edge splits, and reverse periodic edge splits, preserve infinitesimal rigidity. We will show this in two parts, by showing that this move preserves independence of the rows of the rigidity matrix.

Proposition 4.3.4 (Periodic Edge Split). *Let $\langle G, m \rangle$ be a periodic orbit graph, and let $\langle G', m' \rangle$ be an edge split of it. Let $p : V \to \mathcal{T}_0^2$ be a generic realization of $V(G)$ on $\mathcal{T}_0^2$, and let p_0 be chosen generically with respect to p. If the rows of $\mathbf{R}_0(\langle G, m \rangle, p)$ are independent, then the rows of $\mathbf{R}_0(\langle G', m' \rangle, p')$ are also independent, where $p' = p \cup p_0$.*

Proof. Suppose that p is a generic realization of the vertices of G on $\mathcal{T}_0^2$, with no vertex on the boundary of $\mathcal{T}_0^2$, and place p_0 on the edge connecting the vertices v_{i_1} and $v_{i_2} + m_e$, where the segment containing v_{i1} and p_0 lies in $[0,1)^d$. Without loss of generality, suppose that e_1 is the split edge. Let $\mathbf{R}_0(\langle G, m\rangle, p) - e_1$ denote the rigidity matrix of $(\langle G, m\rangle, p)$ without the row corresponding to the edge e_1. The rigidity matrix $\mathbf{R}_0(\langle G', m'\rangle, p')$ is:

$$
\begin{array}{c}
\\
\\
e_2 \\
\vdots \\
e_{|E|} \\
\{v_0, v_{i_1}; 0\} \\
\{v_0, v_{i_2}; m_e\} \\
\{v_0, v_{i_3}; m_{03}\}
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
v_0 & & v_1 & & v_2 & \cdots \quad v_{|V|}
\end{array} \\
\left(
\begin{array}{ccccc}
0 & & & & \\
\vdots & & & \mathbf{R}_0(\langle G, m\rangle, p) - e_1 & \\
0 & & & & \\
\hline
p_0 - p_{i_1} & & p_{i_1} - p_0 & 0 & \cdots \\
p_0 - p_{i_2} - m_e & & 0 & p_{i_2} - p_0 + m_e & \cdots \\
p_0 - p_{i_3} - m_{03} & & 0 & & \cdots
\end{array}
\right)
\end{array}.
$$

Suppose toward a contradiction, that there is a non-trivial dependence among the rows of $\mathbf{R}_0(\langle G', m'\rangle, p')$. That is, suppose that

$$\omega \cdot \mathbf{R}_0(\langle G', m'\rangle, p') = 0$$

for $\omega \neq 0$ where $\omega = \begin{bmatrix} \omega_2 & \cdots & \omega_{|E|} & \omega_{01} & \omega_{02} & \omega_{03} \end{bmatrix}$.

The vector equation describing the first two columns of this expression (the

columns corresponding to v_0) becomes:

$$\omega_{01}(p_0 - p_{i_1}) + \omega_{02}(p_0 - p_{i_2} - m_e) + \omega_{03}(p_0 - p_{i_3} - m_{03}) = 0$$

with not all of $\omega_{01}, \omega_{02}, \omega_{03}$ being 0 (otherwise we would immediately have a non-trivial dependence among the rows of $\mathbf{R}_0(\langle G, m \rangle, p)$, contradicting our hypothesis).

Because we placed p_0 along the edge connecting v_{i_1} and $v_{i_2} + m_e$, the vectors $(p_0 - p_{i_1})$ and $(p_0 - p_{i_2} - m_e)$ are parallel, and $(p_0 - p_{i_3} - m_{03})$ is in a distinct direction, therefore $\omega_{03} = 0$. Since both of these vectors are again parallel to the deleted edge, we have

$$\omega_{01}(p_0 - p_{i_1}) = -\omega_{02}(p_0 - p_{i_2} - m_e) = \omega_{12}(p_{i_1} - p_{i_2} - m_e)$$

for some scalar $\omega_{12} \neq 0$.

But then the coefficients of the rows of $\mathbf{R}_0(\langle G', m' \rangle, p')$ corresponding to the edges in $E \cap E'$, together with ω_{12} form a set of scalars that provide a dependence among the rows of $\mathbf{R}_0(\langle G, m \rangle, p)$, which contradicts our hypothesis.

By the Special Position Lemma (Lemma 3.3.27), we conclude that the edges of $(\langle G', m' \rangle, p')$ are generically independent, since the edges are independent for a special position of p_0. $\qquad\square$

The *reverse periodic edge split* will delete a 3-valent vertex, and add an edge between two of the vertices formerly adjacent to that vertex (Figure 4.3). In particular, if v_0 is the 3-valent vertex adjacent to the vertices v_{i_1}, v_{i_2} and v_{i_3}, where at

113

most two of v_{i_1}, v_{i_2} and v_{i_3} may be the same, then a reverse edge split will add one

of the edges

$$\{v_{i_1}, v_{i_2}; m_{02} - m_{01}\}, \{v_{i_2}, v_{i_3}; m_{03} - m_{02}\}, \{v_{i_3}, v_{i_1}; m_{01} - m_{03}\}.$$
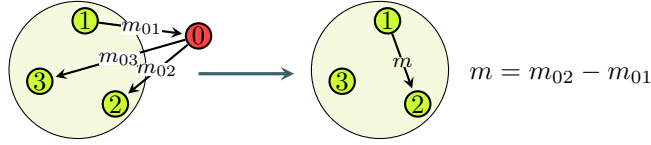
Figure 4.3: Reverse periodic edge split. In this case the edge $\{v_{i_1}, v_{i_2}; m_{02} - m_{01}\}$
is added.

Proposition 4.3.5 (Reverse Periodic Edge Split). *If a 3-valent vertex v_0 is deleted*

from a generically independent periodic orbit graph, then a single edge may be added

between one pair of vertices formerly adjacent to v_0 so that the resulting graph is

also a generically independent periodic orbit graph.

Proof. Suppose the the rows of $\mathbf{R}_0(\langle G', m'\rangle, p')$ are independent for some $p' = p \cup p_0$,

and suppose that the vertex v_0 is connected to vertices v_{i_1}, v_{i_2} and v_{i_3}, where at

most two of these vertices are the same. Let E^* be the edge set created by deleting

vertex v_0 and its adjacent edges. Let G_{12}, G_{23} and G_{31} be the graphs with vertex

set $V \setminus \{v_0\}$, and edge sets $E_{12} = E^* \cup \{v_{i_1}, v_{i_2}; m_{02} - m_{01}\}$ and similarly for E_{23} and

E_{31}. If any of these graphs is independent at p then we are done.

Assume to the contrary that no such graph is independent. Then the rows of

the matrices corresponding to each of these frameworks are dependent. Writing R_e

as the row of the rigidity matrix corresponding to the edge e, we have

$$\alpha_{12} R_{12} = \sum_{e \in E^*} -\alpha_e R_e \quad \text{with} \quad \alpha_{12} \neq 0$$

$$\beta_{23} R_{23} = \sum_{e \in E^*} -\beta_e R_e \quad \text{with} \quad \beta_{23} \neq 0$$

$$\gamma_{31} R_{31} = \sum_{e \in E^*} -\gamma_e R_e \quad \text{with} \quad \gamma_{31} \neq 0$$

We now have two cases:

1. The vertices v_{i_1}, v_{i_2} and v_{i_3} are distinct

2. The vertices v_{i_1}, v_{i_2} and v_{i_3} are not distinct.

In case 1, consider the graph on the vertices $\{v_0, v_{i_1}, v_{i_2}, v_{i_3}\}$ with all of the candidate edges (see Figure 4.4). This has $|E| = 2|V| - 2$. Note that the net gain on any closed path in the graph is $(0, 0)$, and hence this graph is T-gain equivalent to a graph with all gains identically zero. By Lemma 3.3.17 and Theorem 3.3.32, this graph is dependent.
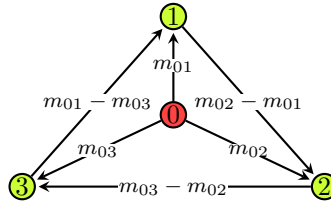
Figure 4.4: This graph, corresponding to Case 1 of Proposition 4.3.5, satisfies $|E| = 2|V| - 2$, and is T-gain equivalent to a graph with all zero gains, therefore a dependence exists among the edges.

We have

$$\omega_{01}R_{01} + \omega_{02}R_{02} + \omega_{03}R_{03} + \omega_{12}R_{12} + \omega_{23}R_{23} + \omega_{31}R_{31} = 0$$

Scaling and substituting the expressions above, we obtain

$$\omega_{01}R_{01} + \omega_{02}R_{02} + \omega_{03}R_{03} + \sum_{e \in E^*} -(\alpha'_e + \beta'_e + \gamma'_e)R_e = 0$$

which is a dependence on the rows of $\mathbf{R}_0(\langle G', m' \rangle, p')$, a contradiction. Therefore, at least one of the graphs G_{12}, G_{23}, G_{31} must be independent.

For case 2, assume without loss of generality that $v_{i_2} = v_{i_3}$. We consider the graph on the vertices $\{v_0, v_{i_1}, v_{i_2}\}$ with all of the candidate edges (see Figure 4.5). This graph has $|E| = 2|V| - 1$, and hence is dependent. The proof of this case now follows the proof of the previous case. $\square$
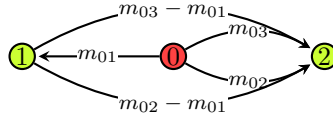
Figure 4.5: This graph, corresponding to Case 2 of Proposition 4.3.5, satisfies $|E| = 2|V| - 1$, therefore a dependence exists among the edges.

Both the vertex-addition and the edge-split preserve the relationship between the number of edges and the number of vertices in the d-periodic orbit graph. If $|E| = 2|V| - 2$ then $|E'| = 2|V'| - 2$ as well. We have the following corollary to the previous propositions:

Corollary 4.3.6. *Periodic vertex additions and edge splits, and their reverse operations, preserve generic minimal rigidity of periodic orbit graphs $\langle G, m \rangle$ on $\mathcal{T}_0^2$.*

The process of deleting a three-valent vertex from $\langle G, m \rangle$ by a reverse edge split, and then performing an edge split will not usually produce a graph that is identical to the original (see Figure 4.6). However, we can ensure that we always produce a graph with an isomorphic space of infinitesimal motions, using the following lemma:

Lemma 4.3.7. *Let $\langle G, m \rangle$ be a periodic orbit graph, and let $\langle G', m' \rangle$ be a reverse edge split of $\langle G, m \rangle$. Then for some edge split $\langle \overline{G}, \overline{m} \rangle$ of $\langle G', m' \rangle$ with $G = \overline{G}$, and some spanning tree T, the resulting graph $\langle G, \overline{m} \rangle$ is T-gain equivalent to $\langle G, m \rangle$.*

Proof. Let v_0 be a 3-valent vertex of $\langle G, m \rangle$, adjacent to vertices v_1, v_2, v_3 (see Figure 4.6. After deleting v_0, suppose without loss of generality that the edge $e = \{v_1, v_2; m_{02} - m_{01}\}$ was added to form the graph $\langle G', m' \rangle$. We perform an edge split on this edge to obtain a graph that differs from our original orbit graph, but whose rigidity matrices have the same rank. In particular, we add to $\langle G', m' \rangle$ the vertex v_0 and the three edges:

$$\{v_0, v_1; (0,0)\}, \quad \{v_0, v_2; m_{02} - m_{01}\}, \quad \{v_0, v_3; m_{03} - m_{01}\}.$$

Let the resulting infinitesimally rigid graph be denoted $\langle G, \overline{m} \rangle$. Note that the gains on the first two edges are determined by the reverse edge split, but the gain on the third edge is a 'free' choice.

117
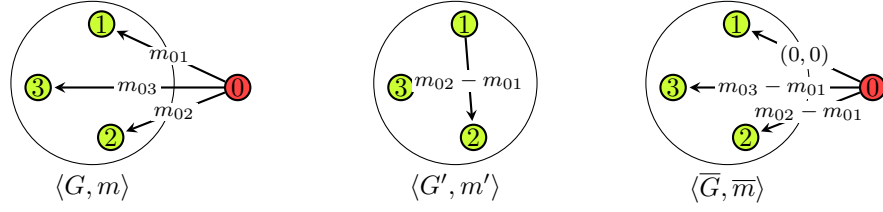
Figure 4.6: Proof of Lemma 4.3.7: Deleting a 3-valent vertex from $\langle G, m \rangle$, followed by an edge split, results in a T-gain equivalent periodic orbit graph $\langle \overline{G}, \overline{m} \rangle$.

Now let T' be a spanning tree in G' with root $u = v_1$ that does *not* include the edge $e = \{v_1, v_2\}$ (which has gain $m_{02} - m_{01}$ in $\langle G, m \rangle'$). It is always possible to select such a tree, since deleting this edge will not disconnect the graph. Let T be the spanning tree of G created by adding the edge $\{v_0, v_1\}$ to T'. This edge has gain m_{01} in $\langle G, m \rangle$, and gain $(0, 0)$ in $\langle G, \overline{m} \rangle$. Performing the T-gain procedure on $\langle G, m \rangle$ and $\langle G, \overline{m} \rangle$ with T, we obtain identical periodic orbit graphs. For example, the edge $e_2 = \{v_0, v_2, m_{02}\} \in \langle G, m \rangle$ has T-gain

$$
\begin{aligned}
m_T(e_2) &= m(v_0, T) + m_{02} - m(v_2, T) \\
&= -m_{01} + m_{02} - m(v_2, T) \\
&= (0, 0) + (m_{02} - m_{01}) - m(v_2, T) \\
&= (0, 0) + (m_{02} - m_{01}) - \overline{m}(v_2, \overline{T}) \\
&= \overline{m}_T(e_2).
\end{aligned}
$$

The same is true of the other edges added in the edge split, and since $T = \overline{T}$ for

all of the edges of $\langle G', m' \rangle$, the orbit graphs are T-gain equivalent. That is,

$$\langle G, m_T \rangle = \langle G, \overline{m}_T \rangle.$$

$\square$

4.3.2 Periodic Henneberg Theorem

Theorem 4.3.8 (Periodic Henneberg Theorem). *A framework $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^2$ is generically minimally rigid if and only if it can be constructed from a single vertex on $\mathcal{T}_0^2$ by a sequence of periodic vertex additions and edge splits.*

Proof. ($\Longleftarrow$) Let $\langle G, m \rangle$ be the periodic orbit graph consisting of a single vertex, $V = \{v_0\}$ and $E = \emptyset$. This is trivially a generically rigid periodic orbit graph. By Lemmas 4.3.1 and 4.3.4 we can perform periodic vertex additions and edge splits to obtain a new generically rigid periodic orbit graph. Since both operations preserve the count $|E| = 2|V| - 2$, the new orbit graph is generically minimally rigid on $\mathcal{T}_0^2$.

($\Longrightarrow$) This direction is proved by induction on the number of vertices, $|V|$. As noted above the single vertex on $\mathcal{T}_0^2$ is generically infinitesimally rigid, which provides the base case.

Now consider a generically minimally rigid periodic orbit graph $\langle G, m \rangle$ with $|V| \geq 2$, and assume that all infinitesimally rigid frameworks on $\mathcal{T}_0^2$ with fewer than $|V|$ vertices satisfy the hypothesis. Since $|E| = 2|V| - 2$, the average valence

119

of any given vertex is

$$\rho = \frac{2|E|}{|V|} = \frac{2(2|V|-2)}{|V|} = \frac{4|V|-4}{|V|} = 4 - 4/|V| < 4.$$

In addition, because the orbit graph is infinitesimally rigid on $\mathcal{T}_0^2$, every vertex has valence at least 2 (any graph with a pendent vertex is *not* infinitesimally rigid). These two facts together imply that G must have a vertex of valence either 2 or 3.

If G has a vertex of valence 2, then $\langle G, m \rangle$ is a (periodic) vertex-addtion of an infinitesimally rigid framework on a graph (V', E') by Corollary 4.3.2.

If G has a vertex of valence 3, then by Lemma 4.3.7 $\langle G, m \rangle$ is T-gain equivalent to an edge split of an infinitesimally rigid framework on a graph (V'', E''). In either case, $|V'| = |V''| < n$, and we may apply the induction hypothesis to the underlying graph. $\qquad\square$

For a periodic orbit graph $\langle G, m \rangle$, we call the sequence of orbit graphs

$$\langle G_1, m_1 \rangle, \langle G_2, m_2 \rangle, \ldots, \langle G_n, m_n \rangle = \langle G, m \rangle$$

beginning with a single vertex $|V_1| = 1$ and ending with $\langle G, m \rangle$ ($|V_n| = n = |V|$) the *(periodic) Henneberg sequence* for $\langle G, m \rangle$. We observe that given a Henneberg sequence for a periodic orbit graph $\langle G, m \rangle$, beginning with a single vertex and concluding with $\langle G, m \rangle$, it can be checked in linear time that $\langle G, m \rangle$ is generically rigid on $\mathcal{T}_0^2$ (with one step per vertex). An example of a Henneberg sequence is shown in Figure 4.7.
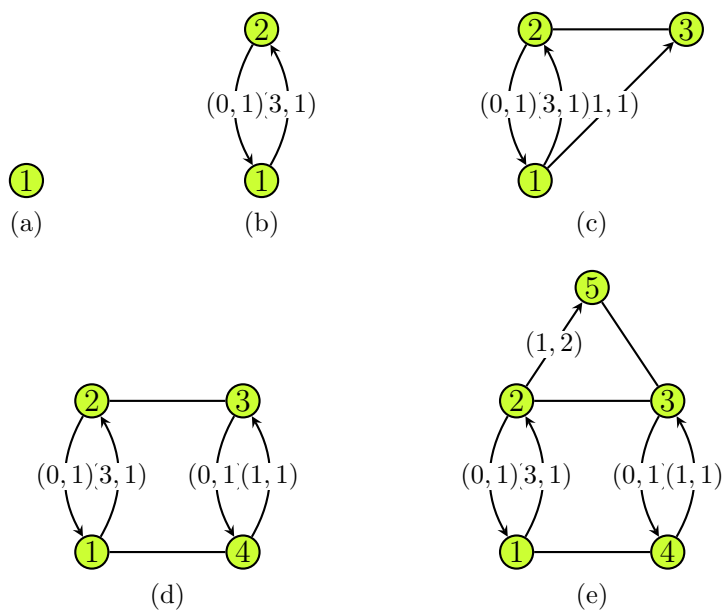
Figure 4.7: An example of a periodic Henneberg sequence. The single vertex (a) becomes a single cycle through a vertex addition (b). Adding a third vertex in (c), then splitting off the edge $\{1, 3; (1, 1)\}$ and adding the fourth vertex (d). The final graph is shown in (e).

4.4 Gain assignments determine rigidity on $\mathcal{T}_0^2$

In this section, we characterize the generic rigidity properties of a framework on the two-dimensional fixed torus $\mathcal{T}_0^2$ by its gain assignment. In Section 4.4.1 we show that only graphs with *constructive gain assignments* can be rigid, and Section 4.4.2 will demonstrate that all such periodic orbit graphs are generically rigid. In Section 4.5.3 we indicate extensions to higher dimensions.

4.4.1 Constructive gain assignments

Let $\langle G, m \rangle$ be a periodic orbit graph. Let C be a closed oriented cycle with no repeated vertices, starting and ending at a vertex u in the multigraph. Recall that the *net cycle gain* is the sum m_C of the gain assignments of the edges of the cycle, where the signs of the edges are determined by the traversal order specified by the orientation. We say the net gain on the cycle is *non-zero* or *non-trivial* if it is non-zero on at least one of the coordinates of $m_C \in \mathbb{Z}^2$.

Let $G = (V, E)$ be a multigraph with $|E| = 2|V| - 2$ edges, and where every subgraph $G' \subset G$ satisfies $|E'| = 2|V'| - 2$. A *constructive gain assignment* on G is a map $m : E \to \mathbb{Z}^2$ such that every subgraph $G' \subset G$ with $G' = (V', E')$ and $|E| = 2|V'| - 2$ contains some cycle of vertices and edges with a non-zero net gain. A cycle C with a non-zero net gain will be called a *constructive cycle*. If $\langle H, m_H \rangle$

is a graph with $|E(H)| > 2|V(H)| - 2$, we say that $\langle H, m_H \rangle$ has a constructive gain assignment if there is some subgraph $G \subset H$ such that $m_H|_G$ is constructive on G.

Proposition 4.4.1. *Let $\langle G, m \rangle$ be a periodic orbit graph with $|E| = 2|V| - 2$, and $|E'| \leq 2|V'| - 2$ for all subgraphs $G' \subset G$. If $(\langle G, m \rangle, p)$ is infinitesimally periodic rigid for some realization p, then m is constructive.*

Proof. We will show the contrapositive. Suppose that m is not constructive, and therefore there exists a subgraph $\langle G', m' \rangle \subseteq \langle G, m \rangle$ with $|E'| = 2|V'| - 2$ and no constructive cycles. Let T' be a spanning tree in G', and expand T' to a spanning tree T of all of G. This is always possible, since G is connected.

Perform the T-gain procedure on $\langle G, m \rangle$. Every edge in T and therefore in T' will have zero gains, and hence no other edge in E' may have non-zero gain, since the T-gain procedure preserves net cycle gains.

Hence $\langle G', m' \rangle$ consists of $2|V'| - 2$ edges with zero gains, which correspond to dependent rows in the rigidity matrix, since at most $2|V'| - 3$ edges without gains can be independent in the rigidity matrix, by Lemma 3.3.17. Therefore,

$$\text{rank}\mathbf{R}_0(\langle G, m \rangle, p) < 2|V| - 2,$$

and $(\langle G, m \rangle, p)$ is infinitesimally flexible. $\square$

Remark 4.4.2. Malestein and Theran [49] independently use a similar idea, and define the $\mathbb{Z}^2$-rank of a periodic orbit graph $\langle G, m \rangle$ to be the number of linearly independent vectors among the cycle gains of the cycle space of the graph. They use the word 'coloured graphs' to describe our gain graphs. □

Remark 4.4.3. A constructive cycle in $(\langle G, m \rangle, p)$ corresponds to an infinite path in the derived periodic framework (G^m, p^m). Let u be a vertex of $(\langle G, m \rangle, p)$, and suppose C is a cycle beginning and ending at u with net gain $(z_1, z_2) \in \mathbb{Z}^2$. Then the edges of C correspond to a finite path connecting the vertices $(u, (0,0))$ and $(u, (z_1, z_2))$ in (G^m, p^m). Repeating the argument we find that all vertices of the form $(u, c(z_1, z_2)), c \in \mathbb{Z}$ are connected along a single (infinite) path. □

The following section will demonstrate that constructive gain assignments are also sufficient for infinitesimal rigidity on $\mathcal{T}_0^2$. In section 4.5.3 we will return to d-dimensional frameworks, and define constructive gain assignments for the d-dimensional setting.

4.4.2 Periodic Laman Theorem on $\mathcal{T}_0^2$

The main result of this section is the following:

Theorem 4.4.4 (Periodic Laman Theorem)**.** *Let $\langle G, m \rangle$ be a periodic orbit graph. Then $(\langle G, m \rangle, p)$ is generically minimally rigid on $\mathcal{T}_0^2$ if and only if $\langle G, m \rangle$ satisfies*

1. $|E| = 2|V| - 2$, and $|E'| \leq 2|V'| - 2$ for all subgraphs $G' \subset G$

2. m is a constructive gain assignment.

Since we have already established that (1) is necessary for minimal rigidity on $\mathcal{T}_0^2$, we will prove the following:

Theorem 4.4.5. *Let $\langle G, m \rangle$ be a periodic orbit graph, with $|E| = 2|V| - 2$, and $|E'| \leq 2|V'| - 2$ for all subgraphs $G' \subset G$. Then $(\langle G, m \rangle, p)$ is generically minimally rigid on $\mathcal{T}_0^2$ if and only if m is a constructive gain assignment.*

The 'only if' part was Proposition 4.4.1. The proof of the 'if' part of this theorem will require a number of technical results, which follow. In particular, we will show

Proposition 4.4.6. *Let $\langle G, m \rangle$ be a periodic orbit graph on $\mathcal{T}_0^2$ satisfying*

(1) m is constructive

(2) $|E| - 2|V| - 2$, and every subgraph $G' \subset G$ satisfies $|E'| \leq 2|V'| - 2$.

Then it is always possible to delete any 2-valent vertex v_0, or perform a reverse edge split on any 3-valent vertex v_0 such that the resulting graph $\langle G_0, m_0 \rangle$ also satisfies the properties (1) – (2) above.

Note that the graph $\langle G, m \rangle$ in Proposition 4.4.6 is not assumed to be rigid, which distinguishes this result from the fact that vertex-deletions and reverse edge-splits preserve infinitesimal rigidity on $\mathcal{T}_0^2$ (Propositions 4.3.1 and 4.3.5).

We delay the proof of Propostion 4.4.6 until after the proof of our main result.

Proof of Theorem 4.4.5. The proof proceeds by induction on the number of vertices, $n = |V|$.

First notice that the hypothesis is true in the case $|V| = |E| = 2$. By the proof of the Periodic Henneberg Theorem (Theorem 4.3.8), any periodic orbit graph $\langle G, m \rangle$ with a constructive gain assignment with 2 vertices can be obtained as a 2-addition to a single vertex (which is minimally rigid on $\mathcal{T}_0^2$). See also Figure 3.4 for an illustration of this case.

Now let $|V|$ be at least 3, and assume the claim holds for all graphs $G = (V, E)$ with $|V| < n$. That is, for a graph G satisfying (2), we assume that for all constructive gain assignments m the framework $(\langle G, m \rangle, p)$ is generically minimally rigid on $\mathcal{T}_0^2$.

Let $G = (V, E)$ be a graph with $|V| \geq 3$, and suppose m is any constructive gain assignment of the edges. By Lemma 4.4.6 we can always delete a 2- or 3-valent vertex in a way that leaves a graph $\langle G', m' \rangle$ satisfying (2) and with m' constructive. Then $|V'| = n-1$, hence the inductive hypothesis applies, and $\langle G', m' \rangle$ is generically minimally rigid on $\mathcal{T}_0^2$.

To obtain the original orbit graph under consideration, $\langle G, m \rangle$, we simply perform the appropriate periodic Henneberg move on the graph $\langle G', m' \rangle$ as follows:

1. If a 2-valent vertex was deleted, simply add back the same edges that were

deleted.

2. If a 3-valent vertex was deleted, then by Lemma 4.3.7 we can edge split the added edge to a obtain the orbit graph $\langle G, \overline{m} \rangle$, which is T-gain equivalent to $\langle G, m \rangle$.

In either case, $\langle G, m \rangle$ is generically minimally rigid on $\mathcal{T}_0^2$. In the second case, it is minimally rigid because $\langle G, \overline{m} \rangle$ is minimally rigid. □

Proposition 4.4.6 can be broken into the following two propositions, which deal with the two cases of deleting 2- and 3-valent vertices respectively. The proof of Proposition 4.4.7, is straightforward. The remainder of this section is devoted to the proof of Proposition 4.4.8.

Proposition 4.4.7. *Let $\langle G, m \rangle$ be an orbit graph on $\mathcal{T}_0^2$ satisfying*

(1) m is constructive

(2) $|E| - 2|V| - 2$, and every subgraph $G' \subset G$ satisfies $|E'| \leq 2|V'| - 2$.

Then it is always possible to delete any 2-valent vertex v_0 such that the resulting graph $\langle G_0, m_0 \rangle$ also satisfies the properties (1) and (2) above.

Proof. Deleting the 2-valent vertex v_0 leaves a graph G' which is a subgraph of the original graph G with $|E'| = 2|V'| - 2$. Since m was constructive, this subgraph $\langle G', m' \rangle$ also has a constructive gain assignment. □

127

Proposition 4.4.8. *Let $\langle G, m \rangle$ be a graph on $\mathcal{T}_0^2$ satisfying*

(i) m is constructive

(ii) $|E| - 2|V| - 2$, and every subgraph $G' \subset G$ satisfies $|E'| \le 2|V'| - 2$.

Then it is always possible to perform a reverse edge split on any 3-valent vertex v_0 such that the resulting graph $\langle G_0, m_0 \rangle$ also satisfies the properties (i) – (ii) above.

Proof. We have two cases:

1. v_0 is adjacent to two distinct vertices

2. v_0 is adjacent to three distinct vertices

<u>Case 1.</u> Suppose v_0 is adjacent to the vertices v_1 and v_2, and that there are two copies of the edge connecting v_0 to v_1, with gain assignments m_a and m_b. Let the gain assignment of the edge connecting v_0 and v_2 be m_{02} Then the two candidates for edges to insert are:

$$\{v_1, v_2; m_{02} - m_a\}$$

$$\{v_1, v_2; m_{02} - m_b\}$$

Lemma 4.4.12 will prove that it is always possible to add one of these two candidate edges, while preserving properties (i) and (ii).

<u>Case 2.</u> Suppose v_0 is adjacent to vertices v_1, v_2, v_3. Suppose the edge connecting v_0 with v_i has gain assignment m_i. Then the three candidates for reverse edge split

are:

$$\{v_1, v_2; m_{02} - m_{01}\}$$

$$\{v_2, v_3; m_{03} - m_{02}\}$$

$$\{v_3, v_1; m_{01} - m_{03}\}$$

Our goal is to prove that, in both cases, there is always at least one edge that can be added while maintaining properties *(i)* and *(ii)*. In particular, we will consider subgraphs $G_{ij} \subset G$ where $v_i, v_j \in V_{ij}$, for $i, j \in \{1, 2, 3\}$. Such a subgraph could prevent the addition of the the edge $e = \{v_i, v_j; m_{0j} - m_{0i}\}$ for one of two reasons. Either the resulting graph would be over-counted (that is, $|E_{ij}| = 2|V_{ij}| - 2$ already), or adding the candidate edge would create a subgraph of G that did not have a constructive gain assignment. Lemmas 4.4.15, 4.4.16 and 4.4.17 will cover all of the possible cases, and demonstrate that it is always possible to add at least one of the candidate edges. $\square$

The remainder of this section builds up the necessary pieces for the proof of Proposition 4.4.8. Lemma 4.4.9 is a straightforward combinatorial result. We obtain several simple and useful corollaries (4.4.10 and 4.4.11). Finally, lemmas 4.4.12 – 4.4.17 cover all of the cases in the proof of Proposition 4.4.8.

Lemma 4.4.9 (Lattice Lemma)**.** *Let* $\langle G, m \rangle$ *be a periodic orbit graph satisfying* $|E| = 2|V| - 2$ *and* $|E'| \leq 2|V'| - 2$ *for all subgraphs* $G' \subseteq G$. *Let* v_0 *be some vertex*

of the graph. Let $\mathcal{G}$ *be the set of all subgraphs* $G' \subseteq G$ *that contain* v_0 *and satisfy*

$|E'| = 2|V'| - 2.$ *Then* $\mathcal{G}$ *is a lattice.*

Proof. Let $G_1, G_2 \in \mathcal{G}$. We know that

$$|E_1 \cup E_2| + |E_1 \cap E_2| \ = \ |E_1| + |E_2|$$

$$|E_1 \cup E_2| + |E_1 \cap E_2| \ = \ 2(|V_1| + |V_2|) - 4$$

$$|E_1 \cup E_2| + |E_1 \cap E_2| \ = \ (2|V_1 \cup V_2| - 2) + (2|V_1 \cap V_2| - 2) \qquad (4.1)$$

Since $V_1 \cap V_2 \neq \emptyset$, the graph $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ is a subgraph of G, and

therefore $|E_1 \cap E_2| \leq 2|V_1 \cap V_2| - 2$. Then (4.1) becomes

$$|E_1 \cup E_2| \ \geq \ 2|V_1 \cup V_2| - 2. \qquad (4.2)$$

Because $(V_1 \cap V_2, E(V_1 \cap V_2))$ is also a subgraph of G, it follows that

$$2|V_1 \cup V_2| - 2 \ \geq |E(V_1 \cup V_2)| \ \geq |E_1 \cup E_2|. \qquad (4.3)$$

(4.2) and (4.3) together imply that $|E_1 \cup E_2| = 2|V_1 \cup V_2| - 2$. Finally, it follows

that $|E_1 \cap E_2| = 2|V_1 \cap V_2| - 2$ too. □

Corollary 4.4.10. *There is a smallest (largest) subgraph* $G' \subseteq G$ $(G'' \subseteq G)$ *containing* v_0 *with* $|E'| = 2|V'| - 2$ $(|E''| = 2|V''| - 2)$.

Corollary 4.4.11. *Let* $\langle G, m \rangle$ *be a graph as in Lemma 4.4.9. The set* $\mathcal{G}''$ *of all*

subgraphs $G'' \subseteq G$ *with* $|E''| = 2|V''| - 2$ *and containing a finite number* $\{v_{i_1}, \ldots, v_{i_k}\}$

of vertices of G *is also a lattice.*

Lemma 4.4.12. *Let $\langle G, m \rangle$ be a periodic orbit graph satisfying (i) and (ii) of Proposition 4.4.8 where v_0 is a 3-valent vertex adjacent to vertices v_1 and v_2 only. After deleting v_0 it is always possible to add one of the edges*

$$\{v_1, v_2; m_{02} - m_{01}\}$$

$$\{v_1, v_2; m_{03} - m_{01}\}$$

so that the resulting graph also satisfies (i) and (ii) of Proposition 4.4.8.

Proof. First notice that we cannot have a subgraph $G' \subset G$ satisfying

1. $|E'| = 2|V'| - 2$

2. $v_0 \notin V'$

3. $v_1, v_2 \in V'$

since this would mean that after adding v_0 and its three adjacent edges, the resulting graph would be overcounted. Therefore, any subgraph G' containing v_1 and v_2 but not v_0 must satisfy $|E'| \leq 2|V'| - 3$.

We now address the question of whether it is possible that after adding one of the candidate edges, a subgraph G^* is created with $|E^*| = 2|V^*| - 2$ but that has no constructive cycles.

Select one of the candidate edges to add, say

$$e = \{v_1, v_2; m_{02} - m_{01}\}.$$

131

Let $\langle G_a, m_a \rangle \subset \langle G, m \rangle$ be a subgraph satisfying the following:

1. $|E_a| = 2|V_a| - 2$

2. $v_0 \notin V_a$

3. $v_1, v_2 \in V_a$ (and therefore $e \in E_a$)

4. all directed paths connecting v_1 to v_2 have net gain $m_{02} - m_{01}$.

Let $\langle G_b, m_b \rangle \subset \langle G, m \rangle$ be another subgraph satisfying 1 - 3, and also

4'. all directed paths connecting v_1 to v_2 (with the exception of the edge e) have net gain $m_{03} - m_{01}$.

Then $\langle G_a, m_a \rangle$ appears to be a subgraph of G with no constructive cycles. However, since G_a and G_b both have $2|V| - 2$ edges, the intersection of these graphs must also have $2|V| - 2$ edges. Therefore, there is at least one other edge in $E_a \cap E_b$ in addition to the edge e. In particular, there is some path from v_1 to v_2 that is distinct from e. Because this path is in G_a, it must have net gain $m_{02} - m_{01}$. But because the path is also in G_b, it must also have net gain $m_{03} - m_{01}$. This is only possible if $m_{03} = m_{02}$, which contradicts the fact that m is constructive.

Therefore, subgraphs $\langle G_a, m_a \rangle$ and $\langle G_b, m_b \rangle$ cannot both exist, and hence it is always possible to add one of the candidate edges. $\qquad\square$

The remainder of this section will be devoted to proving Case 2 of Proposition 4.4.8. Let v_0 be a three-valent vertex adjacent to the edges $(v_0, v_1; m_{01})$, $(v_0, v_2; m_{02})$ and $(v_0, v_3; m_{03})$.

Lemma 4.4.13. *Let $\langle G, m \rangle$ be a periodic orbit graph satisfying the hypotheses of Proposition 4.4.8. Let i, j, k be assigned distinct values from the set $\{1, 2, 3\}$. Let $G_{ij} \subset G$, be a subgraph satisfying:*

(a) $|E_{ij}| = 2|V_{ij}| - 3$

(b) $v_i, v_j \in V_{ij}$, and $v_k, v_0 \notin V_{ij}$

(c) G_{ij} contains no constructive cycles

Let G_{ik} be defined analogously. Then either

1. *$|V_{ij} \cap V_{ik}| = 1$ and $|E_{ij} \cup E_{ik}| = 2|V_{ij} \cup V_{ik}| - 4$ or*

2. *$|V_{ij} \cap V_{ik}| > 1$ and $|E_{ij} \cup E_{ik}| = 2|V_{ij} \cup V_{ik}| - 3$ and $|E_{ij} \cap E_{ik}| = 2|V_{ij} \cap V_{ik}| - 3$*

Proof.

$$
\begin{aligned}
|E_{ij} \cup E_{ik}| + |E_{ij} \cap E_{ik}| &= |E_{ij}| + |E_{ik}| \\
&= 2(|V_{ij}| + |V_{ik}|) - 6 \\
&= (2|V_{ij} \cup V_{ik}| - 3) + (2|V_{ij} \cap V_{ik}| - 3). \quad (4.4)
\end{aligned}
$$

We now have two cases. Either

Case 1. $|V_{ij} \cap V_{ik}| = 1$. In this case $|E_{ij} \cap E_{ik}| = 0$. From (4.4) we get

$$|E_{ij} \cap E_{ik}| = 2|V_{ij} \cap V_{ik}| - 4.$$

Case 2. $|V_{ij} \cap V_{ik}| > 1$. First note that we must have

$$|E_{ij} \cup E_{ik}| \leq 2|V_{ij} \cup V_{ik}| - 3 \qquad (4.5)$$

because this is a subgraph containing all three vertices v_1, v_2, v_3 but not v_0. We must also have

$$|E_{ij} \cap E_{ik}| \leq 2|V_{ij} \cap V_{ik}| - 3 \qquad (4.6)$$

because this is a subgraph of both G_{ij} and G_{ik}, neither of which possess a constructive cycle. Rewriting (4.4) we obtain:

$$|E_{ij} \cup E_{ik}| = (2|V_{ij} \cup V_{ik}| - 3) + \left\{ (2|V_{ij} \cap V_{ik}| - 3) - |E_{ij} \cap E_{ik}| \right\}. \qquad (4.7)$$

By (4.5) we obtain

$$\left\{ (2|V_{ij} \cap V_{ik}| - 3) - |E_{ij} \cap E_{ik}| \right\} \leq 0.$$

It follows that in fact

$$2|V_{ij} \cap V_{ik}| - 3 \leq |E_{ij} \cap E_{ik}|$$

which, together with (4.6) shows that we have equality in this case. The relationship

$$|E_{ij} \cup E_{ik}| = 2|V_{ij} \cup V_{ik}| - 3$$

134

then follows from (4.4). □

Corollary 4.4.14. *The graph $(V_{ij} \cap V_{ik}, E_{ij} \cap E_{ik})$ is connected.*

The following lemma shows that there is at most one choice of edges that will create a subgraph that fails combinatorially (that is, adding an edge would create a subgraph with $|E'| > 2|V'| - 2$). This result also follows from Fekete and Szegő [27].

Lemma 4.4.15. *Let $\langle G, m \rangle$ be a periodic orbit graph satisfying the hypotheses of Proposition 4.4.8. Then G has at most one subgraph $G' \subset G$ that satisfies:*

(a) $|E'| = 2|V'| - 2$,

(b) $v_0 \notin V'$, and

(c) V' contains at least two vertices from the set $\{v_1, v_2, v_3\}$.

Proof. First observe that there can be no subgraph $G' \subseteq G$ such that $v_1, v_2, v_3 \in V'$, $v_0 \notin V'$, and $|E'| = 2|V'| - 2$. Otherwise we could add the vertex v_0 with its three adjacent edges to obtain an over-counted subgraph of G.

Suppose, toward a contradiction that there are two subgraphs satisfying *(a)* - *(c)*. Without loss of generality, suppose these graphs are G_{12} and G_{23}, where $v_i, v_j \in V_{ij}$. Then both graphs are members of the lattice of subgraphs containing

135

the vertex V_2. By Lemma 4.4.9 it follows that

$$|E_{12} \cup E_{23}| = 2|V_{12} \cup V_{23}| - 2.$$

Hence $(V_{12} \cup V_{23}, E_{12} \cup E_{23})$ is a subgraph containing all three vertices v_1, v_2, v_3 but

not v_0, and by the preceding paragraph, this is a contradiction. $\qquad\square$

The following lemma states that if there is one choice of edge whose addition

would cause a combinatorial failure, then there is at most one choice of edge that

would cause a failure to have a constructive gain assignment.

Lemma 4.4.16. *Let $\langle G, m \rangle$ be a periodic orbit graph satisfying the hypotheses of*

Proposition 4.4.8, and that contains a subgraph $G' \subset G$ that satisfies (a) – (c)

of the previous lemma. Then there is at most one pair v_i, v_j of vertices from the

set $\{v_1, v_2, v_3\}$ and distinct from the pair contained in G' that are contained in a

subgraph $G'' \subset G$ satisfying:

(i) $v_0 \notin V''$,

(ii) $|E''| = 2|V''| - 3$, and

(iii) G'' contains no cycle with non-trivial net gain.

Proof. Let G_{ij} be the graph with $v_i, v_j \in V_{ij}$, but $v_k \notin V_{ij}$; $i, j, k \in \{1, 2, 3\}$.

Suppose, toward a contradiction, that there are two subgraphs satisfying *(i) – (iii)*.

136

Without loss of generality suppose that G_{23} and G_{31} are these subgraphs, and that G_{12} satisfies $|E_{12}| = 2|V_{12}| - 2$.

First consider the intersection of G_{12} with some subgraph G^* satisfying (i) – (iii). In this case,

$$
\begin{aligned}
|E^* \cup E_{12}| + |E^* \cap E_{12}| &= |E^*| + |E_{12}| \\
&= (2|V^*| - 3) + (2|V_{12}| - 2) \\
&= 2(|V^*| + |V_{12}|) - 5 \\
&= 2|V^* \cup V_{12}| - 2 + 2|V^* \cap V_{12}| - 3. \qquad (4.8)
\end{aligned}
$$

We now have two cases:

<u>Case A:</u> $|V^* \cap V_{12}| > 1$.

Then, because G^* satisfies property (iii), we have

$$
|E^* \cap E_{12}| \leq 2|V^* \cap V_{12}| - 3,
$$

and hence (4.8) becomes

$$
|E^* \cup E_{12}| \geq 2|V^* \cup V_{12}| - 2.
$$

In fact, since the reverse inequality always holds, we have equality

$$
|E^* \cup E_{12}| = 2|V^* \cup V_{12}| - 2.
$$

<u>Case B:</u> $|V^* \cap V_{12}| = 1$.

137

Then $|E^* \cap E_{12}| = 0$ and hence

$$|E^* \cup E_{12}| = 2|V^* \cup V_{12}| - 3.$$

Let $\overline{G} \subset G$ be the subgraph of G on the vertices $V_{23} \cup V_{31}$. Then $|\overline{E}| \geq |E_{23} \cup E_{31}|$, since there could be edges in $\overline{E}$ that were not part of either E_{23} or E_{31}, but that connect vertices in $V_{23} \cup V_{31}$. By Lemma 4.4.13, we know that either

Case 1. $|V_{23} \cap V_{31}| = 1$ and $|E_{23} \cup E_{31}| = 2|V_{23} \cup V_{31}| - 4$ or

Case 2. $|V_{23} \cap V_{31}| > 1$ and $|E_{23} \cup E_{31}| = 2|V_{23} \cup V_{31}| - 3$.

We will deal with *Case 2* first. In this case $|\overline{E}| = |E_{23} \cup E_{31}| = 2|V_{23} \cup V_{31}| - 3$. In other words, there can't be any edges in $\overline{E}$ that aren't also in $E_{23} \cup E_{31}$, otherwise the graph $(\overline{V}, \overline{E})$ would be over-counted. We now consider the intersection graph $\overline{G} \cap G_{12} = (\overline{V} \cap V_{12}, \overline{E} \cap E_{12})$. Note that $|\overline{V} \cap V_{12}| > 1$, since v_1 and v_2 are in both vertex sets. Hence by *Case A* above, $|\overline{E} \cup E_{12}| = 2|\overline{V} \cup V_{12}| - 2$. But this is a contradiction, because this graph contains all three of the vertices v_1, v_2, v_3, which means that adding v_0 will produce an over-counted subgraph.

We now return to *Case 1*. Here $|\overline{E}| \geq |E_{23} \cup E_{31}| = 2|V_{23} \cup V_{31}| - 4$. If $|\overline{E}| > |E_{23} \cup E_{31}|$, then we are in the situation of *Case 2*. Hence we may assume that $|\overline{E}| = |E_{23} \cup E_{31}|$.

