Limited Feedback Unitary Precoding for Spatial Multiplexing Systems

David J. Love, *Member, IEEE,* and Robert W. Heath, Jr., *Member, IEEE*

***Abstract*—Multiple-input multiple-output (MIMO) wireless systems use antenna arrays at both the transmitter and receiver to provide communication links with substantial diversity and capacity. Spatial multiplexing is a common space–time modulation technique for MIMO communication systems where independent information streams are sent over different transmit antennas. Unfortunately, spatial multiplexing is sensitive to ill-conditioning of the channel matrix. Precoding can improve the resilience of spatial multiplexing at the expense of full channel knowledge at the transmitter—which is often not realistic. This correspondence proposes a quantized precoding system where the optimal precoder is chosen from a finite codebook known to both receiver and transmitter. The index of the optimal precoder is conveyed from the receiver to the transmitter over a low-delay feedback link. Criteria are presented for selecting the optimal precoding matrix based on the error rate and mutual information for different receiver designs. Codebook design criteria are proposed for each selection criterion by minimizing a bound on the average distortion assuming a Rayleigh-fading matrix channel. The design criteria are shown to be equivalent to packing subspaces in the Grassmann manifold using the projection two-norm and Fubini–Study distances. Simulation results show that the proposed system outperforms antenna subset selection and performs close to optimal unitary precoding with a minimal amount of feedback.**

***Index Terms*—Diversity methods, Grassmannian subspace packing, multiple-input multiple-output (MIMO) systems, quantized precoding, Rayleigh channels, spatial multiplexing, vertical Bell Labs layered space–time (V-BLAST) architecture.**

I. Introduction

Multiple-input multiple-output (MIMO) wireless channels, created by exploiting antenna arrays at both the transmitter and receiver, promise high capacity and high-quality wireless communication links [1], [2]. Exploiting the benefits offered by MIMO channels requires choosing a space–time modulation scheme and receiver algorithm that provide a sensible performance and complexity tradeoff. Along these lines, spatial multiplexing, where a bit stream is demultiplexed into multiple substreams that are sent over different antennas, is a practical space–time modulation technique that permits a choice of optimal, near-optimal, and suboptimal receivers. Unfortunately, spatial multiplexing is sensitive to ill-conditioning of the channel matrix.

Premultiplying the transmitted data streams by a precoding matrix, chosen based on channel information, is one way to guard against rank deficiencies in the channel and to improve error rate performance. The basic idea of precoding is to use some form of channel knowledge at the transmitter to customize the transmitted signal to the eigenstructure of the matrix channel. Precoding has been proposed based on knowledge of the full channel state information at the transmitter [3], [4], first-order statistics [5], [6], or second-order statistics of the channel [5], [7]–[11]. The full gains from precoding are achieved with full channel state information since this allows the transmitted signal to be customized based on the eigenstructure of the matrix channel. In time-division duplex systems with suitable ping-pong time, full channel state information may be available. In frequency-division duplex systems, however, full channel state information must be conveyed through a feedback channel. This is impractical, though, due to the number of channel coefficients that need to be quantized and sent back to the transmitter over limited bandwidth control channels.

In this correspondence, we propose a solution to the problem of precoding for spatial multiplexing systems with limited feedback capacity. The essential idea is that the transmit precoder is chosen from a finite set of precoding matrices, called the codebook, known to both the receiver and the transmitter. The receiver chooses the optimal precoder from the codebook as a function of the current channel state information and sends the binary index of this matrix to the transmitter over a limited feedback channel. We address two key problems in this correspondence: i) selection of the optimal precoder from the codebook and ii) design of optimal codebooks. We assume that the channel is statistically described by the narrowband uncorrelated Rayleigh matrix fading model and that there is zero delay in the feedback channel. For codeword selection, we propose selection criteria based on the error probability [3], [12]–[14] and mutual information [15]–[17]. We consider both maximum-likelihood (ML) (optimal but high implementation complexity) and linear receivers (suboptimal but lower implementation complexity). The optimal precoding matrix can be easily chosen using our selection criteria by simply searching through all codebook matrices. We address codebook design for each of the proposed selection criteria. By bounding the average distortion, we show that the codeword selection criteria imply that codebooks should be designed such that the constituent matrices are maximally spaced. The distance measure is not Euclidean distance but rather subspace distance on the Grassmann manifold. Specifically, we show that, depending on the receiver, optimal codebook designs are subspace packings in the Grassmann manifold using either the projection two-norm or the Fubini–Study distance.