Notice that the intersection of V_{12} with either of the other graphs V_{23} or V_{31} may only consist of one element, otherwise we have, by *Case A*, an over-counted subgraph. So the three subgraphs must intersect pair-wise in one of the vertices

v_1, v_2, v_3, and it follows that the intersection of all three of these subgraphs is empty.

$$\begin{aligned}
|\overline{E} \cup E_{12}| + |\overline{E} \cap E_{12}| &= |\overline{E}| + |E_{12}| \\
&= (2|\overline{V}| - 4) + (2|V_{12}| - 2) \\
&= 2(|\overline{V}| + |V_{12}|) - 6 \\
&= (2|\overline{V} \cup V_{12}| - 3) + (2|\overline{V} \cap V_{12}| - 3). \quad (4.9)
\end{aligned}$$

But we know that $|\overline{V} \cap V_{12}| = 2$, and it must be the case that $|\overline{E} \cap E_{12}| = 0$, since the intersection of the three graphs is empty. Hence equation (4.9) becomes

$$|\overline{E} \cup E_{12}| = 2|\overline{V} \cup V_{12}| - 2$$

which is a contradiction, since $v_1, v_2, v_3 \in \overline{V} \cup V_{12}$. Adding v_0 would violate the subgraph property of G. $\square$

This final lemma shows that if there are no bad choices of edges for combinatorial reasons, there is always at least one choice of edge that will produce a constructive gain assignment.

Lemma 4.4.17. *Let $\langle G, m \rangle$ be a periodic orbit graph satisfying the hypotheses of Proposition 4.4.8. Then there are at most two distinct pairs of vertices from the set $\{v_1, v_2, v_3\}$ that are contained in subgraphs $G' \subset G$ satisfying the following:*

(i) $v_0 \notin V'$

(ii) $|E'| = 2|V'| - 3$

(iii) G' *contains no cycle with non-trivial net gain*

(iv) *every path through G' connecting v_i with v_j has net gain $m_{0j} - m_{0i}$.*

In other words, there are at most two minimal subgraphs each containing a distinct pair of vertices from v_1, v_2, v_3, and having properties *(i) – (iv)*.

Proof. Toward a contradiction, suppose that there are three such graphs G_{12}, G_{23}, G_{31}, with $v_i, v_j \in V_{ij}$. It will be presently be shown that the union of these graphs, $G' \subset G$ will always satisfy:

(a) $|E'| = 2|V'| - 3$ and

(b) G' contains no cycle with non-trivial net gain

(c) every path through G' connecting v_i with v_j has net gain $m_j - m_i$.

We do this in two cases:

Case 1. $V_{ij} \cap V_{jk} = \{v_j\}$ for $j \in \{1, 2, 3\}$

In other words, each pair of subgraphs intersects in a single vertex. Here

$$
\begin{aligned}
|E_{12} \cup E_{23} \cup E_{31}| &= |E_{12}| + |E_{23}| + |E_{31}| \\
&= 2(|V_{12}| + |V_{23}| + |V_{31}|) - 9 \\
&= 2|V_{12} \cup V_{23} \cup V_{31}| - 3
\end{aligned}
$$

140

since

$$|V_{12} \cup V_{23} \cup V_{31}| \;=\; |V_{12}| + |V_{23}| + |V_{31}| - |V_{12} \cap V_{23}| - |V_{23} \cap V_{31}|$$

$$-|V_{31} \cap V_{12}| + 2|V_{12} \cap V_{23} \cap V_{31}|$$

$$=\; |V_{12}| + |V_{23}| + |V_{31}| - 3.$$

It is evident in this case that G' contains no non-trivial cycle, since any such cycle would pass through v_1, v_2 and v_3. This would contradict property *(iv)*. It is evident that G' satisfies (c) in this case.

<u>*Case 2.*</u> $|V_{ij} \cap V_{jk}| > 1$ for at least one $j \in \{1, 2, 3\}$.

By a repeated application of Lemma 4.4.13, we find that the union of these three graphs satisfies $|E'| = 2|V'| - 3$. (Let $G^* = G_{12} \cup G_{23}$. Assuming that $|V_{12} \cap V_{23}| > 1$, apply Lemma 4.4.13 to see that $|E^*| = 2|V^*| - 3$. Now it must be the case that $|V^* \cap V_{31}| > 1$ as well, since v_1, v_3 are in both vertex sets. Another application of Lemma 4.4.13 gives the result.) Note further that the intersection of G^* and G_{31} contains at least two vertices (v_1 and v_3), and satisfies $|E^* \cap E_{31}| = 2|V^* \cap V_{31}|$ by Lemma 4.4.13. Furthermore, this intersection is non-empty. Equivalently, the intersection $V_{12} \cap V_{23} \cap V_{31}$ is non-empty.

We now demonstrate that G' contains no non-trivial net gain. We assume that there is a non-trivial net gain in G', and we will obtain a contradiction to condition *(iii)*. We do this in two parts, first by showing that there are no constructive cycles

141
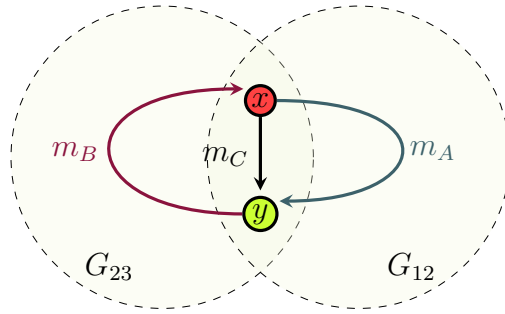
Figure 4.8: Two subgraphs satisfying *(i) – (iv)* of Lemma 4.4.17 whose intersection contains more than one vertex.

in the union of any pair of subgraphs (a), and next showing that there there are no constructive cycles in the union of all three (b).

Case 2a.   Suppose without loss of generality, that there is a non-trivial cycle in the graph $(V_{12} \cup V_{23}, E_{12} \cup E_{23})$. Suppose that $|V_{12} \cap V_{23}| > 1$, and that the non-trivial cycle passes through vertices x and y, where $x, y \in V_{12} \cap V_{23}$. See Figure 4.8.

Let the non-trivial cycle through x and y be broken into two parts $xAyBx$, where A and B are paths through G_{12} and G_{23} respectively that make up the non-trivial cycle. Let m_A and m_B be the net gain of paths A and B respectively. Then $m_A + m_B \neq 0$ by assumption. By Corollary 4.4.14, the graph $(V_{12} \cap V_{23}, E_{12} \cap E_{23})$ is connected. Hence there exists a path through this graph that connects x with y. Let the net gain of this path be m_C. Then

$$m_A - m_C = 0$$

142
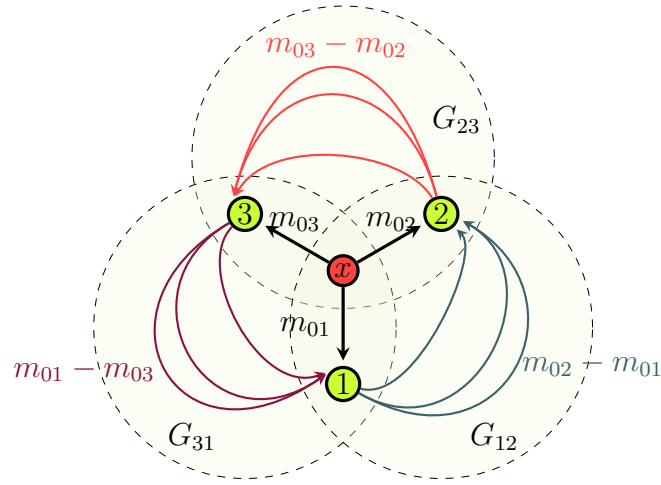
Figure 4.9: Three subgraphs satisfying *(i)* – *(iv)* of Lemma 4.4.17 that intersect in a vertex x

$$\Rightarrow m_A = m_C$$

$$\Rightarrow m_C + m_B \neq 0.$$

But this is a non-trivial net gain in G_{23}, a contradiction.

Case 2b. Now assume that there is a non-trivial cycle in the subgraph of G on the vertices $V_{12} \cup V_{23} \cup V_{31}$. See Figure 4.9. By a similar argument to the previous case, suppose that the non-trivial cycle is written as the sum of three paths, one through each of the graphs. That is, let $x_1 \in V_{31} \cap V_{12}$, $x_2 \in V_{12} \cap V_{23}$, and $x_3 \in V_{23} \cap V_{31}$. If any of the vertices x_1, x_2, x_3 is in the intersection of all three graphs, then we are in the situation described above. So we assume that this is not the case. Let the nontrivial cycle be written $x_1 A x_2 B x_3 C x_1$, where $A \in G_{12}, B \in G_{23}, C \in G_{31}$. Let these paths have cycle gains m_A, m_B, m_C respectively, and our assumption is that

$$m_A + m_B + m_C \neq 0.$$

Let $x \in V_{12} \cap V_{23} \cap V_{31}$. Each intersection $V_{ij} \cap V_{jk}$ is connected, hence for the vertex $x_i \in V_{ki} \cap V_{ij}$ there is a path connecting x to x_i. Let this path have net gain m_i. Similarly we have paths connecting vertices x_j and x_k respectively to the vertex x. Then

$$m_A - m_{01} + m_{03} = 0$$

$$m_B - m_{02} + m_{01} = 0$$

$$m_C - m_{03} + m_{02} = 0.$$

But summing these three expressions gives $m_A + m_B + m_C = 0$, which contradicts our assumption. As in the previous case, the union of the three graphs can have no non-trivial cycle.

To see that G' also satisfies property (c), we consider without loss of generality, all paths P from v_1 to v_2 through G'. If each vertex of the path is in V_{12} then it has net gain $m_{02} - m_{01}$ by hypothesis. If some vertex in P is not in V_{12}, then suppose P has net gain m_P. Then $m_P - (m_{02} - m_{01}) = 0$, since G' has no trivial cycles, by (b). Hence $m_P = m_{02} - m_{01}$, as desired.

In both Case 1 and Case 2, we have a subgraph $G' \subset G$ that contains v_1, v_2, v_3 but not v_0, and satisfies $|E'| = 2|V'| - 3$. Furthermore, this graph contains no cycle with non-trivial net gain, and all paths connecting v_i to v_j have net gain $m_{0j} - m_{0i}$. Let $V_0 = V' \cup \{v_0\}$, and consider the graph $G_0 = (V_0, E_0)$. E_0 will be E' augmented

by the three edges connecting v_0 with v_1, v_2, v_3. Then $|E_0| = 2|V_0| - 2$, and hence this graph must be constructive. But we know that G' contains no cycle with non-trivial net gain, which means that the non-trivial net gain in G_0 must pass through v_0. Hence it must contain two of the edges adjacent to v_0. But any such cycle will have net gain zero, a contradiction. $\qquad\square$

The proofs of these technical results prove Theorem 4.4.5, which in turn establishes the Periodic Laman Theorem (Theorem 4.4.4) and completes the proof of the summary theorem, Theorem 4.2.1.

4.5 Higher dimensions

4.5.1 Inductive constructions on d-dimensional frameworks

The key feature of the inductive constructions presented in this chapter was that they preserved the rank of the rigidity matrix for frameworks on $\mathcal{T}_0^2$. We have recorded a rigidity matrix for frameworks on the d-dimensional fixed torus, and we can modify our proofs of Propositions 4.3.1 and 4.3.4 to prove similar facts for inductive constructions in this higher dimensional setting. For consistency and convenience with subsequent sections, the proofs of Propositions 4.3.1, 4.3.4 and 4.3.5 were presented in the two dimensional case, but the same techniques apply in higher dimensions.

The Periodic Henneberg Theorem (Theorem 4.3.8) *does not* generalize to three

or more dimensions, nor did its finite version (Theorem 2.5.13). When $n = 3$ for

example, we can only establish the existence of vertices of valence 5 or less, through

an argument similar to that in the proof of Theorem 4.3.8. The 3-dimensional

versions of vertex addition and edge split will generate 3- and 4-valent vertices

only.

There are other inductive constructions which preserve generic rigidity of finite

frameworks, namely X-replacement in the plane and vertex splitting in all dimen-

sions. X-replacement allows us to produce 4-valent vertices, while vertex-splitting

may be used to produce vertices of higher valence, in several different ways. The

definition of finite versions of these moves can be found in [74], and periodic adapted

versions appear to be straightforward extensions.

The Periodic Laman Theorem (Theorem 4.4.5) also does not generalize, as we

shall soon see, which simply rests on the fact that the finite Laman Theorem (The-

orem 2.5.10) did not extend to three or more dimensions.

4.5.2 Necessary conditions for infinitesimal rigidity on $\mathcal{T}_0^d$

What is a constructive gain assignment on an orbit graph $\langle G, m \rangle$, where the gain

group is $\mathbb{Z}^d$? To answer this question, we need to establish the necessary conditions

on the gains of a periodic orbit graph $\langle G, m \rangle$ for it to be infinitesimally rigid on $\mathcal{T}_0^d$.

Here is a preliminary necessary condition for infinitesimal rigidity on $\mathcal{T}_0^d$. Recall that for a gain graph $\langle G, m \rangle$ with cycle space $\mathcal{C}(G)$, the gain space $\mathcal{M}_{\mathcal{C}}(G)$ is the vector space (over $\mathbb{Z}$) spanned by the net gains on the cycles of $\mathcal{C}(G)$.

Theorem 4.5.1. *Let $\langle G, m \rangle$ be a d-periodic orbit graph with $|E| = d|V| - d$. If $(\langle G, m \rangle, p)$ is infinitesimally rigid for some realization p, then every subgraph $G' \subseteq G$ with $|E'| = d|V'| - d$ has $|\mathcal{M}_{\mathcal{C}}(G')| \geq d - 1$.*

Proof. Suppose $G' \subseteq G$ has $|\mathcal{M}_{\mathcal{C}}(G')| = k$, where $k < d - 1$. Performing the T-gain procedure if necessary, the gains of the edges of G' are zero on at least two coordinates, say x and y. The basic idea of this proof is that such a framework is disconnected in the xy-plane, and we can apply a rotation in this plane. Suppose without loss of generality that all edges of $\langle G, m \rangle$ have gains $m_e = (0, 0, m_{e3}, \ldots, m_{ed}) \in \mathbb{Z}^d$. Let $p = (p_1, p_2, \ldots, p_d)$ be a point in $\mathcal{T}_0^d$. Let $v = (-p_2, p_1, 0, \ldots, 0)$. Then

$$v \cdot (p_i - (p_j + m_e L_0)) = (-p_2, p_1, 0, \ldots, 0) \cdot (p_{i1} - p_{j1}, p_{i2} - p_{j2}, \ldots)$$

which is a rotation in the plane of the first two coordinates, of a finite (i.e. not periodic) framework. This corresponds to a non-trivial motion of $(\langle G, m \rangle, p)$, since it represents a rotation within the unit cell. $\square$

As motivation for the next result, consider an infinitesimally rigid framework $(\langle G, m \rangle, p)$ on the 3-dimensional fixed torus $\mathcal{T}_0^3$ with $|E| = 3|V| - 3$. The edges of E are therefore independent. By Theorem 4.5.1, every fully-counted subgraph $G' \subseteq G$

satisfying $|E'| = 3|V'| - 3$ has $|\mathcal{M}_{\mathcal{C}}(G')| \geq 2$. On the other hand, by Proposition 3.3.17, any set of edges $E'' \subset E$ with $E'' > 3|V''| - 6$ and $|\mathcal{M}_{\mathcal{C}}(E'')| = 0$ is dependent. Therefore, there must be additional conditions on subsets of edges $E'' \subset E$ with $|E''| = 3|V''| - 5$ and $|E''| = 3|V''| - 4$. The following theorem provides necessary conditions on these *intermediate* subsets of edges.

Theorem 4.5.2. *Let $\langle G, m \rangle$ be a minimally rigid framework on $\mathcal{T}_0^d$. Then for all subsets of edges $Y \subseteq E$,*

$$|Y| \leq d|V(Y)| - \binom{d+1}{2} + \sum_{i=1}^{|\mathcal{M}_{\mathcal{C}}(Y)|} (d-i). \tag{4.10}$$

In essence this says that we can add edges beyond what would normally be independent, provided that we also add cycles with non-trivial gains. Maxwell's condition for finite frameworks in dimension d (Theorem 2.5.9) says that an isostatic framework must satisfy $|E| = d|V| - \binom{d+1}{2}$, and $|E'| \leq d|V'| - \binom{d+1}{2}$ for all induced subgraphs $G' \subseteq G$. Analogously, a minimally rigid periodic framework in dimension d will have $|E| = d|V| - d$ and induced subgraphs will satisfy $|E'| \leq d|V'| - d$ (Corollary 3.3.13).

In addition, we already showed that for a minimally rigid framework $\langle G, m \rangle$ on $\mathcal{T}_0^d$:

(a) all induced subgraphs with $|E'| = d|V'| - d$ must have $|\mathcal{M}_\mathcal{C}(G')| \geq d - 1$

(Theorem 4.5.1)

(b) any connected subset of edges $Y \subset E$ with $|Y| > d|V(Y)| - \binom{d+1}{2}$ must have

$|\mathcal{M}_\mathcal{C}(Y)| > 0$. (Proposition 3.3.17)

Theorem 4.5.2 extends these results. We make use of the following simple fact:

$$\textbf{Fact:} \quad \binom{d}{2} - \sum_{i=1}^{k}(d - i) = \binom{d - k}{2}. \tag{4.11}$$

Proof of the Fact.

$$
\begin{aligned}
\binom{d}{2} - \sum_{i=1}^{k}(d - i) &= \binom{d}{2} - \left(kd - \sum_{i=1}^{k} i\right) \\
&= \binom{d}{2} - \left(kd - \binom{k + 1}{2}\right) \\
&= \frac{d(d - 1)}{2} - \left(\frac{2kd - k^2 - k}{2}\right) \\
&= \frac{(d - k)(d - k - 1)}{2} \\
&= \binom{d - k}{2}.
\end{aligned}
$$

$\square$

Proof of Theorem 4.5.2. Let $\langle G, m \rangle$ be generically minimally rigid on $\mathcal{T}_0^d$, and let

$Y \subseteq E$ be a subset of edges. First note that for any subset Y with $|Y| \leq d|V| - \binom{d+1}{2}$,

Equation (4.10) holds trivially. If $|Y| = d|V(Y)| - d$, then the edges of Y are the

edges of an induced subgraph, and we must have $|\mathcal{M}_\mathcal{C}(Y)| \geq d - 1$ by (a).

149

Suppose then that $|Y| = d|V(Y)| - \binom{d+1}{2} + \ell$, where $0 < \ell < \binom{d}{2}$. Then for some

$0 < k < d - 2$,

$$\sum_{i=1}^{k}(d-i) \leq \ell < \sum_{i=1}^{k+1}(d-i).$$

Toward a contradiction, suppose that $|\mathcal{M}_{\mathcal{C}}(Y)| < k$. We apply the T-gain procedure

to the edges Y, and we obtain gains that are 0 on more than $d - k$ coordinates.

By the arguments of the proof of Theorem 4.5.1 for each pair zero coordinates, we

can obtain a rotation in that plane. Therefore the space of non-trivial infinitesimal

motions of the subset Y on $\mathcal{T}_0^d$ is *strictly* larger than $\binom{d-k}{2}$. Letting $\mathcal{I}_k(Y)$ denote

the space of non-trivial infinitesimal motions of the subset Y, we have shown that

$$|\mathcal{I}_k(Y)| > \binom{d-k}{2}.$$

However, since $|Y| < d|V(Y)| - d$, we expect some non-trivial infinitesimal

motions of the edges Y on $\mathcal{T}_0^d$. Since $\langle G, m \rangle$ is generically rigid, these motions

will disappear when more edges are added to the subset Y. How many non-trivial

infinitesimal motions would we expect? An isostatic finite framework with $|E| = $

$d|V| - \binom{d+1}{2}$ has $\binom{d}{2}$ non-trivial infinitesimal motions when realized as a periodic

orbit framework. Let $\mathcal{I}(Y)$ denote the space of non-trivial infinitesimal motions

we predict based only on the number of edges. Since $\langle G, m \rangle$ is minimally rigid,

and $|Y| = d|V(Y)| - \binom{d+1}{2} + \ell$, the space of non-trivial infinitesimal motions has
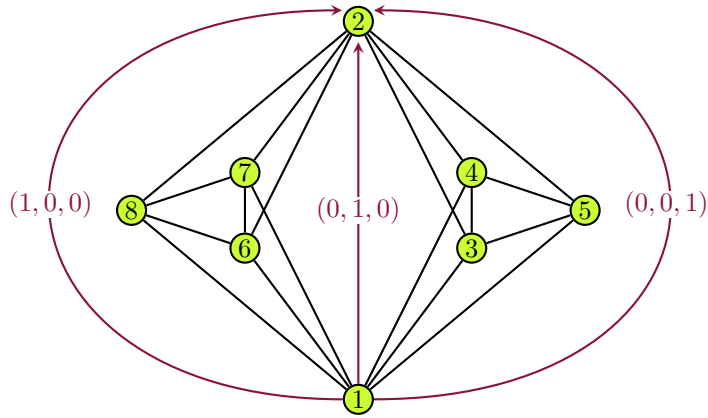
Figure 4.10: An example of a generically flexible periodic orbit graph on $\mathcal{T}_0^3$ with a constructive gain assignment. The black edges form the $3|V|-6$ "double bananas" graph, and here we give them gain $(0,0,0)$. The three coloured edges provide the constructive gains. This graph is flexible on $\mathcal{T}_0^3$.

dimension $|\mathcal{I}(Y)| = \binom{d}{2} - \ell$. Now

$$
\begin{aligned}
|\mathcal{I}(Y)| &= \binom{d}{2} - \ell \\
&\leq \binom{d}{2} - \sum_{i=1}^{k}(d-i) \\
&= \binom{d-k}{2} \quad \text{by (4.11)} \\
&< |\mathcal{I}_k(Y)|.
\end{aligned}
$$

Hence the space of non-trivial infinitesimal motions we expect based on the deficit of edges is smaller than the space predicted by the deficit in the dimension of $\mathcal{M}_{\mathcal{C}}(Y)$, which is our contradiction. $\qquad \square$

4.5.3 Constructive gain assignments for d-periodic orbit frameworks

We say that $\langle G, m \rangle$ has a *constructive gain assignment* if the gain assignment

151

m of $\langle G, m \rangle$ is such that (4.10) is satisfied for every subset Y of edges of $\langle G, m \rangle$. When $d = 2$ this is equivalent to our earlier definition, and we have seen that constructive gain assignments are sufficient for generic minimal rigidity in this case. Unfortunately, the same is not true in higher dimensions. For example, when $d = 3$, we can realize the "double banana" graph as part of a $3|V| - 3$ graph with a constructive gain assignment, as seen in Figure 4.10. This graph is flexible despite having a constructive gain assignment. The two "bananas" consisting of all the edges without gains can be rotated independently about the line through vertices 1 and 2.

There are, however, gain assignments on the edges of this graph that will produce infinitesimally rigid frameworks on $\mathcal{T}_0^3$. Such frameworks will involve the "wrapping" of some of the edges of the bananas around the torus. For example, one possible gain assignment is given by the proof of Theorem 3.3.34 (see Section 3.3.11), in which the edges of each of the 3 edge-disjoint spanning trees are assigned the gains $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ respectively. A similar idea was mentioned in a recent talk of Borcea and Streinu [6]. Notice also that the particular gain assignment produced in the proof of Whiteley's theorem 3.3.34 is constructive.

5 Frameworks on the flexible torus $\mathcal{T}_k^d$

5.1 Introduction

The results of Chapter 4 for frameworks on the 2-dimensional fixed torus lead to natural questions about similar frameworks on the flexible 2-dimensional torus. In the flexible case, we allow the generators of the torus (entries of the lattice matrix) to vary continuously with time, and we consider multiple variations of the problem by selecting how 'flexible' we want the torus to be. A characterization of generically rigid graphs on the 2-dimensional flexible torus has appeared in Malestein and Theran [49]. They note that it is possible to specialize their results to obtain the material presented in Chapter 4. However, they do not use an inductive characterization or Henneberg-type theorem, which we believe is interesting in its own right. In particular, inductive techniques are likely tools for the discussion of global periodic rigidity (see Chapter 8).

In this chapter, we describe some contributions to the study of frameworks on the flexible torus $\mathcal{T}^d$, and the partially flexible torus $\mathcal{T}_k^d$. This work was summarized

in a talk at Lancaster in July 2010 [57]. Borcea and Streinu [7] independently outlined an algebraic theory for such frameworks in d-dimensions, with generic gains. We consider frameworks on the partially flexible torus $\mathcal{T}_k^d$, and record a different rigidity matrix from the Borcea-Streinu presentation. We highlight the connections with their work, where appropriate.

In the 2-dimensional case, Malestein and Theran recently characterized the necessary and sufficient conditions for generic rigidity of a periodic orbit graph on the flexible torus $\mathcal{T}^2$, where graphs have $|E| \geq 2|V| + 1$. Here we consider an intermediate case, namely 2-dimensional frameworks satisfying $|E| = 2|V| - 1$ on an appropriately flexible torus. This intermediate case has some importance in its own right, as it applies to frameworks on cylinders (2-dimensional frameworks that are periodic in one direction only).

As both a summary of frameworks on the fixed torus and an introduction to frameworks on the flexible torus, we open this chapter with a discussion of periodic frameworks on the line (1-periodic frameworks). We will use the results from the line later in the chapter.

The outline of the rest of the chapter is as follows: following the discussion of frameworks on the line (Section 5.2) we introduce frameworks on the flexible torus (Section 5.3). In that section we consider many of the same issues described in Chapter 3, namely motions and infinitesimal motions of frameworks, the d-

dimensional rigidity matrix, and we show that the T-gain procedure preserves independence in this new setting.

Section 5.4 outlines some necessary conditions for rigidity on a d-dimensional flexible torus. In Section 5.5 we elaborate on the necessary conditions in Section 5.4 for the particular case of frameworks on $\mathcal{T}_x^2$, the 2-dimensional torus allowed to scale in the x-direction only, and we use techniques from algebraic geometry to show that these conditions are in fact sufficient. We conclude in Section 5.6 by showing that these results for $\mathcal{T}_x^2$ actually apply to frameworks on a flexible cylinder, which correspond to frameworks which are periodic in one direction only.

5.2   1-dimensional periodic frameworks

The basic ideas of the rigidity of finite graphs on the line can be found in [83] or [31]. The key result is that a graph G is rigid as a 1-dimensional framework if and only if it is connected. We no longer have a genericity requirement (except to avoid edges of length zero). It follows then that rigidity and infinitesimal rigidity are the same for all one-dimensional frameworks.

1-dimensional periodic frameworks are infinite frameworks on the line with periodic structure (translational symmetry). Just as we map 2-periodic frameworks onto the torus, we may view 1-periodic frameworks as graphs on a circle. Such graphs may be on a circle of fixed circumference x (*the fixed circle*), in analogy

with Chapter 4, or they may be on a circle that is allowed to change circumference $x(t)$ (*the flexible circle*). We denote the fixed circle by $\mathcal{T}_0^1$, and the flexible circle by $\mathcal{T}^1$.

In either case, a 1-periodic orbit framework is the pair $(\langle G, m \rangle, p)$, with $m : E \to \mathbb{Z}$, and $p : V \to [0, x)$, where x is either a fixed element of $\mathbb{R}$ for the fixed circle, or $x = x(t)$ is a continuous function of time for the flexible circle. We assume further that p maps the endpoints of any edge to distinct locations in $[0, x)$, thereby avoiding edges of length zero.

5.2.1 Frameworks on the fixed circle $\mathcal{T}_0^1$

Consider a periodic framework on the line, where the size of the fundamental region remains fixed. Equivalently, this can be represented as a periodic orbit framework $(\langle G, m \rangle, p)$ on the fixed circle (see Figure 5.1). The conditions for a graph to be infinitesimally rigid in this case are equivalent to the conditions that a graph be infinitesimally rigid on the regular line (not periodic). In particular, we need the graph to be connected. Hence $|E| = |V| - 1$ and G is a tree.

For consistency with our previous notation, let L_0 be the 1×1 matrix $[x]$, where the length of the fundamental region (equivalently circumference of the fixed circle)

156

$$|E| = |V| - 1$$
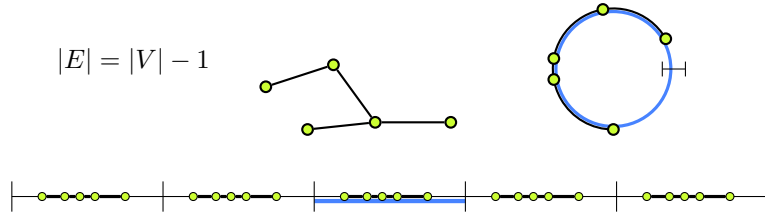
Figure 5.1: A periodic 1-dimensional framework on the line can be realized as a framework on the circle with fixed circumference. The framework is rigid if and only if $\langle G, m \rangle$ is connected (no non-zero gains are required).

is x. The $|E| \times |V|$ rigidity matrix in this case will have rows:

$$
\{i,j;m\}
\begin{array}{c}
 \\
\end{array}
\begin{pmatrix}
& & \vdots & & \\
0 \cdots 0 & p_i - (p_j + mL) & 0 \cdots 0 & (p_j + mL) - p_i & 0 \cdots 0 \\
& & \vdots & & \\
\end{pmatrix},
$$

where $p_i, p_j \in \mathbb{R}$, and $m \in \mathbb{Z}$. Since there is always a 1-dimensional space of trivial infinitesimal motions generated by translation along the line (generated by the vector $(1, \ldots, 1)^T$), the rigidity matrix has maximum rank $|V| - 1$.

Proposition 5.2.1. *The periodic orbit framework $(\langle G, m \rangle, p)$ is (infinitesimally) rigid on $\mathcal{T}_0^1$ if and only if G is connected.*

In other words, $\langle G, m \rangle$ is infinitesimally rigid on $\mathcal{T}_0^1$ if and only if the graph G is infinitesimally rigid as a finite framework (i.e. not as a periodic framework).

Corollary 5.2.2. *The periodic orbit framework $(\langle G, m \rangle, p)$ is (infinitesimally) rigid on $\mathcal{T}_0^1$ if and only if G is infinitesimally rigid in $\mathbb{R}$.*

157

$|E| = |V|$
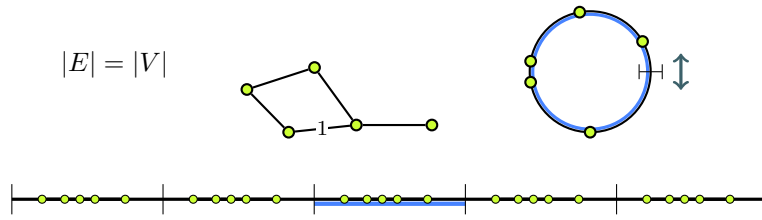
Figure 5.2: A framework on the flexible circle. Clearly $\langle G, m \rangle$ must be connected, and furthermore it must contain a constructive cycle.

(a) (b) (c)
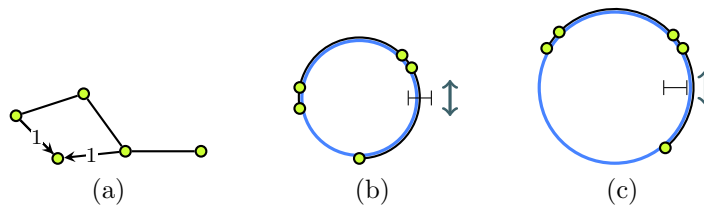
Figure 5.3: A gain graph with non-zero gains but no constructive cycle (a), realized on the variable circle (b). This is flexible (c).

5.2.2 Frameworks on the flexible circle $\mathcal{T}^1$

If we allow the radius of the circle to change size, in addition to connectivity, we now require the graph to "wrap" in a non-trivial fashion around the circle (see Figure 5.2). That is, $\langle G, m \rangle$ must contain a constructive cycle. The framework pictured in Figure 5.3 is a framework which has non-zero gains, but no constructive cycle. This framework is flexible on the variable circle.

One way to see the necessity of a constructive cycle is to perform the T-gain procedure on the edges of a periodic orbit graph $\langle G, m \rangle$ with $|E| = |V|$. All but one edge will have gain $m_T(e) = 0$. Since at most $|V| - 1$ edges can be independent on the fixed circle, at most $|V| - 1$ edges with zero gains can be independent on

158

T^1. It follows that the non-tree edge must have a non-zero gain. Since the non-tree edge represents the net gain on the single cycle in G, we conclude that it is a constructive cycle. So the necessary conditions for rigidity here are $|E| = |V|$ and G contains a constructive cycle (i.e. a cycle with non-zero net gain). The $|E| \times (|V| + 1)$ rigidity matrix in this case has an extra column corresponding to the changing circumference of the circle (as represented by the 1×1 lattice matrix $L = [x(t)]$):

$$
\begin{pmatrix}
 & & \vdots & & & & \vdots & \\
0 \cdots 0 & p_i - (p_j + mL) & 0 \cdots 0 & (p_j + mL) - p_i & 0 \cdots 0 & m(p_i - (p_j + mL)) \\
 & & \vdots & & & & \vdots &
\end{pmatrix},
$$

with a row for every edge $\{i, j; m\}$ in $E\langle G, m \rangle$, where $p_i, p_j \in \mathbb{R}$, and $m \in \mathbb{Z}$. The space of trivial motions of $(\langle G, m \rangle, p)$ on $\mathcal{T}^1$ is generated by the vector $(1, 1, \ldots, 1, 0)^T$. That is, it is an isomorphic space to the space trivial motions on the fixed circle $\mathcal{T}_0^1$. The maximum rank of the rigidity matrix is thus $|V|$.

Proposition 5.2.3. *The periodic orbit framework $(\langle G, m \rangle, p)$ is (infinitesimally) rigid on $\mathcal{T}^1$ if and only if G is connected, and G contains a constructive cycle.*

In this way, when we move from the fixed circle to the flexible one, we add a layer of complexity. Frameworks on the fixed circle have the same requirements for rigidity as finite (not periodic) frameworks on the line. On the other hand,

159

frameworks on the flexible circle require constructive cycles – a non-trivial wrapping

of the edges of the graph around the circle – to fix the circumference of the circle.

Since constructive cycles were also a requirement for 2-dimensional rigidity on the

fixed torus, we can anticipate that when we allow the 2-dimensional torus to change

size and shape, additional requirements will be placed on the cycles.

5.3 Rigidity and infinitesimal rigidity on the d-dimensional flexible torus

5.3.1 The flexible torus $\mathcal{T}_k^d$

The d-dimensional flexible torus $\mathcal{T}^d$ can be obtained from the d-dimensional fixed

torus $\mathcal{T}_0^d$ by allowing the generators to vary continuously with time. That is, as for

the fixed torus, we write a $d \times d$ lattice matrix L, where the rows are the generators

of $\mathcal{T}^d$. $L = L(t)$ is the lower-triangular matrix

$$L(t) = \begin{pmatrix} t_{11}(t) & 0 & 0 & \dots & 0 \\ t_{12}(t) & t_{22}(t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & 0 \\ t_{d1}(t) & t_{d2}(t) & t_{d3}(t) & \dots & t_{dd}(t) \end{pmatrix},$$

where $t_{\ell r}(t)$ are continuous functions of time. There are $\binom{d+1}{2}$ variable entries of

$L(t)$. We denote the initial position of $L(t)$ by L.

160

In this chapter we will also be interested in the study of tori with partial flexibility. That is, we consider tori generated by the rows of the matrix $L_k(t)$, where not all of the $\binom{d+1}{2}$ non-zero entries shown above are variable. We denote the d-dimensional torus with k degrees of freedom by $\mathcal{T}_k^d$, $0 \leq k \leq \binom{d+1}{2}$. Of course, which k pieces of information (equivalently variable entries in the matrix $L_k(t)$) is not specified by this notation, and different choices will yield different necessary conditions for rigidity (see Example 5.3.1 below). However, here we will focus on the common elements of $\mathcal{T}_k^d$. The case $k = 0$ is the fixed torus $\mathcal{T}_0^d$, which was considered in Chapters 3 and 4, and the case $k = \binom{d+1}{2}$ is the fully flexible torus, which we denote $\mathcal{T}^d$ ($k = \binom{d+1}{2}$ is implied). Both of these extremes are uniquely determined. We use the convention that the fully flexible torus, with $\binom{d+1}{2}$ degrees of freedom, does not have a subscript, nor does its corresponding lattice matrix.

Example 5.3.1. In two dimensions, we describe a few special cases of the flexible torus. Let $\mathcal{T}_3^2 = \mathcal{T}^2$ be the fully flexible torus with three degrees of freedom obtained by allowing full motion of the generators (scaling in the x and y directions, and a variable angle between them). Let $\mathcal{T}^2$ be generated by the vectors $x = (x(t), 0)$ and $y = (y_1(t), y_2(t))$ (i.e. $x(t) = t_{11}(t)$, $y_1(t) = t_{12}(t)$ and $y_2(t) = t_{22}(t)$). Then the lattice matrix has the form:

$$L_3 = \begin{pmatrix} x(t) & 0 \\ y_1(t) & y_2(t) \end{pmatrix}.$$

Let $\mathcal{T}_x^2$ be the flexible torus with one degree of freedom obtained by allowing scaling in the x direction:

$$L_x = \begin{pmatrix} x(t) & 0 \\ y_1 & y_2 \end{pmatrix}, \quad y_1, y_2 \in \mathbb{R}.$$

We will call this the *x-scaling torus*. It is the subject of Section 5.5.

Let $\mathcal{T}_2^2$ be the flexible torus with two degrees of freedom obtained by allowing scaling in both the x and y directions, but fixing the angle between the generators:

$$L_2 = \begin{pmatrix} x(t) & 0 \\ \frac{y_2(t)}{C} & y_2(t) \end{pmatrix},$$

for some constant $C \in \mathbb{R}$. As for the fixed torus, the rigidity of frameworks on the flexible torus (in all variations) is invariant under affine transformations, so we eventually assume that the angle between the generators is $\pi/2$. This permits us to write, without loss of generality,

$$L_2 = \begin{pmatrix} x(t) & 0 \\ 0 & y_2(t) \end{pmatrix}.$$

We call the torus $\mathcal{T}_2^2$ generated by L_2 the *scaling torus*, and address the d-dimensional analogue $\mathcal{T}_d^d$ in Section 5.4.

Alternatively we could consider $\mathcal{T}_2^2$ as the torus generated by allowing $x(t)$, $y_1(t)$ and $y_2(t)$ vary freely, but requiring that the area of the torus remain fixed. This forces the relationship $x(t)y_2(t) = C$, for some $C \in \mathbb{R}$. The lattice matrix then has

162

the form

$$L_2 = \begin{pmatrix} x(t) & 0 \\ y_1(t) & \frac{C}{x(t)} \end{pmatrix}.$$

Some modifications of the corresponding rigidity matrix are required to handle this case, and we do not address this in this chapter. More generally, we do not consider algebraic relationships among the generators of the torus. □

5.3.2 Frameworks on the flexible torus $\mathcal{T}_k^d$

Since our definition of periodic orbit framework does not depend on whether or not the torus is flexible, a periodic orbit framework $(\langle G, m \rangle, p)$ on the d-dimensional flexible torus $\mathcal{T}_k^d$ is defined as in Chapter 3 for the fixed torus. The notions of generic outlined also transfer to this setting, by replacing the fixed torus rigidity matrix with the rigidity matrix for the flexible torus, which we shall soon define. In the next sections, we outline the definition of motions and infinitesimal motions on $\mathcal{T}_k^d$, which does depend on the flexibility of the torus. This leads to the definition of the flexible torus rigidity matrix. We also extend a few key results from Chapter 3, namely the affine invariance of rigidity on the flexible torus, and that the T-gain procedure also preserves the rank of the flexible torus rigidity matrix. We also relate our definition with that of Borcea and Streinu for $\mathcal{T}^d$ in [7], but note that the work presented here is independent of their presentation.

The periodic orbit frameworks on the partially flexible torus correspond to periodic frameworks in $\mathbb{R}^d$. In particular, an orbit framework on $\mathcal{T}_k^d = \mathbb{R}^d / L_k(t)\mathbb{Z}^d$ corresponds to the periodic framework $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$, where L_k is the lattice matrix with k degrees of freedom. When $k = \binom{d+1}{2}$ (and the torus is fully flexible, modulo rotation), we write $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ to denote the corresponding periodic framework in $\mathbb{R}^d$. We summarize our notation in Table 5.1.

5.3.3   Motions of frameworks on the flexible torus $\mathcal{T}_k^d$

For completeness, we define here continuous rigidity on the fully flexible torus $\mathcal{T}^d$. We will remark on how to specialize this to the partially flexible torus $\mathcal{T}_k^d$ following the definition on $\mathcal{T}^d$.

Let $(\langle G, m \rangle, p)$ be a periodic orbit framework with $m : E \to \mathbb{Z}^d$ and $p : V \to \mathcal{T}^d$, where $V = \{v_1, v_2, \dots, v_n\}$. Let (P_i, T) be a pair where

- P_i is an indexed family of functions,

$$P_i : [0, 1] \to \mathbb{R}^d, i = 1, \dots, |V|;$$

- and T is a matrix function

$$T : [0, 1] \to GL(d),$$

164

Table 5.1: Summary of notations and frameworks

	Fixed	*Partially flexible*	*flexible*
Lattice matrix	L_0	$L_k(t)$ $L_k(0) = L_k$	$L(t)$ $L(0) = L$
Variable entries in lattice matrix	0	$0 \leq k \leq \binom{d+1}{2}$	$\binom{d+1}{2}$
Torus	$\mathcal{T}_0^d$	$\mathcal{T}_k^d$	$\mathcal{T}^d$
Periodic orbit framework	$(\langle G, m \rangle, p)$	$(\langle G, m \rangle, p)$	$(\langle G, m \rangle, p)$
Periodic framework (in $\mathbb{R}^d$)	$(\langle \widetilde{G}, L_0 \rangle, \widetilde{p}) =$ $(\langle G^m, L_0 \rangle, p^m)$	$(\langle \widetilde{G}, L_k \rangle, \widetilde{p}) =$ $(\langle G^m, L_k \rangle, p^m)$	$(\langle \widetilde{G}, L \rangle, \widetilde{p}) =$ $(\langle G^m, L \rangle, p^m)$

which is composed of the $\binom{d+1}{2}$ non-zero coordinate functions, $T_{\ell r} : [0,1] \to \mathbb{R}$

$$T(t) = \begin{pmatrix} T_{11}(t) & 0 & 0 & \ldots & 0 \\ T_{12}(t) & T_{22}(t) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & 0 \\ T_{d1}(t) & T_{d2}(t) & T_{d3}(t) & \ldots & T_{dd}(t) \end{pmatrix}.$$

A *motion* of the framework on the flexible torus $\mathcal{T}^d$ is a pair (P_i, T), such that:

1. $P_i(0) = p(v_i)$ for all i, and $T(0) = L$.

2. $P_i(t)$ is differentiable on $[0,1]$, for all i, and the $\binom{d+1}{2}$ coordinate functions

   $T_{\ell r}(t)$ are differentiable on $[0,1]$.

3. For all edges $e = \{v_i, v_j; m_e\}$ in $E\langle G, m \rangle$, and for all $t \in [0,1]$,

$$\|P_i - (P_j + m_e T)\| = \|p(v_i) - (p(v_j) + m_e L)\|.$$

The trivial motions of $\mathcal{T}^d$ are the trivial motions of $\mathcal{T}_0^d$, namely translation on the torus. That is P_i is a translation of all of the vertices of $\langle G, m \rangle$, and $T(t)$ is the zero matrix. It follows that there is a d-dimensional space of trivial motions on $\mathcal{T}^d$. Any framework $(\langle G, m \rangle, p)$ for which the only motions on $\mathcal{T}^d$ are the trivial motions is *rigid on* $\mathcal{T}^d$.

We can specialize these definitions to the partially flexible torus $\mathcal{T}_k^d$ by selecting $k < \binom{d+1}{2}$ of the lower triangular entries of $L(t)$ to vary, while fixing the other

166

entries. This forms the matrix $L_k(t)$. The matrix function $T(t)$ may be correspond-ingly modified (with all fixed entries in $L_k(t)$ corresponding to a 0 entry in $T(t)$). As noted previously, $\mathcal{T}_0^d$ is simply this specialization where none of the entries of $L(t)$ vary, and hence $T(t)$ is the zero matrix. The definition of rigidity of $(\langle G, m \rangle, p)$ on $\mathcal{T}_k^d$ is as for $\mathcal{T}^d$, but using the lattice matrix $L_k(t)$, and the matrix function $T_k(t)$.

5.3.4 Infinitesimal motions of frameworks on the two dimensional flexible torus $\mathcal{T}_k^2$.

To develop intuition for the d-dimensional case, we first consider 2-dimensional frameworks.

Consider a periodic orbit graph on the two-dimensional fully flexible torus $\mathcal{T}^2$. Let $e = \{i, j; m_e\}$ be an edge of $\langle G, m \rangle$. Suppose the edge $e = \{i, j; m_e\} \in E\langle G, m \rangle$ has length K. Then

$$\|p_i - (p_j + m_e L)\|^2 = K^2, \text{ where } L(t) = \begin{pmatrix} x(t) & 0 \\ y_1(t) & y_2(t) \end{pmatrix}.$$

Letting $p_i = (p_{i1}, p_{i2})$ and $p_j = (p_{j1}, p_{j2})$, we have

$$(p_{i1} - (p_{j1} + m_1 x + m_2 y_1))^2 + (p_{i2} - (p_{j2} + m_2 y_2))^2 = K^2.$$

To motivate the definition of infinitesimal motion, we differentiate with respect to t, and note that the positions of the vertices, and x, y_1 and y_2 are functions of t.