Limited feedback precoding has been considered in the past extensively for the cases of transmit beamforming [18]–[20], precoding for space–time block codes [21]–[23], and covariance quantization [24], [25]. As well, antenna subset selection can be viewed as an important example of limited feedback precoding where the optimal subset of transmit antennas is computed and conveyed to the transmitter (see, for example, [12], [15], [17], [26]–[28]). The random vector quantization work in [29] is another form of limited feedback precoding. Our correspondence proposes a natural generalization of this existing limtied feedback work.

This correspondence is organized as follows. Section II reviews the precoded spatial multiplexing system model. Criteria for choosing the optimal matrix from the codebook is presented in Section III. Design criteria for creation of the precoder codebook are derived in Section IV. Section V illustrates the performance improvements over no precoding, unquantized precoding, and antenna subset selection using Monte Carlo simulations of the symbol error rate. Conclusions are presented in Section VI.

Notation

We use ^{T} to denote transposition, * to denote conjugate transposition, $^{-1}$ to denote matrix inversion, $^+$ to denote the matrix pseudo-inverse, $\boldsymbol{I}_M$ to denote the $M \times M$ identity matrix, $\|\cdot\|_2$ to denote the matrix two-

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D. J. Love is with the Center for Wireless Systems and Applications, School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907 USA (e-mail: djlove@ecn.purdue.edu).

R. W. Heath, Jr. is with the Wireless Networking and Communications Group, Department of Electrical and Computer Engineering, The University of Texas at Austin, Austin, TX 78712 USA (e-mail: rheath@ece.utexas.edu).

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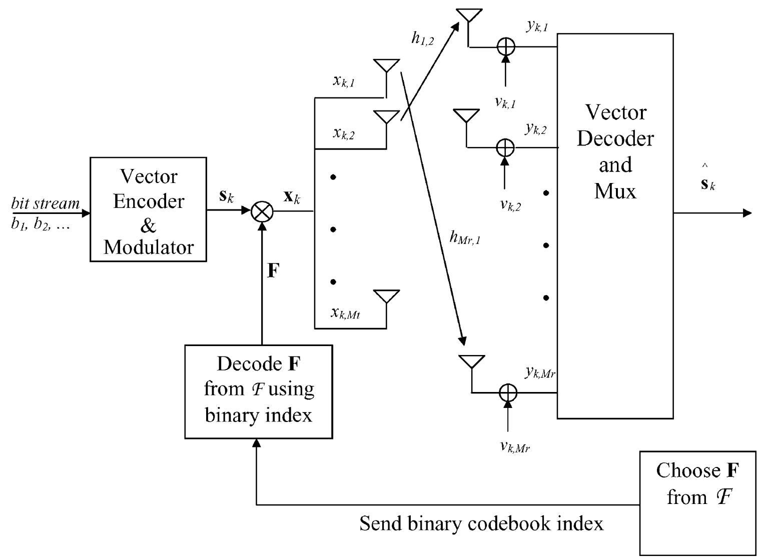

Fig. 1. Block diagram of a limited feedback precoding MIMO system.

norm, $\|\cdot\|_F$ to denote the matrix Frobenius norm, $\mathrm{tr}(\cdot)$ to denote the trace of a matrix, $\det(\cdot)$ to denote the determinant of a matrix, $\mathbb{C}^m$ to denote the m-dimensional complex vector space, $\mathcal{CN}(0,\sigma^2)$ to denote the complex normal distribution with independent real and imaginary parts distributed according to $\mathcal{N}(0,\sigma^2/2)$, $\mathcal{U}(M_t,M)$ to denote the set of $M_t \times M$ matrices with orthonormal columns, $\lambda_i\{\boldsymbol{A}\}$ denotes the ith largest singular value of $\boldsymbol{A}$, $\mathrm{argmin}_{\mathcal{A}}$ ($\mathrm{argmax}_{\mathcal{A}}$) to denote a function that returns a global minimizer (maximizer) over the set $\mathcal{A}$, and $\mathrm{card}(\cdot)$ to denote the cardinality of a set.