We obtain

$$2(p_{i1}-p_{j1}-m_1 x-m_2 y_1)\cdot(p'_{i1}-p'_{j1}-m_1 x'-m_2 y'_1)+2(p_{i2}-p_{j2}-m_2 y_2)\cdot(p'_{i2}-p'_{j2}-m_2 y'_2)=0.$$

We may write

$$(p_i - p_j - m_e L) \cdot (p'_i - p'_j - m_e L') = 0, \tag{5.1}$$

where $L' = \begin{pmatrix} x' & 0 \\ y'_1 & y'_2 \end{pmatrix}$ is the matrix of the derivatives of the entries of $L(t) = \begin{pmatrix} x(t) & 0 \\ y_1(t) & y_2(t) \end{pmatrix}$.

If we consider a partially flexible torus, say T_2^2 where the lattice matrix is given by $L_2(t) = \begin{pmatrix} x(t) & 0 \\ 0 & y_2(t) \end{pmatrix}$, then $L'_2 = \begin{pmatrix} x' & 0 \\ 0 & y'_2 \end{pmatrix}$, and equation (5.1) still represents the result of differentiating the edge length of e with respect to time.

Moreover, when *none* of the entries of the lattice matrix are functions of time, i.e $L(t) = L_0$, then L'_0 is the zero matrix. Then letting $u_i = p'_i$, and $u_j = p'_j$, equation (5.1) becomes

$$(p_i - p_j - m_e L_0) \cdot (u_i - u_j) = 0,$$

which is our familiar condition for the preservation of edge lengths on the fixed torus, $\mathcal{T}_0^d$. (See equation (3.6)).

Example 5.3.2. Let $\langle G, m \rangle$ be the framework pictured in Figure 5.4. Let $\mathcal{T}_2^2$ represent the torus where we allow scaling in the x and y directions, but the angle

168

(a) $(\langle G, m \rangle, p)$ (b) $(\langle G^m, L \rangle, p^m)$ (c) (u, u_L)
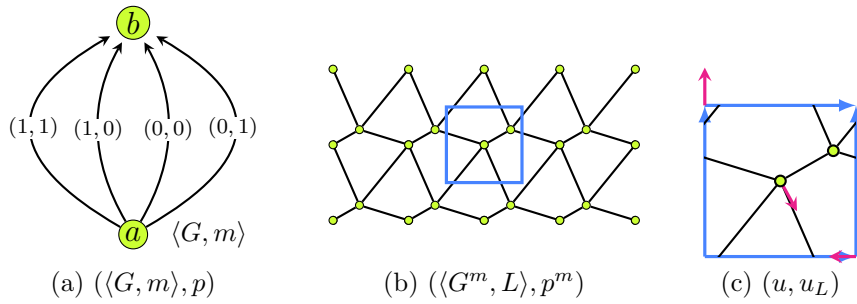
Figure 5.4: A periodic orbit graph $\langle G, m \rangle$ on $\mathcal{T}_2^2$ (a), together with its derived graph for some generic position p. $\langle G, m \rangle$ is generically flexible on $\mathcal{T}_2^2$. An infinitesimal flex of the framework $(\langle G, m \rangle, p)$ on $\mathcal{T}_2^2$ is shown in (c).

between these vectors is fixed at $\pi/2$. The lattice matrix is

$$
L_2(t) = \begin{pmatrix} x(t) & 0 \\ 0 & y_2(t) \end{pmatrix}.
$$

Then $(\langle G, m \rangle, p)$ has a non-trivial infinitesimal flex on $\mathcal{T}_2^2$, which is shown in (c).

Note that this flex also distorts (scales) the torus, as shown in (c).   □

Equation (5.1) motivates the definition of infinitesimal motion, as seen in the next section.

5.3.5 Infinitesimal rigidity of frameworks on the d-dimensional flexible torus $\mathcal{T}_k^d$.

Again we will first define infinitesimal motions for frameworks on the fully flexible torus $\mathcal{T}^d$, before specializing to $\mathcal{T}_k^d$. An infinitesimal motion of a periodic framework on the fully flexible torus $\mathcal{T}^d$ is an assignment of infinitesimal velocities to the

169

vertices of the framework and to the generators of the flexible torus. More precisely, an *infinitesimal motion* of $(\langle G, m \rangle, p)$ on $\mathcal{T}^d$ is an element $(u, u_L) \in \mathbb{R}^{d|V|+\binom{d+1}{2}}$, where

$$u : V \to \mathbb{R}^d, \text{ and } u_L : t_{\ell,r} \to \mathbb{R},$$

such that

$$(p_i - p_j - m_e L) \cdot (u_i - u_j - m_e L') = 0 \text{ for all } \{v_i, v_j; m_e\} \in E\langle G, m \rangle, \qquad (5.2)$$

when we let

$$L' = \begin{pmatrix} u_L(t_{11}) & 0 & 0 & \ldots & 0 \\ u_L(t_{12}) & u_L(t_{22}) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & 0 \\ u_L(t_{d1}) & u_L(t_{d2}) & u_L(t_{d3}) & \ldots & u_L(t_{dd}) \end{pmatrix}.$$

If $u_L = 0$ (i.e. $u_L(t_{\ell r}) = 0$ for all ℓ, r), and u is a trivial infinitesimal motion of $(\langle G, m \rangle, p)$ on the fixed torus $\mathcal{T}_0^d$, then we say that (u, u_L) is *trivial infinitesimal motion* of $(\langle G, m \rangle, p)$ on the flexible torus $\mathcal{T}^d$. In other words, we don't get any new trivial motions when we move to the flexible torus. Any infinitesimal motion where u_L is not identically zero will automatically be a non-trivial motion. Furthermore, every infinitesimal motion u of $(\langle G, m \rangle, p)$ on the fixed torus $\mathcal{T}_0^d$ can be extended to an infinitesimal motion (u, u_L) of $(\langle G, m \rangle, p)$ on the flexible torus $\mathcal{T}^d$ by setting $u_L = (0, \ldots, 0)$.

170

If the only infinitesimal motions of a framework $(\langle G, m \rangle, p)$ on $\mathcal{T}^d$ are trivial, then we say that $(\langle G, m \rangle, p)$ is *infinitesimally rigid* on $\mathcal{T}^d$.

We specialize this definition to the partially flexible torus $\mathcal{T}_k^d$ in the following way. Let $u_L : t_i \to \mathbb{R}$, where $t_1, \ldots, t_k$ are the k variable entries of $L_k(t)$. Let L_k' be the matrix obtained from $L_k(t)$ by replacing all of the variable entries of $L_k(t)$ with their image under u_L, and setting all other entries to zero. Thus $(u, u_L) \in \mathbb{R}^{d|V|+k}$. The definitions of trivial motions and infinitesimal rigidity on $\mathcal{T}_k^d$ are the same as for $\mathcal{T}^d$.

5.3.6   Infinitesimal motions of periodic frameworks $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ in $\mathbb{R}^d$

Let $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ be a d-periodic framework in $\mathbb{R}^d$, with L_k a lower triangular matrix with k variables, $0 < k \le \binom{d+1}{2}$. Then by Theorem 3.2.1, $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ has a representation as the derived periodic framework $(\langle G^m, L_k \rangle, p^m)$ corresponding to $(\langle G, m \rangle, p)$ on $\mathcal{T}_k^d$. We will define an infinitesimal periodic motion of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ in $\mathbb{R}^d$ using the representation of $\widetilde{G}$ as G^m (see Theorem 3.2.2).

Recall that the vertices of G^m are given by pairs (v_i, z), where $v_i \in V(G)$ and $z \in \mathbb{Z}^d$. The position $\widetilde{p}$ of these vertices in $\mathbb{R}^d$ satisfies $\widetilde{p}(v_i, z) = \widetilde{p}(v_i, 0) + x L_k(t)$, and $L_k(t)$ is the lattice matrix with k variable entries, say $t_1(t), \ldots, t_k(t)$.

An *infinitesimal periodic motion* of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ is a pair of functions

$$\widetilde{u} : \quad \widetilde{V} \quad \to \mathbb{R}^d$$

$$\widetilde{u}_L : \quad t_i \quad \to \mathbb{R}, \ i = 1, \ldots k$$

such that the following two conditions hold:

1. For every edge $e = \{(v_i, a), (v_j, b)\} \in \widetilde{E}$

$$(\widetilde{p}(v_i, a) - \widetilde{p}(v_j, b)) \cdot (\widetilde{u}(v_i, a) - \widetilde{u}(v_j, b)) = 0. \tag{5.3}$$

2. Letting L_k' be the matrix obtained from $L_k(t)$ by replacing the k variable entries t_i by $\widetilde{u}(t_i)$, and all other entries by zero,

$$(\widetilde{u}(v_i, a) - \widetilde{u}(v_i, c)) = (a - c)L_k'. \tag{5.4}$$

In other words, condition 1 is simply our usual edge condition (compare Equation (2.3) for finite frameworks) on the (infinite number of) edges of $\widetilde{G}$, and condition 2 is the periodicity condition. That is, condition 2 says that the velocities assigned to any two vertices in an equivalence class are 'the same', up to the flexibility of the lattice, which is specified by L_k'. As a consequence of (5.4), we may write

$$\widetilde{u}(v_i, a) = \widetilde{u}(v_i, 0) + aL_k'. \tag{5.5}$$

An infinitesimal motion of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ is *trivial* if $\widetilde{u}$ assigns the same infinitesimal velocity to all vertices of $\widetilde{G}$, and $\widetilde{u}_L$ is the zero map. That is, it is a translation

172

of the whole framework. Rotation is not a trivial motion because we are fixing the orientation of $L_k(t)$. In particular, any infinitesimal motion in which $\widetilde{u}_L \neq 0$ is non-trivial. We say that $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ is *infinitesimally periodic rigid* in $\mathbb{R}^d$ if the only infinitesimal motions are trivial (i.e. are translations). We emphasize that the infinitesimal motions of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ described above preserve the periodicity of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$. Hence, $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ can be infinitesimally periodic rigid without being infinitesimally rigid.

Proposition 5.3.3. *Let $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ be a periodic framework where $L_k(t)$ has k variable entries, $k \in 0, \ldots, \binom{d+1}{2}$. Then the following are equivalent:*

1. *$(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ is infinitesimally periodic rigid in $\mathbb{R}^d$*

2. *$(\langle G, m \rangle, p)$ is infinitesimally rigid on the flexible torus $\mathcal{T}_k^d$.*

Proof. We show that non-trivial motions of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ in $\mathbb{R}^d$ correspond to non-trivial motions of $(\langle G, m \rangle, p)$ on $\mathcal{T}_k^d$. By Theorem 3.2.1, $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ has a representation as the derived periodic framework $(\langle G^m, L_k \rangle, p^m)$ corresponding to $(\langle G, m \rangle, p)$ on $\mathcal{T}_k^d$. Therefore, the edge $(e, z) = \{(v_i, z), (v_j, z + m_e)\} \in \widetilde{E}$ if and only if $\{v_i, v_j; m_e\} \in E\langle G, m \rangle$.

Let $(\widetilde{u}, \widetilde{u}_L)$ be an infinitesimal motion of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ in $\mathbb{R}^d$. Define an infinitesimal motion $(u, u_L) \in \mathbb{R}^{d|V|+k}$ of $(\langle G, m \rangle, p)$ by

$$u(v_i) = \widetilde{u}(v_i, (0, \ldots, 0)), \text{ and } u_L = \widetilde{u}_L.$$

173

By (5.3) and (5.5), this new motion satisfies

$$
\begin{aligned}
& \big(p_i - (p_j + m_e L_k)\big) \cdot \big(u_i - (u_j + m_e L'_k)\big) \\
=\ & \big(p(v_i) - (p(v_j) + m_e L_k)\big) \cdot \big(u(v_i) - (u(v_j) + m_e L'_k)\big) \\
=\ & \big(\widetilde{p}(v_i, 0) - \widetilde{p}(v_j, m_e)\big) \cdot \big(\widetilde{u}(v_i, 0) - \widetilde{u}(v_j, m_e)\big) \\
=\ & 0,
\end{aligned}
$$

for each edge $e = \{v_i, v_j; m_e\}$, which is exactly the edge condition for an infinitesimal

motion of a framework on $\mathcal{T}_k^d$ (see Equation (5.2)).

On the other hand, suppose that we have an infinitesimal motion $(u, u_L) \in$

$\mathbb{R}^{d|V|+k}$ of $(\langle G, m \rangle, p)$ on $\mathcal{T}_k^d$. Let $\widetilde{u} : \widetilde{V} \to \mathbb{R}^d$ be given by

$$
\widetilde{u}(v_i, a) = u(v_i) + a L'_k.
$$

We claim this is an infinitesimal motion of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$. To first see that $\widetilde{u}$ satisfies

2 above, consider

$$
\begin{aligned}
\widetilde{u}(v_i, a) - \widetilde{u}(v_j, c) \ &=\ (u(v_i) + a L'_k) - (u(v_i) + c L'_k) \\
&=\ (a - c) L'_k,
\end{aligned}
$$

174

as desired. Now let $(e, a) = \{(v_i, a), (v_j, b)\}$ be an edge of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$.

$$
\begin{aligned}
&\big(\widetilde{p}(v_i, a) - \widetilde{p}(v_j, b)\big) \cdot \big(\widetilde{u}(v_i, a) - \widetilde{u}(v_j, b)\big) \\
={}& \big(p(v_i) + aL_k - (p(v_j) + bL_k)\big) \cdot \big(\widetilde{u}(v_i, 0) - (\widetilde{u}(v_j, 0) + (b - a)L'_k)\big) \\
={}& \big(p_i - (p_j + (b - a)L_k)\big) \cdot \big(u_i - (u_j + (b - a)L'_k)\big) \\
={}& 0,
\end{aligned}
$$

since $(e, a) = \{(v_i, a), (v_j, b)\}$ is an edge of $\widetilde{G} = G^m$ if and only if $e = \{v_i, v_j; b - a\}$ is an edge of $\langle G, m \rangle$.

That trivial motions are equivalent to trivial motions is obvious (both assign the same infinitesimal velocity to all vertices, and the lattice flex is zero). $\qquad \square$

Remark 5.3.4. As for the fixed torus, we could prove a version of Proposition 5.3.3 for continuous (not infinitesimal) rigidity of a framework on $\mathcal{T}_k^d$. The argument follows the same reasoning as the above proof, and we therefore omit it since our focus is on infinitesimal rigidity for the remainder of the chapter.

Remark 5.3.5. In the case where $k = 0$ and we are considering a fixed torus, an infinitesimal motion of $(\langle G^m, L_0 \rangle, p^m)$ was simply an assignment of the *same* infinitesimal velocity u_i to every vertex in the the fibre of vertices (v_i, z), $z \in \mathbb{Z}^d$. In contrast, an infinitesimal motion of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ is given by $\widetilde{u}(v_i, z) = u(v_i) + zL'_k$. It follows, therefore, that the infinitesimal velocity of a particular vertex (v_i, z) in $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ will depend on $z \in \mathbb{Z}^d$, and hence for vertices that are arbitrarily "far

away" from $(0, \ldots, 0)$, their velocities may be arbitrarily large when $L_k' \neq 0$. This is evidence to suggest that the fixed torus model may be more relevant than it seems at first.

The presentation of frameworks as gain graphs thus allows for an explicit representation of the infinite periodic frameworks (in the form of the derived graph) and their motions, which to our knowledge, has not appeared elsewhere. It is related to the work of Owen and Power [53], who study infinite frameworks with 'vanishing flexibility'. This would be a framework which becomes *less* flexible away from the 'centre'. In that case, the authors use an infinite dimensional rigidity matrix. $\square$

Remark 5.3.6. We could also define a (continuous) periodic motion of $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ in $\mathbb{R}^d$. It is an easy exercise similar to the proof of Proposition 5.3.3 to show that $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ is periodic rigid in $\mathbb{R}^d$ if and only if $(\langle G, m \rangle, p)$ is rigid on $\mathcal{T}_k^d = \mathbb{R}^d / L_k(t) \mathbb{Z}^d$. The fact that infinitesimal rigidity implies rigidity on $\mathcal{T}_k^d$ will apply to show that $(\langle \widetilde{G}, L_k \rangle, \widetilde{p})$ is periodic rigid whenever it is infinitesimally periodic rigid in $\mathbb{R}^d$. $\square$

5.3.7 An infinitesimal motion of $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ is an infinitesimal motion of $(\langle G, m \rangle, p)$

For completeness, we now demonstrate that the periodic infinitesimal motions of an infinite periodic framework given by $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ in the notation of Borcea and

Streinu are equivalent to the infinitesimal motions of the periodic orbit framework on the flexible torus.

We first define infinitesimal motion for the d-periodic framework $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$. Let $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ be a periodic framework, and choose an isomorphism $\Gamma \to \mathbb{Z}^d$. Let $x_i = p(v_i), i = 1, \ldots, |V|$ be the position of the vertices of $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$, and $\mu_1, \ldots, \mu_d$ be period vectors for the representation of the translation vectors of Γ. Borcea and Streinu define a vector $(y_1, \ldots, y_{|V|}, \nu_1, \ldots, \nu_d) \in \mathbb{R}^{d|V|+d^2}$ to be an *infinitesimal motion* of $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ if

$$\langle (x_j + \mu(\beta)) - x_i, (y_j + \nu(\beta)) - y_i \rangle = 0, \text{ where } \beta = 1, \ldots, |E|. \qquad (5.6)$$

Here the angle brackets represent the inner product, and

$$\mu(\beta) = \sum_{n=1}^{d} c_\beta^n \mu_n, \text{ and } \nu(\beta) = \sum_{n=1}^{d} c_\beta^n \nu_n, \quad c_\beta^n \in \mathbb{Z}.$$

The authors say that a periodic framework $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ is *infinitesimally (periodic) rigid* in $\mathbb{R}^d$ if the only infinitesimal motions are trivial. However, they consider a $\binom{d+1}{2}$ dimensional space of trivial motions, namely the usual space of trivial motions of the framework $(\widetilde{G}, \widetilde{p})$ generated by d translations and $\binom{d}{2}$ rotations.

Proposition 5.3.7. *Let $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ be a d-periodic framework. Let $(\langle G, m \rangle, p)$ be the corresponding periodic orbit framework given by Proposition 3.2.3. Then the following are equivalent:*

(i) $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ is infinitesimally (periodic) rigid in $\mathbb{R}^d$

(ii) $(\langle G, m\rangle, p)$ is infinitesimally rigid on $\mathcal{T}^d$.

Proof. We show that a non-trivial infinitesimal motion of $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ is an non-trivial infinitesimal motion of $(\langle G, m\rangle, p)$, and vice versa. We have already seen (Proposition 3.2.3) that any periodic framework $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ can be viewed as rotation-equivalent to the framework $(\langle \widetilde{G}, L\rangle, \widetilde{p})$ where $L(t)$ is a lower triangular matrix, and it is thus represented as a periodic orbit framework $(\langle G, m\rangle, p)$ on $\mathcal{T}^d$. We assume without loss of generality that $(\widetilde{G}, \Gamma, \widetilde{p}, \pi) = (\langle \widetilde{G}, L\rangle, \widetilde{p})$ is the rotated framework such that $L(t)$ is lower triangular.

The proof of this proposition is a straightforward translation of notation from [7]. In our language, for $\beta = 1, \ldots, |E|$, the row vector $(c_\beta^n), n = 1, \ldots, d$ is the gain of the edge e_β, written $m(e_\beta)$. Then $\mu(\beta)$ can be rewritten simply as $m(e_\beta)L$, where L is the matrix whose rows are the translation vectors $\mu_1, \ldots, \mu_d$.

If (u, u_L) is an infinitesimal motion of $(\langle G, m\rangle, p)$ on $\mathcal{T}^d$, then simply letting $y_i = u(v_i)$, and letting ν_i be given by the ith row of L' will define an infinitesimal motion on $(\langle \widetilde{G}, L\rangle, \widetilde{p})$. It is evident that the linear system (5.6) is equivalent to the simultaneous solution of (5.1) for each edge $e_\beta \in E$.

On the other hand, suppose that $(y_1, \ldots, y_{|V|}, \nu_1, \ldots, \nu_d) \in \mathbb{R}^{d|V|+d^2}$ is an infinitesimal motion of $(\langle \widetilde{G}, L\rangle, \widetilde{p})$ in $\mathbb{R}^d$. Let $L(t)$ be the matrix whose rows are the

translation vectors $\mu_1, \dots, \mu_d$, and by assumption $L(t)$ is lower triangular. Similarly, $\nu(\beta)$ can be written $m(e_\beta)L'$ where L' is the matrix whose rows are $\nu_1, \dots, \nu_d$. Moreover, if we demand that the infinitesimal motion preserve this orientation of the lattice, then L' will be lower triangular too. Finally, let $u(v_i) = y_i$, and $u_L(t_{\ell,r})$ be the (ℓ,r)-th entry of L' to define an infinitesimal motion (u, u_L) of $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$.

The last remaining justification is that trivial motions are equivalent to trivial motions. Because we have eliminated rotation as a trivial motion by fixing the orientation of the lattice, rotations are no longer trivial motions of $(\langle \widetilde{G}, L \rangle, \widetilde{p})$. Therefore, the only trivial motions are the translations, which assign all vertices the same velocity, and fix the lattice. These are identical in either representation. $\square$

This result can be specialized to the fixed torus, by fixing the generators of Γ to be the generators of $\mathcal{T}_0^d$, and setting $\nu_1 = \dots = \nu_d = 0$.

Remark 5.3.8. While the notation of [7] is precise and general, we feel that the simplifications we make in our presentation allow for a more transparent manipulation of the periodic objects. In addition, it permits us to easily consider partial variations of the lattice. As we shall see in the next section, the notation for the rigidity matrix using our representation has exactly one column for each variable entry in $L_k(t)$. That is, the rigidity matrix has dimension $|E| \times (d|V| + k)$, where $0 \le k \le \binom{d+1}{2}$. In Borcea and Streinu, the rigidity matrix has dimension

179

$|E| \times (d|V| + d^2)$. Note that $d^2 > \binom{d+1}{2}$ for $d > 1$. The maximal rank in both representations is the same. □

As before, we will focus on infinitesimal rigidity for the remainder of the chapter, noting that the key results of Chapter 3 transfer to this setting with small modifications. In particular, the averaging technique will again apply to show that infinitesimal rigidity implies rigidity on $\mathcal{T}_k^d$ (Theorem 3.3.25). In this setting we will also need to average the generators of the flexible torus. In addition, the arguments of Asimow and Roth can be used to prove that for generic realizations on $\mathcal{T}_k^d$, infinitesimal rigidity and rigidity are equivalent (Theorem 3.3.30).

5.3.8 The rigidity matrix for the 2-dimensional flexible torus $\mathcal{T}^2$

As a preamble to a discussion of the d-dimensional rigidity matrix for the flexible torus, we first consider the two-dimensional rigidity matrix, together with an example of a framework in this setting. When we allow the torus to change size and shape, we add extra columns to the rigidity matrix. In particular, if $(x, 0)$ and (y_1, y_2) are the generators of the torus (the rows of $L(t)$), we can add up to three columns to the rigidity matrix for each of the three variables x, y_1 and y_2.

We write the $|E| \times (2|V| + 3)$ rigidity matrix as follows:

- The first $2|V|$ columns are indexed by the vertices, with a column for each of their coordinates. The last three columns correspond to x, y_1 and y_2.

- For the edge $\{i, j; m_e\}$, the entries under each vertex i, j correspond to their coefficient in (5.2).

- The columns corresponding to x, y_1, and y_2 will also have entries corresponding to their coefficient in (5.2).

The rigidity matrix $\mathbf{R}(\langle G, m\rangle, p)$ is thus:

$$
\begin{array}{c}
\phantom{\{i,j;m\}} \quad\quad i \quad\quad\quad\quad\quad\quad\quad j \quad\quad\quad\quad\quad\quad L \\
\begin{array}{cc}
\{i,j;m\} \\
\\
\{j,j;m\}
\end{array}
\left(\begin{array}{cccccc}
& \vdots & & & & \vdots \\
0\cdots 0 & p_i - (p_j + mL) & 0\cdots 0 & p_j + mL - p_i & 0\cdots 0 & \mathcal{L}\{i,j;m\} \\
& \vdots & & & & \vdots \\
0\cdots 0 & 0 & 0\cdots 0 & 0 & 0\cdots 0 & \mathcal{L}\{j,j;m\} \\
& \vdots & & & & \vdots
\end{array}\right),
\end{array}
$$

where $\mathcal{L}\{i,j;m\}$ is given by the 3-tuple

$$
(-m_1(p_{i1} - (p_{j1} + m_1 x + m_2 y_1)), -m_2(p_{i1} - (p_{j1} + m_1 x + m_2 y_1)), -m_2(p_{i2} - (p_{j2} + m_2 y_2)))
$$

for non-loop edges, and

$$
\mathcal{L}\{j,j;m\} = (-m_1(m_1 x + m_2 y_1), -m_2(m_1 x + m_2 y_1), -m_2(m_2 y_2))
$$

for a loop edge. More generally we may write

$$
\mathcal{L}\{i,j;m\} = (p_i - (p_j + m(e)L)) \begin{pmatrix} m_1 & m_2 & 0 \\ 0 & 0 & m_2 \end{pmatrix}.
$$

181

Notice that the row of the matrix corresponding to the loop edge is independent of the vertex at which the loop is located, since all non-lattice entries are identically zero. This means that for a two-dimensional framework on the fully flexible torus $\mathcal{T}^2$, at most three loops may appear as independent rows of the matrix. We will return to this observation in more generality in the next few sections.

Example 5.3.9. We consider a two-dimensional periodic orbit framework with two vertices and five edges.

(a) $((\widetilde{G}, L), \widetilde{p})$ (b) $\langle G, m \rangle$
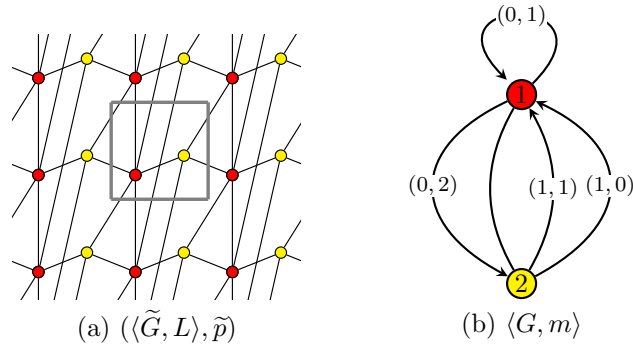
Figure 5.5: A two vertex example in $\mathbb{R}^2$.

The rigidity matrix $\mathbf{R}(\langle G, m \rangle, p)$ for this framework has 5 rows and 7 columns. Below it is broken into two sections: the four columns corresponding to variables $p_1 = (a, b), p_2 = (c, d)$ (represented as two columns of 2-tuples), and the three columns corresponding to the three non-zero variables in L: x, y_1, y_2. Note that the edge $\{1, 2; (0, 0)\}$ has the entry $(0, 0, 0)$ in the (x, y_1, y_2) columns, but is non-zero elsewhere. In contrast, the loop edge $\{1, 1; (1, 0)\}$ has zero entries everywhere

except in the columns corresponding to L. The rigidity matrix $\mathbf{R}(\langle G, m\rangle, p)$ for this framework is:

$$
\begin{array}{c}
\begin{array}{ccc}
p_1 = (a, b) & p_2 = (c, d) & (x, y_1, y_2)
\end{array}\\
\begin{array}{c}
\{1, 2; (0, 0)\}\\
\{1, 2; (0, 2)\}\\
\{2, 1; (1, 0)\}\\
\{2, 1; (1, 1)\}\\
\{1, 1; (0, 1)\}
\end{array}
\left(
\begin{array}{ccc}
(p_1 - p_2) & (p_2 - p_1) & (0, 0, 0))\\
(p_1 - [p_2 + (0, 2)L]) & (p_2 - [p_1 - (0, 2)L]) & (*, *, *)\\
(p_1 - [p_2 - (1, 0)L]) & (p_2 - [p_1 + (1, 0)L]) & (*, *, *)\\
(p_1 - [p_2 - (1, 1)L]) & (p_2 - [p_1 + (1, 1)L]) & (*, *, *)\\
0 & 0 & (0, 0, 1)
\end{array}
\right).
\end{array}
$$

The three columns corresponding to (x, y_1, y_2) (the variable entries of $L(t)$) are given by:

$$
\begin{array}{c}
\begin{array}{ccc}
x & y_1 & y_2
\end{array}\\
\left(
\begin{array}{ccc}
0 & 0 & 0\\
0 & 0 & -2(b - (d + 2y_2))\\
-(c - (a + x)) & 0 & 0\\
-(c - (a + x + y_1)) & -(c - (a + x + y_1)) & -(d - (b + ty_2))\\
0 & 0 & 1
\end{array}
\right).
\end{array}
$$

Two trivial motions of $(\langle G, m\rangle, p)$ are represented by the column vectors $u_x = (1, 0, 1, 0, 0, 0, 0)^T$ and $u_y = (0, 1, 0, 1, 0, 0, 0)^T$, which are always solutions to the linear system $\mathbf{R}(\langle G, m\rangle, p) \cdot u = 0$. These are translations of the whole periodic orbit framework on $\mathcal{T}^2$, and they generate the vector space of trivial infinitesimal motions

183

of $(\langle G, m \rangle, p)$. If we are only interested in the infinitesimal motions of $(\langle G, m \rangle, p)$ on the fixed torus $\mathcal{T}_0^2$, we may omit the columns corresponding to (x, y_1, y_2) and the translations are the same. $\qquad\square$

5.3.9 The rigidity matrix for the d-dimensional flexible torus $\mathcal{T}^d$

For a periodic orbit graph $\langle G, m \rangle$ on the flexible torus $\mathcal{T}^d$, let e be an edge of $\langle G, m \rangle$, with $m(e) = (m_1, \ldots, m_d)$. Let $M(e)$ be the $d \times \binom{d+1}{2}$ matrix:

$$
M(e) = \begin{pmatrix}
m_1 & m_2 & 0 & m_3 & 0 & 0 & \cdots & m_d & 0 & \cdots & & 0 \\
0 & 0 & m_2 & 0 & m_3 & 0 & & 0 & m_d & & & 0 \\
\vdots & 0 & 0 & 0 & 0 & m_3 & & \vdots & & & & \vdots \\
& \vdots & \vdots & 0 & 0 & 0 & & & & \ddots & & \\
& & & \vdots & \vdots & \vdots & & & & 0 & m_d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & m_d
\end{pmatrix}.
$$

The matrix can be broken horizontally into $d \times \ell$ blocks, where $\ell = 1, \ldots, d$. The ℓ-th block consists of $m_\ell I_\ell$, where I_ℓ is the $\ell \times \ell$ identity matrix, followed by $d - \ell$ rows of zeros. Then $\mathcal{L}\{i, j; m_e\}$ is the $\binom{d+1}{2}$-tuple given by:

$$
\mathcal{L}\{i, j; m\} = -(p_i - (p_j + m(e)L))M(e).
$$

The coefficients of $\mathcal{L}\{i, j; m\}$ therefore have a natural correspondence with $t_{\ell r}(t)$, the variable entries of $L(t)$.

184

The *d-dimensional flexible torus rigidity matrix* $\mathbf{R}(\langle G, m \rangle, p)$ is the $|E| \times d|V| + \binom{d+1}{2}$ dimensional matrix with one row for each edge $\{i, j; m\}$ or loop $\{j, j; m\}$:

$$
\begin{array}{c}
\phantom{\{i,j;m\}} \qquad\quad i \qquad\qquad\qquad\qquad j \qquad\qquad\qquad L \\
\begin{array}{c}
\\
\{i,j;m\} \\
\\
\{j,j;m\} \\
\\
\end{array}
\left(
\begin{array}{cccccc}
& \vdots & & & & \vdots \\
0 \cdots 0 & p_i - (p_j + mL) & 0 \cdots 0 & p_j + mL - p_i & 0 \cdots 0 & \mathcal{L}\{i,j;m\} \\
& \vdots & & & & \vdots \\
0 \cdots 0 & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 & \mathcal{L}\{j,j;m\} \\
& \vdots & & & & \vdots \\
\end{array}
\right).
\end{array}
$$

When we consider a partially flexible torus $\mathcal{T}_k^d$ where $k < \binom{d+1}{2}$, we modify the matrix $M(e)$ by deleting the columns of $M(e)$ corresponding to fixed elements in L. The entry $t_{\ell r}$ in L corresponds to the r-th column of the ℓ-th vertical block of $M(e)$. We denote the resulting $|E| \times (d|V| + k)$-dimensional rigidity matrix $\mathbf{R}_k(\langle G, m \rangle, p)$, where $k \in \{0, \ldots, \binom{d+1}{2}\}$. We call the columns of $\mathbf{R}_k(\langle G, m \rangle, p)$ corresponding to the coefficients of L the *lattice columns*. Their entries will be referred to as the *lattice elements*. The essential idea when moving from the flexible torus to the partially flexible torus is that we strike out columns of $\mathbf{R}_k$ corresponding to the lattice.

We remark again that loops in a framework on $\mathcal{T}_k^d$ are independent of their vertex of application, and as a result, at most k loops may be independent as rows in the rigidity matrix $\mathbf{R}_k$.

185

The key result regarding the rigidity matrix is the following extension of Theorem 3.3.9.

Theorem 5.3.10. *A d-periodic orbit framework $(\langle G, m \rangle, p)$ is infinitesimally rigid on the flexible torus $\mathcal{T}_k^d$ if and only if the rigidity matrix $\mathbf{R}_k(\langle G, m \rangle, p)$ has rank $d|V| - d + k$, $(k = 0, \ldots, \binom{d+1}{2})$.*

Proof. By construction, the kernel of the flexible torus rigidity matrix corresponds to the space of infinitesimal motions of $(\langle G, m \rangle, p)$ on $\mathcal{T}_k^d$. As in the case of the fixed torus $(k = 0)$, the space of trivial infinitesimal motions for a framework $(\langle G, m \rangle, p)$ on $\mathcal{T}_k^d$ has dimension d (it is independent of the variability of the torus indicated by k). Therefore the maximum rank of $\mathbf{R}_k(\langle G, m \rangle, p)$ is $d|V| - d + k$, $k = 0, \ldots, \binom{d+1}{2}$.

Since $(\langle G, m \rangle, p)$ is infinitesimally rigid on $\mathcal{T}_k^d$ if and only if its only infinitesimal motions are trivial motions (i.e. translations), it follows that $(\langle G, m \rangle, p)$ is infinitesimally rigid on $\mathcal{T}_k^d$ if and only if the kernel of the rigidity matrix has dimension d. $\qquad\square$

In light of Theorem 5.3.10, we say that a periodic orbit framework $(\langle G, m \rangle, p)$ is *minimally rigid* on $\mathcal{T}_k^d$ if it is infinitesimally rigid, and $|E| = d|V| - d + k$.

Combining this with Theorem 5.3.7, we recover a result of Borcea and Streinu [7], which relates the rigidity of the d-periodic framework $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ with the rigidity matrix for the fully flexible torus. Note here that this is a fully flexible torus,

$k = \binom{d+1}{2}$.

Corollary 5.3.11 ([7]). *The periodic framework $(\widetilde{G}, \Gamma, \widetilde{p}, \pi)$ is infinitesimally periodic rigid in $\mathbb{R}^d$ if and only if the rank of the rigidity matrix $\mathbf{R}(\langle G, m \rangle, p)$ for the corresponding (rotation-equivalent) periodic orbit framework $(\langle G, m \rangle, p)$ on the flexible torus $\mathcal{T}^d$ is $d|V| - d + \binom{d+1}{2}$.*

As in the fixed torus case, we describe edges of the periodic orbit graph $\langle G, m \rangle$ to be *independent* (resp. *dependent*) on the flexible torus $\mathcal{T}_k^d$ if the corresponding rows of the rigidity matrix $\mathbf{R}_k$ are linearly independent (resp. linearly dependent).

We have already observed that loops are independent of the vertex to which they are attached. Moreover,

Proposition 5.3.12. *Let $(\langle G, m \rangle, p)$ be a minimally rigid framework on the flexible torus, $\mathcal{T}_k^d$, $0 \le k \le \binom{d+1}{2}$. Then $\langle G, m \rangle$ has at most k loops.*

Proof. Rows of the rigidity matrix corresponding to loops are zero in all columns except the lattice columns. When $k = 1$, $\mathbf{R}_k(\langle G, m \rangle, p)$ has one lattice column, and hence only one loop. When we add flexibility to the torus, we add columns, up to $\binom{d+1}{2}$ of them, which allows for up to $\binom{d+1}{2}$ loops. $\qquad\square$

If a framework is dependent on the flexible torus, then it is also dependent on the fixed torus.

Proposition 5.3.13. *If periodic orbit framework $(\langle G, m \rangle, p)$ is dependent on $\mathcal{T}_k^d$, $1 \leq k \leq \binom{d+1}{2}$ then it is dependent on $\mathcal{T}_0^d$.*

Proof. If the edges of $(\langle G, m \rangle, p)$ are dependent on $\mathcal{T}_k^d$, then they are dependent on $\mathcal{T}_0^d$, since the rigidity matrix of $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$ can be obtained from the rigidity matrix of $(\langle G, m \rangle, p)$ on $\mathcal{T}_k^d$ by deleting the k columns corresponding to the k degrees of flexibility of $\mathcal{T}_k^d$. $\square$

Corollary 5.3.14. *Let $\mathcal{T}_k^d$ be the flexible torus generated by the matrix $L_k(t)$, and let $L_\ell(t)$ be the matrix obtained from $L_k(t)$ by fixing $k - \ell$, of the variable entries of $L_k(t)$, where $0 \leq \ell < k$. Let $\mathcal{T}_\ell^d$ be the flexible torus generated by $L_\ell(t)$. If the periodic orbit framework $(\langle G, m \rangle, p)$ is dependent on $\mathcal{T}_k^d$, $1 \leq k \leq \binom{d+1}{2}$ then it is dependent on $\mathcal{T}_\ell^d$.*

There are several other observations we can record as a consequence of Theorem 5.3.10. However we delay their discussion until Section 5.4, after we describe the affine invariance of rigidity on $\mathcal{T}_k^d$, and the T-gain procedure.

5.3.10 Affine invariance on $\mathcal{T}_k^d$

As for the fixed torus, infinitesimal rigidity on the flexible torus is invariant under affine transformations. This was noted independently by Borcea and Streinu in [7] for the fully flexible torus, $\mathcal{T}^d$, where $k = \binom{d+1}{2}$.

Theorem 5.3.15. *Let $(\langle G, m \rangle, p)$ be a d-periodic orbit framework on $\mathcal{T}_k^d$. Let $L_k(t)$ be the $d \times d$ matrix whose rows are the generators of $\mathcal{T}_k^d$. Let A be an affine transformation of $\mathbb{R}^d$, with $A(x) = xB + t$. Then the edges of $(\langle G, m \rangle, A(p))$ are independent on $\mathbb{R}^d / L_k(t)B\mathbb{Z}^d$ if and only if the edges of $(\langle G, m \rangle, p)$ are independent on $\mathbb{R}^d / L_k(t)\mathbb{Z}^d$.*

The proof is nearly identical to the proof of the analogous result in Chapter 4, so we omit it. The first $d|V|$ columns of $\mathbf{R}_k(\langle G, m \rangle, p)$ are the same as the $d|V|$ columns of $\mathbf{R}_0(\langle G, m \rangle, p)$. The dependence of the remaining k columns follows by the same argument. We use this result to assume that $\mathcal{T}_k^d$ is initialized to the unit torus, $[0, 1)^d$. That is, the matrix $L_k(t)$ is initialized to the $d \times d$ identity matrix, $L_k = I_{d \times d}$. Hence expression (5.1) becomes

$$(p_i - p_j - m_e) \cdot (u_i - u_j - m_e L'_k) = 0,$$

eliminating L_k from the expression.

5.3.11 *T*-gain procedure preserves infinitesimal rigidity on $\mathcal{T}_k^d$

In Chapter 3 we described the T-gain procedure, and we showed that the rigidity matrices corresponding to two T-gain equivalent periodic orbit frameworks have the same rank. We now extend this to the flexible torus case.

Theorem 5.3.16. *Let $(\langle G, m \rangle, p)$ be a periodic orbit framework on $\mathcal{T}^d$. Then*

$\mathrm{rank}\mathbf{R}_k(\langle G, m \rangle, p) = \mathrm{rank}\mathbf{R}_k(\langle G, m_T \rangle, p')$, where $p' : V \to \mathbb{R}^d$ is given by $p'_i = p_i + m_T(v_i)$.

Proof. Throughout the proof we present the case $k = \binom{d+1}{2}$. If $k < \binom{d+1}{2}$, we may simply strike out the corresponding columns in $\mathbf{R}(\langle G, m \rangle, p)$ to obtain $\mathbf{R}_k(\langle G, m \rangle, p)$. The basic row dependence in the matrix is unchanged by this operation.

Suppose that a set of rows is dependent in $\mathbf{R}(\langle G, m \rangle, p)$. Then there exists a vector of scalars, say $\omega = \begin{bmatrix} \omega_1 & \cdots & \omega_{|E|} \end{bmatrix}$ such that

$$\omega \cdot \mathbf{R}(\langle G, m \rangle, p) = 0.$$

For each vertex $v_i \in V$ the column sum of $\mathbf{R}((\langle G, m \rangle, p))$ becomes

$$\sum_{e_\alpha \in E_+} \omega_{e_\alpha}(p_i - (p_j + m_{e_\alpha})) + \sum_{e_\beta \in E_-} \omega_{e_\beta}(p_i - (p_k - m_{e_\beta})) = 0, \qquad (5.7)$$

where E_+ and E_- are the edges directed out from and into vertex i repectively. In Chapter 4, we demonstrated that (5.7) is equivalent to the following:

$$\sum_{e_\alpha \in E_+} \omega_e \Big(p_i + m_T(v_i) - (p_j + m_T(v_j)) - m_T(e) \Big) +$$

$$\sum_{e_\beta \in E_-} \omega_e \Big(p_i + m_T(v_i) - (p_j + m_T(v_j)) + m_T(e) \Big) = 0. \quad (5.8)$$

which is the column sum of the column of $\mathbf{R}(\langle G, m_T \rangle, p')$ corresponding to the vertex v_i.

Since we are working with the flexible torus, we have $\binom{d+1}{2}$ additional columns corresponding to the lattice elements. We will show that if there exists a vector of

190

scalars $\omega = \begin{bmatrix} \omega_1 & \cdots & \omega_{|E|} \end{bmatrix}$ such that

$$\omega \cdot \mathbf{R}(\langle G, m \rangle, p) = 0$$

then $\omega \cdot \mathbf{R}(\langle G, m_T \rangle, p) = 0$ too. Since the first $2|V|$ columns are exactly as in the fixed torus case, we need only show this holds for the new columns.

Consider the column sum corresponding to the columns of the lattice elements in $\mathbf{R}(\langle G, m_T \rangle, p')$:

$$\sum_{e \in E} \omega_e \mathcal{L}(i, j; m_T(e)) = \sum_{e \in E} \omega_e \Big[(p_i + m_T(v_i)) - (p_j + m_T(v_j)) - m_T(e) \Big] M_T(e), \quad (5.9)$$

where $M_T(e)$ is defined as for $M(e)$ using the T-gains, $m_T(e)$ for each edge e. Recall that $m_T(e) = m_T(v_i) + m(e) - m_T(v_j)$, where $m_T(v_i)$ represents the T-potential of the vertex v_i (the T-potential of a vertex v_i is the net gain on the direct path along T from the root vertex). Let $M_T(v_i)$ be the $\binom{d+1}{2} \times d$ matrix defined as $M(e)$, but using the T-potentials of the vertex v_i. Expanding (5.9), we obtain

$$\sum_{e \in E} \omega_e \Big([\cdots] M(e) + [\cdots] M_T(v_i) - [\cdots] M_T(v_j) \Big), \quad (5.10)$$

191

where $\left[\cdots\right] = \left[(p_i + m_T(v_i)) - (p_j + m_T(v_j)) - m_T(e)\right]$. We know that

$$\sum_{e \in E} \omega_e \left[(p_i + m_T(v_i)) - (p_j + m_T(v_j)) - m_T(e)\right] M(e)$$

$$= \sum_{e \in E} \omega_e \left[p_i - p_j - m(e)\right] M(e)$$

$$= \sum_{e \in E} \omega_e \mathcal{L}(i, j; m(e))$$

$$= 0, \quad \text{since } \omega \cdot \mathbf{R}(\langle G, m \rangle, p) = 0.$$

Now note that $m_T(v_i)$ and $m_T(v_j)$ have one of $|V|$ different values, and therefore there are at most $|V|$ distinct matrices $M_T(v_i)$. Grouping (5.10) according to these matrices, we obtain

$$\sum_{i=1}^{|V|} \left[\sum_{j:(i,j) \in E} \omega_e (p_i + m_T(v_i) - (p_j + m_T(v_j)) - m_T(e)) \right] M_T(v_i),$$

where each edge is counted exactly twice, once for its initial vertex and once for its terminal vertex, with sign depending on the orientation of the edge. But by (5.8), the sum inside the square brackets is zero, since it represents the column sum at any vertex. Hence (5.9) is also zero. The same argument also works in reverse, which proves the claim. $\qquad\square$

5.4 Necessary conditions for infinitesimal rigidity of frameworks on $\mathcal{T}_k^d$

We now establish some necessary conditions for rigidity on $\mathcal{T}_k^d$. As a direct consequence of Theorem 5.3.10 and the rigidity matrix,

Proposition 5.4.1. *Let* $(\langle G, m \rangle, p)$ *be minimally infinitesimally rigid on* $\mathcal{T}_k^d$, *where* $0 \leq k \leq \binom{d+1}{2}$. *Then for all subgraphs* $\langle G', m' \rangle \subseteq \langle G, m \rangle$,

$$|E'| \leq d|V'| - d + k.$$

The following is a result of Nash-Williams and Tutte:

Theorem 5.4.2 ([51]). *A graph* $G = (V, E)$ *is the union of* r *edge-disjoint forests if and only if* G *satisfies* $|E'| \leq r|V'| - r$ *for all subgraphs* $G' \subseteq G$.

For an r-connected graph, we may replace forests by trees. Recall that a *map-graph* is a graph in which each connected component contains exactly one cycle (see Figure 2.3). Each connected component of a map-graph is composed of a tree plus one edge. In the appendix of [84], White and Whiteley build on Theorem 5.4.2 to prove the following:

Theorem 5.4.3 ([84]). *Let* G *be a graph with* $|E| = r|V| - \ell$, *and* $|E'| \leq r|V'| - \ell$ *for all subgraphs* $G' \subseteq G$, *and* $0 \leq \ell \leq r$. *Then* E *is a disjoint union of* ℓ *spanning trees and* $r - \ell$ *spanning map-graphs.*

Returning our attention to the partially flexible torus $\mathcal{T}_k^d$, this time with a more restricted range of flexibility: $0 \le k \le d$, we obtain:

Proposition 5.4.4. *Let $(\langle G, m \rangle, p)$ be minimally infinitesimally rigid on $\mathcal{T}_k^d$, where $0 \le k \le d$. Then the edges of $\langle G, m \rangle$ admit a decomposition into $d - k$ spanning trees and k spanning map-graphs.*

In fact, more is true: when $(\langle G, m \rangle, p)$ is infinitesimally rigid on on a torus which is only allowed to scale, $\mathcal{T}_k^d, 0 \le k \le d$, then the map-graphs must be connected. In other words, each of the k map-graphs specified by Proposition 5.4.4 must contain exactly one cycle, which is the content of the next theorem. Let $\mathcal{T}_d^d$ be the torus generated by the diagonal matrix

$$
L_d(t) = \begin{pmatrix} t_{11}(t) & 0 & \ldots & 0 \\ 0 & t_{22}(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & t_{dd}(t) \end{pmatrix}.
$$

In this case, for an edge $e = \{i, j; m_e\}$ the matrix $M(e)$ becomes the $d \times d$ matrix

$$
M(e) = \begin{pmatrix} m_1 & 0 & \ldots & 0 \\ 0 & m_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & m_d \end{pmatrix}.
$$

194

We take $\mathcal{T}_k^d$ to be the scaling torus obtained from $\mathcal{T}_d^d$ by allowing only k variable entries in the diagonal of $L_k(t)$, and fixing the other entries. We call $\mathcal{T}_k^d$ for this particular choice of k the *scaling torus*.

Theorem 5.4.5. *Let* $(\langle G, m \rangle, p)$ *be a minimally rigid framework on the scaling torus* $\mathcal{T}_k^d$, *where* $0 \le k \le d$. *Then the edges of* $\langle G, m \rangle$ *admit a decomposition into* $d - k$ *spanning trees and* k *connected spanning map-graphs.*

Proof. This proof is similar to the proof of Theorem 2.18 in [79]. Let $(\langle G, m \rangle, p)$ be a minimally rigid framework on $\mathcal{T}_k^d$. The rigidity matrix, $\mathbf{R}_k(\langle G, m \rangle, p)$ has rank $d|V| - d + k$, and dimension $(d|V| - d + k) \times (d|V| + k)$, with $d|V|$ columns corresponding to the vertices, and k columns corresponding to the lattice elements. Adding d rows

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \cdots$$

has the effect of eliminating the d-dimensional space of infinitesimal translations. This "tie down" is described in [79], and is equivalent to pinning one vertex on the torus. The resulting square matrix has $d|V| + k$ independent rows, and hence a non-zero determinant.

Reorder the columns of $\mathbf{R}_k(\langle G, m \rangle, p)$ by coordinates, with the k lattice columns for the generators of the flexible torus separated as well. Regard the determinant as a Laplace decomposition where the terms are products of the determinants of k

195

square blocks M_i with dimension $(|V|+1) \times (|V|+1)$, and $d-k$ square blocks N_i with dimension $|V| \times |V|$. Each block is composed of entries from the columns of the $i-th$ coordinates, and each block contains a single tie-down row.

There must be at least one nonzero product $\prod_{i=1}^{k} \det(M_i) \prod_{j=1}^{d-k} \det(N_j)$. By the Laplace decomposition, the rows used in each M_i and N_j form disjoint subgraphs, and each M_i or N_j will contain one of the tie-down rows. In addition, each of the k M_i blocks will contain one column from the lattice columns. The $|V|$ rows of M_i that are not tie-down rows have rank $|V|$, and this submatrix corresponds to the rigidity matrix of a 1-dimensional graph on the flexible circle (the periodic line). By Proposition 5.2.3, we know that such a graph must be connected, and must contain a constructive cycle. Hence the $|V|$ edges of any M_i block form a spanning connected map-graph.

Similarly, the $|V|-1$ edges of a N_j block that are not tie-down edges correspond to the rigidity matrix of a graph on the fixed circle. Since each block N_j has rank $|V|$, the edges that are not tie-down edges are independent on the fixed circle, and hence by Proposition 5.2.1 the graph is connected. Therefore the edges of each N_j block form a spanning tree of G. $\qquad \square$

We will make use of this result when we turn our attention to frameworks on $\mathcal{T}_x^2$ – the torus allowed only to scale in one direction – in Section 5.5.

The proof of the following lemma is a straightforward argument similar to the

proof of Lemma 4.4.9. See also Fekete and Szegő [27], Lemma 2.3.

Lemma 5.4.6. *Let $0 \leq \ell \leq r$. Let $G = (V, E)$ be a multigraph with $|E| = r|V| - \ell$, and satisfying $|E'| \leq r|V'| - \ell$ for all subgraphs $G' \subseteq G$. Let $X, Y \subset V$, and let $b(X) = r|X| - \ell - |E(X)|$, where $E(X)$ are the induced edges of G. If $b(X) = b(Y) = 0$ and $X \cap Y \neq \emptyset$, then $b(X \cup Y) = b(X \cap Y) = 0$.*

In other words, the set of all intersecting subgraphs $G' \subseteq G$ with $|E'| = r|V'| - \ell$ form a lattice. Let G have $|E| = r|V| - \ell$. We call a subgraph $G' \subset G$ with $|E'| = r|V'| - \ell$ *fully counted*.

Lemma 5.4.7. *Suppose G has $|E| = d|V| - d + k$, where $0 < k \leq d$. If the edges of G admit a decomposition into $d - k$ (edge-disjoint) spanning trees and k connected spanning map-graphs, then G contains at most one fully-counted subgraph G^* that itself contains no proper fully-counted subgraphs.*

Proof. Suppose that G has a decomposition into $d - k$ (edge-disjoint) spanning trees and k connected spanning map-graphs. Toward a contradiction, suppose that G has two subgraphs G_1 and G_2 with $|E_i| = d|V_i| - d + k$, and which contain no smaller fully-counted subgraphs. It must be the case the $V_1 \cap V_2 = \emptyset$, otherwise their intersection would be a smaller fully counted subgraph by Lemma 5.4.6. The decomposition of G has exactly k cycles. However, since neither G_1 nor G_2 admits a decomposition into edge-disjoint forests (they contain too many edges by Theorem

5.4.2), any decomposition of G must contain at least $2k$ cycles, k from each of G_1 and G_2, a contradiction. $\qquad\square$

Corollary 5.4.8. *Let G be a graph satisfying the hypotheses of Lemma 5.4.7. Then G contains a unique smallest fully-counted subgraph G^* which contains no smaller fully-counted subgraphs.*

We shall soon seen that in the two-dimensional case, it is this smallest subgraph which "rigidifies" the flexible torus, and the rest of the graph can be viewed as essentially living on the fixed torus.

We conclude this section with a further observation about the two-dimensional flexible torus. When we move from the fixed torus to the flexible torus, we add columns to the rigidity matrix. As a result, it seems possible that edges that were dependent on the fixed torus become independent on the flexible torus. When $d = 2$, and our dependent subgraphs are of size $2|V| - 2$, this is not the case.

Proposition 5.4.9. *Let $\langle G, m \rangle$ be an orbit graph with $|E| = 2|V| - 2$ and $|E'| \leq 2|V'| - 2$ for all subgraphs $\langle G', m' \rangle \subseteq \langle G, m \rangle$. If $\langle G, m \rangle$ is dependent on $\mathcal{T}_0^2$ then $\langle G, m \rangle$ is also dependent on $\mathcal{T}_k^2$, $1 \leq k \leq 3$.*

Proof. Suppose that $\langle G, m \rangle$ is dependent on $\mathcal{T}_0^2$. Then there is some subgraph $\langle G', m' \rangle \subseteq \langle G, m \rangle$ with $|E'| = 2|V'| - 2$, and no constructive cycle. Therefore, all gains on this subgraph are T-gain equivalent to $(0, 0)$. Then the entries in the lattice

columns of the rigidity matrix corresponding to these edges will be zero, since $M(e)$ is the zero matrix for all e, and therefore the edges continue to be dependent on $\mathcal{T}_k^2$. □

As in Chapter 4, we would like to find necessary and sufficient conditions for rigidity on the (partially) flexible torus $\mathcal{T}_k^d$. The necessary conditions outlined above are not sufficient in general, for the reasons stated in Section 4.5, so we focus our attention on the case $d = 2$. In the next section we find necessary and sufficient conditions for the case when $k = 1$. That is, we will develop *further* necessary conditions, and show that a key set of these are also sufficient in this special case.

5.5 Necessary and sufficient conditions for infinitesimal rigidity of frameworks on $\mathcal{T}_x^2$

We now consider the case of the flexible torus $\mathcal{T}_x^2$, which is flexible in the x-direction only. Of course, these results can be translated directly to the torus which flexes in the y direction only. Furthermore, we may expand this characterization to any flexible torus with one degree of freedom, such as the torus with a flexible angle. We elaborate on this idea in Section 5.5.3. Using affine invariance of infinitesimal

rigidity on the torus, the lattice matrix L_x is

$$L_x = \begin{pmatrix} x(t) & 0 \\ 0 & 1 \end{pmatrix},$$

and $\mathcal{T}_x^2 = \mathbb{R}^d / L_x \mathbb{Z}^d$. The basic idea of this section is that we will show that frameworks that are infinitesimally rigid on $\mathcal{T}_x^2$ contain a smallest rigid sub-framework, which "rigidifies" the flexible torus.

We will need the following stronger form of Lemma 5.4.7 for this case.

Lemma 5.5.1. *Let G be a graph with $|E| = 2|V| - 1$. Then the edges of G admit a decomposition into a spanning tree and a connected spanning map-graph, if and only if G contains at most one fully counted subgraph G^* which does not contain any smaller fully counted subgraphs, and is 2-connected.*

Proof. The necessity of this lemma is Lemma 5.4.7. For the other direction, let G^* be the subgraph with $|E^*| = 2|V^*| - 1$ that does not contain any smaller fully-counted subgraphs. Delete any edge $e \in E^*$. The resulting graph $G^* - e$ must have $|E'| \leq 2|V'| - 2$ for all subgraphs $G' \subseteq G^*$ (any subgraph with more than $2|V'| - 2$ edges would contradict the minimality of G^*). Such a graph has a decomposition into two spanning trees by Theorem 5.4.2. Adding back the edge e creates exactly one cycle, which forms the connected spanning map-graph. $\square$

Remark 5.5.2. The d-dimensional analogue of this result is not clear. Deleting any

edge from G^*, where $|E^*| = d|V^*| - d + k$ does not necessarily leave a graph with $|E'| = d|V'| - d + k - 1$ that has a unique smallest fully counted subgraph. $\square$

Corollary 5.5.3. *Let G be a graph with $|E| = 2|V| - 1$ whose edges admit a decomposition into a spanning tree and a connected spanning map-graph. Then G has a unique smallest subgraph G^* with $|E^*| = 2|V^*| - 1$.*

We call the periodic orbit graph $\langle G^*, m^* \rangle$ induced by this unique smallest subgraph G^* the *critical subgraph*. In the remainder of this section we will show that the generic rigidity of a periodic orbit framework $\langle G, m \rangle$ on $\mathcal{T}_x^2$ is determined by the rigidity of this critical subgraph (Theorem 5.5.5). The following theorem determines sufficient conditions for critical subgraphs to be infinitesimally rigid on $\mathcal{T}_x^2$. The majority of this section is devoted to the proof of this result.

Theorem 5.5.4. *Let $\langle G, m \rangle$ be a periodic orbit graph with $|E| = 2|V| - 1$, that contains no fully counted proper subgraphs. That is, $|E'| \leq 2|V'| - 2$ for all proper subgraphs $G' \subseteq G$. Then $\langle G, m \rangle$ is generically minimally rigid on $\mathcal{T}_x^2$ if and only if $\langle G, m \rangle$ satisfies:*

1. *Any subset E_0 of edges of $\langle G, m \rangle$ satisfying $|E_0| = 2|V_0| - 2$ has a constructive cycle.*

2. *$\langle G, m \rangle$ contains an x-constructive cycle. That is, there is a cycle in $\langle G, m \rangle$ that has a non-zero net gain in the x-coordinate.*

201

Proof. The necessity of the two conditions is clear. Part 1 follows from Proposition 5.3.9. To see Part 2, suppose that $\langle G, m \rangle$ contains no x-constructive cycle. Applying the T-gain procedure to any tree in G will produce a T-gain assignment m_T, where all x-coordinates are zero. Then all of the entries of the single lattice column of the rigidity matrix will be zero. Hence we effectively have $2|V| - 1$ edges in the fixed torus rigidity matrix, which has maximum rank $2|V| - 2$, a contradiction.

The remainder of this chapter is devoted to showing the sufficiency of 1 and 2 for rigidity. The proof will be continued in Section 5.5.2. $\qquad\square$

We are now ready to state a more general version of Theorem 5.5.4, which is the main result of this section. If $\langle G, m \rangle$ contains subgraphs G' with $|E'| = 2|V'| - 1$, then the following result states that we need to find the critical subgraph $\langle G^*, m^* \rangle$ that must itself be generically rigid on $\mathcal{T}_x^2$. The basic idea is that the critical subgraph rigidifies the flexible torus, and then the rest of the framework behaves as a framework on the fixed torus.

Theorem 5.5.5. *Let $\langle G, m \rangle$ be a periodic orbit graph satisfying $|E| = 2|V| - 1$, and $|E'| \leq 2|V'| - 1$ for all $G' \subseteq G$. Then $\langle G, m \rangle$ is generically minimally rigid on $\mathcal{T}_x^2$ if and only if*

1. *The critical subgraph $\langle G^*, m^* \rangle$ of $\langle G, m \rangle$ with $|E^*| = 2|V^*| - 1$ is generically infinitesimally rigid on $\mathcal{T}_x^2$*

2. *for any* $e \in E\langle G^*, m^* \rangle$, $\langle G - e, m - m(e) \rangle$ *is infinitesimally rigid on* $\mathcal{T}_0^2$.

Proof. The necessity of Part 1 is immediate. The necessity of Part 2 follows from the fact that deleting any edge of $\langle G^*, m^* \rangle$ leaves a graph with $|E| = 2|V| - 2$, and all subgraphs having $|E'| \leq 2|V'| - 2$. By Proposition 5.3.9, if these edges are dependent on $\mathcal{T}_0^2$, then they will also be dependent on $\mathcal{T}_x^2$.

For sufficiency, we claim that $\langle G, m \rangle$ is generically rigid on $\mathcal{T}_x^2$ if we can construct $\langle G, m \rangle$ from $\langle G^*, m^* \rangle$ by a sequence of (fixed-torus) Henneberg moves. First note that the fixed torus inductive constructions also preserve independence on $\mathcal{T}_x^2$. Hence if we begin with the rigid framework $\langle G^*, m^* \rangle$, and perform 2-valent vertex additions and edge-splits, the resulting periodic orbit graph is also generically rigid on $\mathcal{T}_x^2$. In addition, By Part 2, $(\langle G, m \rangle, p)$ is infinitesimally rigid on $\mathcal{T}_0^2$. Therefore, deleting 2- and 3-valent vertices will also preserve rigidity, provided that those deletions occur in $G \backslash G^*$. It is possible to lose independence through reverse inductive constructions if working within $\langle G^*, m^* \rangle$, by deleting the x-constructive cycle. So we need to show that we can always construct $\langle G, m \rangle$ from $\langle G^*, m^* \rangle$ without changing $\langle G^*, m^* \rangle$.

Let $|V| = |V^*| + n$. We use induction on n, the number of vertices we need to add to V^* to get V. When $n = 1$, $\langle G, m \rangle$ must be a periodic vertex addition on $\langle G^*, m^* \rangle$. It can't be an edge split, since in that case, $\langle G^*, m^* \rangle$ is no longer a subset of $\langle G, m \rangle$. Suppose the claim holds for $n \geq 1$, and let $|V| = |V^*| + (n + 1)$. By
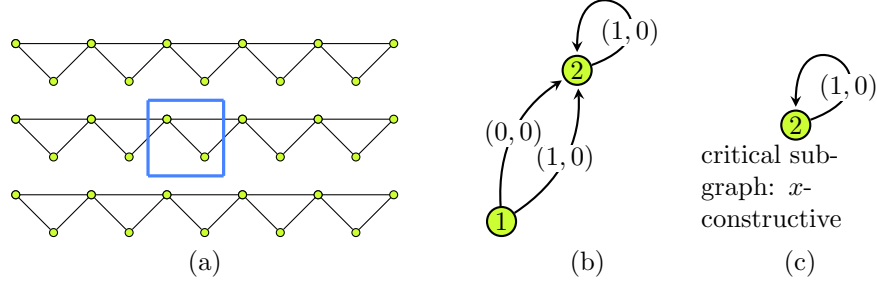
203

Figure 5.6: A periodic framework on $\mathcal{T}_x^2$ (a). The periodic orbit graph is shown in (b) and the critical subframework is the single loop shown in (c).

the same argument as in the proof of the Periodic Henneberg Theorem (Theorem 4.3.8), there must be some vertex of valence 2 or 3. The rest of the argument is the same as for the proof of Theorem 4.4.5: we may delete this vertex in a reverse edge split or 2-valent deletion, as long as no edges are added to $\langle G^*, m^* \rangle$ in the process. Furthermore we may always do this in such a way that preserves constructive cycles and x-constructive cycles, and the fact that Part 2 holds. The resulting graph on $|V^*| + n$ vertices is infinitesimally rigid on $\mathcal{T}_x^2$, by the inductive hypothesis. Performing the relevant periodic Henneberg move yields the result. $\square$

Remark 5.5.6. If $\langle G, m \rangle$ contains a loop, then the loop edge and the loop vertex must form the critical subgraph $\langle G^*, m^* \rangle$. It must therefore be the case that loop gain is x-constructive (see Figure 5.6). $\square$

Corollary 5.5.7. *Let $\langle G, m \rangle$ be a periodic orbit graph satisfying $|E| = 2|V| - 1$, and having exactly one loop, e_0. Then $\langle G, m \rangle$ is generically minimally infinitesimally*

204

rigid on $\mathcal{T}_x^2$ if and only if

1. *the loop has an x-constructive gain.*

2. *$\langle G, m \rangle - e_0$ is minimally infinitesimally rigid on $\mathcal{T}_0^2$*

Corollary 5.5.8. *Let $\langle G, m \rangle$ be a periodic orbit graph where the edges $E\langle G, m \rangle$ admit a decomposition into a spanning tree and a connected spanning map-graph , where the cycle part of the map-graph is non-zero in the x-direction. Then $\langle G, m \rangle$ is infinitesimally rigid on $\mathcal{T}_x$.*

Proof. If $\langle G, m \rangle$ has a decomposition into a connected spanning map-graph and a tree, then it must be the case that the cycle part of the map-graph is in the unique smallest subgraph $\langle G^*, m^* \rangle$. $\square$

5.5.1 Algebraic geometry preliminaries

We now introduce some basic notions from algebraic geometry which will play a role in the proof of Theorem 5.5.4. Recall the following basic definitions from algebraic geometry, see Cox, Little and O'Shea [17] for a basic reference.

Let F be an arbitrary field. A subset $I \subset F[x_1, \ldots, x_n]$ is an *ideal* if it satisfies:

(i) $0 \in I$.

(ii) If $f, g \in I$, then $f + g \in I$.

(iii) If $f \in I$ and $h \in F[x_1, \ldots, x_n]$, then $hf \in I$.

Let $f_1, \ldots, f_s$ be polynomials in $F[x_1, \ldots, x_n]$. Then let

$$\mathbf{V}(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in F^n \; : \; f_i(a_1, \ldots, a_n) = 0 \;\; \text{for all} \;\; 1 \leq i \leq s\}.$$

We call $\mathbf{V}(f_1, \ldots, f_s)$ the *affine variety* defined by $f_1, \ldots, f_s$. In other words, the affine variety is the set of all solutions of the system of equations $f_1(x_1, \ldots, x_n) = \cdots = f_s(x_1, \ldots, x_n) = 0$.

Let $f_1, \ldots, f_s$ be polynomials in $F[x_1, \ldots, x_n]$. Then let

$$\langle f_1, \ldots, f_s \rangle = \left\{ \sum_{i=1}^{s} h_i f_i \; : \; h_1, \ldots, h_s \in F[x_1, \ldots, x_n] \right\}.$$

Lemma 5.5.9. *If $f_1, \ldots, f_s \in F[x_1, \ldots, x_n]$, then $\langle f_1, \ldots, f_s \rangle$ is an ideal of $F[x_1, \ldots, x_n]$.*

We will call $\langle f_1, \ldots, f_s \rangle$ the *ideal generated by* $f_1, \ldots, f_s$. An ideal I is *finitely generated* if there exist $f_1, \ldots, f_s \in F[x_1, \ldots, x_n]$ such that $I = \langle f_1, \ldots, f_s \rangle$. In this case we call the polynomials $f_1, \ldots, f_s$ a *basis* of I.

Proposition 5.5.10 (Proposition 4 of 1.4, [17]). *If $f_1, \ldots, f_s$ and $g_1, \ldots, g_t$ are bases of the same ideal I in $F[x_1, \ldots, x_n]$, so that $\langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$, then $\mathbf{V}(f_1, \ldots, f_s) = \mathbf{V}(g_1, \ldots, g_t)$.*

In what follows we will be talking about the variety corresponding to ideals. We write $\mathbf{V} = \mathbf{V}(I)$ to denote $\mathbf{V}(f_1, \ldots, f_s)$, where $f_1, \ldots, f_s$ are a basis for I, and

206

the previous proposition assures us that $\mathbf{V}$ is actually independent of the choice of basis.

We need a few more special ideals. These definitions appear in [18]:

Let I, J be ideals in $F[x_1, \ldots, x_n]$, and define $I \cdot J$ to be the ideal generated by all the product polynomials $f \cdot g$ where $f \in I$ and $g \in J$. We call $I \cdot J$ the *product* of the ideals I and J, and it is not hard to show that $I \cdot J$ is itself an ideal.

The *intersection* $I \cap J$ of two ideals I and J in $F[x_1, \ldots, x_n]$ is the set of polynomials which belong to both I and J. Note that $I \cdot J \subseteq I \cap J$, since elements of $I \cdot J$ are sums of polynomials of the form $f \cdot g$ with $f \in I$ and $g \in J$. Because $f \in I$ and $g \in J$, these polynomials are in both I and J, and hence are in the intersection ideal. The opposite inclusion is not necessarily true, however.

There is also a notion of the *quotient ideal* of I by J:

$$(I : J) = \{f \in F[x_1, \ldots, x_n] \ : \ fg \in I \text{ for all } g \in J\}. \qquad (5.11)$$

Lemma 5.5.11 ([17], Theorem 7 page 184)**.** *If I and J are ideals in $F[x_1, \ldots, x_n]$, then*

$$\mathbf{V}(I \cdot J) = \mathbf{V}(I) \cup \mathbf{V}(J).$$

Lemma 5.5.12 ([17], Theorem 5 page 189)**.** *If I and J are ideals in $F[x_1, \ldots, x_n]$, then*

$$\mathbf{V}(I \cap J) = \mathbf{V}(I) \cup \mathbf{V}(J).$$

In other words, the varieties corresponding to $I \cdot J$ and $I \cap J$ are the same. Both results can be extended by induction to finite products and finite intersections respectively.

5.5.2 Proof of Theorem 5.5.4

Let $\langle G, m \rangle$ be a periodic orbit graph with $|E| = 2|V| - 1$, and containing no smaller fully-counted subgraphs. Suppose that $\langle G, m \rangle$ additionally satisfies the two conditions of Theorem 5.5.4, namely

1. Any subset E_0 of edges of $\langle G, m \rangle$ satisfying $|E_0| = 2|V_0| - 2$ has $|\mathcal{M}_{\mathcal{C}}(E_0)| \geq 1$ (i.e. it has a constructive cycle).

2. $\langle G, m \rangle$ contains an x-constructive cycle. That is, there is a cycle in $\langle G, m \rangle$ that has a non-zero net gain in the x-coordinate.

We will show that $\langle G, m \rangle$ is generically rigid on $\mathcal{T}_x^2$. We will denote the rigidity matrix of $\langle G, m \rangle$ on $\mathcal{T}_x^2$ by $\mathbf{R}_x(\langle G, m \rangle, p)$, and note that the properties of $\mathbf{R}_x$ are as for $\mathbf{R}_1$.

Let a spanning tree T be selected at random, and find the T-gain equivalent graph $\langle G, m_T \rangle$. This graph will have at most $|V|$ edges with non-zero gains, since every tree edge has gain $(0, 0)$. Label the edges of $\langle G, m_T \rangle$ so that the the tree edges are denoted $t_1, \ldots, t_{|V|-1}$, and the $|V|$ non-tree edges are labeled $e_1, \ldots, e_{|V|}$.

Let the rigidity matrix of this periodic orbit graph be denoted $\mathbf{R}_x(\langle G, m_T\rangle, p)$, and recall that $\mathrm{rank}\mathbf{R}_x(\langle G, m\rangle, p) = \mathrm{rank}\mathbf{R}_x(\langle G, m_T\rangle, p)$ for generic p.

We select some tie-down (we can think of this as pinning a single vertex of the framework). This method is adapted from [79]. The augmented matrix $\mathbf{R}'_x(\langle G, m_T\rangle, p)$ corresponding to this tied-down graph will be as follows:

$$
\begin{array}{c}
\\
t_1 \\
\vdots \\
t_{|V|-1} \\
\hline
e_1 \\
e_2 \\
\vdots \\
e_{|V|} \\
\hline
\\
\\
\end{array}
\begin{array}{c}
\overset{p_j}{}\quad\quad\quad\overset{p_k}{}\quad\quad\quad\quad\overset{x(t)}{}\\
\left(
\begin{array}{ccccccc}
 & & & & & & 0 \\
0\ldots 0 & p_j - p_k & 0\ldots 0 & p_k - p_j & 0\ldots 0 & & \vdots \\
 & & & & & & 0 \\
\hline
 & & & & & & [m_1(\quad\ldots\quad)]_x \\
0\ldots 0 & p_j - (p_k + m_2) & 0\ldots 0 & (p_k + m_2) - p_j & 0\ldots 0 & [m_2(p_j - (p_k + m_2))]_x \\
 & & & & & & \vdots \\
 & & & & & [m_{|V|}(\quad\ldots\quad)]_x \\
\hline
1\ 0\ 0 & 0 & 0 & \ldots & & & 0 \\
0\ 1\ 0 & 0 & 0 & \ldots & & & 0
\end{array}
\right)
\end{array}
$$

We take the determinant of this $(2|V|+1) \times (2|V|+1)$ matrix, using a Laplace decomposition along the last column to obtain:

$$
\begin{aligned}
\det \mathbf{R}'_x(\langle G, m_T\rangle, p) &= \sum_{i=1}^{|V|} (-1)^i [m(e_i)\,(o(e_i) - (t(e_i) + m(e_i)))]_x \det \mathbf{R}_i \\
&= \sum_{i=1}^{|V|} (-1)^i [m(e_i)o(e_i) - m(e_i)t(e_i) - m(e_i)^2]_x \det \mathbf{R}_i \quad (5.12)
\end{aligned}
$$

209

where e_i is the edge $e_i = \{o(e_i), t(e_i); m(e_i)\}$, and $\det \mathbf{R}_i$ is the subdeterminant of $\mathbf{R}'_x(\langle G, m_T \rangle, p)$ obtained by deleting the row corresponding to the edge e_i and the final column of $\mathbf{R}'_x(\langle G, m_T \rangle, p)$ (corresponding to the generator $x(t)$). By Lemma 5.5.1, deleting any edge leaves a subset with $|E| = 2|V| - 2$, and $|E'| \le 2|V'| - 2$ for all $G' \subset G$. Moveover, by Part 1 of the theorem, any subset with $|E'| = 2|V'| - 2$ has $|\mathcal{M}_\mathcal{C}(E')| \ge 1$. It follows that any such subset is therefore generically rigid on $\mathcal{T}_0^2$, and hence $\det \mathbf{R}_i \ne 0$ for all i.

Then $\det \mathbf{R}'_x(\langle G, m_T \rangle, p)$ is a polynomial whose indeterminants are the generic coordinates of the vertices $v_1, \ldots, v_{|V|}$, and whose coefficients are themselves polynomials in the polynomial ring $M = \mathbb{Z}[m(e_i)_x, m(e_i)_y]$, for $i = 1, \ldots, |V|$. In other words, this is the ring of polynomials with integer coefficients whose indeterminants are the non-zero gains on the edges of $\langle G, m_T \rangle$.

Let I be the ideal generated by the coefficients of $\det \mathbf{R}'_x(\langle G, m_T \rangle, p)$, and let $I_{m(e_i)} = I_i$ be the ideal generated by the coefficients of $\det \mathbf{R}_i$. Let

$$I^* = \prod_{i=1}^{|V|} I_i.$$

Note therefore that

$$I^* \subseteq \bigcap_{i=1}^{|V|} I_i.$$

The goal of the remainder of the proof is to demonstrate that if $\langle G, m \rangle$ satisfies the conditions of the theorem, then m_T is not on the variety corresponding to the

ideal I, $\mathbf{V}(I)$. That is, $m_T \notin \mathbf{V}(I)$. This will imply that $\det \mathbf{R}'_x(\langle G, m_T \rangle, p) \neq 0$, and hence $(\langle G, m \rangle, p)$ is infinitesimally rigid on $\mathcal{T}^2_x$.

First observe that $\mathbf{V}(I) \subseteq \mathbf{V}(I : I^*) \cup \mathbf{V}(I^*)$. Why?

$$(I : I^*) \cdot I^* \subseteq I,$$

since for any $f \in (I : I^*)$, $fg \in I$ for all $g \in I^*$ by the definition of the quotient ideal. It follows that

$$\mathbf{V}(I) \subseteq \mathbf{V}((I : I^*) \cdot I^*)$$

because the operations of finding ideals and finding the varieties is inclusion-reversing. Finally, by Lemma 5.5.11, it follows that

$$\mathbf{V}(I) \subseteq \mathbf{V}(I : I^*) \cup \mathbf{V}(I^*).$$

Iterating this result, we obtain

$$\mathbf{V}(I) \subseteq \mathbf{V}(I : I^*) \cup \left[\bigcup_{i=1}^{|V|} \mathbf{V}(I_i) \right].$$

We know that $m_T \notin \mathbf{V}(I_i)$, since $\det \mathbf{R}_i \neq 0$ for all i, which guarantees that deleting any edge leaves a rigid graph on $\mathcal{T}^2_0$. Therefore we need only show that $m_T \notin \mathbf{V}(I : I^*)$ to conclude that $m_T \notin \mathbf{V}(I)$ either.

211

To this end, consider $(I : I^*) = \{f \in M : fg \in I \text{ for all } g \in I^*\}$, and let

$g \in I^*$. Then $g \in \prod I_i$, and hence

$$g \in \prod I_i \implies g \in \bigcap_{i=1}^{|V|} I_i$$
$$\implies g \in I_i, \text{ for } i = 1, \ldots, |V|.$$

Now, for any ideal I_i, $m(e_i)_x^2 \notin I_i$, since I_i is generated by the coefficients of

$\det \mathbf{R}_i$, and the row corresponding to e_i is not in $\mathbf{R}_i$. However, by considering

equation (5.12), it is clear that for any $g \in I^*$,

$$m(e_i)_x^2 \cdot g \in I.$$

Putting these facts together, by (5.11) we see that $m(e_i)_x^2 \in (I : I^*)$, for any

$i = 1, \ldots, |V|$.

Condition 2 of the theorem is equivalent to demanding that for some $i \in$

$1, \ldots, |V|$, $m(e_i)_x \neq 0$. Since $m(e_i)_x^2 \in (I : I^*)$, we may conclude, finally, that

$m_T \notin \mathbf{V}(I : I^*)$, which proves the claim.

$\square$

Remark 5.5.13. The techniques of this proof rest on the fact that when we delete

any edge in the minimal subgraph, we are left with a framework that is independent

on $\mathcal{T}_0^2$. When we are working on $\mathcal{T}_2^2$ or $\mathcal{T}^2$, the same is not true. $\square$

5.5.3 Other variations of the flexible torus with one degree of freedom

As mentioned in the introduction to this section, it is possible to adapt the preceding proof technique to other situations in which the two dimensional torus has one degree of freedom. For instance, let $\mathcal{T}_\theta^2$ be the torus generated by the lattice matrix

$$
L_\theta = \begin{pmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{pmatrix}.
$$

This is the torus that has generators with fixed lengths, and a variable angle between them. Then the rigidity matrix for frameworks on $\mathcal{T}_\theta^2$ has a column corresponding to the variable $\theta(t)$. Instead of requiring an x-constructive cycle, the necessary condition for rigidity can be seen to be that the critical subgraph contains a cycle with net gain (m_1, m_2) satisfying $m_1 m_2 \neq 0$. The other requirements for rigidity are as for frameworks on $\mathcal{T}_x^2$.

Similarly, we could consider frameworks on a torus where the area scales. This would be generated by the lattice matrix

$$
L_{vol} = \begin{pmatrix} x & 0 \\ 0 & kx \end{pmatrix}.
$$

A framework can easily seen to be infinitesimally rigid on this torus if and only if it is rigid on $\mathcal{T}_x^2$ or $\mathcal{T}_y^2$.
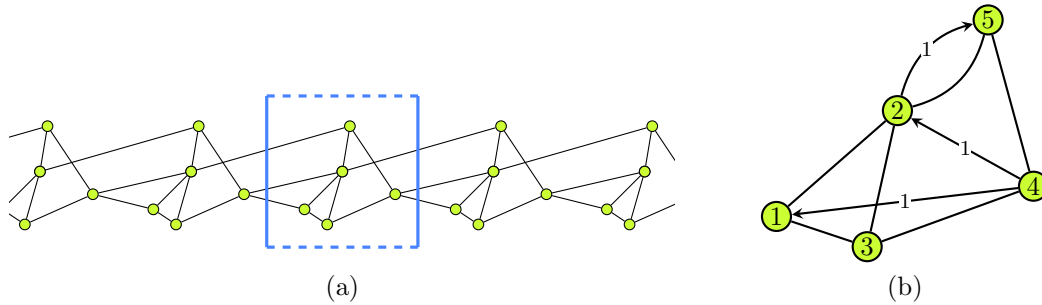
Figure 5.7: A framework which is periodic in one direction only (a), and its gain graph (b) which is labeled by elements of $\mathbb{Z}$.

5.6 Discussion

5.6.1 The cylinder

As a direct consequence of Theorem 5.5.5 we obtain a characterization of the generic rigidity of frameworks which are periodic in one direction only. That is, Theorem 5.5.5 provides necessary and sufficient conditions for the rigidity of frameworks on the cylinder with variable circumference. See Figure 5.7 for an example. Such frameworks are similar to *frieze patterns*, although we assume that such patterns exhibit *only* translational symmetry (in one direction), and do not have any of the other symmetries of frieze patterns.

Let $\langle G, m \rangle$ be a gain graph with $m : E \to \mathbb{Z}$. For each edge $e = \{i, j; m_e\}$, the integer m_e now represents the number of times the edge e "wraps" around the cylinder. Let $p : V \to \mathbb{R}^2/(\mathbb{Z} \times Id)$. That is, for $v \in V$, $p(v) \in [0, 1) \times \mathbb{R}$. Let $\mathcal{C} = [0, 1) \times \mathbb{R}$, and we call this the *flexible cylinder*.

214

Let $\langle G, m \rangle$ be a gain graph with gain assignments from $\mathbb{Z}$. Let $\hat{m} : E \to \mathbb{Z}^2$ be the gain assignment on G given by $\hat{m}(e) = (m(e), 0)$. Since the rigidity matrix for the cylinder and the rigidity matrix for $\mathcal{T}_x^2$ are identical (each has exactly one lattice column), we have the following proposition.

Proposition 5.6.1. *A periodic orbit graph $\langle G, m \rangle$ is generically rigid on the flexible cylinder $\mathcal{C}$ if and only if $\langle G, \hat{m} \rangle$ is generically rigid on $\mathcal{T}_x^2$.*

5.6.2 Inductive techniques for the flexible torus

One might ask whether it is possible to obtain the necessary and sufficient conditions on $\mathcal{T}_x^2$ using inductive techniques. The fixed torus inductive moves of Chapter 4 immediately extend to the flexible torus, and preserve independence in this setting. However, deleting three-valent vertices could eliminate the x-constructive cycle, so we cannot completely characterize the $2|V| - 1$ graphs with the existing moves. A careful modification of the existing rules seems possible, and could provide an alternative proof of Theorem 5.5.5.

The inductive techniques become more problematic as we move to the fulling scaling torus $\mathcal{T}_2^2$ and the fully flexible torus $\mathcal{T}^2$. In both cases, we may have periodic orbit graphs were all vertices have valence ≥ 4. This means that we need another inductive technique to generate the four-valent vertices. This is possible using x-replacement, and it is not hard to show that x-replacement preserves rigidity on

$\mathcal{T}^2$. At this point, it is not clear that reversing the operation of x-replacement preserves rigidity for periodic orbit frameworks.

5.6.3 Results in context

Recently, in independent work, Malestein and Theran completely characterized generic rigidity on the two-dimensional flexible torus. They find

Theorem 5.6.2 ([49])**.** *The periodic orbit graph* $\langle G, m \rangle$ *is generically rigid on* $\mathcal{T}^2$ *if and only if, for all subsets of edges* $Y \subset E$,

$$|Y| \leq 2|V(Y)| - 3 + 2|\mathcal{M}_{\mathcal{C}}(Y)| - 2(c - 1),$$

where c is the number of connected components of the subset.

The proof uses periodic direction networks. They do not, however, address the case $\mathcal{T}_x^2$. It should be noted that the result for the scaling torus $\mathcal{T}_x^2$ builds on the characterization for $\mathcal{T}_0^2$, which was proved in Chapter 4 using inductive methods. Malestein and Theran have obtained the same result using non-inductive methods, as a corollary to their result stated above. Since the extension of the characterization of the fixed torus to $\mathcal{T}_x^2$ presented in the proof of Theorem 5.5.5 is a non-inductive matrix argument, it follows that $\mathcal{T}_x^2$ is characterized non-inductively.

6 Algorithms for frameworks on the fixed torus $\mathcal{T}_0^2$

6.1 Introduction

In this chapter we outline an algorithm for testing the generic rigidity of graphs on the two-dimensional fixed torus $\mathcal{T}_0^2$. This tests whether a periodic orbit graph $\langle G, m \rangle$ satisfies the hypothesis of the periodic Laman theorem of Chapter 4, namely Theorem 4.4.4. This algorithm is a modification of the existing *pebble game* algorithm for finite rigidity.

6.2 Background: the pebble game algorithm

The pebble game is an algorithm that was first introduced by Jacobs and Hendrickson in [41] to identify rigid regions in network glasses. These were modelled on two-dimensional graphs, and hence the pebble game was used to find graphs satisfying the conditions of Laman's Theorem (Theorem 2.5.10), which characterizes

generic two-dimensional rigidity.

In [47], Lee and Streinu address pebble game algorithms for a broader class of graphs. We use their terminology here. Let $G = (V, E)$ be a multigraph. Recall that we say that a vertex set $V' \subset V$ spans the edges $E' \subset E$ if (V', E') is the subgraph of G induced by V'. We call G (r, ℓ)-*sparse* if every subset of vertices $V' \subseteq V$ spans at most $r|V'| - \ell$ edges. If, in addition, $|E| = r|V| - \ell$, we say that G is (r, ℓ)-*tight*. G is (r, ℓ)-*spanning* if it contains a spanning subgraph that is (r, ℓ)-tight. Lee and Streinu present pebble game algorithms to test for (r, ℓ)-sparsity, $0 \leq \ell < 2r$.

The (r, ℓ)-pebble game algorithm is detailed in Algorithm 1. The presentation is similar to that of Adnan Sljoka in [71], and Lee and Streinu in [47].

Algorithm 1 (**The (r, ℓ)-pebble game**).

Input: Multigraph $G = (V, E)$.

Output: "well-constrained", "over-constrained", "under-constrained" or "other".

Setup: Place r pebbles on each vertex. Initialize $I(G)$ and $R(G)$ to be empty sets of edges which will represent the independent and redundant edges respectively.

Edge-acceptance condition: An edge between two vertices is added to $I(G)$ when a total of $\ell + 1$ free pebbles are found on the two end points of the edge.

Algorithm: Test the edges of G in an arbitrary order.

1. As long as all of the edges of G are not tested, take any edge e and go to step 2.

2. Count the number of free pebbles on the end vertices of e, say $e = \{u, v\}$.

   a) If u and v have at least $\ell + 1$ free pebbles, then place a pebble on e and direct the edge e away from the source of the pebble. Add this directed edge e to the set $I(G)$, and return to 1.

   b) Else, u and v are not covered by $\ell + 1$ pebbles. Starting at the endpoints of the edge e, search the vertices along the directed edges of $I(G)$ for a free pebble (e.g. by depth-first search).

      i) If a free pebble is found on some vertex at the end of a directed path $P \in I(G)$ from u or v, perform a sequence of direction swaps of the edges of P (a *cascade*), reversing the entire path until the free pebble appears on u or v. Return to 2.

      ii) Else, no free pebble is found. Place e in the set of redundant edges $R(G)$. Return to 1.

3. Stop when there are no more edges to be tested. If there are exactly ℓ pebbles remaining on the vertices of G, and $R(G) = \emptyset$ (no edge was rejected), return "well-constrained", and "over-constrained" if $R(g) \neq \emptyset$. If there are more than ℓ pebbles remaining on the vertices of G, return "under-constrained" if

219

no edge was rejected, and "other" if an edge was rejected.

In [47], Lee and Streinu show that
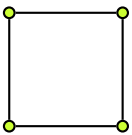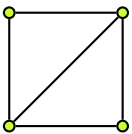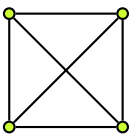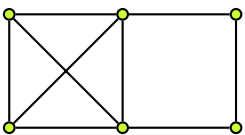
Theorem 6.2.1 ([47], Theorem 8)**.** *The class of under-constrained pebble game graphs coincides with the class of (r, ℓ)-sparse graphs. The class of well-constrained pebble game graphs coincides with the class of (r, ℓ)-tight graphs. The class of over-constrained graphs corresponds with the class of (r, ℓ)-spanning graphs, and all other pebble game graphs correspond to graphs which are neither spanning nor sparse.*

In the special case of $(2, 3)$-sparse graphs, the pebble game tells us more due to Laman's theorem, as the following example illustrates.

Example 6.2.2 (The $(2, 3)$-pebble game)*.* Letting $r = 2$, $\ell = 3$ we obtain the $(2, 3)$-pebble game. The game is initialized with two pebbles on each vertex. An edge is accepted (pebbled, directed, and placed in $I(G)$) if four free pebbles are on the endpoints of the edge.