II. System Overview

The proposed system is illustrated in Fig. 1. A bit stream is sent into a vector encoder and modulator block where it is demultiplexed into M different bit streams. Each of the M bit streams is then modulated independently using the same constellation $\mathcal{W}$. This yields a symbol vector at time k of $\boldsymbol{s}_k = [s_{k,1}\ s_{k,2}\ \ldots\ s_{k,M}]^T$. For convenience, we will assume that $E_{\boldsymbol{s}_k}[\boldsymbol{s}_k\boldsymbol{s}_k^*] = \boldsymbol{I}_M$.

The symbol vector $\boldsymbol{s}_k$ is then multiplied by an $M_t \times M$ precoding matrix $\boldsymbol{F}$ (which is chosen as a function of the channel using criteria to be described) producing a length M_t vector $\boldsymbol{x}_k = \sqrt{\frac{\mathcal{E}_s}{M}}\boldsymbol{F}\boldsymbol{s}_k$ where $\mathcal{E}_s$ is the total transmit energy, M_t is the number of transmit antennas, and $M_t > M$. We assume throughout the correspondence that $M_r \geq M$. Assuming perfect timing, synchronization, sampling, and a memoryless linear matrix channel, this formulation allows the baseband, discrete-time equivalent received signal to be written as

$$\boldsymbol{y}_k = \sqrt{\frac{\mathcal{E}_s}{M}}\boldsymbol{H}\boldsymbol{F}\boldsymbol{s}_k + \boldsymbol{v}_k \tag{1}$$

where $\boldsymbol{H}$ is the channel matrix and $\boldsymbol{v}_k$ is the noise vector. We assume that the entries of $\boldsymbol{H}$ are independent and identically distributed (i.i.d.) according to $\mathcal{CN}(0,1)$ and the entries of $\boldsymbol{v}_k$ are independent and distributed according to $\mathcal{CN}(0,N_0)$. The received vector is then decoded by a vector decoder, assuming perfect knowledge of $\boldsymbol{H}\boldsymbol{F}$, that produces a hard decoded symbol vector $\widehat{\boldsymbol{s}}_k$.

In this correspondence, the receiver chooses a precoding matrix $\boldsymbol{F}$ from a finite set of possible precoding matrices $\mathcal{F} = \{\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_N\}$ and conveys the index of the chosen precoding matrix back to the transmitter over a limited capacity, zero-delay feedback link. We assume that each $\boldsymbol{F} \in \mathcal{F}$ has unit column vectors that are orthogonal. This assumption is not especially restrictive since it follows from the form of the optimal, full channel knowledge precoders derived in [3] assuming a maximum singular value constraint on $\boldsymbol{F}$. Thus, the proposed codebook will satisfy $\mathcal{F} \subset \mathcal{U}(M_t,M)$. To simplify implementation, we will typically assume that B bits of feedback are available; thus, the codebook consists of $N = 2^B$ matrices in $\mathcal{U}(M_t,M)$. The fact that the set $\mathcal{F}$ is discrete allows the receiver to solve for $\boldsymbol{F}$ by computing the selection metric of interest for each of the $N = 2^B$ codebook entries. The limitation of the codebook to 2^B matrices allows the system designer to constrain the precoding overhead and to take full advantage of the limited feedback channel.

To illustrate the concept, consider a codebook that corresponds to antenna subset selection [12]. Such a codebook would consist of the $\binom{M_t}{M}$ matrices consisting of M columns of $\boldsymbol{I}_{M_t}$. Notice that each set of M columns of $\boldsymbol{I}_{M_t}$ is an element of $\mathcal{U}(M_t,M)$. Naturally, antenna selection precoding can be directly implemented in a limited feedback system because a total feedback of only $\lceil \log_2 \binom{M_t}{M} \rceil$ bits is required. Unfortunately, the performance is highly limited because i) the columns of $\boldsymbol{F}$ are restricted to being M columns of $\boldsymbol{I}_{M_t}$ and ii) the size of the codebook is limited by M and M_t. It is of interest to remove any restrictions about the nature of the elements of the codebook as well as the number of elements in an effort to come closer to the gains of approximately optimal precoding.