Recall that Laman's Theorem (Theorem 2.5.10) states that a graph G is generically minimally rigid in $\mathbb{R}^2$ if and only if G is $(2, 3)$-tight. It follows that the well-constrained, over-constrained and under-constrained outputs of the $(2, 3)$-pebble game correspond respectively to (generically) minimally rigid, rigid and flexible finite frameworks. We illustrate these correspondences in Table 6.1. $\qquad\square$

Table 6.1: The (2, 3)-pebble game

Example	Pebble game output	Sparsity	Generic rigidity
	under-constrained	(2, 3)-sparse	independent, generically flexible
	well-constrained	(2, 3)-tight	minimally rigid (independent and generically rigid)
	over-constrained	(2, 3)-spanning	generically rigid
	other	none of the above	neither independent nor generically rigid

Example 6.2.3 (The $(2,2)$-pebble game). This game is set up in the same way as the $(2,3)$-pebble game, with two vertices on each vertex. An edge is accepted (placed in the set $I(G)$) if three free pebbles are found on its endpoints. By Theorem 6.2.1 if the pebble game returns "well-constrained" then the graph G is $(2,2)$-tight. By a theorem of Nash-Williams [51], a graph G is $(2,2)$-tight if and only if the edges of G admit a decomposition into two edge-disjoint spanning trees. $\square$

6.2.1 Properties of the (r, ℓ)-pebble game

Complexity analysis of Algorithm 1: Each edge is considered exactly once, and requires at most $\ell + 1$ depth-first searches through $I(G)$ (since we are looking for $\ell + 1$ pebbles), for a total of $O(|V||E|)$ time. $\square$

In the remainder of the chapter, we will be concerned with pebble games where $r = 2$ and $\ell = 2, 3$. The collection of all $(2, \ell)$-tight graphs, $\ell = 2, 3$ on n vertices can be shown to be the set of bases of a matroid [47, 84]. It follows that pebble game algorithms for the detection of such graphs are *greedy*. That is, the order of the edges tested does not determine the output of the algorithm. In fact, one can make similar statements about (r, ℓ)-sparse graphs in general, but with some further restrictions on r, ℓ, n, see Lee and Streinu [47].

We will make use of the following lemma from [47]. Since loops are always dependent for these counts ($|E| = 2|V| - 3$ and $|E| = 2|V| - 2$), we exclude them

here.

Lemma 6.2.4 (Invariants of the $(2, \ell)$-pebble game, $\ell = 2, 3$)**.** *Let $G = (V, E)$ be a multigraph without loops. Let $\ell = 2$ or 3. At every stage of the pebble game, the following invariants are maintained:*

a) *For each vertex, the number of free pebbles (pebbles lying on that vertex) plus the number of outgoing edges in $I(G)$ is exactly 2.*

b) *There are at least ℓ free pebbles on the vertices of G.*

c) *Every subset $V' \subset V$ of vertices spans at most $2|V'| - \ell$ edges in $I(G)$ (pebbled edges).*

d) *For every subgraph $G' \subset G$, with $G' = (V', E')$, (the number of free pebbles on V') + (the number of pebbled edges in E') + (the number of outgoing edges out of G') $= 2|V'|$.*

Remark 6.2.5. The first invariant of the above lemma reflects the idea that pebbles are "yo-yo-ing" on and off vertices. Two pebbles remain associated with any particular vertex, either in the form of pebbles on the vertex, or directed edges away from the vertex. □

Lemma 6.2.6. *Let G be a multigraph. At any stage of the $(2, \ell)$-pebble game on G, 2 pebbles may be recovered at any vertex v^* of G.*

The subset of edges $\hat{E}_e \subseteq I(G)$ that we search over in step 2.b.ii) determines the *failed search region* corresponding to the edge e, which we denote by $\hat{F}_e$. Let $\hat{F}_e = (\hat{V}_e, \hat{E}_e)$ where $\hat{V}_e$ is the set of vertices adjacent to the edges $\hat{E}_e$ of the failed search region. Such regions play a special role in the periodic pebble game on the fixed torus, as we shall soon see.

By Lemma 6.2.6, a failed search region of the $(2, 3)$-pebble game will have at most three free pebbles, since it would not be a failed search region if it had four or more. Furthermore, it will have no outgoing edges (since if there were outgoing edges, we would search over them too). If a failed search region $\hat{F}_e$ of the $(2, 3)$-pebble game has *exactly* three free pebbles, then as a consequence of Lemma 6.2.4d, it must satisfy $|\hat{E}_e| = 2|\hat{V}_e| - 3$. Similarly, a failed search region $\hat{F}_e$ of the $(2, 2)$-pebble game with exactly three free pebbles must satisfy $|\hat{E}_e| = 2|\hat{V}_e| - 3$, while a failed search region with exactly two free pebbles will satisfy $|\hat{E}_e| = 2|\hat{V}_e| - 2$.

For both the $(2, 3)$- and $(2, 2)$-games, in the case that there are exactly three free pebbles on the failed search region $\hat{F}_e$, we will find it useful to establish notation to refer to the failed search region *together with the edge e*. Let $V_e = \hat{V}_e$, and let $E_e = \hat{E}_e \cup \{e\}$ be the edges of the failed search region plus the edge e. Now put $F_e = (V_e, E_e)$, and call F_e the $(2, 2)$-*critical subgraph induced by e*. In other words F_e is simply the failed search region $\hat{F}_e$ together with the edge e. The edges $\hat{E}_e$ of the failed search region $\hat{F}_e$ form a subset of $I(G)$. On the other hand, e is not

pebbled, and therefore $E_e \not\subseteq I(G)$.

By the observations above, since a failed search region of the $(2, \ell)$-pebble game with exactly three free pebbles is a subgraph $\hat{F}_e$ with $|\hat{E}_e| = 2|\hat{V}_e| - 3$, then the $(2, 2)$-critical subgraph induced by e must satisfy $|E_e| = 2|V_e| - 2$. Moreover, by Lemma 6.2.4d, F_e is the *smallest* $(2, 2)$-tight subgraph of G containing the edge e.

Recall that if a periodic orbit graph $\langle G, m \rangle$ is minimally generically rigid on $\mathcal{T}_0^2$, then it is $(2, 2)$-tight. Therefore, it would be possible to use the $(2, 2)$-pebble game to find graphs that are $(2, 2)$-tight, and therefore possibly rigid on $\mathcal{T}_0^2$. However, we will find it more useful to use the $2|V| - 3$ pebble game, and use a special pebbling criterion for the $(2, 2)$-tight subgraphs. We will make use of the failed search regions of the $(2, 3)$-pebble game to identify the subgraphs with *exactly* $2|V| - 2$ edges, then we will check the gains on these subgraphs. In this way, we will simultaneously play the $2|V| - 3$ and $2|V| - 2$ pebble games.

6.3 Periodic adapted pebble game for frameworks on $\mathcal{T}_0^2$

We review our characterization of generic rigidity on the fixed torus $\mathcal{T}_0^2$. Let $\langle G, m \rangle$ be a periodic orbit graph where G is $(2, 2)$-tight. Recall that we called the gain assignment m on $\langle G, m \rangle$ *constructive* if every subgraph $\langle G', m' \rangle$ of $\langle G, m \rangle$ where G' is $(2, 2)$-tight contains some cycle with non-zero net gain. We called any cycle with non-zero net gain a *constructive cycle*.

Theorem 6.3.1 (see Theorem 4.4.4). *The periodic orbit graph $\langle G, m \rangle$ is generically minimally rigid on $\mathcal{T}_0^2$ if and only if G is $(2,2)$-tight, and m is constructive.*

The basic idea of the fixed torus pebble game algorithm (Algorithm 4) is that we play the $(2,3)$-pebble game on the underlying (undirected) multigraph G of our periodic orbit graph, $\langle G, m \rangle$, and pay special attention to the failed search regions which have exactly three pebbles remaining. As discussed above, the failed search regions of G with exactly three free pebbles remaining are $(2,2)$-tight subgraphs of G. We then develop a subroutine `gain-check` which checks these $(2,2)$-tight subgraphs of $\langle G, m \rangle$ for constructive cycles.

The pebble game described in Algorithm 1 is played on an undirected multigraph. The periodic version of the pebble game is played on a periodic orbit graph $\langle G, m \rangle$, where G is a directed multigraph, and m is a gain assignment on the edges of G. However, the main step of this periodic-adapted game is played on the underlying undirected multigraph induced by G. Since every cycle of the underlying undirected graph induces a cycle in the directed graph G, we do not distinguish between the directed and undirected versions of this graph in our notation.

We use a subroutine `gain-check` to check for constructive cycles on (labeled, directed) subgraphs of $\langle G, m \rangle$ induced by subgraphs of G. In particular, we define the $(2,2)$-*critical orbit graph of $\langle G, m \rangle$ induced by e*, denoted $\langle F_e, m_e \rangle$, to be the subgraph of $\langle G, m \rangle$ induced by the $(2,2)$-critical subgraph F_e induced by e in G.

226

The gain assignment m_e is the gain assignment of our original orbit graph $\langle G, m \rangle$ restricted to the edges E_e. Note that the edge e is contained in E_e, and hence the graph F_e is connected.

The question of testing the cycles of a gain graph has been well-studied for example in [38, 40] and [61]. The general problem of determining whether all of the cycles in a gain graph are *balanced*, that is, have zero net gain, is difficult. In our case, however, since we are working with connected graphs, we can always find a spanning tree, and therefore we can always find a fundamental system of cycles. We can use this to detect constructive cycles. We will describe `gain-check` after the main algorithm.

Algorithm 2 (**Fixed torus pebble game** (`FixTor`)).

Input: Periodic orbit graph $\langle G, m \rangle$ where $|E| \leq 2|V| - 2$.

Output: "minimally rigid on $\mathcal{T}_0^{2}$", "independent and flexible on $\mathcal{T}_0^{2}$", "neither independent nor rigid on $\mathcal{T}_0^{2}$". The directed subgraph of independent edges, $I(G) \subset G$ and the number of free pebbles remaining on the vertices of G.

Setup: Intialize $I(G)$ and $R(G)$ to be empty sets of edges.

Edge-acceptance condition. An edge e between two vertices is added to $I(G)$ when either

a) a total of 4 free pebbles are found on the two end points of the edge

227

b) a total of 3 free pebbles are found on the end points of the edge e, and `gain-check`
   on the failed search region F_e corresponding to e returns "constructive".

Algorithm:

1. As long as all of the edges of G are not tested, take any edge e and go to step 2.

2. Count the number of free pebbles on the end vertices of e, say $e = (u, v)$.

   a) If u and v have at least 4 free pebbles, then place a pebble on e and direct the edge e away from the source of the pebble. Add the edge e to the set $I(G)$, and return to 1.

   b) Else perform a depth-first search for a free pebble on the directed edges of $I(G)$.

      i) If a free pebble is found, perform a sequence of direction swaps of the edges of $I(G)$ to cascade the free pebble to u or v. Return to 2.

      ii) Else, no free pebble is found.

         A) If there are exactly 3 pebbles on u and v, run `gain-check` on the $(2, 2)$-critical orbit graph induced by e, $\langle F_e, m_e \rangle$.

         If gain-check returns "constructive", pebble the edge, add e to $I(G)$, return to 1.

228

Else gain-check returns "dependent", add the edge to $R(G)$, return
to 1.

B) Else there are exactly 2 pebbles on u and v, add edge to $R(G)$,
return to 1.

3. Stop when there are no more edges to be tested. If there are exactly 2 pebbles
remaining on the vertices of $\langle G, m \rangle$, and $R(G) = \emptyset$, return "minimally rigid
on $\mathcal{T}_0^{2}$". If there are more than 2 pebbles remaining, return "independent
and flexible on $\mathcal{T}_0^{2}$" if $R(G) = \emptyset$, and return "neither independent nor rigid
on $\mathcal{T}_0^2$ if $R(G) \neq \emptyset$. In all cases, return $I(G)$ and the number of free pebbles
remaining.

Algorithm 3 (`gain-check`).

Input: $(2,2)$-critical orbit graph $\langle F_e, m_e \rangle$ induced by e (corresponding to $\hat{F}_e$, a
failed search region for e with three free pebbles)

Output: "constructive", "dependent"

Algorithm:

Find a spanning tree T in F_e. Run the T-gain procedure.

a) If there is some (non-tree) edge with non-zero gain, return "constructive".

b) Else, if no edge has non-zero gain, return "dependent".

It remains to be shown that the outputs of the fixed torus pebble game actually correspond to the qualities they advertise. That is, we must show that periodic orbit graphs labelled as "minimally rigid on $\mathcal{T}_0^2$" by the fixed torus pebble game actually correspond to the characterization of these graphs in Theorem 6.3.1. We will do this in Section 6.3.2. We first present an example of the fixed torus pebble game.

6.3.1 Example

We perform the fixed torus pebble game on the periodic orbit graph shown in Figures 6.1 and 6.2. At any step the edge being tested (the *test edge*) is shown in blue. Note that the set $I(G)$ of directed edges created by the pebble game (shown in pink) are independent of the directions of the edges of the underlying graph. The direction of the edges of the underlying graph are only used at the `gain-check` stage. The steps of the game are as follows:

A. The graph is initialized with two pebbles on each vertex. The test edge e is in blue. Four pebbles are found on the endpoints of e, we move a pebbles off one of the adjacent vertices and onto the edge, assigning it a direction, shown in pink in (B).

B. Four pebbles are found on the endpoints of the new test edge e, we pebble it.

C. Four pebbles are found on the endpoints of the new test edge e, we pebble it.

D. Only three pebbles are found on the endpoints of the test edge. We search through the tested edges of the graph (in pink) from the endpoints of the test edge until a free pebble is found (shown in pink).

E. The free pebble is cascaded to the endpoint of the test edge, reversing the direction of the connecting edge. Four free pebbles are on the endpoints of the test edge: we pebble it.

F. Only three free pebbles are found on the endpoints of the new test edge. The search region contains only the two blue vertices, and the two edges between them, since there are no outgoing pink edges. Since there are only three free pebbles on this search region, it is a failed search region. The failed search region, together with the test edge, form a $(2, 2)$-critical subgraph. We run `gain-check` on the corresponding $(2, 2)$-critical orbit graph, which performs the T-gain procedure, and returns "constructive". We pebble the edge.

G. Only one pebble is on the endpoints of the test edge, we search through the directed edges and find two free pebbles (in pink).

H. Drawing the free pebbles onto the end points of our test edge, we search again

to find a fourth pebble, but the search fails. There are exactly three free pebbles on the failed search region (in light blue), we run `gain-check` on the associated $(2, 2)$-critical subgraph (the failed search region, plus the test edge). This performs the T-gain procedure, which detects a non-zero cycle and returns "constructive". We pebble the edge.

I. Only three pebbles are on the end points of the test edge. We search for a free pebble, and find it (in pink). Drawing it back to the endpoint of the test edge through a cascade of edge reversals, we have four free pebbles on the end points, hence we pebble the edge.

J. The last edge to be tested. We search for free pebbles, one is found and we draw it back to the endpoint of the test edge.

K. Only three free pebbles are on the endpoints of the test edge. We search again, in this case the directed edges lead us throughout the whole graph. The failed search region is the whole graph, and there are three free pebbles, so we run `gain-check` on the associated $(2, 2)$-critical graph (the whole graph). The T-gain procedure finds a non-zero edge, `gain-check` returns "constructive", and we pebble the edge.

L. No edges remain to be tested. No edges were rejected (labeled redundant), and exactly two pebbles remain on the graph. The fixed torus pebble game returns

"Minimally rigid on $\mathcal{T}_0^2$".

6.3.2 Correctness of the fixed torus pebble game

We begin by making a few observations. First observe that edges are accepted by the fixed torus pebble game *only if* there are at least three pebbles available on their endpoints. That is, the fixed torus pebble game is a special case of the $(2, 2)$ pebble game, where there is a special criterion for pebbling the $(2, 2)$-tight subgraphs. By Theorem 6.2.1, the outputs of the fixed torus pebble game (minimally rigid on $\mathcal{T}_0^2$, independent on $\mathcal{T}_0^2$, neither rigid nor independent on $\mathcal{T}_0^2$) correspond to $(2, 2)$-tight, $(2, 2)$-sparse and "other" respectively.

Let $\langle G, m \rangle$ be a periodic orbit graph. Recall that a gain assignment m is constructive if every $(2, 2)$-tight subgraph of $\langle G, m \rangle$ contains some cycle with non-zero net gain. By the discussion following Lemma 6.2.6, the inputs to `gain-check` are $(2, 2)$-tight graphs.

Lemma 6.3.2. *Let $\langle G, m \rangle$ be a periodic orbit graph, and let $\langle F_e, m_e \rangle$ be a $(2, 2)$-critical orbit graph induced by the edge e. Then m_e is a constructive gain assignment on F_e if and only if `gain-check` returns "constructive".*

Proof. `gain-check` will perform the T-gain procedure on $\langle F_e, m_e \rangle$. The T-gain procedure preserves the net gains of the cycles of F_e, while making all tree edges
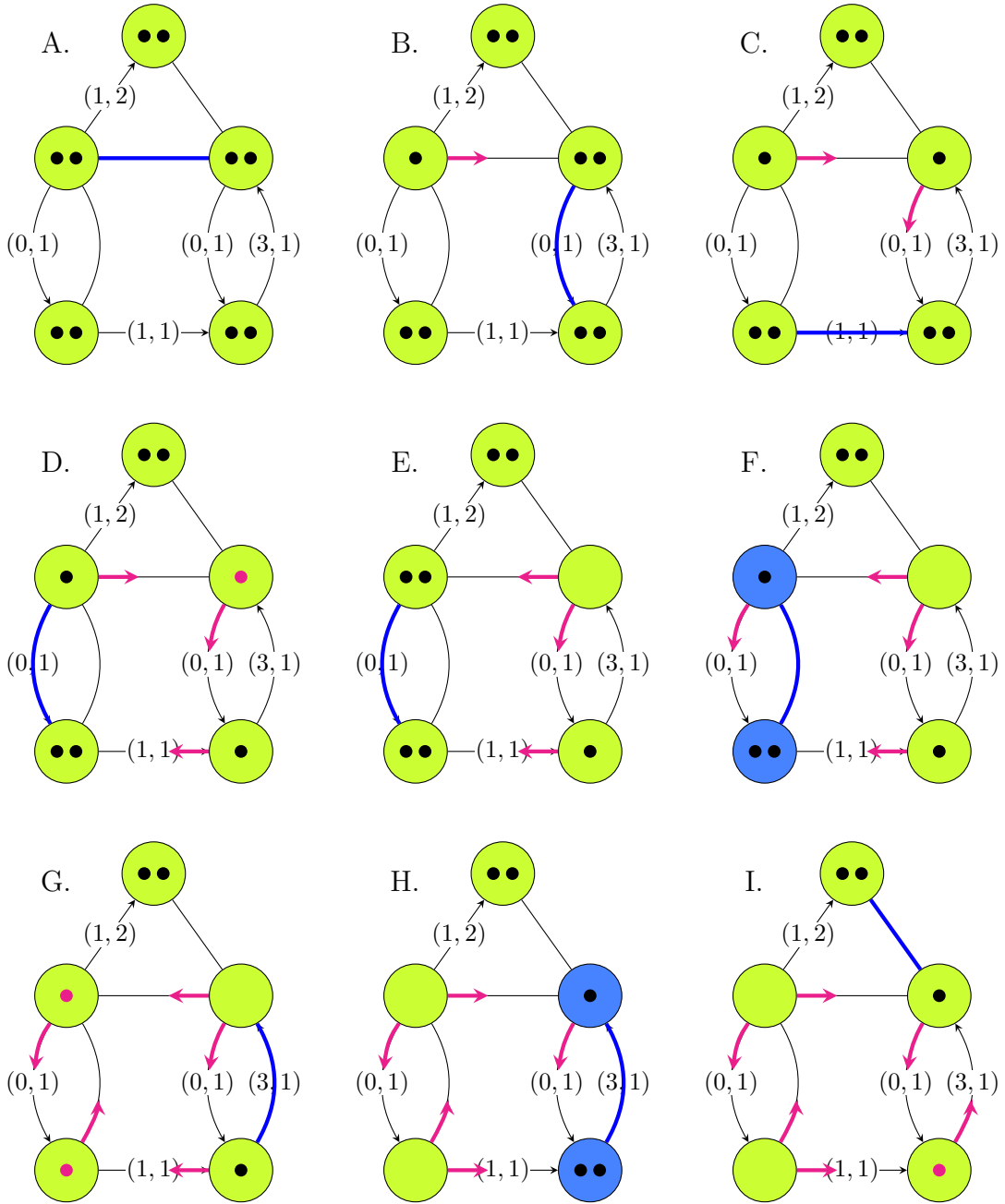
Figure 6.1: The periodic pebble game. See Example 6.3.1 for an explanation of each move. Game continues in Figure 6.2.

234
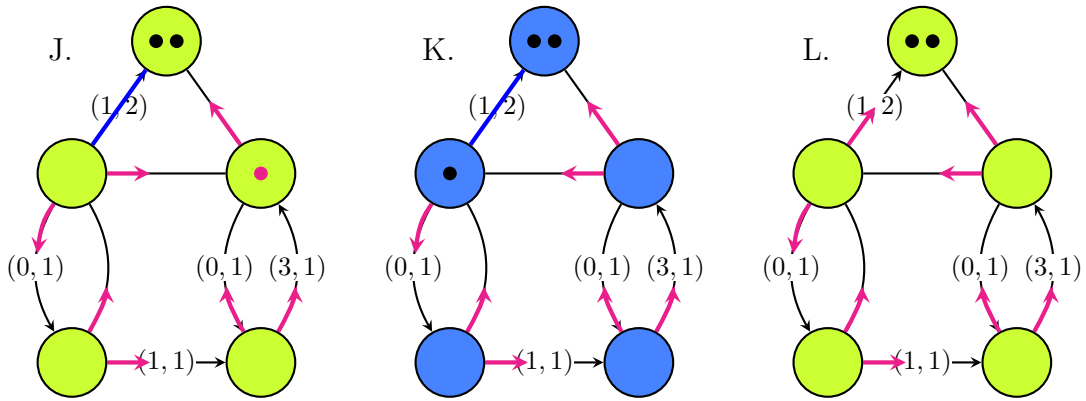
Figure 6.2: A continuation of the pebble game shown in Figure 6.1.

have gain $(0,0)$. The non-tree edges therefore correspond to a fundamental system

of cycles, the gains of which span the gain space of $\langle G, m \rangle$. The T-gain procedure

finds a non-zero edge if and only if there is some cycle in $\langle F_e, m_e \rangle$ with non-zero

net gain. □

We now state the main result of this section, which confirms that the outputs

of the fixed torus pebble game correspond with the appropriate generic rigidity

classifications of frameworks on the fixed torus $\mathcal{T}_0^2$.

Theorem 6.3.3. *Let G be a graph with $|E| = 2|V| - 2$. Then the following are*

equivalent:

1. *$\langle G, m \rangle$ is generically minimally rigid on $\mathcal{T}_0^2$*

2. *G is $(2,2)$-tight and m is constructive*

3. *The fixed torus pebble game ends with all edges covered by pebbles ($I(G) = E$),*

235

and exactly two free pebbles remaining.

Proof. The equivalence of 1 and 2 is the periodic Laman theorem of Chapter 4, namely Theorem 4.4.4. We show the equivalence of 2 and 3.

$2 \to 3$: We need to show that any failed search region with exactly three pebbles induces a $(2, 2)$-critical orbit graph which is labeled "constructive" by `gain-check`, no matter what edge is added last. Let $\hat{F}_e$ be such a failed search region. By Lemma 6.2.4 (and the discussion thereafter), $\hat{F}_e$ induces a $(2, 2)$-critical orbit graph $\langle F_e, m_e \rangle$ which is $(2, 2)$-tight. Therefore, by hypothesis, m_e is a constructive gain assignment on F_e, and so by Lemma 6.3.2, `gain-check` returns "constructive". Therefore, *any* edge e for which the failed search region has exactly three pebbles will be pebbled (accepted into $I(G)$). Since G is $(2, 2)$-tight, the rest follows from the fact that `FixTor` is a special case of the $(2, 2)$-pebble game.

$3 \to 2$: Let $\langle G, m \rangle$ be a periodic orbit graph for which the pebble games ends with all edges pebbled, and exactly two free pebbles remaining. Since the fixed torus pebble game is a special case of the $(2, 2)$-pebble game, by Theorem 6.2.1 any output of the periodic pebble game in which all edges are pebbled, and two free pebbles remain is $(2, 2)$-tight. We now show that m is a constructive gain assignment. That is, we show that any $(2, 2)$-tight subgraph of $\langle G, m \rangle$ contains a constructive cycle.

Let G' be a $(2, 2)$-tight subgraph of G (possibly all of G). We say a $(2, 2)$-tight

236

subgraph G' of G is *proper* if $1 < |V'| < |V|$ (note that all single vertices are trivial $(2,2)$-tight subgraphs). Then we have two cases:

I) G' contains no proper $(2,2)$-tight subgraph, or

II) G' contains a proper $(2,2)$-tight subgraph.

In both cases, the edges of G are naturally ordered by the order in which they were tested and accepted into $I(G)$.

Case I) Draw two pebbles onto G'. By Lemma 6.2.4d, the number of outgoing edges from G' must be zero. Let e^* be the final edge added to G' in the pebbling of the edges of G'. Then, $G' - e^*$ has exactly three pebbles (the pebble on e^* is added to the vertices of G'), and hence $G' - e^*$ is the failed search region for e^*. Then $\langle G', m' \rangle$ is the $(2,2)$-critical orbit graph of $\langle G, m \rangle$ induced by the edge e^*. Since e^* was accepted by `FixTor`, it must be the case that `gain-check` on $\langle G', m' \rangle$ returned constructive. By Lemma 6.3.2, G' has a constructive gain assignment.

Case II) In this case G' contains at least one proper $(2,2)$-tight subgraph. Let e^* be the final edge added to G' in the pebbling of G'. Draw two free pebbles to the endpoints of e^*, and let G^* be the smallest $(2,2)$-tight subgraph of G' containing e^* (possibly all of G', see Figure 6.3). Then by Lemma 6.2.4d, $G^* - e^*$ is the failed search region for e^* with exactly three free pebbles. The

237
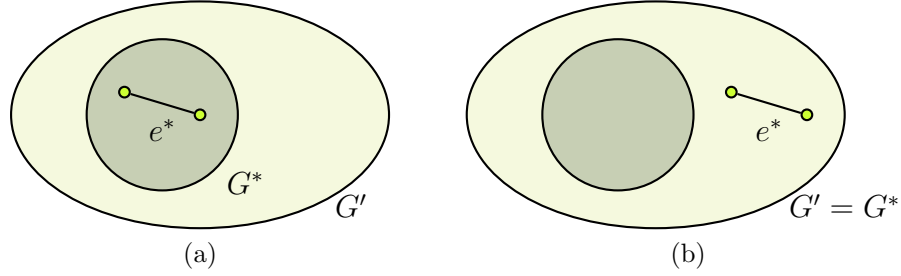
Figure 6.3: G^* is the smallest $(2,2)$-tight subgraph of G' containing e^*. If e^* is contained in a proper $(2,2)$-tight subgraph of G', then $G^* \neq G'$ (a).

failed search region for e^* cannot be larger than G^*, since that would mean that there were outgoing edges, which would violate Lemma 6.2.4d. That is, $\hat{F}_{e^*} = G^* - e^*$. Since e^* was accepted by `FixTor`, it must be the case that `gain-check` on $\langle G^*, m_{e^*} \rangle$ returned "constructive". By Lemma 6.3.2, $\langle G^*, m_{e^*} \rangle$ has a constructive cycle. Since $G^* \cup e^* \subset G'$, it follows that $\langle G', m' \rangle$ has a constructive cycle too.

$\square$

It is possible that some careful bookkeeping could speed up the run-time of the fixed torus pebble game algorithm. Let G be a graph, and let v_0 be a vertex of G. Recall that the set of $(2,2)$-tight subgraphs G containing v_0 form a lattice (see Lemma 4.4.9). Now suppose that $\langle G_1, m_1 \rangle \subset \langle G, m \rangle$ is a constructive $(2,2)$-tight subgraph. If $\langle G_2, m_2 \rangle$ is a subgraph of $\langle G, m \rangle$ with $|E_2| = 2|V_2| - 2$, and $v_0 \in V_2$, then $G_1 \cap G_2$ is $(2,2)$-tight. Moreover, if the intersection is non-trivial (i.e. $G_1 \cap G_2$

238

has more than one vertex) then the gain assignment on $G_1 \cap G_2$ is also constructive, since $\langle G_1, m_1 \rangle$ is constructive.

Using this idea in `FixTor`, the vertices of the tested, accepted $(2,2)$-tight subgraphs could be labeled, with a unique label for each tested subgraph. For example, if F_1 is the $(2,2)$-critical subgraph corresponding to e_1, and the edge e_1 is accepted by `gain-check`, label the vertices of F_1 with x_1. The vertices of a subsequent $(2,2)$-critical subgraph F_e could then be compared to the list of previously checked vertices. Whenever at least two vertices of V_e are labeled with a single label, say x_i, we know that the intersection $F_e \cap F_i$ is $(2,2)$-tight, and therefore has already been checked by `gain-check`.

Such a bookkeeping system would also have the advantage of recording the *rigid regions* of a particular graph. While the set $I(G)$ will contain all of the independent edges of the orbit graph $\langle G, m \rangle$, any set of vertices with a single label induces a $(2,2)$-tight subgraph of $\langle G, m \rangle$ which has a constructive gain assignment. This is a rigid region.

It follows from Theorem 6.3.3 that

Corollary 6.3.4. *Let $\langle G, m \rangle$ be a periodic orbit graph. The following are equivalent:*

1. $\langle G, m \rangle$ is $(2,2)$-sparse, but has $|E| < 2|V| - 2$

2. $\langle G, m \rangle$ is generically flexible on $\mathcal{T}_0^2$,

3. *the fixed torus pebble game will conclude with more than 2 pebbles remaining on the edges of G, and $R(G) = \emptyset$.*

Proof. The equivalence of 1 and 3 follows directly from Theorem 6.2.1. The equivalence of 1 and 2 follows from Theorem 6.3.1. □

Corollary 6.3.5. *Let $\langle G, m \rangle$ be a periodic orbit graph with $|E| = 2|V| - 2$. The following are equivalent:*

1. *$\langle G, m \rangle$ is not $(2, 2)$-tight*

2. *$\langle G, m \rangle$ is generically flexible on $\mathcal{T}_0^2$,*

3. *the fixed torus pebble game will conclude with more than 2 pebbles remaining on the edges of G, and $R(G) \neq \emptyset$.*

Proof. The equivalence of 1 and 3 follows directly from Theorem 6.2.1. The equivalence of 1 and 2 follows from Theorem 6.3.1. □

6.3.3 Features of the fixed torus pebble game algorithm

Complexity analysis of `gain-check`*:* In the worst case, we perform a single depth-first search to find a spanning tree, which is $O(|V| + |E|)$. We then perform one additional operations on each of the edges to find the T-gains, and finally we compare at most $n + 1$ non-tree gains to $(0, 0)$ (since the $|V| - 1$ tree edges will have zero gains). Hence gain check is as fast as the depth first search: $O(|V| + |E|)$. □

Complexity analysis of `FixTor`*:* In the worst case, we test $|E|$ edges, and for each edge we perform at most 4 depth-first searches to find free pebbles, as for the usual pebble game. For each edge, we also perform at most one more depth first search on the corresponding failed search region (if it exists) as part of `gain-check`. So the fixed torus pebble game is also $O(|V||E|)$. Note that in fact the failed search for the fourth pebble generates a spanning tree of the failed search region, which we can use in `gain-check`. □

Corollary 6.3.6 (Corollary to Theorem 6.3.3). *The fixed torus pebble game (*`FixTor`*) is a greedy algorithm.*

Proof. By Theorem 6.3.3, the fixed torus pebble game algorithm on $\langle G, m \rangle$ is checking the independence of the rows of the rigidity matrix $R_0 \langle G, m \rangle$. Finding sets of linearly independent rows is greedy. □

It follows that it does not matter in what order we test the edges of $\langle G, m \rangle$.

We remark that the periodic pebble game *does not* produce an easily checked "certificate" of independence or rigidity on $\mathcal{T}_0^2$. An example of such a certificate would be a sequence of Henneberg constructions illustrating that a given graph can be built up from a single vertex by a sequence of periodic Henneberg moves. For a graph $\langle G, m \rangle$ with $|V| = n$, this would be a sequence of n graphs beginning with a single vertex and ending with $\langle G, m \rangle$ itself. Finding Henneberg sequences are, in

general, more computationally expensive than the pebble game algorithm. In the worst case, every time we remove a three-valent vertex, we may need to check 3 candidate edges, making this process exponential ($O(3^{|V|})$).

6.3.4 Fixed torus pebble game for graphs with too many edges

Let $\langle G, m \rangle$ be a periodic orbit graph where $|E| > 2|V| - 2$. A naive algorithm to test the rigidity of $\langle G, m \rangle$ is recorded below. `FixTorII` first checks the graph using the $(2,2)$-pebble game to make sure that it contains *some* $(2,2)$-tight spanning subgraph. If it does, then it runs `FixTor` on all spanning subgraphs $\langle G_i, m_i \rangle$ of $\langle G, m \rangle$ where $|E_i| = 2|V| - 2$. If it finds one such spanning subgraph that is minimally rigid on $\mathcal{T}_0^2$, then the graph $\langle G, m \rangle$ is declared rigid on $\mathcal{T}_0^2$.

Algorithm 4 (**Fixed torus pebble game for large graphs** (`FixTorII`)).

Input: Periodic orbit graph $\langle G, m \rangle$ where $|E| > 2|V| - 2$.

Output: "rigid on $\mathcal{T}_0^2$", "flexible on $\mathcal{T}_0^2$".

Setup: Intialize $T(G)$ to be an empty set of subgraphs of G which will record the tested subgraphs of $\langle G, m \rangle$.

Algorithm:

1. Run the $(2,2)$-pebble game on G.

    (a) If the result is "over-constrained", Go to step 2.

(b) Else, the result is "other". Stop, and return "flexible on $\mathcal{T}_0^{2}$".

2. Pick a subgraph $\langle G_i, m_i \rangle$ of $\langle G, m \rangle$, where $G_i \notin T(G)$, and $E_i = 2|V| - 2$ (i.e G_i is an untested spanning subgraph of G). Run `FixTor` on $\langle G_i, m_i \rangle$.

(a) If `FixTor` returns "minimally rigid on $\mathcal{T}_0^{2}$" then stop, and return "rigid on $\mathcal{T}_0^2$.

(b) Else add the subgraph G_i to $T(G)$, and return to 2.

3. If all spanning subgraphs G_i of G with $|E_i| = 2|V| - 2$ have been tested, then return "flexible on $\mathcal{T}_0^{2}$".

As a consequence of Theorem 6.3.3 we obtain the following:

Theorem 6.3.7. *Let G be a graph with $|E| > 2|V| - 2$. Then the following are equivalent:*

1. *$\langle G, m \rangle$ is generically rigid on $\mathcal{T}_0^2$*

2. *$\langle G, m \rangle$ contains a spanning subgraph that is minimally rigid on $\mathcal{T}_0^2$*

3. `FixTorII` *returns "rigid on $\mathcal{T}_0^2$".*

Working out the details of a better algorithm for over-braced graphs is a topic for future research.

6.4 Other algorithmic issues

6.4.1 Periodic adapted pebble game for frameworks on $\mathcal{T}_x^2$

We can build, in analogy to the fixed torus pebble game, an algorithm to check for minimal rigidity on the scaling torus, $\mathcal{T}_x^2$. However, the scaling torus algorithm requires that we run the fixed torus pebble game algorithm numerous times, and is therefore not optimal. The creation of an algorithm with a similar time complexity to the fixed torus pebble game depends on the answer to the following question:

Question 6.4.1. Let $\langle G, m \rangle$ be a periodic orbit graph with $G = (V, E)$. Let $e = \{u, v; m_e\}$ be an edge of $\langle G, m \rangle$, and let $e_0 = \{i, j; m_0\}$ be an edge between two vertices of V that is *not* in E. Let $\langle G_0, m_0 \rangle$ be the periodic orbit graph obtained from $\langle G, m \rangle$ be replacing e with e_0. Under what conditions will this replacement preserve the generic minimal rigidity of $\langle G_0, m_0 \rangle$?

At the time of writing we do not have an answer to this question that does not involve running the fixed torus pebble game on $\langle G_0, m_0 \rangle$. This is another topic for future investigation.

6.4.2 Computing generic periodic rigidity

The algorithms above all test for *generic* rigidity on $\mathcal{T}_0^2$. It follows that we would like to have a testable, algorithmic definition of generic. The definition provided in

Section 3.3.8 does not fall into this category, since it requires testing a countable number of polynomial conditions. A "better" definition of generic will only check a finite number of polynomial conditions.

We can obtain such a definition by simply considering the existing edges of our periodic orbit graph $\langle G, m \rangle$. After Schulze [63] we call a configuration p $\langle G, m \rangle$-*generic* if the determinant of any submatrix of $R(\langle G, m \rangle, p)$ is zero if and only if the determinant of the corresponding submatrix of $R(\langle G, m \rangle, x)$ is zero, where x is a $d|V|$-dimensional vector of indeterminants.

In fact, it may be possible that more is true. In [80], White and Whiteley develop the idea of the *pure condition* of a framework. This is, roughly, a single polynomial of "bad positions" for the joints of the framework. It is likely that a similar property holds here, and instead of checking a finite number of polynomial conditions to find $\langle G, m \rangle$-generic positions, we could check only a single polynomial.

7 Periodic frameworks with additional symmetry

7.1 Introduction

This chapter outlines the background and principal findings of recent joint work with Bernd Schulze and Walter Whiteley [58]. These results describe surprising predictions of flexibility for some periodic frameworks with crystallographic symmetry, and reflect a recognition that Schulze's research on symmetric frameworks (see for example [66]) and the present research on periodic frameworks admit a common representation, namely gain graphs. In particular, we can describe a periodic framework which possesses additional symmetry within its fundamental region using a gain graph $\langle G, g \rangle$. The edges of the graph are labeled by elements of a group $\mathbb{Z}^d \rtimes \mathcal{S}$, where $\mathcal{S}$ is a symmetry group. For example, $\mathcal{S}$ may be the group generated by an inversion, a half-turn rotation, a mirror, or some combination of these symmetries.

Many crystal structures combine both periodic structure and symmetry within the unit cells, which motivates this work. For example, symmetry often appears in zeolites, a type of mineral whose flexibility contributes to their physical and chemical properties [42]. It follows that predicting the flexibility of theoretical zeolites may be a criterion for selecting which compounds should be synthesized for laboratory testing.

For motivation we restate Maxwell's Rule about flexibility in *finite* frameworks (see Theorem 2.5.8 in Chapter 2), which provides us with an easily checked necessary condition for rigidity.

Theorem 7.1.1. *Let (G, p) be a d-dimensional framework whose joints span an affine subspace of $\mathbb{R}^d$ of dimension at least $d-1$. If*

$$|E| < d|V| - \binom{d+1}{2},$$

then (G, p) has an infinitesimal flex. Furthermore, if the joints of (G, p) are in generic positions, then (G, p) has a continuous flex.

The final sentence is a consequence of Theorem 2.5.7, and in fact holds for all regular points, not just generic ones. The remainder of this chapter will be devoted to developing similar statements for periodic frameworks with additional symmetry, which will provide us with easy counting methods for detecting flexible frameworks. For a symmetric periodic framework with symmetry group $\mathbb{Z}^d \rtimes \mathcal{S}$, this will depend

on

- the number of edge and vertex orbits under $\mathbb{Z}^d \rtimes \mathcal{S}$,

- the number of symmetry-preserving lattice deformations, (only some types of lattice deformation may preserve a particular symmetry)

- the dimension of the space of trivial motions (translations) which preserve the symmetry $\mathcal{S}$.

The key result of this chapter (and of the paper [58]) is that for some symmetry groups $\mathcal{S}$, adding symmetry to a periodic framework will cause additional flexibility beyond what the original graph without symmetry would have exhibited in the periodic setting.

We build up the analysis by first recalling the key results from previous chapters of this thesis on periodic rigidity (Section 7.2). In Section 7.3 we sketch some basic ideas from the work of Schulze on symmetric finite frameworks. In Section 7.4 we describe the current object of study: symmetric periodic frameworks, together with their gain graphs and orbit matrices. Sections 7.5 and 7.6 are quoted directly from the paper [58], and contain the "results" of our analysis of certain classes of symmetric periodic frameworks.

7.2 Review of background on periodic frameworks

We here restate the relevant results from the theory of periodic frameworks described in Chapters 3, 4 and 5.

Recall that a *d-periodic framework* is the pair $(\langle \widetilde{G}, L\rangle, \widetilde{p})$, where $L(t)$ is a matrix of translations, possible variable. As in previous chapters we assume that $L(t)$ has the lower triangular form. The infinite framework $(\widetilde{G}, \widetilde{p})$ is invariant with respect to the translations given by the rows of $L(t)$. We refer to $L(t)$ as the *lattice*, and the variable entries of $L(t)$ are the *lattice parameters*.

Theorem 7.2.1 (Corollary 5.3.11, see also [7])**.** *The periodic framework* $(\langle \widetilde{G}, L\rangle, \widetilde{p})$ *is infinitesimally periodic rigid in* $\mathbb{R}^d$ *if and only if the rank of the rigidity matrix for the corresponding periodic orbit framework* $\mathbf{R}(\langle G, m\rangle, p)$ *is* $d|V| - d + \binom{d+1}{2}$.

We also note a periodic Maxwell-type rule for detecting continuous periodic flexes:

Theorem 7.2.2 ([7])**.** *Let* $(\langle \widetilde{G}, L\rangle, \widetilde{p})$ *be a periodic framework in dimension d with a corresponding orbit framework* $(\langle G, m\rangle, p)$*, where* $v = |V(G)|$ *and* $e = |E(G)|$*. If*

$$e < dv - d + \binom{d+1}{2} = dv + \binom{d}{2},$$

then $(\langle G, m\rangle, p)$ *has an infinitesimal flex on* $\mathcal{T}^d$*, which corresponds to a periodic infinitesimal flex of* $(\langle \widetilde{G}, L\rangle, \widetilde{p})$ *in* $\mathbb{R}^d$.

Furthermore, for generic positions of the vertices of G relative to the generating lattice L, $(\langle G, m \rangle, p)$ has a continuous flex on $\mathcal{T}^d$, which corresponds to a periodic continuous flex of $(\langle \widetilde{G}, L \rangle, \widetilde{p})$.

Theorem 7.2.2 can be adapted for the variations of the flexible torus $\mathcal{T}_k^d, k = 0, \ldots, \binom{d+1}{2}$ addressed in Chapter 5. We emphasize several special cases for two and three dimensions in Table 7.1, which shows the number of lattice parameters corresponding to each of the lattice variants in the following list:

(i) *fully flexible lattice*: all variations of the lattice shape are permitted;

(ii) *distortional change*: keep the volume fixed but allow the shape of the lattice to change;

(iii) *scaling change*: keep the angles fixed but allow the scale of the translations to change independently;

(iv) *hydrostatic change*: keep the shape of the lattice unchanged but scale to change the volume;

(v) *fixed lattice*: allow no change in the lattice.

We now outline some background for symmetric frameworks, both finite and (infinite) periodic.

Table 7.1: Number of parameters corresponding to types of lattice deformations with no added symmetry, in two and three dimensions.

$LatticeDef$	$2-D$	$3-D$
flexible	3	6
distortional	2	5
scaling	2	3
hydrostatic	1	1
fixed	0	0

7.3 Background on symmetric frameworks

The work of Schulze (see [63, 65, 66], for example) addressed finite frameworks with symmetry. In this section we summarize the pertinent definitions and results that we will need for the study of symmetric periodic frameworks. We here emphasize that this section concerns only *finite* frameworks, such as those discussed in Chapter 2, and does not describe infinite frameworks, or frameworks on a torus.

7.3.1 Symmetric frameworks and motions

Recall that an isometry of $\mathbb{R}^d$ is a map $s : \mathbb{R}^d \to \mathbb{R}^d$ such that $\|s(a) - s(b)\| = \|a - b\|$ for all $a, b \in \mathbb{R}^d$. Let G be a finite, simple graph, with $V = \{1, \ldots, n\}$, and automorphism group $Aut(G)$. Let $p : V \to \mathbb{R}^d$. A *symmetry operation* of the framework (G, p) in $\mathbb{R}^d$ is an isometry s of $\mathbb{R}^d$ such that for some $\alpha_s \in Aut(G)$, we have

$$s(p_i) = p_{\alpha_s(i)} \text{ for all } i \in V.$$

The set of all symmetry operations of a framework forms a group under composition, called the *point group* of (G, p) [5, 34]. Recall that translation does not change the rigidity properties of a framework. We therefore assume without loss of generality that the point group of a framework is always a subgroup of the orthogonal group $O(\mathbb{R}^d)$, and is hence a *symmetry group* [64, 65].

We will use the Schoenflies notation for the symmetry groups of a framework, and will focus on only two and three dimensional frameworks. The groups we will address in our examples and tables are denoted by $\mathcal{C}_s$, $\mathcal{C}_n$, $\mathcal{C}_{nv}$, $\mathcal{C}_{nh}$, $\mathcal{C}_i$, $\mathcal{D}_n$, and $\mathcal{D}_{nh}$, which we now describe.

For dimension 2, there are exactly three types of symmetry groups:

- $\mathcal{C}_s$ is a symmetry group consisting of the identity Id and a single reflection s,

- $\mathcal{C}_n$ is a cyclic group generated by an n-fold rotation C_n,

- $\mathcal{C}_{nv}$ is a dihedral group generated by a pair $\{C_n, s\}$.