The primary goal of this precoding is to improve the overall system performance using a suboptimal receiver (though we do consider optimal ML decoding as well for completeness). For example, precoding can allow an easily implemented linear receiver to outperform optimal decoding at the expense of i) using more transmit antennas and ii) requiring channel information at the transmitter. It should be noted that depending on the signal-to-noise ratio (SNR) and the antenna configuration, feedback may not be needed. Using feedback to improve the data rate of the forward data path can cause a significant overhead on the reverse path. When the SNR is high, it is well known that feedback is not needed to achieve capacity and the feedback overhead will be detrimental. We do not consider the effect of feedback overhead on the system.

As well, we assume that M is fixed. We do not vary the dimensionality of the transmitted data vector as a function of the channel matrix. This kind of adaptation will cause a large increase in both transmit and receive complexity. It should be noted, however, that varying M can dramatically improve the system performance. Readers are referred to [30] for discussion on this kind of adaptation.

Problem Statement

The objective of this correspondence is to solve the two key problems that are needed to effectively design and implement a limited feedback precoding system as proposed in Fig. 1. The first is to develop algorithms for selecting the optimal $\boldsymbol{F}$ from $\mathcal{F}$ as a function of the error probability or mutual information. This is the codeword or precoder selection problem. The second is to determine how to select a good codebook $\mathcal{F}$, based on a distortion measure that accounts for the fact that the channel is uncorrelated Rayleigh fading. This is the codebook design problem.

III. Precoding Criteria

In this section, we discuss the criteria used for choosing the optimal precoding matrix from a given codebook. We outline criteria based on minimizing the error rate for the ML or linear decoder and on maximizing the mutual information. When illustrative, we derive the optimal matrix over $\mathcal{U}(M_t, M)$.

A. ML Receiver

The ML receiver solves the optimization problem

$$\widehat{\boldsymbol{s}}_k = \underset{\boldsymbol{s}\in\mathcal{W}^M}{\operatorname{argmin}} \left\| \boldsymbol{y}_k - \sqrt{\frac{\mathcal{E}_s}{M}} \boldsymbol{H}\boldsymbol{F}\boldsymbol{s} \right\|_2^2. \tag{2}$$

A closed-form expression for the probability of symbol vector error is difficult to derive. One approach is to observe that the probability of symbol vector error can be upper bounded for high SNRs using the vector Union Bound [31]. This approach is motivated by the fact that the Union Bound provides an adequately tight prediction of the probability of error for large SNR. Since we assume $\mathcal{E}_s/N_0$ to be fixed, the Union Bound is solely a function of the receive minimum distance $d_{\min,R}$ of the multidimensional constellation $\mathcal{W}^M$ [13], which is given by

$$d_{\min,R} = \min_{\boldsymbol{s}_1,\boldsymbol{s}_2\in\mathcal{W}^M:\boldsymbol{s}_1\neq\boldsymbol{s}_2} \sqrt{\frac{\mathcal{E}_s}{M}} \| \boldsymbol{H}\boldsymbol{F}(\boldsymbol{s}_1 - \boldsymbol{s}_2) \|_2. \tag{3}$$

The computation of $d_{\min,R}$ requires a search over $\binom{\operatorname{card}(\mathcal{W}^M)}{2}$ vectors.

Using (3), the minimum Euclidean distance criterion is to pick $\boldsymbol{F}$ from the codebook $\mathcal{F}$ for a given $\boldsymbol{H}$ assuming that $\mathcal{W}$ and $\mathcal{E}_s/N_0$ are fixed according to the following criterion.

ML Selection Criterion (ML-SC): Pick $\boldsymbol{F}$ such that

$$\boldsymbol{F} = \underset{\boldsymbol{F}_i\in\mathcal{F}}{\operatorname{argmax}}\, d_{\min,R}. \tag{4}$$

Deriving a closed-form solution to *ML-SC* is difficult since the minimum distance depends on the constellation as well as the channel realization.