In dimension 3,

- $\mathcal{C}_s$ is a symmetry group consisting of the identity Id and a single reflection s (as for $d = 2$),

- $\mathcal{C}_{nv}$ is a symmetry group that is generated by a rotation C_n and a reflection s whose corresponding mirror contains the rotational axis of C_n,

- $\mathcal{C}_{nh}$ is generated by a rotation C_n and the reflection s whose corresponding mirror is perpendicular to the C_n-axis,

- $\mathcal{C}_i$ consists of the identity Id and an inversion i in 3-space,

- $\mathcal{D}_n$ is generated by an n-fold rotation C_n and a 2-fold rotation C_2 whose rotational axes are perpendicular to each other,

- $\mathcal{D}_{nh}$ is generated by the generators C_n and C_2 of a group $\mathcal{D}_n$ and by a reflection s whose mirror is perpendicular to the C_n-axis.

We outline a few more concepts from [63, 64, 65]. Let G be a graph and $\mathcal{S}$ be a symmetry group in dimension d. Let $\mathscr{R}_{(G,\mathcal{S})}$ denote the set of all d-dimensional realizations of G whose point group is either equal to $\mathcal{S}$ or contains $\mathcal{S}$ as a subgroup. That is, $\mathscr{R}_{(G,\mathcal{S})}$ contains all framework (G, p) for which for which there exists a map $\Phi : \mathcal{S} \to \mathrm{Aut}(G)$ so that

$$s(p_i) = p_{\Phi(s)(i)} \text{ for all } i \in V(G) \text{ and all } s \in \mathcal{S}. \tag{7.1}$$

If a framework $(G, p) \in \mathscr{R}_{(G,\mathcal{S})}$ satisfies the equations in (7.1) for the map $\Phi : \mathcal{S} \to \mathrm{Aut}(G)$, we say that (G, p) is *of type* Φ. Roughly speaking, $\mathscr{R}_{(G,\mathcal{S})}$ is the set of frameworks on a particular graph G that have the specified symmetry Φ.

Results of Schulze [63, 65] demonstrate that if the map p of a framework $(G, p) \in \mathscr{R}_{(G,\mathcal{S})}$ is injective, then (G, p) is of a unique type, and further that Φ

is a homomorphism. Throughout the remainder of this chapter we therefore assume that p is injective, i.e. that $p_i \neq p_j$ if $i \neq j$. When the type Φ is clear from context we write $s(i)$ in place of $\Phi(s)(i)$.

In the previous chapters we addressed periodic frameworks, and considered infinitesimal motions of these frameworks which preserved their periodicity. Similarly, we now consider infinitesimal motions of a (finite) symmetric framework which are themselves symmetric (forced symmetry). An infinitesimal motion u of a framework $(G, p) \in \mathscr{R}_{(G,\mathcal{S})}$ is $\mathcal{S}$-symmetric if

$$s\big(u_i\big) = u_{s(i)} \text{ for all } i \in V(G) \text{ and all } s \in \mathcal{S}. \tag{7.2}$$

In other words, u is unchanged under all symmetry operations in $\mathcal{S}$. This is illustrated in Figure 7.1(a) and (b).
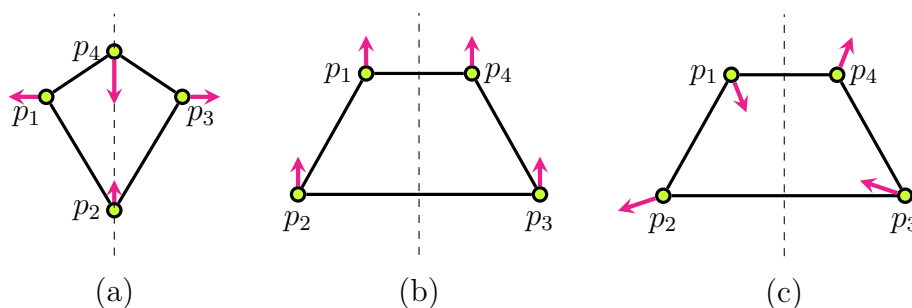
Figure 7.1: Infinitesimal motions of frameworks in the plane: (a) a $\mathcal{C}_s$-symmetric infinitesimal flex; (b) a $\mathcal{C}_s$-symmetric infinitesimal rigid motion; (c) an infinitesimal flex which is not $\mathcal{C}_s$-symmetric.

Let $(G, p) \in \mathscr{R}_{(G,\mathcal{S})}$, and choose a set of vertex representatives $\{1, \ldots, v_0\}$ for the orbits $\mathcal{S}(i) = \{s(i) : s \in \mathcal{S}\}$ of the vertices of G under the group action of $\mathcal{S}$.

Then the positions of all joints of (G, p) are uniquely determined by the positions of the joints corresponding to $\{1, \ldots, v_0\}$, namely $p_1, \ldots, p_{v_0}$, and the symmetry constraints of $\mathcal{S}$. In a similar way, an $\mathcal{S}$-symmetric infinitesimal motion u of the framework (G, p) is uniquely determined by the velocity vectors $u_1, \ldots, u_{v_0}$ on the vertex representatives.

In analogy to Theorems 2.5.7 and 3.3.30, which prove the correspondence between infinitesimal and continuous rigidity for generic frameworks (finite and periodic respectively), we have the following symmetric version, which appears in [65, 67]. This depends on an appropriate definition of generic position for symmetric frameworks, which is developed in the work of Schulze. For a framework (G, p) and a symmetry group $\mathcal{S}$, this means the vertices of a set of representatives for the vertex orbits under the action of $\mathcal{S}$ are placed 'as generically as possible' with respect to the symmetry (see [65, 67, 68] for details).

Theorem 7.3.1. *Let $\mathcal{S}$ be a symmetry group in dimension d, and let $(G, p) \in \mathscr{R}_{(G,\mathcal{S})}$ be a framework whose joints span all of $\mathbb{R}^d$, in an affine sense. If (G, p) is generic modulo the symmetry group $\mathcal{S}$, and (G, p) also possesses an $\mathcal{S}$-symmetric infinitesimal flex, then (G, p) also has a continuous flex which preserves all the symmetries in $\mathcal{S}$ throughout the path.*

7.3.2 Orbit rigidity matrices for symmetric frameworks

Just as the periodic rigidity matrix was a matrix of the orbits of a periodic frame-work under the symmetry group $\mathbb{Z}^d$, we can record an orbit matrix for a finite symmetric framework. This was recently described in Schulze and Whiteley [68]. Here we consider a somewhat simplified case, in which a framework (G, p) has no joint that is 'fixed' by a non-trivial symmetry operation in $\mathcal{S}$. That is, (G, p) has no joint p_i with $s(p_i) = p_i$ for some $s \in \mathcal{S}, s \neq id$. The treatment of this case is contained in [68], along with the following definition.

Let $\mathcal{S}$ be a symmetry group in dimension d and let $(G, p) \in \mathscr{R}_{(G,\mathcal{S})}$ be a framework which has no joint that is 'fixed' by a non-trivial symmetry opera-tion in $\mathcal{S}$. Further, let $\mathscr{O}_V = \{1, \ldots, v_0\}$ be a set of representatives for the orbits $\mathcal{S}(i) = \{s(i) \mid s \in \mathcal{S}\}$ of vertices of G. For each edge orbit $\mathcal{S}(e) = \{s(e) \mid s \in \mathcal{S}\}$ of G, the *orbit matrix* $\mathbf{O}(G, p, \mathcal{S})$ of (G, p) has the following corresponding (dv_0-dimensional) row vector:

Case 1: If the two end-vertices of the edge e lie in distinct vertex orbits, then there exists an edge in $\mathcal{S}(e)$ that is of the form $\{a, s(b)\}$ for some $s \in \mathcal{S}$, where $a, b \in \mathscr{O}_V$. The row we write in $\mathbf{O}(G, p, \mathcal{S})$ is:

$$
\begin{array}{cc}
a & b \\
\left(0 \ldots 0 \quad \left(p_a - s(p_b) \right) \quad 0 \ldots 0 \quad \left(p_b - s^{-1}(p_a) \right) \quad 0 \ldots 0 \right). &
\end{array}
$$

Case 2: If the two end-vertices of the edge e lie in the same vertex orbit, then there exists an edge in $\mathcal{S}(e)$ that is of the form $\{a, s(a)\}$ for some $s \in \mathcal{S}$, where $a \in \mathcal{O}_V$. The row we write in $\mathbf{O}(G, p, \mathcal{S})$ is:

$$\begin{array}{c} a \\ \begin{pmatrix} 0 \ldots 0 & \left(2p_a - s(p_a) - s^{-1}(p_a)\right) & 0 \ldots 0 \end{pmatrix} \end{array}.$$

Example 7.3.2. Consider the 2-dimensional framework (G, p) with point group $\mathcal{C}_2 = \{id, C_2\}$ depicted in Figure 7.2. If we denote $p_1 = (a, b)$, $p_2 = (c, d)$, $p_3 = (-a, -b)$, and $p_4 = (-c, -d)$, then the rigidity matrix of (G, p) is

$$\begin{array}{ccccc} & 1 & 2 & 3 = C_2(1) & 4 = C_2(2) \\ \{1, 2\} & \begin{pmatrix} (a - c, b - d) & (c - a, d - b) & 0\ 0 & 0\ 0 \\ \{1, C_2(2)\} & (a + c, b + d) & 0\ 0 & 0\ 0 & (-a - c, -b - d) \\ C_2(\{1, 2\}) & 0\ 0 & 0\ 0 & (c - a, d - b) & (a - c, b - d) \\ C_2(\{1, C_2(2)\}) & 0\ 0 & (a + c, b + d) & (-a - c, -b - d) & 0\ 0 \end{pmatrix} \end{array}$$

In contrast, the orbit matrix $\mathbf{O}(G, p, \mathcal{C}_2)$ of (G, p) will only have two rows, one for each representative of the edge orbits under the action of $\mathcal{C}_2$. Further, $\mathbf{O}(G, p, \mathcal{C}_2)$ will have only four columns, because G has only two vertex orbits under the action of $\mathcal{C}_2$, represented by the vertices 1 and 2, for example, and each of the joints p_1 and p_2 has two degrees of freedom in the plane. Since both edge orbits satisfy Case

257

2 in the definition of the orbit matrix, $\mathbf{O}(G, p, \mathcal{C}_2)$ has the following form:

$$
\begin{array}{c}
\phantom{\{1,2\}} \quad\;\; 1 \qquad\qquad\qquad 2 \\
\begin{array}{c} \{1,2\} \\[1.2em] \{1, C_2(2)\} \end{array}
\left(
\begin{array}{cc}
(p_1 - p_2) & (p_2 - p_1) \\[1em]
\left(p_1 - C_2(p_2)\right) & \left(p_2 - C_2^{-1}(p_1)\right)
\end{array}
\right)
\end{array}
=
\left(
\begin{array}{cc}
(a - c, b - d) & (c - a, d - b) \\[1em]
(a + c, b + d) & (c + a, d + b)
\end{array}
\right)
$$

(a)   (b)
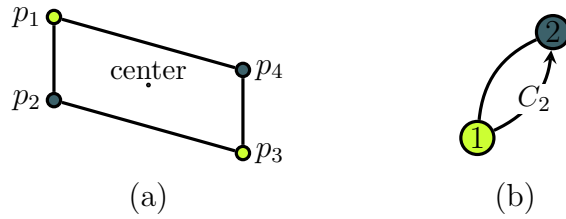
Figure 7.2: The framework $(G, p) \in \mathscr{R}_{(G, \mathcal{C}_2)}$ (a) and its corresponding symmetric orbit graph (b).

$\square$

As in the periodic case, we can use gain graphs to describe finite symmetric frameworks. The gains will be elements of the symmetry group $\mathcal{S}$. More precisely, the *symmetric orbit graph* $G_\mathcal{S}$ of a framework $(G, p) \in \mathscr{R}_{(G, \mathcal{S})}$ is a labeled multigraph (it may contain loops and multiple edges) whose vertex set $\{1, \ldots, v_0\}$ is a set of representatives of the vertex orbits of G under the action of $\mathcal{S}$, and whose edge set is defined as follows. For each edge orbit of G under the action of $\mathcal{S}$, there exists one edge in $G_\mathcal{S}$: for an edge orbit satisfying Case 1 of the definition of the symmetric orbit matrix, $G_\mathcal{S}$ has a directed edge connecting the vertices a and b. If the edge is directed from a to b, it is labeled with s, and if the edge is directed from b to a, it is labeled with s^{-1}. For simplicity we omit the label and the direction of

the edge if $s = id$. Similarly, for an edge orbit satisfying Case 2 of the definition of the symmetric orbit matrix, $G_\mathcal{S}$ has a loop at the vertex a which is labeled with s. Figure 7.2(b) illustrates the symmetric orbit graph for the framework discussed in Example 7.3.2.

The key result for the orbit matrix is as follows:

Theorem 7.3.3. *[68] Let S be a symmetry group and let (G, p) be a framework in $\mathscr{R}_{(G,\mathcal{S})}$. Then the solutions to $\mathbf{O}(G, p, \mathcal{S})u = 0$ are isomorphic to the space of $\mathcal{S}$-symmetric infinitesimal motions of the original framework (G, p).*

As an immediate consequence of Theorems 7.3.1 and 7.3.3 we obtain:

Theorem 7.3.4. *[63, 67, 68] Let $\mathcal{S}$ be a symmetry group in dimension d and let (G, p) be a framework in $\mathscr{R}_{(G,\mathcal{S})}$ which has no joint that is 'fixed' by a non-trivial symmetry operation in S. Further, let e_0 and v_0 denote the number of edge orbits and vertex orbits under the action of $\mathcal{S}$, respectively, and let $triv_\mathcal{S}$ denote the dimension of the space of $\mathcal{S}$-symmetric infinitesimal rigid motions of (G, p). If*

$$e_0 < dv_0 - triv_\mathcal{S}, \tag{7.3}$$

then (G, p) has an $\mathcal{S}$-symmetric infinitesimal flex. If the joints of (G, p) also span all of $\mathbb{R}^d$ (in an affine sense) and are in generic position modulo $\mathcal{S}$, then there also exists a continuous flex of (G, p) which preserves the symmetries in $\mathcal{S}$ throughout the path.

The dimension $triv_S$ of the space of S-symmetric infinitesimal rigid motions of $(G,p) \in \mathscr{R}_{(G,S)}$ can be deduced from character tables [16], or using techniques described in the work of Schulze [63, 64]. Thus, in order to check condition (7.3), we need only determine the size of the orbit matrix $\mathbf{O}(G,p,S)$. This in turn requires only a simple count of the vertex orbits and edge orbits of the graph G under the action of S, that is, the number of vertices and edges in the symmetric orbit graph (see Table 7.2 for examples).

We note that in some cases it is easy to determine the dimension of the space $triv_S$. For example, consider the symmetry group $C_s = \{id, \sigma\}$ in dimension d consisting of a single reflection. It is easy to see that the space of C_s-symmetric infinitesimal translations is of dimension $(d-1)$, since it consists of those translations whose velocity vectors are elements of the $(d-1)$-dimensional mirror-plane corresponding to σ (see also Figure 7.1(b)). On the other hand, making such heuristic arguments for the dimension of the space of rotational symmetries becomes increasingly difficult, if not impossible, in dimensions > 3. In such cases, further techniques are required [63, 64].

Example 7.3.5. We continue Example 7.3.2, and apply Theorem 7.3.4 to the framework (G,p) we considered there (see also Figures 7.2 (a) and 7.3). Since the periodic orbit graph has two vertices and two edges, we clearly have $dv_0 = 2 \cdot 2 = 4$ and $e_0 = 2$. The only trivial infinitesimal motions that are C_2-symmetric are the ones

Figure 7.3: A $\mathcal{C}_2$-symmetric infinitesimal flex of the framework from Example 3.2.1 (a) and the path taken by the joints of the framework under the corresponding symmetry-preserving continuous flex (b).

that correspond to rotations about the origin. Therefore, $triv_{\mathcal{C}_2} = 1$ (see [63, 64] for details). Thus, we have

$$e_0 = 2 < 3 = dv_0 - triv_{\mathcal{C}_2}.$$

So, by Theorem 7.3.4, we may conclude that any realization of G which is 'generic' modulo the half-turn symmetry has a symmetry-preserving continuous flex (Figure 7.3). $\square$

In fact the framework discussed in Examples 7.3.2 and 7.3.5 can be seen to be flexible by the standard (non-symmetric) Maxwell count (Theorem 7.1.1). For the graph G, we have $e = 4 < 2v - 3$. However, note that the flex predicted by the symmetric version of this result (Theorem 7.3.4) is a *symmetric* flex, while the flex predicted by Maxwell's original count may or may not be symmetric.

Table 7.2 shows the symmetric Maxwell type counts for a selection of point groups $\mathcal{S}$ in 3-space. We record the following quantities:

$k_{\mathcal{S}}$: The size of the vertex and edge orbits under the action of $\mathcal{S}$. For simplicity at this stage, we assume that no joint and no bar is fixed by a non-trivial element in $\mathcal{S}$, so that all vertex orbits and edge orbits under the action of $\mathcal{S}$ have the same size $k_{\mathcal{S}}$. (Recall that a joint p_i is fixed by $s \in \mathcal{S}$ if $s(p_i) = p_i$; a bar $\{p_i, p_j\}$ is fixed by $s \in \mathcal{S}$ if either $s(p_i) = p_i$ and $s(p_j) = p_j$ or $s(p_i) = p_j$ and $s(p_j) = p_i$). So, in particular, both the number of joints, v, and the number of bars, e are divisible by $k_{\mathcal{S}}$.

e: The least number of edges for the framework to be rigid without symmetry and to be compatible with the symmetry constraints given by $\mathcal{S}$. Recall that a necessary condition for rigidity in 3-space is $e \geq 3v - 6$ (Theorem 7.1.1). Thus for each group $\mathcal{S}$ in Table 7.2, e is chosen to be the smallest number which satisfies $e \geq 3v - 6$ and is divisible by $k_{\mathcal{S}}$.

$f_{\mathcal{S}}$: The dimension of the space of $\mathcal{S}$-symmetric infinitesimal flexes if $f_{\mathcal{S}} > 0$.

The final column of Table 7.2 indicates that at 'generic' configurations, the frameworks with $\mathcal{C}_2$ symmetry always have a finite flex, while those with $\mathcal{C}_4$ symmetry are always dependent (stressed).

Table 7.2: Impact of some 3-space point groups on counts for rigidity.

$\mathcal{S}$	$k_{\mathcal{S}}$	$triv_{\mathcal{S}}$	e	e_0	$3v_0 - triv_{\mathcal{S}}$	$f_{\mathcal{S}}$
$\mathcal{C}_1$	1	6	$3v - 6$	$3v_0 - 6$	$3v_0 - 6$	0
$\mathcal{C}_i$	2	3	$3v - 6$	$3v_0 - 3$	$3v_0 - 3$	0
$\mathcal{C}_2$	2	2	$3v - 6$	$3v_0 - 3$	$3v_0 - 2$	1
$\mathcal{C}_s$	2	3	$3v - 6$	$3v_0 - 3$	$3v_0 - 3$	0
$\mathcal{C}_{2h}$	4	1	$3v - 4$	$3v_0 - 1$	$3v_0 - 1$	0
$\mathcal{D}_{2h}$	8	0	$3v$	$3v_0$	$3v_0 - 0$	0
$\mathcal{C}_4$	4	2	$3v - 4$	$3v_0 - 1$	$3v_0 - 2$	(-1)
$\mathcal{C}_3$	3	2	$3v - 6$	$3v_0 - 2$	$3v_0 - 2$	0

7.4 Periodic frameworks with symmetry

For both (infinite) periodic frameworks and (finite) symmetric finite frameworks,
counting the number of rows and columns of the corresponding orbit matrices led to
simple necessary conditions for the framework to be generically rigid or minimally
rigid (Maxwell's rule). Our goal in the remainder of this chapter is to build up
Maxwell-type results for periodic frameworks that exhibit additional symmetry. In
particular, we study frameworks which have orbit graphs (gain graphs) whose edges
are labeled by elements of a group $\mathbb{Z}^d \rtimes \mathcal{S}$ (the gain group). An example of such a
framework is shown in Figure 7.4.

Let $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ be a periodic framework that has additional symmetry within
the unit cell, such that the $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ has the symmetry group $\mathbb{Z}^d \rtimes \mathcal{S}$, where $\mathbb{Z}^d$ is
the group of translations of the framework, $\mathcal{S}$ is the group of additional symmetries

of the framework, and $\rtimes$ denotes the semi-direct product of $\mathcal{S}$ acting on $\mathbb{Z}^d$. We call such a framework a *symmetric periodic framework*. Every symmetry operation in such a group can be written as a unique product of an element of $\mathbb{Z}^d$ and an element of $\mathcal{S}$. However, $\mathcal{S}$ is typically not normal in $\mathbb{Z}^d \rtimes \mathcal{S}$, and therefore the groups $\mathbb{Z}^d \rtimes \mathcal{S}$ are in general also not direct products. For details on the semi-direct product, we refer to [26], or any abstract algebra text.

The *symmetric periodic orbit graph* $\langle G, g \rangle$ corresponding to this framework is the labelled multigraph with one representative for each equivalence class of edges and vertices under the action of $\mathbb{Z}^d \rtimes \mathcal{S}$. The labelling of the edges $g : E(G) \to \mathbb{Z}^d \rtimes \mathcal{S}$ is determined in the manner described in the definition of the symmetric orbit graph (Section 7.3.2), and the periodic orbit graph (Section 3.2.2). The edge $\{a, b; g\}$ in $\langle G, m \rangle$, where $g \in \mathbb{Z}^d \rtimes \mathcal{S}$ indicates that the vertex a is connected to $g(b)$. If $g = (z, s), z \in \mathbb{Z}^d, s \in \mathcal{S}$, then $(p_a, s(p_b) + z)$ is a bar of the framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$.

Let $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ be a symmetric-periodic framework with symmetry group $\mathbb{Z}^d \rtimes \mathcal{S}$. An infinitesimal motion u of $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ is a *symmetric-periodic infinitesimal motion* if

$$g(u_i) = u_{g(i)} \text{ for all } i \in V, \text{ and all } g \in \mathbb{Z}^d \rtimes \mathcal{S}.$$

The *crystal system* of a symmetric periodic framework is a characterization of the parameters which determine the unit cell. This is determined by the symmetry group $\mathbb{Z}^d \rtimes \mathcal{S}$ of the framework, and is usually defined by the number and arrange-

264

(a)   (b)

$$g_1 = ((-1,0), id)$$
$$g_2 = ((0,0), C_2)$$
$$g_3 = ((0,1), C_2)$$

(c)
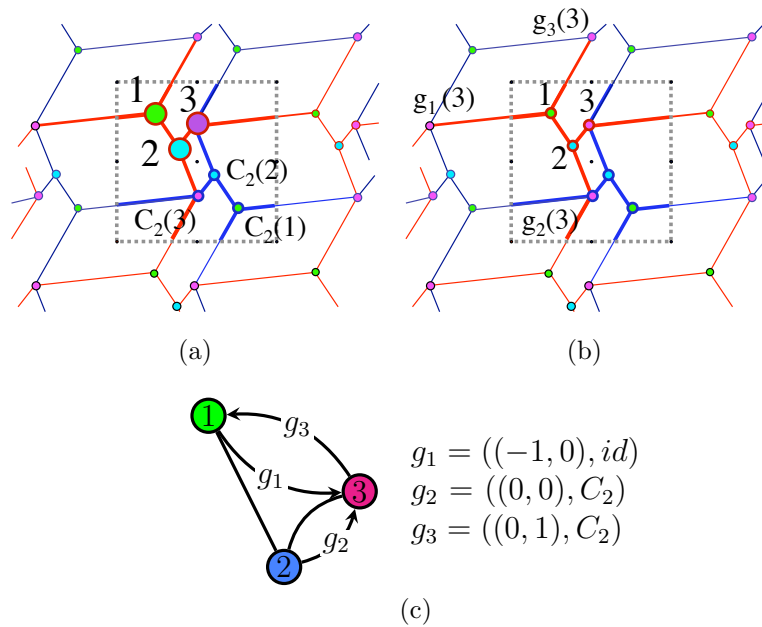
Figure 7.4: A plane framework with $\mathbb{Z}^2 \rtimes \mathcal{C}_2$ symmetry can be labeled with the elements of the group (a), or in short hand with gains (b) as in the gain graph (c)

ment of lengths and angles determining the unit cell. These parameters represent the variations of lattice shapes which preserve the given symmetry $\mathbb{Z}^d \rtimes \mathcal{S}$ [39, 88], and we call them the *lattice parameters*. Figures 7.5 and 7.6 show the crystal systems under consideration here, four in the plane and six in space.

(a) square   (b) rectangle   (c) rhombus   (d) parallelogram
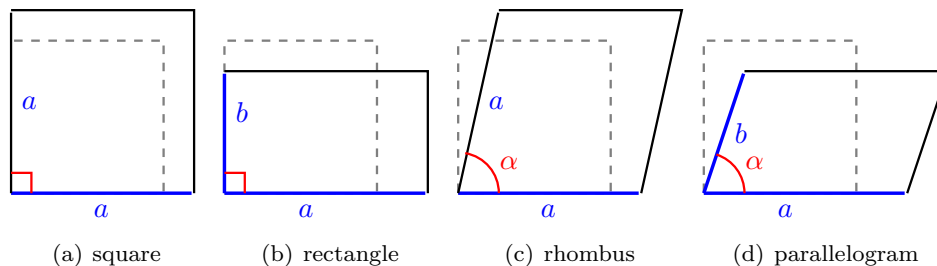
Figure 7.5: The four planar crystal systems. The number of lattice parameters are (a) 1, (b) 2, (c) 2, (d) 3.

(a) cubic   (b) tetragonal   (c) orthorhombic

(d) trigonal   (e) monoclinic   (f) triclinic
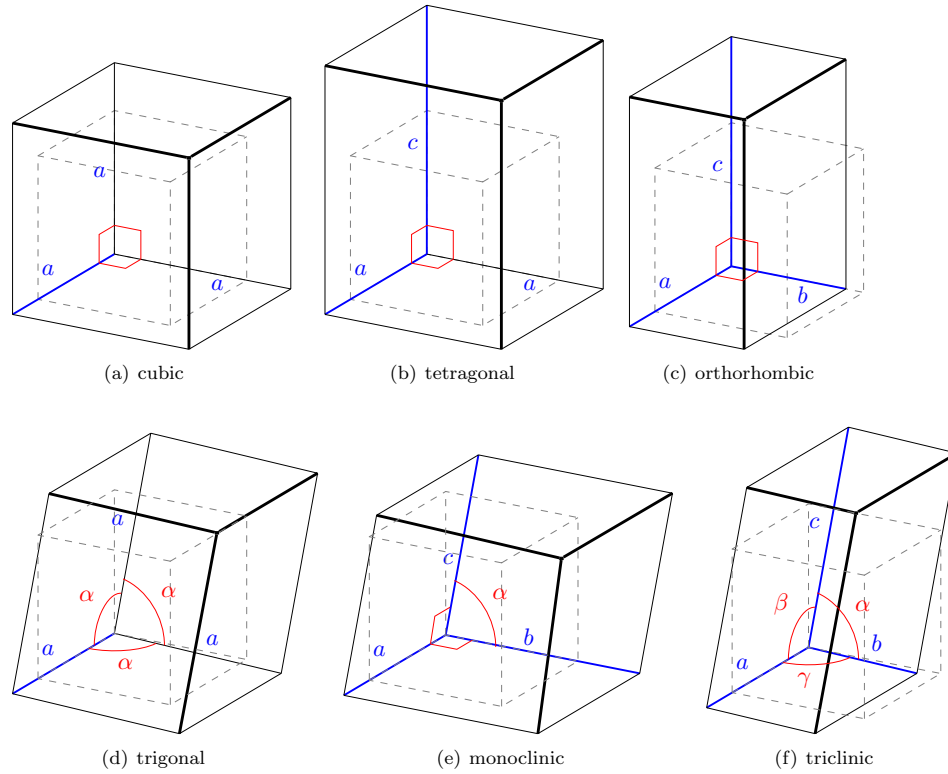
Figure 7.6: The six crystal systems discussed in this chapter. The number of lattice parameters are (a) 1, (b) 2, (c) 3, (d) 2, (e) 4, (f) 6.

Let v_0 and e_0 represent the number of vertices and edges in the orbit graph G. The *symmetric-periodic orbit matrix* is a $e_0 \times (dv_0 + \ell_{\mathcal{S}})$ matrix, with $\ell_{\mathcal{S}}$ being the number of columns corresponding to the lattice parameters. The rows of the matrix are determined in analogy with the determination of the rows of the symmetric or periodic orbit matrices. Let $e = \{a, b; g\}$ be an edge of $(\langle G, g \rangle, p)$, where $g = (z, s) \in \mathbb{Z}^d \rtimes \mathcal{S}$. We have three cases.

Case 1 If $z = (0, \ldots, 0)$, then the row of $\mathbf{R}(\langle G, g \rangle, p)$ corresponding e is as described for the symmetric orbit matrix, in Section 7.3.2, with two subcases

266

depending on whether the endpoints of e are in the same vertex orbit or not.

Case 2 If $s = Id$, then the row of $\mathbf{R}(\langle G, g\rangle, p)$ corresponding e is as described for the periodic orbit matrix, in Chapter 5, again with two subcases.

Case 3 Finally, if $z \neq (0,\ldots,0)$ and $s \neq Id$, we have two subcases.

a) If $a \neq b$ (i.e. a and b are in distinct vertex orbits), then the row corresponding to $e = \{a, b; g\}$ is

$$
\begin{array}{cccccc}
& a & & b & & L \\
\left(0\ldots 0 \right. & \left(p_a - (s(p_b) + zL)\right) & 0\ldots 0 & \left(p_b - (s^{-1}(p_a) - zL)\right) & 0\ldots 0 & \left. \left(p_a - (s(p_b) + zL)\right)M_e \right),
\end{array}
$$

where M_e is the $d \times \binom{d+1}{2}$ matrix defined in Section 5.3.9. That is, for $z = (z_1, z_2) \in \mathbb{Z}^2$ and $z = (z_1, z_2, z_3) \in \mathbb{Z}^3$ respectively,

$$
M_e = \left(\begin{array}{cc|c} z_1 & z_2 & 0 \\ 0 & 0 & z_2 \end{array} \right) \quad \text{and} \quad M_e = \left(\begin{array}{c|cc|ccc} z_1 & z_2 & 0 & z_3 & 0 & 0 \\ 0 & 0 & z_2 & 0 & z_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_3 \end{array} \right).
$$

b) On the other hand, if $a = b$, that is, the endpoints of e are in the same vertex orbit under the action of $\mathbb{Z}^d \rtimes \mathcal{S}$, then the corresponding row of the rigidity matrix is

$$
\begin{array}{cccc}
& a & & L \\
\left(0\ldots 0 \right. & \left(2p_a - s(p_a) - s^{-1}(p_a)\right) & 0\ldots 0 & \left. \left(2p_a - s(p_a) - s^{-1}(p_a)\right)M_e \right).
\end{array}
$$

267

For an example, see Section 7.5.

The maximum number of parameters that determine the lattice is specified by the crystal system of a framework. This in turn determines the number of lattice columns of our orbit rigidity matrix. We may further reduce the number of lattice columns by changing the type of lattice that we are considering: flexible, distortional, scaling, hydrostatic or fixed. It should be noted, however, that the lattice system will partially determine these choices. For instance, for a two-dimensional framework with a rhombus unit cell, scaling and hydrostatic will be identical.

The key result that we need here (which is implicit in [58]) is that the symmetric periodic orbit graph, and the corresponding matrix, describe the properties of original symmetric periodic framework.

Theorem 7.4.1. *Let $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ be a symmetric periodic framework with symmetry group $\mathbb{Z}^d \rtimes \mathcal{S}$, and let $\langle G, g \rangle$ be its corresponding symmetric periodic orbit graph, where $v_0 = |V|$, $e_0 = |E|$ and the positions of the vertices of G are given by p. Then an element $u \in \mathbb{R}^{dv_0 + \ell_\mathcal{S}}$ lies in the kernel of $\mathbf{R}(\langle G, g \rangle, p)$ if and only if u is the restriction to v_0 of a symmetric-periodic infinitesimal motion of $(\langle \widetilde{G}, L \rangle, \widetilde{p})$.*

The proof of Theorem 7.4.1 follows from Theorem 6.1 in [68], together with the results on periodic frameworks in Chapter 5, or in Borcea and Streinu [7]. The essential idea is that we view a periodic framework as a (finite) framework on the torus, with the associated rigidity matrix as described in previous chapters. We may

then apply the results of Schulze and Whiteley about finite symmetric frameworks to this setting.

Recall that for d-periodic frameworks, there is always a d-dimensional space of trivial infinitesimal motions, which is generated by d linearly independent translations. In contrast, for symmetric *finite* frameworks, the space of trivial motions which preserve a particular symmetry group $\mathcal{S}$ depends on the group itself. Similarly, for a symmetric-periodic framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$, the space of trivial motions exhibited by $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ will be a subspace of $\mathbb{R}^d$ determined by $\mathcal{S}$. We denote by $t_{\mathcal{S}}$ the dimension of the space of points which are fixed by all elements of the group $\mathcal{S}$. In three dimensions, $t_{\mathcal{S}}$ can only be $0, 1, 2$ or 3, corresponding to a point, a line, a plane or all of 3-space, respectively.

Theorem 7.4.2. *Let $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ be a symmetric periodic framework with symmetry group $\mathbb{Z}^d \rtimes \mathcal{S}$, and let $\langle G, g \rangle$ be its corresponding symmetric periodic orbit graph where $v_0 = |V|$ and $e_0 = |E|$. If*

$$e_0 < dv_0 + \ell_{\mathbb{Z}^d \rtimes \mathcal{S}} - t_{\mathbb{Z}^d \rtimes \mathcal{S}}$$

then $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ has a symmetric-periodic infinitesimal flex.

Furthermore, for generic positions of the vertices of $\widetilde{G}$ relative to the generating lattice L and the symmetry group $\mathcal{S}$, the symmetric periodic framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ has a continuous symmetric periodic flex.

269

The proof of Theorem 7.4.2 follows from Theorem 7.4.1, together with the symmetric and periodic versions of the result of Asimow and Roth (see Theorems 2.5.7, and 3.3.30, together with the results in Schulze [63]). We note that the word "continuous" in the final statement of Theorem 7.4.2 could be replaced by "differentiable" due to the results of Roth and Whiteley [60].

In the next section, we describe symmetric-periodic frameworks in the plane, with two detailed samples of the symmetry group $\mathcal{S}$. We also include tables summarizing our analysis for other groups. Section 7.6 will describe 3-dimensional frameworks, together with tables. These sections (7.5 and 7.6) are quoted directly from the paper [58].

Throughout this chapter we assume that our frameworks have symmetry group $\mathbb{Z}^d \rtimes \mathcal{S}$. As a consequence, we do not consider the full range of wallpaper groups in 2-dimensions, or space groups in 3-dimensions. In particular, absent are the groups which contain glide reflections, or which have 6-fold rotational symmetry. Although these other groups would also have orbit matrices, they require an alternative analysis which has not been completed yet. This is described more fully in Chapter 8 as a topic of future research.

7.5 2-D periodic frameworks with symmetry: $\mathbb{Z}^2 \rtimes \mathcal{S}$

7.5.1 $\mathbb{Z}^2 \rtimes \mathcal{C}_2$ - half-turn symmetry in the plane lattice

Half-turn symmetry in the plane is equivalent to inversion in the point axis. This symmetry fits an arbitrary parallelogram for the lattice (Figure 7.5(d)), and $\ell_{\mathcal{C}_2} = 3$. We will consider periodic plane frameworks with symmetry $\mathbb{Z}^2 \rtimes \mathcal{C}_2$ for two variations of the lattice: (1) a fully flexible lattice; (2) a fixed lattice.

Example 7.5.1 (Fully flexible lattice $\mathbb{Z}^2 \rtimes \mathcal{C}_2$). The original (non-symmetric) necessary count for a periodic framework on the fully flexible lattice to be minimally rigid is $e = 2v + 1$ (recall Theorem 7.2.2). To permit half-turn symmetry, with no vertex or edge fixed by the half-turn, we will need to start with the modified count $2e_0 = 2(2v_0) + 2$, where v_0 and e_0 are the numbers of vertices and edges of the orbit graph, respectively. Dividing by 2, this gives $e_0 = 2v_0 + 1$.

Under the half-turn symmetry with a fully flexible lattice, the orbit matrix has 2 columns under each orbit of vertices, plus $\ell_{\mathcal{C}_2} = 3$ columns for the three parameters for the lattice deformations. Further, we clearly have $t_{\mathcal{C}_2} = 0$ since there are no infinitesimal trivial motions which preserve the half-turn symmetry along with the periodic lattice. This creates the necessary symmetric Maxwell condition

$$e_0 \;\geq\; 2v_0 + 3 - 0$$

271
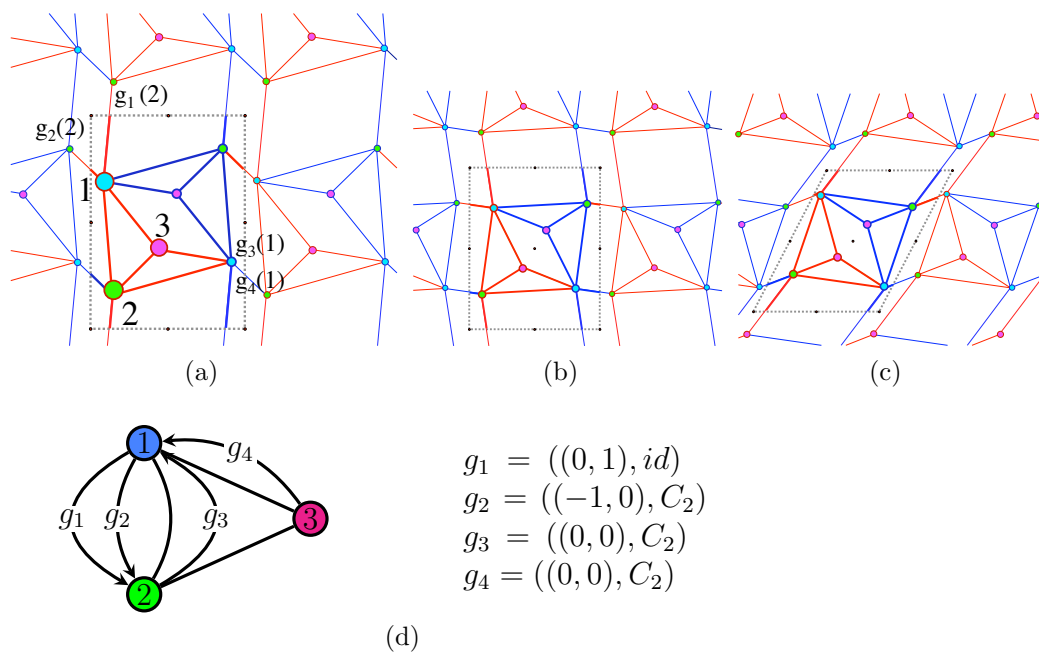
Figure 7.7: A generically rigid graph on a fully flexible lattice, realized with 2-fold symmetry has several non-trivial flexes changing the lattice. Its periodic symmetric orbit graph is pictured in (d).

for periodic rigidity. However, as shown above, for a graph that was previously minimally rigid without the symmetry, we have $e_0 = 2v_0 + 1 < 2v_0 + 3$. This gap predicts that a graph which counted to be minimally rigid without symmetry, realized generically with half-turn symmetry on a fully flexible lattice, now has two degrees of (finite) flexibility. As an example, consider the snapshots of three configurations with the same edge lengths but changing angles and lengths of the unit cell in Figure 7.7. Together these snap shots confirm the predicted two degrees of freedom.

The orbit matrix corresponding to the framework pictured in Figure 7.7 has the

following form:

$$
\begin{array}{c}
\begin{array}{cccc}
p_1 & p_2 & p_3 & (t_{11}, t_{21}, t_{22})
\end{array} \\
\begin{array}{c}
\{1,2\} \\
\{2,3\} \\
\{3,1\} \\
\{1,2;g_1\} \\
\{1,2;g_2\} \\
\{2,1;g_3\} \\
\{3,1;g_4\}
\end{array}
\left(
\begin{array}{cccc}
(p_1 - p_2) & (p_2 - p_1) & 0 & (0,0,0) \\
0 & (p_2 - p_3) & (p_3 - p_2) & (0,0,0) \\
(p_1 - p_3) & 0 & (p_3 - p_1) & (0,0,0) \\
(p_1 - g_1(p_2)) & (p_2 - g_1^{-1}(p_1)) & 0 & (*,*,*) \\
(p_1 - g_2(p_2)) & (p_2 - g_2^{-1}(p_1)) & 0 & (*,*,*) \\
(p_1 - g_3^{-1}(p_2)) & (p_2 - g_3(p_1)) & 0 & (*,*,*) \\
(p_1 - g_4^{-1}(p_3)) & 0 & (p_3 - g_4(p_1)) & (*,*,*)
\end{array}
\right)
\end{array}.
$$

$\square$

Example 7.5.2 (Fixed lattice $\mathbb{Z}^2 \rtimes \mathcal{C}_2$.). The original (non-symmetric) necessary count for any minimally rigid periodic framework on the fixed lattice is $e = 2v - 2$. With added $\mathcal{C}_2$ symmetry, we have $e = 2e_0$ and $v = 2v_0$ (all orbits have $k_{\mathcal{C}_2} = 2$), so a minimally rigid orbit graph, realized with $\mathcal{C}_2$ symmetry, will have $2e_0 = 2(2v_0) - 2$, or $e_0 = 2v_0 - 1$.

As the example below illustrates, with a fixed lattice, the orbit matrix has 2 columns under each orbit of vertices. Further, there are no translations which preserve the half-turn symmetry along with the periodic lattice, and hence we have

$t_{\mathcal{C}_2} = 0$. This creates the necessary symmetric Maxwell condition

$$e_0 \geq 2v_0$$

for periodic rigidity. However, as shown above, if the graph was chosen to be minimally rigid without the symmetry, we have $e_0 = 2v_0 - 1$. The gap $e_0 = 2v_0 - 1 < 2v_0$ shows that with the added half-turn symmetry, a minimally rigid graph will become flexible within the fixed lattice. Figure 7.8 shows the sample graph already presented in Figure 7.4 with $v_0 = 3$, $e_0 = 5$ and $e_0 = 5 < 6 = 2v_0$, with two realizations with the same edge lengths - illustrating snapshots of a non-trivial motion, as predicted.

(a)   (b)
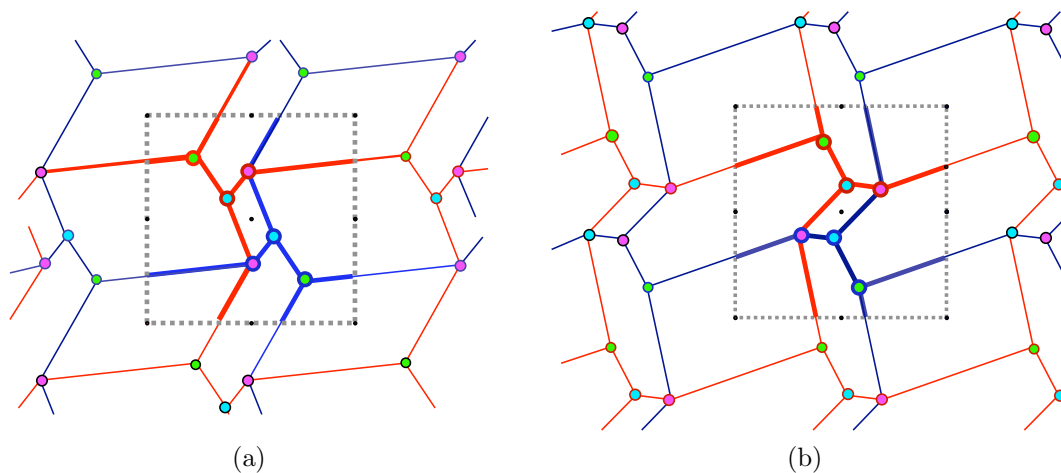
Figure 7.8: A plane framework with $\mathbb{Z}^2 \rtimes \mathcal{C}_2$ symmetry has a non-trivial flex on the fixed lattice.

Here is the orbit matrix of the framework depicted in Figure 7.8 on the fixed

lattice, with joints p_1, p_2, p_3:

$$
\begin{array}{c}
 \\
\{1,2\} \\
\{2,3\} \\
\{1,3; g_1\} \\
\{2,3; g_2\} \\
\{1,3; g_3\}
\end{array}
\begin{pmatrix}
p_1 & p_2 & p_3 \\
\left(p_1 - p_2\right) & \left(p_2 - p_1\right) & 0 \\
0 & \left(p_2 - p_3\right) & \left(p_3 - p_2\right) \\
\left(p_1 - g_1(p_3)\right) & 0 & \left(p_3 - g_1^{-1}(p_1)\right) \\
0 & \left(p_2 - g_2(p_3)\right) & \left(p_3 - g_2^{-1}(p_2)\right) \\
\left(p_1 - g_3(p_2)\right) & \left(p_2 - g_3^{-1}(p_1)\right) & 0
\end{pmatrix}.
$$

$\square$

In Table 7.3 we summarize the $(\mathbb{Z}^2 \rtimes \mathcal{C}_2)$-symmetric Maxwell type counts for each of the lattice variants. For simplicity at this stage, we again assume that no joint and no bar is fixed by the half-turn, so that all vertex orbits and edge orbits of the periodic orbit graph under the action of the group have the same size $k_{\mathcal{C}_2} = 2$. For each type of lattice deformation, we always assume that e is chosen to be the least number of edges for the framework to be rigid without symmetry and to be compatible with the symmetry constraints given by $\mathbb{Z}^2 \rtimes \mathcal{C}_2$. The number $f_{\mathcal{C}_2}$ in the final column denotes the dimension of the space of $(\mathbb{Z}^2 \rtimes \mathcal{C}_2)$-symmetric infinitesimal flexes in each case. For 'generic' configurations, these extend to finite symmetry-preserving flexes.