B. Linear Receiver

Linear receivers apply an $M \times M_r$ matrix $\boldsymbol{G}$, chosen according to some criterion, to produce $\widehat{\boldsymbol{s}}_k = \boldsymbol{Q}(\boldsymbol{G}\boldsymbol{y}_k)$ where $\boldsymbol{Q}(\cdot)$ is a function that performs single-dimensional ML decoding for each entry of a vector. Criteria will be presented for two different forms of $\boldsymbol{G}$: zero-forcing and minimum mean-square error (MMSE). For a zero-forcing (ZF) linear decoder, $\boldsymbol{G} = (\boldsymbol{H}\boldsymbol{F})^+$. When a MMSE linear decoder is used

$$\boldsymbol{G} = [\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{F} + (MN_0/\mathcal{E}_s)\boldsymbol{I}_M]^{-1}\boldsymbol{F}^*\boldsymbol{H}^*.$$

1) Minimum Singular Value: We will characterize the average probability of symbol vector error performance using the substream with the minimum SNR following the results given in [12]. It was shown in [12] that the SNR of the kth substream is given by

$$\text{SNR}_k^{(\text{ZF})} = \frac{\mathcal{E}_s}{MN_0[\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{F}]^{-1}_{k,k}} \tag{5}$$

for the ZF decoder and

$$\text{SNR}_k^{(\text{MMSE})} = \frac{\mathcal{E}_s}{MN_0[\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{F} + (MN_0/\mathcal{E}_s)\boldsymbol{I}_M]^{-1}_{k,k}} - 1 \tag{6}$$

for the MMSE decoder, where $\boldsymbol{A}^{-1}_{k,k}$ is entry (k,k) of $\boldsymbol{A}^{-1}$. In [12], it is shown that in order to minimize a bound on the average probability of a symbol vector error, the minimum substream SNR must be maximized.

Using a selection criterion based on the minimum SNR requires the computation of the SNR of each of the M substreams and the estimation of $\mathcal{E}_s/N_0$. The computational complexity combined with the possibility of estimation error makes the minimum cumbersome to implement. For this reason, [12] shows that the minimum SNR for ZF can be bounded using

$$\begin{aligned} \text{SNR}_{\min}^{(\text{ZF})} &= \min_{1\le k\le M} \text{SNR}_k^{(\text{ZF})} & (7)\\ &\ge \lambda^2_{\min}\{\boldsymbol{H}\boldsymbol{F}\}\frac{\mathcal{E}_s}{MN_0} & (8)\end{aligned}$$

where $\lambda_{\min}\{\boldsymbol{H}\boldsymbol{F}\}$ is the minimum singular value of $\boldsymbol{H}\boldsymbol{F}$.

We use (8) to obtain a requirement for choosing $\boldsymbol{F}$ from $\mathcal{F}$ for a given $\boldsymbol{H}$. We have assumed that $\mathcal{F}$ and $\mathcal{E}_s/N_0$ are fixed.

Minimum Singular Value Selection Criterion (MSV-SC): Pick $\boldsymbol{F}$ such that

$$\boldsymbol{F} = \underset{\boldsymbol{F}_i\in\mathcal{F}}{\operatorname{argmax}}\, \lambda_{\min}\{\boldsymbol{H}\boldsymbol{F}_i\}. \tag{9}$$

This criterion provides a close approximation to maximizing the minimum SNR for dense constellations. The reason for this is that as $\operatorname{card}(\mathcal{W})$ grows large, the probability of an error vector lying collinear to the minimum singular value direction goes to one.

Optimal Unquantized Precoder:

For comparison purposes, we also derive $\boldsymbol{F}_{\text{opt}} \in \mathcal{U}(M_t, M)$ that maximizes $\lambda_{\min}\{\boldsymbol{H}\boldsymbol{F}_{\text{opt}}\}$. Note that when the feasible set[1] is $\mathcal{U}(M_t, M)$, $\boldsymbol{F}_{\text{opt}}$ is not unique. For example, if $\boldsymbol{F}_{\text{opt}}$ maximizes $\lambda_{\min}\{\boldsymbol{H}\boldsymbol{F}_{\text{opt}}\}$ then so does $\boldsymbol{F}_{\text{opt}}\boldsymbol{U}$ for any $\mathcal{U}(M, M)$.