275

Table 7.3: Plane lattice deformations with $\mathcal{C}_2$ symmetry.

$LatticeDef$	$\mathcal{S}$	$k_{\mathcal{S}}$	$t_{\mathcal{S}}$	$\ell_{\mathcal{S}}$	rows	columns $-t_{\mathcal{S}}$	$f_{\mathcal{S}}$
flexible	$\mathcal{C}_2$	2	0	3	$e_0 = 2v_0 + 1$	$2v_0 + 3$	2
distortional	$\mathcal{C}_2$	2	0	2	$e_0 = 2v_0$	$2v_0 + 2$	2
hydrostatic	$\mathcal{C}_2$	2	0	1	$e_0 = 2v_0 - 1$	$2v_0 + 1$	2
fixed	$\mathcal{C}_2$	2	0	0	$e_0 = 2v_0 - 1$	$2v_0$	1

7.5.2 $\mathbb{Z}^2 \rtimes \mathcal{C}_s$ - mirror symmetry in the plane lattice

The mirror parallel to a side of the lattice restricts the possible lattices to rectangles. This mirror symmetry is preserved by translation along the line of the mirror, so $t_{\mathcal{C}_s} = 1$.

We will consider periodic plane frameworks with symmetry $\mathbb{Z}^2 \rtimes \mathcal{C}_s$ again in two layers: (1) a fully flexible lattice; (2) a fixed lattice.

Example 7.5.3 (Fully flexible lattice $\mathbb{Z}^2 \rtimes \mathcal{C}_s$.). The original (non-symmetric) necessary count for any minimally rigid periodic framework on the fully flexible lattice is $e = 2v + 1$ (recall Theorem 7.2.2). To permit mirror symmetry, with no vertex or edge fixed by the mirror, we will need to start with the shifted count $2e_0 = 2(2v_0) + 2$ or equivalently $e_0 = 2v_0 + 1$.

Under the mirror symmetry with a flexible lattice which preserves the symmetry, the orbit matrix has 2 columns under each orbit of vertices, plus $\ell_{\mathcal{C}_s} = 2$ lattice scaling columns for the mirror preserving flexes of the lattice. Since $t_{\mathcal{C}_s} = 1$, we

276

have the necessary symmetric Maxwell condition

$$e \geq 2v_0 + 2 - 1 = 2v_0 + 1$$

for periodic rigidity. This inequality, together with the previous condition for minimal rigidity without the mirror symmetry, suggests that there is no added flexibility from this mirror symmetry. The example in Figure 7.9 illustrates such a situation with $v_0 = 3$, $e_0 = 7$, and $e_0 = 7 = 2v_0 + 1$. It is indeed rigid on a flexible lattice, up to vertical translation along the mirror line. □
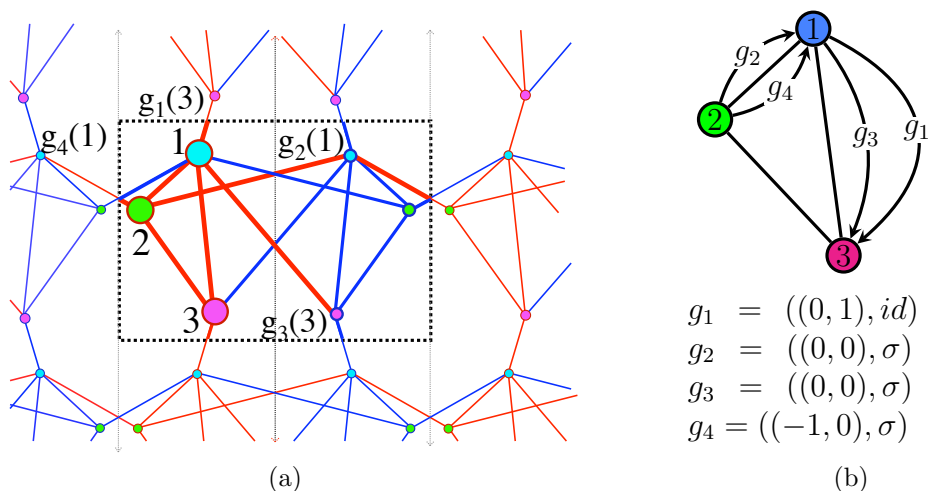
Figure 7.9: The mirrors (vertical lines in (a)) fit only with the two scalings and this framework prevents those scalings. The orbit graph corresponding to this framework is shown in (b).

Example 7.5.4 (Fixed lattice $\mathbb{Z}^2 \rtimes \mathcal{C}_s$.). The original necessary count for a periodic framework on the fixed lattice to be minimally rigid is $e = 2v - 2$. With added

mirror symmetry, we have $e = 2e_0$ and $v = 2v_0$, so a minimally rigid orbit graph, realized with $\mathcal{C}_s$ symmetry, will have $2e_0 = 2(2v_0) - 2$, or $e_0 = 2v_0 - 1$.

Under the mirror with a fixed lattice, the orbit matrix has 2 columns under each orbit of vertices. Moreover, we have $t_{\mathcal{C}_s} = 1$ since the translation along the axis preserves the mirror symmetry along with the periodic lattice. This creates the necessary symmetric Maxwell condition

$$e_0 \geq 2v_0 - 1$$

for periodic rigidity. Together with the count for minimal rigidity without symmetry, this suggests that there is no added flexibility from this symmetry. $\square$

It turns out that for mirror symmetry, all of the variants of lattice deformations produce no added motions.

7.5.3 Table of groups for the fully flexible lattice in 2-dimensions

Examples 7.5.1 and 7.5.2 indicate a process that can be applied to other plane symmetries which preserve the lattice. Each row in Table 7.4 presents the calculation for a given plane wall-paper group which is presented as $\mathbb{Z}^2 \rtimes \mathcal{S}$. Recall that we are not including the plane wall-paper groups that have core glide reflections or 3-fold and 6-fold rotations, since they require some significant modifications of the simple pattern presented here (see also Section 8.2.2). Thus, we do not have 17 lines in

278

the table.

In each row of Table 7.4, the calculation has several parts, each producing an integer:

1. the number of edge orbits, e_0, so that $k_\mathcal{S} e_0 \geq 2(k_\mathcal{S} v_0) + 1$, which guarantees that we have at least the number of edges needed for a rigid periodic framework without symmetry. This means we need to add a modified constant $\lceil \frac{1}{k_\mathcal{S}} \rceil$. For Table 7.4, this value is always 1, and the number of rows is always $e_0 = 2v_0 + 1$.

2. $t_\mathcal{S}$ which is the dimension of the space of translations contained in the symmetry element of $\mathcal{S}$. This will be 2 for the identity group, 1 for a single mirror, and 0 otherwise.

3. $\ell_\mathcal{S}$ which is the dimension of the space of lattice deformations which preserve all the symmetries in $\mathcal{S}$ or equivalently, the number of independent parameters in the lattice system (edge lengths and angles).

4. the comparison of these numbers as the number of rows e_0 compared to the number of columns minus $t_\mathcal{S}$: $2v_0 + \ell_\mathcal{S} - t_\mathcal{S}$.

5. the difference $f_\mathcal{S} = 2v_0 + \ell_\mathcal{S} - t_\mathcal{S} - (2v_0 + 1) = \ell_\mathcal{S} - t_\mathcal{S} - 1$ which is the dimension of the guaranteed extra non-trivial motions of the symmetric framework, over the rigidity which the original count without symmetry promised.

279

Table 7.4: The added flexibility induced by basic symmetries on a fully flexible 2-D lattice for $\mathbb{Z}^2 \rtimes \mathcal{S}$.

Lat	$\text{Sch}_\mathcal{S}$	$H\text{-}M_\mathcal{S}$	$orb_\mathcal{S}$	$k_\mathcal{S}$	$t_\mathcal{S}$	$\ell_\mathcal{S}$	$rows$	$\text{columns} - t_\mathcal{S}$	$f_\mathcal{S}$
par	$\mathcal{C}_1$	1	○	1	2	3	$2v+1$	$2v+3-2$	0
"	$\mathcal{C}_2$	2	2222	2	0	3	$2v_0+1$	$2v_0+3-0$	2
"	$\mathcal{C}_{2v}$	$2m$	$2*22$	4	0	2	$2v_0+1$	$2v_0+2-0$	1
rect	$\mathcal{C}_s$	m	$**$	2	1	2	$2v_0+1$	$2v_0+2-1$	0
"	$\mathcal{C}_{2v}$	$2/m$	$*2222$	4	0	2	$2v_0+1$	$2v_0+2-0$	1
square	$\mathcal{C}_4$	4	442	4	0	1	$2v_0+1$	$2v_0+1-0$	0
"	$\mathcal{C}_{4v}$	$4m$	$*442$	4	0	1	$2v_0+1$	$2v_0+1-0$	0

7.5.4 Table of groups for the fixed lattice in 2-dimensions

As we mentioned earlier, it can be of interest to consider periodic frameworks where the lattice is fixed. In the following table, each row will present the corresponding calculation for a given plane wall-paper group which is presented as $\mathbb{Z}^2 \rtimes \mathcal{S}$. As above, this analysis does not include rows for the hexagonal tiling or groups which include glide reflections.

As in the previous section, in each row of Table 7.5, the calculation has several parts, each producing an integer.

Note that the number of rows (edge orbits), e_0, is now such that $k_\mathcal{S} e_0 \geq 2(k_\mathcal{S} v_0) - 2$. This guarantees that we have at least the number of edges needed for a rigid periodic

Table 7.5: The added flexibility induced by basic symmetries on a fixed 2-D lattice for $\mathbb{Z}^2 \rtimes \mathcal{S}$.

Lat	$Sch_\mathcal{S}$	$H\text{-}M_\mathcal{S}$	$orb_\mathcal{S}$	$k_\mathcal{S}$	$t_\mathcal{S}$	rows	columns $- t_\mathcal{S}$	$f_\mathcal{S}$
par	$\mathcal{C}_1$	1	$\circ$	1	2	$2v - 2$	$2v - 2$	0
$''$	$\mathcal{C}_2$	2	2222	2	0	$2v_0 - 1$	$2v_0$	1
$''$	$\mathcal{C}_{2v}$	$2m$	$2 * 22$	4	0	$2v_0$	$2v_0 - 0$	0
rect	$\mathcal{C}_s$	m	$**$	2	1	$2v_0 - 1$	$2v_0 - 1$	0

framework on the fixed lattice without symmetry. This means we need to subtract a modified constant $c = \lfloor \frac{2}{k_\mathcal{S}} \rfloor$. For Table 7.5, c is 2, 1, or 0.

Since for a fixed lattice, we clearly have $\ell_\mathcal{S} = 0$ for each group $\mathcal{S}$, the corresponding column is omitted in Table 7.5.

Analogously to Table 7.4, the final column of Table 7.5 shows the difference $f_\mathcal{S} = 2v_0 - t_\mathcal{S} - (2v_0 - c) = c - t_\mathcal{S}$ which is the dimension of the guaranteed extra non-trivial motions of the symmetric framework, over the rigidity which the original count without symmetry promised.

Note that Table 7.5 does not include all the point groups from Table 7.4. The groups we omitted only produce 0's in the last column.

7.6   3-D periodic frameworks with symmetry: $\mathbb{Z}^3 \rtimes \mathcal{S}$

We now apply the basic patterns of the previous sections to investigate the types of counts which arise for periodic structures with added symmetry in 3-space. As

happened in the plane, these symmetries can have three impacts:

(a) the symmetry can restrict the possible shapes of the lattice cell or equivalently, the symmetry constraints leave a specific subset of $\ell_\mathcal{S}$ flexes of the lattice structure which preserve the desired symmetry.

(b) the symmetry can block some, or all, of the translations of the lattice structure, altering the basic count of $t_\mathcal{S}$;

(c) the symmetry determines the order of the group, that is, the size $k_\mathcal{S}$ of the orbits.

7.6.1 $\mathbb{Z}^3 \rtimes \mathcal{C}_i$ - inversive symmetry in space

Consider the inversive symmetry in 3-space with the center of symmetry at the origin. This operation (which in the Schoenflies notation is called i) takes a joint p to a joint $-p$. In many tables of crystal symmetry, this symmetry operation is called central symmetry, and the crystals are called centrosymmetric. All shapes of lattices are possible, and these fit into the triclinic lattice system (three angle choices). In the Schoenflies notation, if inversion is the only non-trivial symmetry operation, the group is written as $\mathcal{C}_i$. In the Hermann-Mauguin notation, it is written as $\bar{1}$, and in the orbifold notation, it is written as $1x$.

As in the plane, if we have a center of inversion c, and a translation vector t then

there is another inversion centered at $c + \frac{1}{2}t$. So, given the lattice of translations $\mathbb{Z}^3$ and one center of inversion at the origin, there is a full lattice of inversions, with translations $\frac{1}{2}\mathbb{Z}^3$, and the group of operations on the framework is written $\mathbb{Z}^3 \rtimes \mathcal{C}_i$ (see also Figure 7.10(a)).

$$g_1 = ((0,0,0), i)$$
$$g_2 = ((-1,0,0), id)$$
$$g_3 = ((0,0,-1), id)$$
$$g_4 = ((0,1,0), id)$$
$$g_6 = ((-1,0,0), i)$$
$$g_7 = ((0,1,0), i)$$
$$g_8 = ((0,0,-1), i)$$

(a)　　　　　　　(b)　　　　　　　(c)

(d)　　　　　　　(e)
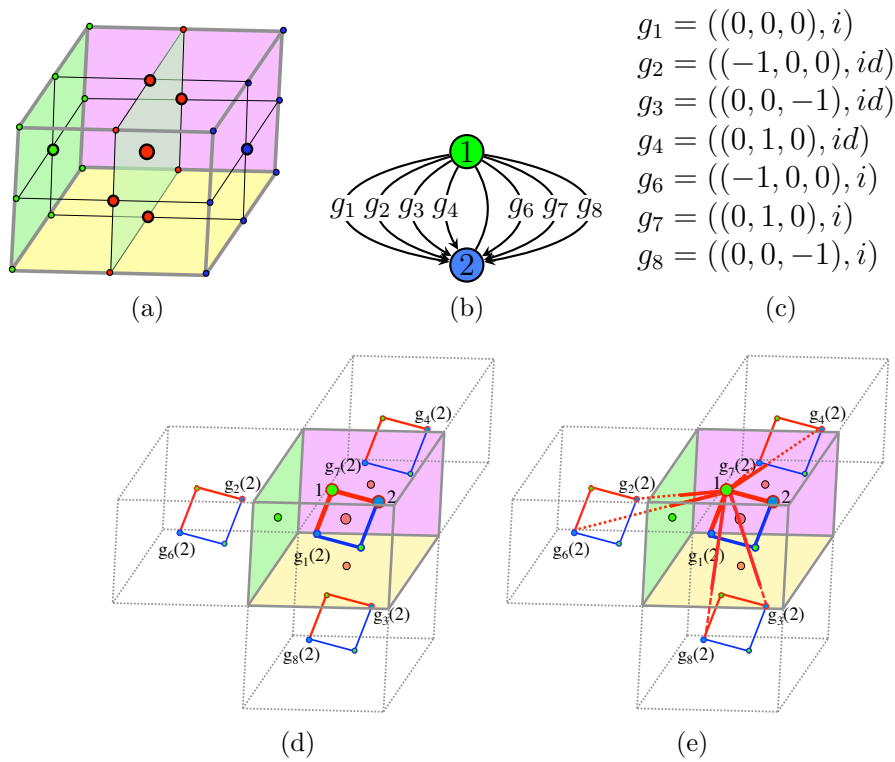
Figure 7.10: In 3-D, one center of inversion repeats with half the period (a). An orbit framework with 2 orbits of vertices is shown in (b), with the group elements associated with the directed edges listed in (c). Parts (d) and (e) illustrate building up the corresponding symmetric-periodic framework, moving from 2 to 8 orbits of edges (d).

Example 7.6.1 (Fully flexible lattice $\mathbb{Z}^3 \rtimes \mathcal{C}_i$.). The necessary count for any minimally rigid non-symmetric periodic framework on the fully flexible lattice is $e = 3v + 3$

(recall Theorem 7.2.2). To permit inversion symmetry ($k_{\mathcal{C}_i} = 2$) we will need to start with the shifted count $2e_0 = 3(2v_0) + 4$ or equivalently $e_0 = 3v_0 + 2$.

Since the full flexibility of the lattice fits with the inversive symmetry, we still have $\ell_{\mathcal{C}_i} = 6$. Further, when we move to the symmetric periodic orbit matrix under inversive symmetry, all of the infinitesimal translations disappear from the kernel, so that $t_{\mathcal{C}_i} = 0$. This gives rise to the symmetric Maxwell condition

$$e_0 \geq 3v_0 + 6$$

for periodic rigidity. The gap $e_0 = 3v_0 + 2 \; < \; 3v_0 + 6$ implies that a graph which counted to be minimally rigid without symmetry, realized generically with inversive symmetry on a fully flexible lattice now has a space of (finite) flexes of dimension 4. $\qquad \square$

Example 7.6.2 (Fixed lattice $\mathbb{Z}^3 \rtimes \mathcal{C}_i$.). The necessary count for any minimally rigid non-symmetric periodic framework on the fixed lattice to be minimally rigid is $e = 3v - 3$. To permit inversive symmetry, we will need to start with the shifted count $2e_0 = 3(2v_0) - 2$ or equivalently $e_0 \; = \; 3v_0 - 1$.

Since we again have $t_{\mathcal{C}_i} = 0$, we obtain the necessary symmetric Maxwell condition

$$e_0 \geq 3v_0$$

for periodic rigidity. The gap $e_0 = 3v_0 - 1 \; < \; 3v_0$ predicts a non-trivial finite flex

in generic realizations with inversive symmetry on the fixed lattice. □

As a summary, here is the impact of inversive symmetry for each of the variants

of lattice flexibility introduced in Section 7.2:

Table 7.6: 3-D lattice deformations with $\mathcal{C}_i$ symmetry.

$Lattice Def$	$\mathcal{S}$	$k_\mathcal{S}$	$t_\mathcal{S}$	$\ell_\mathcal{S}$	rows	columns $-t_\mathcal{S}$	$f_\mathcal{S}$
flexible	$\mathcal{C}_i$	2	0	6	$e_0 = 3v_0 + 2$	$3v_0 + 6$	4
distortional	$\mathcal{C}_i$	2	0	5	$e_0 = 3v_0 + 1$	$3v_0 + 5$	4
scaling	$\mathcal{C}_i$	2	0	3	$e_0 = 3v_0$	$3v_0 + 3$	3
hydrostatic	$\mathcal{C}_i$	2	0	1	$e_0 = 3v_0 - 1$	$3v_0 + 1$	2
fixed	$\mathcal{C}_i$	2	0	0	$e_0 = 3v_0 - 1$	$3v_0$	1

7.6.2 $\mathbb{Z}^3 \rtimes \mathcal{C}_2$ and $\mathbb{Z}^3 \rtimes \mathcal{C}_s$ - half-turn and mirror symmetry in space

Assume we have a 2-fold rotational axis along the z direction. This places the

pattern into the monoclinic crystal system: one face of the lattice is a parallelogram

(perpendicular to the axis) and two faces are parallel to the axis and perpendicular

to the parallelogram face. For this type of lattice, there are 4 lattice parameters:

the scale of each of the generating translations, and the one angle between the two

generating translations of the parallelogram.

Example 7.6.3 (Fully flexible lattice $\mathbb{Z}^3 \rtimes \mathcal{C}_2$.). With a fully flexible lattice, the

necessary minimal number of edges for a periodic framework to be rigid and to be

compatible with half-turn symmetry is $2e_0 = 3(2v_0) + 4$, or $e_0 = 3v_0 + 2$. In the

orbit matrix, there are four columns corresponding to the lattice deformations, so the necessary symmetric Maxwell type count for periodic rigidity is

$$e_0 \geq 3v_0 + 4 - 1 = 3v_0 + 3.$$

Since we started with $e_0 = 3v_0 + 2 < 3v_0 + 3$, we predict a non-trivial symmetry preserving finite flex for generic realizations with half-turn symmetry on the flexible lattice.   □

Example 7.6.4 (Fixed lattice $\mathbb{Z}^3 \rtimes \mathcal{C}_2$.). With a fixed lattice, the necessary minimal number of edges for a periodic framework to be rigid and to be compatible with half-turn symmetry is $2e_0 \geq 3(2v_0) - 2$, or $e_0 \geq 3v_0 - 1$. The necessary symmetric Maxwell type count for periodic rigidity on the fixed lattice is

$$e_0 \geq 3v_0 - 1.$$

Thus, we do not detect any added motions in this case.   □

In Table 7.7 we present the $(\mathbb{Z}^3 \rtimes \mathcal{C}_2)$-symmetric Maxwell type counts for each type of lattice deformation.

Consider a periodic framework in space with mirror symmetry. For this new group, there are only two key calculations to be done:

1. $t_{\mathcal{C}_s} = 2$, since the two translations on directions within the mirror will (instantaneously) preserve the mirror.

Table 7.7: 3-D lattice deformations with $\mathcal{C}_2$ symmetry.

$LatticeDef$	$\mathcal{S}$	$k_{\mathcal{S}}$	$t_{\mathcal{S}}$	$\ell_{\mathcal{S}}$	rows	columns -$t_{\mathcal{S}}$	$f_{\mathcal{S}}$
flexible	$\mathcal{C}_2$	2	1	4	$e_0 = 3v_0 + 1$	$3v_0 + 4 - 1$	2
distortional	$\mathcal{C}_2$	2	1	3	$e_0 = 3v_0$	$3v_0 + 3 - 1$	2
scaling	$\mathcal{C}_2$	2	1	3	$e_0 = 3v_0$	$3v_0 + 3 - 1$	2
hydrostatic	$\mathcal{C}_2$	2	1	1	$e_0 = 3v_0 - 1$	$3v_0 + 1 - 1$	1
fixed	$\mathcal{C}_2$	2	1	0	$e_0 = 3v_0 - 1$	$3v_0 - 1$	0

2. $\ell_{\mathcal{C}_s} = 4$. Although there initially appear to be two alignments for the mirror: (i) parallel to two translation axes and perpendicular to another or (ii) containing an axis of translation, these turn out to be two variations of the same larger space tiling, and crystallographers only consider the first version. In this case we have an orthorhombic lattice system, and we have four parameters, $\ell_{\mathcal{C}_s} = 4$.

Example 7.6.5 (Fully flexible lattice $\mathbb{Z}^3 \rtimes \mathcal{C}_s$.). As before, we start with the following initial count without symmetry: $2e_0 = 3(2v_0)+4$, or $e_0 = 3v_0+2$. From the periodic symmetric orbit matrix we obtain the following necessary symmetric Maxwell type count for periodic rigidity:

$$e_0 \geq 3v_0 + 4 - 2 = 3v_0 + 2.$$

This suggests that there is no additional flexibility in the structure when mirror symmetry is added. □

It turns out that for mirror symmetry, all of the variants of lattice deformations

produce no added motions.

7.6.3 Table of groups for the fully flexible lattice in 3-dimensions

Following the process illustrated in the previous examples, we can track the necessary increases in flexibility which follow from minimal generically rigid periodic frameworks for various symmetry groups $\mathbb{Z}^3 \rtimes \mathcal{S}$ in 3-space. As before, this does not include rows for the groups with 6-fold rotational symmetry, or any patterns with glide reflections. They will require some significant modifications of the simple pattern presented here.

Analogous to the tables in Sections 7.5.3 and 7.5.4, in each row of Table 7.8, the calculation has several parts - each producing an integer.

The number of rows (edge orbits), e_0, is such that $k_{\mathcal{S}} e_0 \geq 3(k_{\mathcal{S}} v_0) + 3$, which guarantees that we have at least the number of edges needed for a rigid periodic framework without symmetry. This means we need to add a modified constant $c = \lceil \frac{3}{k_{\mathcal{S}}} \rceil$. For Table 7.8, $c = 3$ for $k_{\mathcal{S}} = 1$, $c = 2$ for $k_{\mathcal{S}} = 2$, and $c = 1$ for all bigger orbit sizes.

As usual, $t_{\mathcal{S}}$ is the dimension of the space of translations contained in the symmetry element of $\mathcal{S}$. This will be $t_{\mathcal{S}} = 3$ for the identity group, $t_{\mathcal{S}} = 2$ for a single mirror, $t_{\mathcal{S}} = 1$ for a single rotation (with or without a mirror along the axis), and $t_{\mathcal{S}} = 0$ if only a point is fixed.

In Table 7.8 we compare the number of rows, e_0, with the number of columns minus $t_{\mathcal{S}}$, $3v_0 + \ell_{\mathcal{S}} - t_{\mathcal{S}}$; the difference $f_{\mathcal{S}} = 3v_0 + \ell_{\mathcal{S}} - t_{\mathcal{S}} - (3v_0 + c) = \ell_{\mathcal{S}} - t_{\mathcal{S}} - c$ is the dimension of the guaranteed extra non-trivial motions of the symmetric framework over the rigidity which the original count without symmetry promised.

7.6.4 Table of groups for the fixed lattice in 3-dimensions

In Table 7.9, we track the necessary increases in flexibility which follow from minimal generically rigid periodic frameworks on a fixed lattice for various symmetry groups in 3-space. This analysis is analogous to the one in the previous section. We simply remove the column for $\ell_{\mathcal{S}}$ which is always 0, and work with the modified counts.

The entries (-1) in Table 7.9 indicate that, for this group, the symmetry guarantees that there is a symmetric self-stress in the symmetric framework (see also Section 8.2.4). Because the patterns of 0 and occasional (-1) become clear quickly, we do not fill in all rows of the table.

Table 7.8: The added flexibility induced by basic symmetries on a fully flexible 3-D lattice for $\mathbb{Z}^3 \rtimes \mathcal{S}$.

Lat. System	$Sch_\mathcal{S}$	$H\text{-}M_\mathcal{S}$	$orb_\mathcal{S}$	$k_\mathcal{S}$	$t_\mathcal{S}$	$\ell_\mathcal{S}$	rows	columns $-t_\mathcal{S}$	$f_\mathcal{S}$
triclinic	$\mathcal{C}_1$	1	11	1	3	6	$3v+3$	$3v+6-3$	0
”	$\mathcal{C}_i$	$\bar{1}$	$1x$	2	0	6	$3v_0+2$	$3v_0+6-0$	4
monoclinic	$\mathcal{C}_2$	2	22	2	1	4	$3v_0+2$	$3v_0+4-1$	1
”	$\mathcal{C}_s$	m	$1*$	2	2	4	$3v_0+2$	$3v_0+4-2$	0
”	$\mathcal{C}_{2h}$	$2/m$	$2*$	4	0	4	$3v_0+1$	$3v_0+4-1$	2
orthorhom	$\mathcal{C}_{2v}$	222	$*22$	4	1	3	$3v_0+1$	$3v_0+3-1$	1
”	$\mathcal{D}_2$	$mm2$	222	4	0	3	$3v_0+1$	$3v_0+3-0$	2
”	$\mathcal{D}_{2h}$	mmm	$*222$	8	0	3	$3v_0+1$	$3v_0+3-0$	2
tetragonal	$\mathcal{C}_4$	4	44	4	1	2	$3v_0+1$	$3v_0+2-1$	0
”	$\mathcal{S}_4$	$\bar{2}$	$2x$	4	0	2	$3v_0+1$	$3v_0+2-0$	1
”	$\mathcal{C}_{4h}$	$4/m$	$4*$	8	0	2	$3v_0+1$	$3v_0+2-0$	1
”	$\mathcal{C}_{4v}$	$4mm$	$*44$	8	1	2	$3v_0+1$	$3v_0+2-1$	0
”	$\mathcal{D}_{2d}$	$\bar{4}2m$	$2*2$	8	0	2	$3v_0+1$	$3v_0+2-0$	1
”	$\mathcal{D}_4$	422	422	8	0	2	$3v_0+1$	$3v_0+2-0$	1
”	$\mathcal{D}_{4h}$	$4/mmm$	$*422$	16	0	2	$3v_0+1$	$3v_0+2-0$	1
trigonal	$\mathcal{C}_3$	3	33	3	1	2	$3v_0+1$	$3v_0+2-1$	0
”	$\mathcal{S}_6$	$\bar{3}$	$3x$	6	0	2	$3v_0+1$	$3v_0+2-0$	1
”	$\mathcal{D}_3$	32	322	6	0	2	$3v_0+1$	$3v_0+2-0$	1
”	$\mathcal{C}_{3v}$	$3m$	$*33$	6	1	2	$3v_0+1$	$3v_0+2-1$	0
”	$\mathcal{D}_{3d}$	$\bar{3}m$	$2*3$	12	0	2	$3v_0+1$	$3v_0+2-0$	1
cubic	$\mathcal{T}$	23	332	12	0	1	$3v_0+1$	$3v_0+1-0$	0
”	$\mathcal{T}_h$	$m\bar{3}$	$3*2$	24	0	1	$3v_0+1$	$3v_0+1-0$	0
”	$\mathcal{T}_d$	$\bar{4}3m$	$*332$	24	0	1	$3v_0+1$	$3v_0+1-0$	0
”	$\mathcal{O}$	432	432	24	0	1	$3v_0+1$	$3v_0+1-0$	0
”	$\mathcal{O}_h$	$m\bar{3}m$	$*432$	48	0	1	$3v_0+1$	$3v_0+1-0$	0

Table 7.9: The added flexibility induced by symmetries on a fixed 3-D lattice for $\mathbb{Z}^3 \rtimes \mathcal{S}$.

Lat. System	$Sch_\mathcal{S}$	$H\text{-}M_\mathcal{S}$	$orb_\mathcal{S}$	$k_\mathcal{S}$	$t_\mathcal{S}$	rows	columns -$t_\mathcal{S}$	$f_\mathcal{S}$
triclinic	$\mathcal{C}_1$	1	11	1	3	$3v - 3$	$3v - 3$	0
"	$\mathcal{C}_i$	$\bar{1}$	$1x$	2	0	$3v_0 - 1$	$3v_0 - 0$	1
monoclinic	$\mathcal{C}_2$	2	22	2	1	$3v_0 - 1$	$3v_0 - 1$	0
"	$\mathcal{C}_s$	m	$1*$	2	2	$3v_0 - 1$	$3v_0 - 2$	(-1)
"	$\mathcal{C}_{2h}$	$2/m$	$2*$	4	0	$3v_0$	$3v_0 - 0$	0
orthorhomb.	$\mathcal{C}_{2v}$	222	$*22$	4	1	$3v_0$	$3v_0 - 1$	(-1)
"	$\mathcal{D}_2$	$mm2$	222	4	0	$3v_0$	$3v_0 - 0$	0
"	$\mathcal{D}_{2h}$	mmm	$*222$	8	0	$3v_0$	$3v_0 - 0$	0
tetragonal	$\mathcal{C}_4$	4	44	4	1	$3v_0$	$3v_0 - 1$	(-1)
"	$\mathcal{S}_4$	$\bar{2}$	$2x$	4	0	$3v_0$	$3v_0 - 0$	0

8 Conclusions and Further Work

8.1 Discussion

We return briefly to a topic mentioned in the introduction, namely the study of zeolites. Since zeolites can be modelled as a system of corner-sharing tetrahedra, as orbit graphs they satisfy $|E| = d|V|$. Clearly then, zeolites will be generically flexible on the flexible torus, where we need at least $|E| = d|V| + \binom{d}{2}$ for minimal rigidity. However, recall that an infinitesimal flex of a periodic orbit framework on a flexible torus translates into an infinitesimal flex of a d-periodic framework in $\mathbb{R}^d$ where some vertices may have arbitrarily large velocity (see Section 5.3.6).

So we might ask: where is the natural "home" for zeolites? Is it the flexible torus, the fixed torus, or something in between? Perhaps zeolites are most accurately modelled on the fixed torus, where they are overbraced (on the fixed torus we only require $|E| = d|V| - d$ for minimal rigidity), but their special symmetries means that they have extra flexibility, as was the case for several examples in Chapter 7.

Zeolites remain an interesting source of examples and inspiration for the study

of periodic frameworks. As described in the introduction, the work in this thesis is a contribution to a rapidly developing, and expanding set of work on periodic frameworks. The results and the methods contained here open up a number of immediate extensions and suggest a number of avenues for further investigations within this field. In the section below we collect some of the unanswered questions from previous sections, and outline a few other topics for further work.

8.2 Further work

8.2.1 Periodic bar-body frameworks

One natural extension of the work in this thesis is to *periodic bar-body frameworks*. The generic rigidity of *finite* bar-body frameworks is completely characterized in d-dimensions with polynomial time algorithms [72], and a recent proof of the Molecular Conjecture expands this characterization to molecular frameworks [44]. That is, unlike bar-joint frameworks for $d \geq 3$, the generic rigidity of bar-body frameworks for all d can be understood through combinatorial methods alone. See Table 8.1 (compare Table 1.1), and Theorem 8.2.2 below for a statement of the characterization for finite bar-body frameworks.

As in the periodic bar-joint case, periodic bar-body frameworks can be defined using gain graphs.

Question 8.2.1. Which gain graphs admit a realization as a generically rigid periodic bar-body framework in d-dimensions?
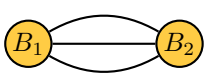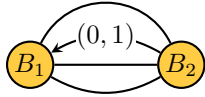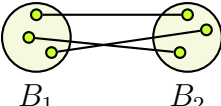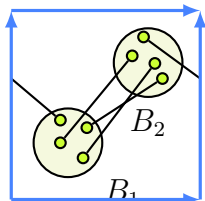
The investigation to date has indicated that a periodic version of bar-body rigidity theory would generalize the well-understood model for finite (not periodic) bar-body frameworks. There are available inductive techniques [27], which are promising candidates for extending the previous results for planar bar-body frameworks, but will need to be modified to describe the topological and geometric elements of the periodic setting. We could also ask how to extend these (proposed) results to the "geometrically special" class of molecular frameworks [44]. Describing a full rigidity theory of periodic bar-body frameworks may enhance our understanding of zeolites. In particular, when the characterizations proposed above is applied to zeolites, this would provide efficient algorithms of the type now used for finite materials such as proteins [75].

Table 8.1: Bar-body frameworks

Type	finite	periodic
Torus	–	fixed $\mathcal{T}_0^d$
$d = 1, 2$	necessary & sufficient ✓	$\mathcal{T}_0^2$ ✓ (from bar-joint)
$d \geq 3$	necessary & sufficient ✓	necessary conditions, conjecture they are sufficient

In fact, this research is well underway for bar-body frameworks on the fixed

Table 8.2: Examples of bar-body frameworks in the finite and periodic cases

	Finite	Periodic
Counts	$|E| = 3|V| - 3$	$|E| = 3|V| - 2$
(Gain) Graph	$H = (V, E)$	$\langle H, m \rangle$
Framework	(H, q) in $\mathbb{R}^2$	$(\langle H, m \rangle, q)$ on $\mathcal{T}_0^2$

torus, and we summarize the results here, following a few basic definitions. Informally, a bar-body framework consists of a number of rigid bodies linked together with bars. Each rigid body can move independently, and the bars place constraints on the motions of the individual bodies. We will first consider the infinitesimal motions of a single rigid body, before turning our attention to infinitesimal motions of bar-body frameworks.

Let $B \subset \mathbb{R}^d$ be a set of points whose affine span is $\mathbb{R}^d$. We call B a *rigid body*. Let $u : B \to \mathbb{R}^d$ be a map such that for every pair of points $p_1, p_2 \in B$,

$$(p_1 - p_2) \cdot (u(p_1) - u(p_2)) = 0.$$

Thus u preserves the distance between any pair of points on B. We call the map u an *infinitesimal motion* of B, and the vector $u(p)$ will be the *instantaneous velocity*

of the point p. The infinitesimal motions of B form a vector space of dimension $\binom{d+1}{2}$ over $\mathbb{R}$, which is spanned by d infinitesimal translations and $\binom{d}{2}$ infinitesimal rotations. That is, any element of this vector space can be written uniquely as a linear combination of infinitesimal rotations and translations of $\mathbb{R}^d$. The infinitesimal rotations and translations of a rigid body may in turn be coordinatized using screw centres, which are given by certain $\binom{d+1}{2}$-tuples. These vectors are described in [79] as $(d-1)$-extensors in projective d-space.

A *d-periodic bar-body framework* $(\langle H, m \rangle, q)$ is a gain graph $\langle H, m \rangle$, together with a map q which associates a line segment $q(e) = q_e$ of $\mathbb{R}^d$ to each edge $e \in E\langle H, m \rangle$:

$$q_e : (A, B; m) \to (a, b + m),$$

where $a, b \in \mathcal{T}_0^d$. In other words, to each edge $e \in E(H)$, q associates a pair of points $a, b \in \mathcal{T}_0^d$. The vertices of H represent bodies in $\mathbb{R}^d$, and we denote elements of $V(H)$ by $A, B, \ldots$ etc. Points on the bodies we denote by $a, b, \ldots$ etc., where $a \in A$, $b \in B$, and so on. We assume the points $a, b, \ldots$ are distinct from one another, but we allow loops in the gain graph, which connect distinct points on a single body.

An *infinitesimal motion* of $(\langle H, m \rangle, q)$ is a map $S : V(H) \to \mathbb{R}^{\binom{d+1}{2}}$ which assigns a screw centre to each body of $\langle H, m \rangle$ in such a way that the lengths of all of the edges of $\langle H, m \rangle$ are preserved instantaneously. We omit the details.

296

Let $T(p, p') = (t_{i,j})$ be the $\binom{d+1}{2}$-tuple given by taking all possible 2×2 determinants $t_{i,j}$ of the matrix

$$
t_{p,p'} = \begin{pmatrix} p_1 & p_2 & \cdots & p_d & 1 \\ p'_1 & p'_2 & \cdots & p'_d & 1 \end{pmatrix},
$$

where $t_{i,j}$ is obtained from $t_{p,p'}$ by deleting all but the columns i and j. Again, we take these determinants in lexicographical order. We can think of $T(p, p')$ as representing the line through the points p and p', in the sense that $T(a, b) = \lambda T(c, d)$ if a, b, c, d are points on a line. In [79], $T(p, p')$ is written as $p \vee p'$, reflecting the fact that exterior algebra (specifically Grassmann-Cayley algebra) can be used to describe these geometric objects. Note that $T(p, p') = -T(p', p)$.

The *rigidity matrix* $\mathbf{R}(\langle H, m \rangle, q)$ of the d-periodic bar-body framework $(\langle H, m \rangle, q)$ has one row for each bar and $\binom{d+1}{2}$ columns for each body (vertex). If $(A, B; m)$ is an edge of $\langle H, m \rangle$, and $q_e = (a, b + m)$, then the row corresponding to this edge has $T(a, b + m)$ in the columns under A, $T(b, a - m)$ in the columns under B, and 0 in all other columns. A loop edge $(B, B; m)$ will have $T(b - b', m)$ in the columns under B, and 0 in all other columns. Rewriting, the bar-body rigidity matrix is recorded as:

$$\begin{array}{c}
 \quad\quad\quad\quad A \quad\quad\quad\quad\quad\quad\quad B \\
\begin{array}{cc}
\text{edge} \ \ \{A, B; m\} \\
\\
\text{loop} \ \ \{B, B; m\}
\end{array}
\left(
\begin{array}{cccccc}
& \vdots & & & \\
0 \cdots 0 & T(a, b+m) & 0 \cdots 0 & T(b, a-m) & 0 \cdots 0 \\
& \vdots & & & \\
0 \cdots 0 & 0 & 0 \cdots 0 & T(b-b', m) & 0 \cdots 0 \\
& \vdots & & & \\
\end{array}
\right).
\end{array}$$

Let S be an infinitesimal motion of $(\langle H, m \rangle, q)$, and write $S = (S_A, S_B, \ldots, S_{|V|})$.

Then

$$\mathbf{R}(\langle H, m \rangle, q) \cdot S^T = 0.$$

In fact, by the construction of the bar-body rigidity matrix, $S \in \mathbb{R}^{\binom{d+1}{2}|V|}$ is an

infinitesimal motion of $(\langle H, m \rangle, q)$ if, and only if $\mathbf{R}(\langle H, m \rangle, q) \cdot S^T = 0$.

For finite frameworks, the rigidity of bar-body frameworks is given by the following theorem of Tay:

Theorem 8.2.2 (Tay's Theorem [72]). *For a multigraph $H = (V, E)$, the following are equivalent:*

1. *The graph H is generically rigid as a bar-body framework in $\mathbb{R}^d$*

2. *H contains $\binom{d+1}{2}$ edge-disjoint spanning trees.*

We build on this for the periodic case.

Theorem 8.2.3. *Let $\langle H, m \rangle$ be a generically minimally rigid bar-body framework on $\mathcal{T}_0^d$, with $|E| = \binom{d+1}{2}|V| - d$. Then for all nonempty subsets $Y \subseteq E$ of edges,*

$$|Y| \leq \binom{d+1}{2}|V(Y)| - \binom{d+1}{2} + \sum_{i=1}^{|\mathcal{M}_\mathcal{C}(Y)|} (d - i). \qquad (8.1)$$

Note that one direction of Tay's theorem is implied by this one. If all edges have zero gains, and therefore $|\mathcal{M}_\mathcal{C}(Y)| = 0$ for all subsets $Y \subset E$, then by the result of Nash-Williams, this sparsity condition implies the existence of $\binom{d+1}{2}$ edge-disjoint spanning trees.

The proof of Theorem 8.2.3 is analogous to the proof of Theorem 4.5.2, which makes a similar statement for bar-joint frameworks. In contrast to that result, however, we believe that the conditions of Theorem 8.2.3 may also be sufficient:

Conjecture 8.2.4. *Let $\langle H, m \rangle$ be a bar-body orbit framework, with $|E| = \binom{d+1}{2}|V| - d$. Then $\langle H, m \rangle$ is generically minimally rigid on $\mathcal{T}_0^d$ if, and only if for all nonempty subsets $Y \subseteq E$ of edges,*

$$|Y| \leq \binom{d+1}{2}|V(Y)| - \binom{d+1}{2} + \sum_{i=1}^{|\mathcal{M}_\mathcal{C}(Y)|} (d - i).$$

A proof of the sufficiency of this statement would develop in much the same way as the proof of the periodic Laman theorem in the plane (Theorem 4.4.5). It

would involve carefully modifying existing inductive constructions, as outlined in Fekete and Szegő [27], to incorporate gain assignments.

There is another question we might ask related to bar-body frameworks. In particular, suppose we relax the definition of a bar-body framework so that the bodies are allowed to be either finite (isostatic) frameworks *or* minimally rigid periodic frameworks on the fixed torus. When is such a framework rigid on $\mathcal{T}_0^d$? We anticipate that the (proposed) results for periodic bar-body frameworks as defined above will extend to this larger setting with minor modifications.

8.2.2 More on symmetric periodic frameworks

There are numerous questions for further work that arose in the discussion of periodic frameworks with additional symmetry (see Chapter 7). We outline several of them here. More details and questions are found in the final section of [58].

First recall that we assumed the frameworks we analyzed did not have any bars or joints that were fixed by the symmetry operations in $\mathcal{S}$. Incorporating fixed elements requires small modifications to the counts, but are not a significant barrier to using our methods. This is elaborated on in [58].

As is clear from the tables in Sections 7.5 and 7.6, we have only discussed some of the many possible symmetry groups. We specifically considered frameworks with symmetry given by the group $\mathbb{Z}^d \rtimes \mathcal{S}$.

Question 8.2.5. What are the necessary conditions for rigidity (a Maxwell type rule) for frameworks whose symmetry group includes glide reflections (in 2-space), screw symmetry (in 3-space), or 6-fold rotations (in 2- or 3-space)?

There are straightforward extensions of the methods we used in Chapter 7 to frameworks whose symmetry group involves a single glide reflection or screw symmetry, but we have not considered the more complex examples whose symmetry groups involve combinations of these generators.

An additional question of interest concerns the topology of the orbit frameworks. In this thesis, we have considered periodic frameworks by developing a theory of rigidity for the (finite) orbit frameworks of these structures, on a topological torus. One could take a similar approach to periodic frameworks with additional symmetry. For example, a periodic framework with mirror symmetry in the plane or space (given by $\mathbb{Z}^2 \rtimes \mathcal{C}_\mathcal{S}$ and $\mathbb{Z}^3 \rtimes \mathcal{C}_\mathcal{S}$ respectively) can be regarded as frameworks on 2- or 3-spheres, but with a flat metric. Frameworks with inversive symmetry in space $(\mathbb{Z}^2 \rtimes \mathcal{C}_i)$ "live" in projective three space. In all cases, the orbit matrix represents the rigidity of the orbit framework on this orbifold.