Let the singular value decomposition of $\boldsymbol{H}$ be given by

$$\boldsymbol{H} = \boldsymbol{V}_L\boldsymbol{\Sigma}\boldsymbol{V}_R^* \tag{10}$$

where $\boldsymbol{V}_L \in \mathcal{U}(M_r, M_r)$, $\boldsymbol{V}_R \in \mathcal{U}(M_t, M_t)$, and $\boldsymbol{\Sigma}$ is an $M_r \times M_t$ diagonal matrix with $\lambda_k\{\boldsymbol{H}\}$ denoting the kth largest singular value of $\boldsymbol{H}$ at entry (k, k).

Lemma 1: An optimal precoder over $\mathcal{U}(M_t, M)$ for *MSV-SC* is $\boldsymbol{F}_{\text{opt}} = \overline{\boldsymbol{V}}_R$ where $\overline{\boldsymbol{V}}_R$ is a matrix constructed from the first M columns of $\boldsymbol{V}_R$.

Proof: Let $\tilde{\boldsymbol{F}} = [\boldsymbol{F}_{\text{opt}}\ \tilde{\boldsymbol{f}}_1\ \dots \tilde{\boldsymbol{f}}_{M_t-M}]$ where $\tilde{\boldsymbol{F}}^*\tilde{\boldsymbol{F}} = \boldsymbol{I}_{M_t}$. It is clear that the matrix $\tilde{\boldsymbol{F}}^*\boldsymbol{H}^*\boldsymbol{H}\tilde{\boldsymbol{F}}$ is a Hermitian matrix and $\boldsymbol{F}_{\text{opt}}^*\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{F}_{\text{opt}}$ is obtained from $\tilde{\boldsymbol{F}}^*\boldsymbol{H}^*\boldsymbol{H}\tilde{\boldsymbol{F}}$ by simply taking the principle submatrix corresponding to the first M rows. By the *Inclusion Principle* [32]

$$\begin{aligned} \lambda_{\min}\{\boldsymbol{H}\} &= \lambda_{M_t}\{\boldsymbol{H}\}\\ &\le \lambda_{\min}\{\boldsymbol{H}\boldsymbol{F}_{\text{opt}}\}\\ &= \lambda_M\{\boldsymbol{H}\boldsymbol{F}_{\text{opt}}\}\\ &\le \lambda_M\{\boldsymbol{H}\}.\end{aligned}$$

Here the Mth singular value refers to the singular value at entry (M, M) of the ordered diagonal matrix $\boldsymbol{\Sigma}$. For the case where $M_r \le M_t$, there will be M_r different singular values. Because $M \le M_r$, the Mth singular value is well defined. This upper bound can thus be achieved if $\boldsymbol{F}_{\text{opt}} = \overline{\boldsymbol{V}}_R$. □

[1]The feasible set of an optimization is the domain that the cost function is optimized over.

2) MMSE: Previous work [3] has considered improving the overall system performance by minimizing some function of the mean squared error (MSE) matrix

$$\overline{\text{MSE}}(\boldsymbol{F},\boldsymbol{G}) = E\left[\left(\boldsymbol{G}\boldsymbol{y}_k - \sqrt{\frac{\mathcal{E}_s}{M}}\boldsymbol{s}_k\right)\left(\boldsymbol{G}\boldsymbol{y}_k - \sqrt{\frac{\mathcal{E}_s}{M}}\boldsymbol{s}_k\right)^*\right]$$

where the expectation is taken over $\boldsymbol{s}_k$ and $\boldsymbol{v}_k$. When MMSE linear decoding is used, we express the MSE as

$$\overline{\text{MSE}}(\boldsymbol{F}) = \frac{\mathcal{E}_s}{M}\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{F}\right)^{-1}. \tag{11}$$

Using (11), we derive a selection criterion for choosing $\boldsymbol{F}$ from $\mathcal{F}$.

Mean Squared Error Selection Criterion (MSE-SC): Pick $\boldsymbol{F}$ such that

$$\boldsymbol{F} = \underset{\boldsymbol{F}_i \in \mathcal{F}}{\operatorname{argmin}}\, m\left(\overline{\text{MSE}}(\boldsymbol{F}_i)\right) \tag{12}$$

where $m(\cdot)$ is either $\text{tr}(\cdot)$ or $\det(\cdot)$.