Finally the results of Chapter 7 do not provide any sufficient conditions for generic rigidity. Given that sufficient conditions for planar periodic rigidity exist in several cases (Chapters 4, 5 and [49]), and that some classes of symmetric frameworks have been similarly characterized, it is natural to expect that sufficient

conditions for the rigidity of some symmetric-periodic frameworks in the plane are accessible. Furthermore, we may be able to characterize special classes of 3-dimensional symmetric periodic frameworks, based on inductive techniques.

8.2.3 Discrete scaling of the fundamental region

Throughout this thesis, we have implicitly assumed that we are working with the smallest possible fundamental region (unit cell). That is, for a periodic framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$, the translations forming the rows of L are minimal, in that no smaller translations will generate the same framework. What happens if we relax this assumption? That is, suppose we choose a unit cell which is a discrete multiple of the *smallest* unit cell. We will refer to this operation as *discrete scaling*, and the periodic orbit framework on the torus generated by the larger unit cell as the *scaled framework*. More precisely, let $(\langle G, m \rangle, p)$ be the periodic orbit framework on $\mathcal{T}_k^d = \mathbb{R}^d / L\mathbb{Z}^d$ corresponding to the periodic framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$, and let L' be a sublattice of L. Then we denote by $(\langle G', m' \rangle, p')$ the periodic orbit framework on the torus $\mathcal{T}_k^d = \mathbb{R}^d / L'\mathbb{Z}^d$ corresponding to the periodic framework $(\langle \widetilde{G}, L' \rangle, \widetilde{p})$. The periodic orbit framework $(\langle G', m' \rangle, p')$ is the scaled framework.

There are two questions which arise (See also Table 8.3 for an example):

Question 8.2.6. Under what conditions will discrete scaling of the fundamental region of a periodic framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ maintain its *generic* rigidity properties

as an orbit framework on the torus?

Note that, geometrically, discrete scaling will always take a generic periodic orbit framework on the torus to a non-generic periodic orbit framework on a larger torus. Perhaps a more interesting question is the following:

Question 8.2.7. Under what conditions will discrete scaling of the fundamental region of a periodic framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ maintain its *geometric* rigidity properties as an orbit framework on the torus? In other words, given a framework that is infinitesimally rigid with some fundamental region, when is the (non-generic) scaled framework still infinitesimally rigid?

This question was considered for a number of 2-dimensional zeolites in [43]. The authors of that paper find that the number of deformations of a flexible framework increases as the size of the fundamental region increases by discrete scaling. That is, the dimension of the space of infinitesimal motions increases with the size of the fundamental region, and the authors assumed that the fundamental region was continuously flexible.

We can make a few immediate observations. In terms of combinatorics alone, the scaled framework may be over- or under-braced:

- On the fixed torus, if the initial framework $(\langle G, m \rangle, p)$ has $|E| = d|V| - d$ and is minimally rigid, then the scaled framework $(\langle G', m' \rangle, p')$ will be underbraced, and hence is generically flexible.

303
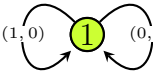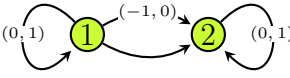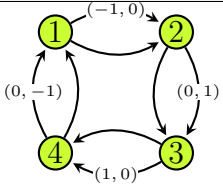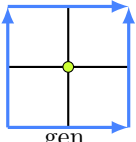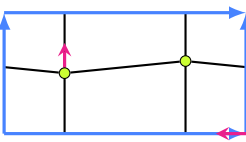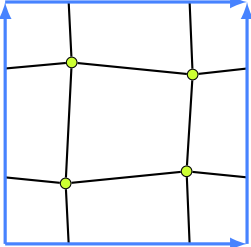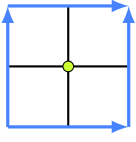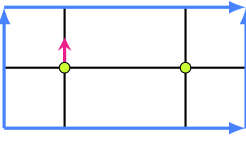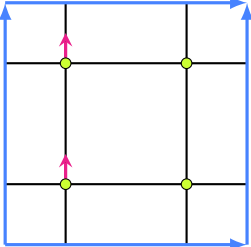
- On the fully flexible torus, if the initial framework $(\langle G, m \rangle, p)$ is minimally rigid with $|E| = d|V| + \binom{d}{2}$, then the scaled framework $(\langle G', m' \rangle, p')$ will be overbraced. It is still possible to introduce motions through this scaling, and the scaled framework may or may not be infinitesimally rigid.

- In the case that $(\langle G, m \rangle, p)$ has $|E| = d|V|$, the scaled framework $(\langle G', m' \rangle, p')$ will also have $|E'| = d|V'|$. Interestingly, zeolites are typically modelled with these counts (as systems of corner-sharing d-simplices). However, any such framework will be overbraced on the fixed torus, and underbraced on the flexible torus.

Bob Connelly has a conjecture related to scaling:

Conjecture 8.2.8. *If a framework $(\langle G, m \rangle, p)$ is infinitesimally rigid on the flexible torus generated by the lattice L, then $(\langle G', m' \rangle, p')$ is infinitesimally rigid on the flexible torus generated by any sub-lattice L'.*

Of course if we want to scale the unit cell of a framework, we would like our frameworks to be connected. We have seen some examples of frameworks that are infinitesimally rigid as frameworks on the torus, but that are *not* connected as derived periodic frameworks. For example, the zig-zag framework on $\mathcal{T}_0^2$ had a countably infinite number of connected components (the zig-zags). Figure 8.1 pictures another such example, where the two components are coloured in black

Table 8.3: Example of generic and geometric scaling on the scaling torus $\mathcal{T}_2^2$

Per. Orbit Graph			
Generic	gen. rigid	gen. flexible	gen. rigid
Geometric	rigid	flexible	flexible

and red. A result from topological graph theory gives us tools to apprehend the connectivity of a derived framework.

Theorem 8.2.9 ([32]). *The number of components of the derived periodic framework* $(\langle G^m, L \rangle, p^m)$ *equals the index in the gain group* $\mathbb{Z}^d$ *of the local gain group of* $\langle G, m \rangle$.

The example in Figure 8.1 has local gain group $2\mathbb{Z} \times \mathbb{Z}$, which has index 2 in $\mathbb{Z}^2$. With respect to discrete scaling of the unit cell, this tells us that we can expect flexibility when the size of the unit cell doubles, since the unit cell will then "see" the disconnection of the framework, and move the pieces relative to one another.

Yet another point of view on scaling would be to consider the scaled framework as a periodic framework with special symmetry within its unit cell, in the style of Chapter 7. In particular, the scaled framework would posses translational symmetry within its unit cell. For example, if the scaled copy of the fundamental region contained two copies of the original fundamental region, then we could consider the scaled framework as having the additional symmetry C_2, representing translation.

8.2.4 Statics

We have made passing reference in several places to the idea of *stresses* in a framework. In fact, the theory of *static rigidity* for finite frameworks can be thought

(a)

(b) infinitesimally rigid

(c) infinitesimally flexible
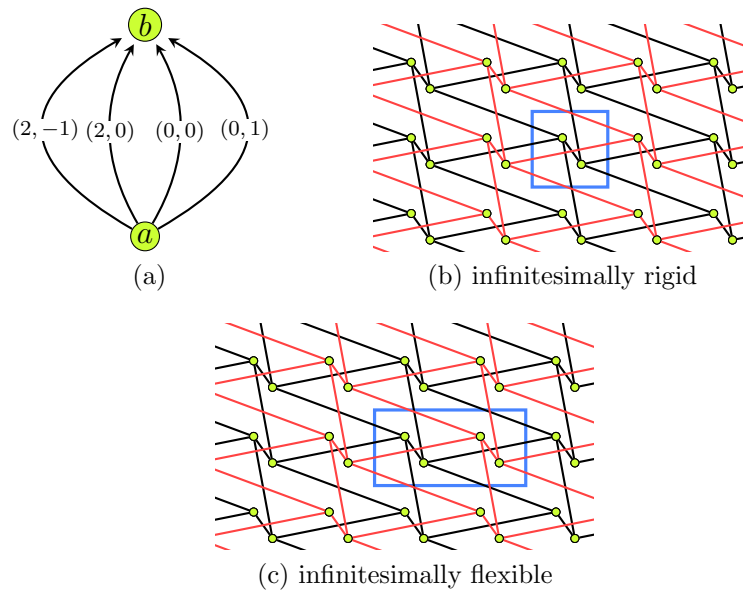
Figure 8.1: A periodic orbit graph with local gain group $\mathcal{A}(u) = 2\mathbb{Z} \times Z$ (a). The local gain group $\mathcal{A}(u)$ has index 2 in the gain group $\mathcal{A} = \mathbb{Z}^2$, and indeed there are two connected components. This periodic framework will be rigid in (b), but flexible in (c), since the two components can move with respect to one another. In fact the periodic orbit graph for the framework in (c) will consist of two (disjoint) copies of the graph in (a).

of as a companion theory to the theory of *kinematic rigidity* (the study of motions and infinitesimal motions). In addition, the idea of independence that has been so central in our discussions of minimal rigidity thus far has additional meaning in the context of static rigidity. We briefly outline a theory of static rigidity for finite and periodic frameworks. For further details on finite static rigidity we refer to [20] or [84], to name just two examples.

Let (G, p) be a finite framework in $\mathbb{R}^d$. Recall that the we said that the edges of G were *dependent* if the corresponding rows in the (finite) rigidity matrix $\mathbf{R}(G, p)$ were dependent. We can equivalently define a dependence on a framework to be an assignment $\omega : E \to \mathbb{R}$, with $\omega\{i, j\} = \omega_{i,j} = \omega_{j,i}$ such that for each vertex i:

$$\sum_{j \mid \{i,j\} \in E} \omega_{i,j}(p_i - p_j) = 0. \tag{8.2}$$

We call ω a *self-stress* of the framework, and it can be thought of as the resolution by the bars of the framework of a zero load on the framework (see Crapo and Whiteley [20]). A framework is called *independent* if it has only the trivial self-stress, and dependent otherwise (it is easily seen that this is equivalent to our earlier definition of independence). Frameworks that are both rigid and independent are called *isostatic* or *minimally rigid*. The key idea linking infinitesimal rigidity with independence is that for a finite graph G with $|E| = d|V| - \binom{d+1}{2}$, (G, p) is infinitesimally rigid if and only if (G, p) is independent. This follows directly from the rigidity matrix, and the fact that the row rank equals the column rank. The

kernel of the rigidity matrix corresponds to the space of infinitesimal motions on the framework, and the cokernel of the rigidity matrix is the space of stresses on the framework.

Remark 8.2.10. We can define a notion of *static rigidity*. A *load* on the framework (G, p) is a function which applies a force F_i to each vertex i. An equilibrium load is a load without any net rotation or translation. A resolution of an equilibrium load is a function $\omega : E \to \mathbb{R}$ such that

$$\sum_{j | \{i,j\} \in E} \omega_{ij}(p_i - p_j) = F_i, \tag{8.3}$$

where we can think of the scalars ω_{ij} on the edges as representing tension or compression as $\omega_{ij} < 0$ or $\omega_{ij} > 0$ respectively. The equations (8.3) say that the forces are in equilibrium at every vertex i. A framework is called *statically rigid* if every equilibrium load has a resolution by the bars of the framework. If $F_i = 0$ for all $i \in V$, then the load is the zero load. Importantly, static rigidity can be shown to be equivalent to infinitesimal rigidity. The basic argument uses the fact that the row rank is equal to the column rank of the rigidity matrix. $\square$

The basic definitions for self-stresses on periodic frameworks are given in Borcea and Streinu [7], together with a relationship between infinitesimal motions and self-stresses. We will describe these ideas in our vocabulary, and note once again that these ideas were developed independently.

Let $(\langle G, m \rangle, p)$ be a periodic orbit framework on the fixed torus $\mathcal{T}_0^d$. It is not hard to see how to extend the idea of a stress to this setting. Recall that an edge $e \in E\langle G, m \rangle$ is a labeled, directed edge, $e = \{i, j; m_e\}$. For a vertex $i \in V$, let E_+ denote the set of edges directed *out* from the vertex i and let E_- denote the set of edges directed *into* the vertex i. Let $\omega : E \to \mathbb{R}$, with $\omega(e) = \omega_e$ such that for each vertex $i \in V$:

$$\sum_{e_\alpha \in E_+} \omega_{e_\alpha} (p_i - (p_j + m_{e_\alpha})) + \sum_{e_\beta \in E_-} \omega_{e_\beta} (p_i - (p_k - m_{e_\beta})) = 0. \tag{8.4}$$

We call ω a *self-stress* of the periodic orbit framework $(\langle G, m \rangle, p)$.

For $(\langle G, m \rangle, p)$ on the flexible torus $\mathcal{T}_k^d$, we must also incorporate a stress on the lattice elements. In other words, a stress on $(\langle G, m \rangle, p)$ on $\mathcal{T}_k^d$ is a function $\omega : E \to \mathbb{R}$ such that equations (8.4) are satisfied, and ω satisfies the additional condition that

$$
\begin{aligned}
0 &= \sum_{e = \{i, j; m_e\} \in E\langle G, m \rangle} \omega_e \mathcal{L}\{i, j; m_e\} \\
&= \sum_{e = \{i, j; m_e\} \in E\langle G, m \rangle} \omega_e \big(p_i - (p_j + m_e)\big) M_e,
\end{aligned}
\tag{8.5}
$$

where M_e is as defined in Chapter 5.

Theorem 8.2.11. *Let $(\langle G, m \rangle, p)$ be a periodic orbit framework on the fixed torus. Then the following are equivalent:*

1. *$(\langle G, m \rangle, p)$ is minimally rigid on $\mathcal{T}_0^d$*

2. $(\langle G, m\rangle, p)$ *is infinitesimally rigid on* $\mathcal{T}_0^d$ *with* $|E| = d|V| - d$

3. $(\langle G, m\rangle, p)$ *is independent on* $\mathcal{T}_0^d$ *with* $|E| = d|V| - d$

4. $(\langle G, m\rangle, p)$ *is infinitesimally rigid on* $\mathcal{T}_0^d$, *and removing any single bar (without removing vertices) leaves an infinitesimally flexible framework on* $\mathcal{T}_0^d$.

We could also define periodic loads, and hence a notion of *static rigidity* for frameworks on the torus. Then we could add to the theorem above:

5. $(\langle G, m\rangle, p)$ *is statically rigid on* $\mathcal{T}_0^d$ *with* $|E| = d|V| - d$.

The analogous statement for the flexible torus $\mathcal{T}_k^d$ can be obtained by replacing the count $|E| = d|V| - d$ with $|E| = d|V| - d + k$, where $k = 1, \ldots, \binom{d+1}{2}$.

Example 8.2.12. Consider the periodic orbit graph $\langle G, m\rangle$ shown in Figure 8.2(a), and its realization on the torus (b). If $\langle G, m\rangle$ is realized on the fixed torus $\mathcal{T}_0^2$, then all three edges are dependent (being loops), and the framework is stressed. In fact, any assignment $\omega_1, \omega_2, \omega_3$ to the edges of $\langle G, m\rangle$ will be a valid stress, meaning that we have a three-dimensional space of stresses. On the other hand, if the framework is realized on the fully flexible torus $\mathcal{T}^3$, then the three loops are independent. In particular, the rigidity matrix for this framework on $\mathcal{T}^3$ becomes

$$\mathbf{R}(\langle G, m\rangle, p) = - \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

311
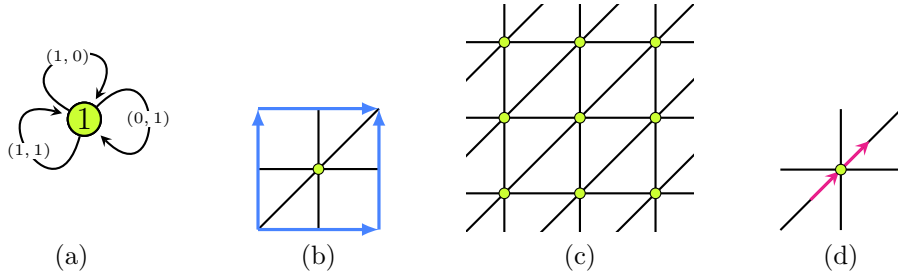
Figure 8.2: (a) A periodic orbit graph with three loops is generically dependent on the fixed torus $\mathcal{T}_0^2$, and generically independent on the flexible torus $\mathcal{T}^2$ (b). Viewed as an infinite periodic framework $(\widetilde{G}, \widetilde{p})$ the framework is rigid, but stressed (d). Every stress of $(\widetilde{G}, \widetilde{p})$ corresponds to a stress of $\langle G, m \rangle$ on $\mathcal{T}_0^2$.

(The rigidity matrix for the framework on the fixed torus, $\mathbf{R}_0(\langle G, m \rangle, p)$, is simply composed of the first two columns of $\mathbf{R}(\langle G, m \rangle, p)$, which are all zero.) However, consider now the periodic framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ shown in (c). As an infinite framework, it is infinitesimally rigid whether it is viewed as a periodic framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ or an infinite framework $(\widetilde{G}, \widetilde{p})$ that happens to be periodic (as a triangulated structure, it is rigid). Furthermore, it is stressed, since for each line entering and exiting each vertex, any stress may be placed on the edge (d). Therefore, any stress of the infinite framework $(\widetilde{G}, \widetilde{p})$ corresponds to a stress on $(\langle G, m \rangle, p)$ on the fixed torus. Incidentally, it follows that this is an example of a framework where the choice of unit cell is unimportant – see Section 8.2.3. $\qquad \square$

We have used the concept of self-stresses in frameworks numerous times throughout this thesis, without making explicit reference to it. In particular, since a self-stress on a framework corresponds to a row dependence in the rigidity matrix, the

312

proofs of several results can be translated into statements about stresses. The proof of the fact that the rank of the rigidity matrix is invariant under affine transformations of the framework (for both the fixed and flexible torus) is one such example, as is the the proof of the fact that the T-gain procedure preserves the rank of the rigidity matrix.

It should be noted that Theorem 8.2.11 holds only for frameworks on the torus, and is a direct consequence of the fact that we are recording a *finite* rigidity matrix. In general for a framework $(\widetilde{G}, \widetilde{p})$ where $\widetilde{G}$ is an infinite graph, this statement does not hold, since the row rank is no longer equal to the column rank of the infinite rigidity matrix. This is the essential observation contained in the paper of Guest and Hutchinson [69], who say that an infinite periodic framework cannot be both statically and kinetically determinate (i.e. cannot be both minimally statically rigid and minimally infinitesimally rigid).

We make one final observation about periodic self-stresses. Suppose we have a periodic framework $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ that we are viewing as an infinite framework $(\widetilde{G}, \widetilde{p})$, *not* as a graph on the torus. Then a stress on this framework is defined by a function $\widetilde{\omega} : E \to \mathbb{R}$ such that, at each vertex v_i of $\widetilde{G}$, (8.2) holds. That is, we have a *local* equilibrium condition at each vertex of the infinite framework. This suggests that if the stress $\widetilde{\omega}$ is periodic, then it must be a stress of $(\langle G, m \rangle, p)$ on the fixed torus $\mathcal{T}_0^d$. The stress does not "see" the fixed nature of the lattice. In other words, we

313

claim that

Proposition 8.2.13. *Let* $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ *be a d-periodic framework, and let* $(\langle G, m \rangle, p)$ *be its periodic orbit framework. Then the following are equivalent*

1. $(\langle \widetilde{G}, L \rangle, \widetilde{p})$ *has a self-stress in* $\mathbb{R}^d$

2. $(\langle G, m \rangle, p)$ *has a self-stress on* $\mathcal{T}_0^d$.

In contrast, the definition of self-stress for the flexible torus provided in Borcea and Streinu [7] does not admit this equivalence. We do not pursue this idea further here, but believe that a full development of the theory of static rigidity for periodic frameworks would be illuminating. Stresses will play a key role in the possible development of a periodic theory of tensegrity frameworks and global rigidity.

8.2.5 Periodic tensegrity frameworks

A (finite) *tensegrity framework* consists of a graph G, the edges of which are partitioned into three sets:

1. the *cables* C, whose length is allowed only to decrease,

2. the *struts* S, whose length is allowed only to increase,

3. the *bars* B, whose length must remain fixed (as for the usual bar-joint frameworks),

314

together with a configuration $p : V \to R^d$ of its vertices, as for bar-joint frameworks. Then a *continuous motion* (or *flex*) of the tensegrity framework (G, p) is a map $p(t) = (p_1(t), \ldots, p_{|V|}(t))$ such that $p_i(t)$ is continuous for $t \in [0, 1]$, $p_i(0) = p_i$ and the lengths of all members of the framework are preserved. If (G, p) admits only the identity flex, then the tensegrity is *rigid*.

An *infinitesimal motion* of (G, p) is an assignment of instantaneous velocities to each of the vertices of G. In particular, an infinitesimal motion of (G, p) is a map $u : V \to \mathbb{R}^d$ such that

$$(p_i - p_j)(u_i - u_j) \begin{cases} = 0 & \text{for } \{i, j\} \in B, \\ \leq 0 & \text{for } \{i, j\} \in C, \\ \geq 0 & \text{for } \{i, j\} \in S. \end{cases}$$

An infinitesimal motion is *trivial* if there is a $d \times d$ skew-symmetric matrix S and a vector $t \in \mathbb{R}^d$ such that $u_i = Sp_i + t$ for all $v_i \in V$. If the only infinitesimal motions of (G, p) are trivial, then the tensegrity framework is *infinitesimally rigid* [81].

We define a *self-stress* on the tensegrity to be an assignment of scalars to the edges of G: $\omega : E \to \mathbb{R}^d$ with

1. $\omega_{ij} \geq 0$ if $\{i, j\} \in C$,

2. $\omega_{ij} \leq 0$ if $\{i, j\} \in S$, and

3. ω_{ij} arbitrary for $\{i, j\} \in B$.

315

A *proper self-stress* is a self-stress $\omega : E \to \mathbb{R}^d$ with strict inequalities for the cables

and struts: $\omega_{ij} > 0$ if $\{i, j\} \in C$, $\omega_{ij} < 0$ if $\{i, j\} \in S$, and ω_{ij} arbitrary for

$\{i, j\} \in B$.

A key characterization of infinitesimal rigidity of finite tensegrity frameworks

is:

Theorem 8.2.14 ([60]). *A tensegrity framework (G, p) is infinitesimally rigid if*

and only if the induced bar framework $(\bar{G}, p)$ is infinitesimally rigid and there exists

a proper self stress on (G, p).

The question is to identify how this translates into the periodic setting.

Question 8.2.15. When is a periodic tensegrity framework rigid on the fixed torus

$\mathcal{T}_0^d$? On the flexible torus $\mathcal{T}_k^d$?

Some of this translation is straightforward. We may define a *periodic tensegrity*

framework to be a periodic orbit framework $(\langle G, m \rangle, p)$, where the edges of $\langle G, m \rangle$

are partitioned as for the finite tensegrities, into bars, cables and struts. An in-

finitesimal motion of $(\langle G, m \rangle, p)$ on the fixed torus $\mathcal{T}_0^d$ is a map $u : V \to \mathbb{R}^d$ such

that

$$(p_i - (p_j + m_e))(u_i - u_j) \begin{cases} = 0 & \text{for } \{i, j; m_e\} \in B, \\ \leq 0 & \text{for } \{i, j; m_e\} \in C, \\ \geq 0 & \text{for } \{i, j; m_e\} \in S. \end{cases}$$

316

An infinitesimal motion of $(\langle G, m \rangle, p)$ is *trivial* if $u_i = t$ for some $t \in \mathbb{R}^d$, and all vertices $v_i \in V$. That is, a trivial infinitesimal motion is a translation. If the only infinitesimal motions of the tensegrity $(\langle G, m \rangle, p)$ on $\mathcal{T}_0^d$ are trivial, then the tensegrity is *infinitesimally rigid* on $\mathcal{T}_0^d$. The definition of self-stresses on periodic tensegrities is the same as for finite tensegrities.

With these definitions in place, the arguments of Roth and Whiteley [60] transfer directly to show:

Theorem 8.2.16. *The tensegrity $(\langle G, m \rangle, p)$ is infinitesimally rigid on the fixed torus $\mathcal{T}_0^d$ if, and only if, the underlying bar joint framework $(\langle \overline{G}, m \rangle, p)$ is infinitesimally rigid on $\mathcal{T}_0^d$, and there exists a proper self-stress of $(\langle G, m \rangle, p)$.*

The question remains about how to transfer these results to tensegrity frameworks on the flexible torus.

When all members of the periodic tensegrity framework are struts, this is the situation of sphere packing (spheres are not allowed to overlap, but may move further apart). However, if we do not insist on a fixed torus, such a framework could always expand.

Periodic tensegrity frameworks have also been studied as *spiderwebs*, tensegrity frameworks formed by pinning some vertices (to the torus), and allowing only cables [1].

8.2.6 Periodic global rigidity

Let G be a finite graph, and let p and q be two realizations of its vertices in $\mathbb{R}^d$. We say write $G(p) \equiv G(q)$ if for each edge $e = \{i, j\}$ of G, $\|p_i - p_j\| = \|q_i - q_j\|$. In other words, p and q are two configurations with the same edge lengths. We say that p and q are *congruent*, and write $p \equiv q$, if for *all* $i, j \in \{1, \ldots, |V|\}$, $\|p_i - p_j\| = \|q_i - q_j\|$. The framework (G, p) is called *globally rigid* if $G(p) \equiv G(q)$ implies that $p \equiv q$ [13]. In this way global rigidity is concerned with the unique realization of a graph with particular edge lengths. In contrast, the rigidity theory outlined in the earlier chapters of this thesis is a type of *local* rigidity, which looks for unique realizations of a graph within a neighbourhood of the initial position. If a framework is globally rigid, then it is certainly locally rigid, but the converse does not hold.

One can define an analogous definition of global rigidity for periodic frameworks, and ask:

Question 8.2.17. When is a periodic orbit framework globally rigid on the fixed torus $\mathcal{T}_0^d$? On the flexible torus $\mathcal{T}_k^d$?

We anticipate that the answers to these questions will involve connectivity considerations, and periodic versions of redundant rigidity.

A variation of the global rigidity problem is concerned with the rigidity of tenseg-

rity frameworks. A result of R. Connelly finds that

Proposition 8.2.18 ([15])**.** *Let* $(\langle G, m \rangle, p)$ *be a periodic tensegrity framework with all edges cables and one pinned vertex on the fixed torus* $\mathcal{T}_0^2$*. Let* ω *be a stress on the edges of* $\langle G, m \rangle$ *with* $\omega_e > 0$ *for all* $e \in E \langle G, m \rangle$*. Then* $(\langle G, m \rangle, p)$ *is globally rigid on* $\mathcal{T}_0^2$*.*

The proof uses the idea of the energy function. A finite version appears in [9].

8.2.7   Other spaces and metrics

Although we have been working exclusively in Euclidean space, we could also set the problem of periodic rigidity into other geometric spaces equipped with a metric. Saliola and Whiteley [87] prove the equivalence of the theory of infinitesimal rigidity for frameworks in Euclidean, Hyperbolic and Spherical spaces. More generally, a recent talk of Whiteley [86] showed a further equivalence in Minkowski space. There is no reason to expect that periodic rigidity would be substantially different. For example in Hyperbolic space, a group of translations exists, which we could use to define the gains on the periodic orbit graphs.

8.2.8   Connection to tilings

Another rich source of interesting problems related to periodic rigidity lies in the connection with tilings, coverings and packings. This is a topic that has been

considered by Bezdek, Bezdek and Connelly [3], and is well outlined as a problem of interest in *Research Problems in Discrete Geometry* [8]. There are other questions related to the tilings I studied in my Master's thesis [55], for example:

Question 8.2.19. Given a non-periodic tiling (such as a Penrose tiling) built from a finite number of tile shapes, let each shape be decorated with a bar-joint framework in such a way that the matching rules of the tiling generate an infinite bar-joint framework. When is such a framework rigid? That is, what are the conditions on the decorations of the tile types such that the resulting framework is rigid?

Related questions were considered by Kenyon in [45], and by Losev and Babalievski in [48].

8.2.9 Incidentally periodic frameworks

Throughout this thesis, we have been concerned with the topic of *forced periodicity*. That is, we have considered periodic frameworks and asked about their rigidity with respect to periodicity-preserving motions. A natural question is about relaxing this restriction to consider *incidentally periodic* frameworks, which are infinite frameworks which happen to be periodic, but where we do not require that the periodicity be preserved by infinitesimal motions of the structure.

Question 8.2.20. When is a periodic framework flexible, where the flexes may or may not preserve the periodicity of the structure?

We now return to the conjecture of Bob Connelly (Conjecture 8.2.8), and present a somewhat stronger form pertaining to incidentally periodic frameworks:

Conjecture 8.2.21. *If a framework $(\langle G, m \rangle, p)$ is infinitesimally rigid on the flexible torus, then it is infinitesimally rigid as an incidentally periodic (infinite) framework $(\widetilde{G}, \widetilde{p})$.*

As an example, consider the periodic orbit graph on $\mathcal{T}^2$ consisting of a single vertex and three edges, labeled by the gains $(1,0), (0,1)$ and $(1,1)$. The derived periodic framework for this graph will be the triangulated grid, which is infinitesimally rigid as both a periodic framework and an incidentally periodic framework. Now we may perform periodic inductive constructions on the periodic orbit graph to obtain other frameworks which we claim have the same property: they are infinitesimally rigid on $\mathcal{T}^2$, but are also infinitesimally rigid as incidentally periodic frameworks in $\mathbb{R}^2$.

Of course, as described earlier, not all infinitesimally rigid frameworks on $\mathcal{T}^2$ can be created through the inductive constructions of Chapter 4. This therefore highlights the gaps that still exist between the characterizations of frameworks on the fixed torus appearing in this thesis and the results of Malestein and Theran [49]. Inductive characterizations of frameworks on $\mathcal{T}^2$ would potentially help to settle the conjecture above.

List of Notation

$\widetilde{L}$ $d \times d$ lattice matrix, page 36

L_0 fixed lower triangular lattice matrix, page 36

$L(t)$ flexible lower triangular lattice matrix, page 36

$L_k(t)$ partially flexible lattice matrix, page 36

L'_k the image of $L_k(t)$ under $u_L = \widetilde{u}_L$, page 172

$\mathcal{L}\{i,j;m\}$ lattice entries in the flexible torus rigidity matrix, page 184

m gain assignment, page 24

m_T T-gain assignment, page 30

$M(e) = M_e$ the $d \times \binom{d+1}{2}$ matrix specifying $\mathcal{L}(e)$, page 184

$\mathcal{M}_{\mathcal{C}}(G)$ gain space of $\langle G, m \rangle$, page 27

$\mathbf{O}(G, p, \mathcal{S})$ orbit matrix (symmetric framework), page 257

$\mathbf{R}(G, p)$ rigidity matrix of (G, p), page 42

$\mathbf{R}_0(\langle G, m \rangle, p)$ fixed torus rigidity matrix, page 77

$\mathbf{R}(\langle G, m \rangle, p)$ flexible torus rigidity matrix, page 185

$\mathbf{R}_k(\langle G, m \rangle, p)$ partially flexible torus rigidity matrix, page 185

$\mathbf{R}(\langle H, m \rangle, q)$ rigidity matrix of the body-bar framework, page 298

$\mathcal{S}$ symmetry group, page 253

$k_{\mathcal{S}}$ size of vertex and edge orbits under $\mathcal{S}$, page 263

$f_{\mathcal{S}}$ dimension of space of symmetric inf. flexes, page 263

$\ell_{\mathcal{S}}$ number of lattice parameters, page 267

$t_{\mathcal{S}}$ dimension of space of fixed points under $\mathcal{S}$, page 270

$\mathcal{T}_0^d$ fixed torus, page 54

$\mathcal{T}^d$ flexible torus generated by $L(t)$, page 160

$\mathcal{T}_k^d$ partially flexible torus, generated by $L_k(t)$, page 161

$\mathcal{T}_d^d$ scaling torus, generated by a diagonal matrix $L_d(t)$, page 195

$\mathcal{T}_x^2$ scaling 2-dimensional torus, page 200

Index

328

Bibliography

[1] P. Ash, E. Bolker, H. Crapo, and W. Whiteley. Convex polyhedra, Dirichlet tessellations, and spider webs. In *Shaping space (Northampton, Mass., 1984)*, Design Sci. Collect., pages 231–250. Birkhäuser Boston, Boston, MA, 1988.

[2] L. Asimow and B. Roth. The rigidity of graphs. *Trans. Amer. Math. Soc.*, 245:279–289, 1978.

[3] A. Bezdek, K. Bezdek, and R. Connelly. Finite and uniform stability of sphere coverings. *Discrete Comput. Geom.*, 13:313 – 139, 1995.

[4] A. Bezdek, K. Bezdek, and R. Connelly. Finite and uniform stability of sphere packings. *Discrete Comput. Geom.*, 20(1):111–130, 1998.

[5] D. Bishop. *Group Theory and Chemistry*. Clarendon Press, Oxford, 1973.

[6] C. Borcea and I. Streinu. Minimally rigid periodic graphs. Lecture presented at the LMS Workshop: "Rigidity of Frameworks and Applications", Lancaster University, UK, July 12-15, 2010, 2010.

[7] C. S. Borcea and I. Streinu. Periodic frameworks and flexibility. *Proc. R. Soc. A*, 466(2121):2633 – 2649, 2010.

[8] P. Brass, W. Moser, and J. Pach. *Research Problems in Discrete Geometry*. Springer, 2005.

[9] R. Connelly. Rigidity and energy. *Invent. Math.*, 66:11–33, 1982.

[10] R. Connelly. Chapter 2: Basic concepts. From an unpublished book on the theory of rigid structures, March 1987.

[11] R. Connelly. Rigid circle and sphere packings I: Finite packings. *Structural Topology*, (14):43 – 60, 1988.

[12] R. Connelly. Rigid circle and sphere packings II: Infinite packings with finite motion. *Structural Topology*, (16):57–76, 1990.

[13] R. Connelly. Generic global rigidity. *Discrete Comput. Geom.*, 33(4):549–563, 2005.

[14] R. Connelly. Rigidity of packings. *European J. Combin.*, 29(8):1862–1871, 2008.

[15] R. Connelly. Personal communication, July 2009.

[16] R. Connelly, P. Fowler, S. Guest, B. Schulze, and W. Whiteley. When is a symmetric pin-jointed framework isostatic? *Int. J. Solids Struct.*, 46:762 – 773, 2009.

[17] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties and Algorithms.* Undergraduate Texts in Mathematics. Springer, 1992.

[18] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry.* Graduate Texts in Mathematics. Springer, New York, 2nd edition, 2005.

[19] H. S. M. Coxeter. *Regular Polytopes.* Dover, 1973.

[20] H. Crapo and W. Whiteley. Statics of frameworks and motions of panel structures, a projective geometric introduction. *Structural Topology*, (6):43–82, 1982. With a French translation.

[21] V. S. Deshpande, M. F. Ashby, and N. A. Fleck. Foam topology bending versus stretching dominated architectures. *Acta materialia*, 49:1035 – 1040, 2001.

[22] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, third edition, 2005.

[23] M. P. do Carmo. *Differential Geometry of Curves and Surfaces.* Prentice Hall, 1976.

[24] A. Donev and S. Torquato. Energy-efficient actuation in infinite lattice structures. *Journal of the mechanics and physics of solids*, 51:1459–1475, 2003.

[25] A. Donev, S. Torquato, F. H. Stillinger, and R. Connelly. A linear programming algorithm to test for jamming in hard-sphere packings. *J. Computational Physics*, 197(1):139 – 166, 2004.

[26] D. S. Dummit and R. M. Foote. *Abstract Algebra.* Prentice Hall, 1991.

[27] Z. Fekete and L. Szegő. A note on $[k, l]$-sparse graphs. In *Graph Theory*, pages 169 – 177. Birkhäuser, 2006.

[28] P. W. Fowler and S. D. Guest. Symmetry and states of self-stress in triangulated toroidal frames. *Internat. J. Solids Structures*, 39(17):4385–4393, 2002.

[29] H. Gluck. Almost all simply connected closed surfaces are rigid. In *Geometric topology (Proc. Conf., Park City, Utah, 1974)*, volume 438 of *Lecture Notes in Mathematics*, pages 225–239. Springer, Berlin, 1975.

[30] J. Graver, B. Servatius, and H. Servatius. *Combinatorial rigidity*, volume 2 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1993.

[31] J. E. Graver. *Counting on Frameworks*. Math. Assoc. America, 2001.

[32] J. L. Gross and T. W. Tucker. *Topological graph theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1987.

[33] B. Grünbaum and G. C. Shephard. *Tilings and patterns*. W. H. Freeman and Company, New York, 1987.

[34] L. Hall. *Group Theory and Symmetry in Chemistry*. McGraw-Hill Inc., 1969.

[35] D. W. Henderson and D. Taimina. *Experiencing Geometry*. Prentice Hall, third edition, 2004.

[36] L. Henneberg. *Die Graphische Statik der starren Systeme*. (Johnson Reprint), 1911.

[37] A. Heppes. Densest circle packing on the flat torus. *Periodica Mathematica Hungarica*, 39(1 − 3):129 − 134, 1999.

[38] F. Höfting and E. Wanke. Polynomial-time analysis of toroidal periodic graphs. *J. Algorithms*, 34(1):14–39, 2000.

[39] International Union of Crystallography. International Tables for Crystallography Volume A: Space-group Symmetry, 2006. first online edition. doi: 0.1107/97809553602060000100.

[40] K. Iwano and K. Steiglitz. Testing for cycles in infinite graphs with periodic structure. In *STOC '87: Proceedings of the nineteenth annual ACM symposium on Theory of computing*, pages 46–55, New York, NY, USA, 1987. ACM.

[41] D. J. Jacobs and B. Hendrickson. An algorithm for two-dimensional rigidity percolation: the pebble game. *J. Comput. Phys.*, 137(2):346–365, 1997.

[42] V. Kapko, C. Dawson, M. J. Treacy, and M. F. Thorpe. Flexibility of ideal zeolite frameworks. *Phys. Chem. Chem. Phys.*, 12:8531 – 8541, 2010.

[43] V. Kapko, M. M. J. Treacy, M. F. Thorpe, and S. D. Guest. On the collapse of locally isostatic networks. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 465(2111):3517–3530, 2009.

[44] N. Katoh and S. ichi Tanigawa. A proof of the molecular conjecture. arXiv:0902.0236v2.

[45] R. Kenyon. Rigidity of planar tilings. *Invent. Math.*, 107(3):637–651, 1992.

[46] G. Laman. On graphs and rigidity of plane skeletal structures. *J. Engrg. Math.*, 4:331 – 340, 1970.

[47] A. Lee and I. Streinu. Pebble game algorithms and sparse graphs. *Discrete Math.*, 308(8):1425–1437, 2008.

[48] A. Losev and F. Babalievski. Rigidity percolation on aperiodic lattices. *Physica A: Statistical Mechanics and its Applications*, 252(1-2):1 – 9, 1998.

[49] J. Malestein and L. Theran. Generic combinatorial rigidity of periodic frameworks. preprint, arXiv:1008.1837, 2010.

[50] J. R. Munkres. *Topology*. Prentice Hall, 2 edition, 2000.

[51] C. S. J. Nash-Williams. Edge-disjoint spanning trees of finite graphs. *J. London Math. Soc.*, 2(36):445 – 450, 1961.

[52] A. Nixon, J. Owen, and S. Power. Rigidity of frameworks supported on surfaces. arXiv:1009.3772v1, September 2010.

[53] J. Owen and S. Power. Infinite bar-joint frameworks, crystals and operator theory. 2010.

[54] S. Power. Crystal frameworks, symmetry and affinely periodic flexes. arXiv:1103.1914v1, March 2011.

[55] E. Ross. The Non-periodic Growth of Penrose Tilings. Master's thesis, University of British Columbia, 2005.

[56] E. Ross. Periodic rigidity. Lecture presented at the 2009 Spring Eastern Section AMS Meeting, April 2009. Abstract number 1050-52-71.

[57] E. Ross. The rigidity of graphs on a flexible torus. Lecture presented at the LMS Workshop: "Rigidity of Frameworks and Applications", Lancaster University, UK, July 12-15, 2010, 2010.

[58] E. Ross, B. Schulze, and W. Whiteley. Finite motions from periodic frameworks with added symmetry. 48:1711–1729, 2011. International Journal of Solids and Structures.

[59] G. Rote, F. Santos, and I. Streinu. Pseudo-Triangulations - a Survey. *ArXiv Mathematics e-prints*, Dec. 2006.

[60] B. Roth and W. Whiteley. Tensegrity frameworks. *Trans. Amer. Math. Soc.*, 265(2):419–446, 1981.

[61] K. Rybnikov and T. Zaslavsky. Cycle and circle tests of balance in gain graphs: forbidden minors and their groups. *J. Graph Theory*, 51(1):1–21, 2006.

[62] A. Sartbaeva, S. S. Wells, M. Treacy, and M. Thorpe. The flexibility window in zeolites. *Nature Materials*, 5(12):962–965, December 2006.

[63] B. Schulze. *Combinatorial and geometric rigidity with symmetry constraints.* PhD thesis, York University, 2009.

[64] B. Schulze. Block-diagonalized rigidity matrices of symmetric frameworks and applications. *Beiträge Algebra Geom.*, 51(2):427 – 466, 2010.

[65] B. Schulze. Injective and non-injective realizations with symmetry. *Contrib. Discrete Math.*, 5:59 – 89, 2010.

[66] B. Schulze. Symmetric versions of Laman's Theorem. *Discrete & Computational Geometry*, 44(4):946 – 974, 2010.

[67] B. Schulze. Symmetry as a sufficient condition for a finite flex. *SIAM J. Discrete Math.*, pages 1291–1312, 2010.

[68] B. Schulze and W. Whiteley. The orbit rigidity matrix of a symmetric framework. *Discrete Comput. Geom.*, 2010. in press.

[69] S.D.Guest and J.W.Hutchinson. On the determinancy of repetitive structures. *Journal of the mechanics and physics of solids*, 51:383–391, 2003.

[70] B. Servatius and H. Servatius. Combinatorial 2d zeolites. In preparation, August 2008.

[71] A. Sljoka. Counting for rigidity, flexibility and extensions via the pebble game algorithm. Master's thesis, York University, 2006.

[72] T.-S. Tay. Rigidity of multigraphs i: linking rigid bodies in n-space. *J. Combinatorial Theory B*, 26:95 – 112, 1984.

[73] T.-S. Tay. A new proof of Laman's theorem. *Graphs Combin.*, 9(4):365–370, 1993.

[74] T.-S. Tay and W. Whiteley. Generating isostatic frameworks. *Structural Topology*, (11):21–69, 1985. Dual French-English text.

[75] M. Thorpe. http://flexweb.asu.edu/.

[76] M. Thorpe. Private Communication, August 2010.

[77] W. T. Tutte. On the problem of decomposing a graph into n connected factors. *J. London Math. Soc.*, 2(36):221 – 230, 1961.

[78] D. B. West. *Introduction to Graph Theory*. Prentice Hall, 2 edition, 2001.

[79] N. White and W. Whiteley. The algebraic geometry of motions of bar-and-body frameworks. *SIAM J. Algebraic Discrete Methods*, 8(1):1–32, 1987.

[80] N. L. White and W. Whiteley. The algebraic geometry of stresses in frameworks. *SIAM J. Algebraic Discrete Methods*, 4(4):481–511, 1983.

[81] W. Whiteley. Chapter 10: Tensegrity. From an unpublished book on the theory of rigid structures, May 1987.

[82] W. Whiteley. The union of matroids and the rigidity of frameworks. *SIAM J. Discrete Math.*, 1(2):237–255, 1988.

[83] W. Whiteley. Matroids and rigid structures. In N. White, editor, *Matroid Applications*. Cambridge University Press, 1992.

[84] W. Whiteley. Some matroids from discrete applied geometry. In *Matroid theory (Seattle, WA, 1995)*, volume 197 of *Contemp. Math.*, pages 171–311. Amer. Math. Soc., Providence, RI, 1996.

[85] W. Whiteley. Rigidity and scene analysis. In *Handbook of discrete and computational geometry*, CRC Press Ser. Discrete Math. Appl., pages 893–916. CRC, Boca Raton, FL, 1997.

[86] W. Whiteley. Transferring rigidity: Coning, flat metrics, and some perspectives on transfers. Lecture presented at the LMS Workshop: "Rigidity of Frameworks and Applications", Lancaster University, UK, July 12-15, 2010, 2010.

[87] W. Whiteley and F. Saliola. Some notes on the equivalence of first-order rigidity in various geometries. arXiv:0709.3354v1, 2007.

[88] Wikipedia. Crystal systems, 2010. Retrieved October 11, 2010, from http://en.wikipedia.org/wiki/Crystal_system.

[89] T. Zaslavsky. A mathematical bibliography of signed and gain graphs and allied areas. *Electron. J. Combin.*, 5:Dynamic Surveys 8, 124 pp. (electronic), 1998. Manuscript prepared with Marge Pratt.