Note that minimizing the MSE does not specifically mean a reduction in the probability of error. In general, if the goal is to minimize the probability of error *MSV-SC* should be chosen.

Optimal Unquantized Precoder:

Again, we present the optimal precoder over the unquantized set $\mathcal{U}(M_t, M)$ for subsequent comparisons. In [3], various constraints on $\boldsymbol{F}_{\text{opt}}$ were considered along with various mean squared-error cost functions based on $\overline{\text{MSE}}(\boldsymbol{F}_{\text{opt}})$. Since we restrict our search to $\boldsymbol{F}_{\text{opt}} \in \mathcal{U}(M_t, M)$, we will consider the constraint in [3] where the maximum eigenvalue of $\boldsymbol{F}_{\text{opt}}\boldsymbol{F}_{\text{opt}}^*$ is unity. Note that all matrices in $\mathcal{U}(M_t, M)$ satisfy this constraint, but belonging to $\mathcal{U}(M_t, M)$ is not a necessary condition for this constraint.

It was shown in [3] that $\boldsymbol{F}_{\text{opt}}$ that minimizes $\text{tr}(\overline{\text{MSE}}(\boldsymbol{F}_{\text{opt}}))$ or $\det(\overline{\text{MSE}}(\boldsymbol{F}_{\text{opt}}))$ under this maximum eigenvalue constraint is $\boldsymbol{F}_{\text{opt}} = \overline{\boldsymbol{V}}_R$. Therefore, we can state the following lemma as a consequence.

Lemma 2: (Scaglione *et al.* [3]) A matrix $\boldsymbol{F}_{\text{opt}} \in \mathcal{U}(M_t, M)$ that minimizes either of the two cost functions $\text{tr}(\overline{\text{MSE}}(\boldsymbol{F}_{\text{opt}}))$ and $\det(\overline{\text{MSE}}(\boldsymbol{F}_{\text{opt}}))$ is $\boldsymbol{F}_{\text{opt}} = \overline{\boldsymbol{V}}_R$.

Once again $\boldsymbol{F}_{\text{opt}}$ is not unique because $\text{tr}(\overline{\text{MSE}}(\boldsymbol{F}_{\text{opt}})) = \text{tr}(\overline{\text{MSE}}(\boldsymbol{F}_{\text{opt}}\boldsymbol{U}))$ and $\det(\overline{\text{MSE}}(\boldsymbol{F}_{\text{opt}})) = \det(\overline{\text{MSE}}(\boldsymbol{F}_{\text{opt}}\,\boldsymbol{U}))$ for any $\boldsymbol{U} \in \mathcal{U}(M, M)$.

C. Capacity

In the context of antenna subset selection for spatial multiplexing systems, the mutual information (or capacity) has been used to formulate a precoder selection criterion [15], [16]. When the transmitter precodes with $\boldsymbol{F}$ before transmission, the equivalent channel is $\boldsymbol{H}\boldsymbol{F}$. Thus, the mutual information assuming an uncorrelated complex Gaussian source given $\boldsymbol{H}$ and a fixed $\boldsymbol{F}$ is

$$I(\boldsymbol{F}) = \log_2 \det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{F}\right). \tag{13}$$

Therefore, we can state a capacity inspired selection criterion as follows.

Capacity Selection Criterion (Capacity-SC): Pick $\boldsymbol{F}$ such that

$$\boldsymbol{F} = \underset{\boldsymbol{F}_i \in \mathcal{F}}{\operatorname{argmax}}\, I(\boldsymbol{F}_i). \tag{14}$$

Note that we call this selection criterion "*Capacity-SC*" for consistency with previous works [15], [16].

Optimal Unquantized Precoder:

It is possible to find the optimal unquantized precoder $\boldsymbol{F}_{\text{opt}} \in \mathcal{U}(M_t, M)$ for the *Capacity-SC* criterion.

Lemma 3: A precoder matrix $\boldsymbol{F}_{\text{opt}} \in \mathcal{U}(M_t, M)$ that maximizes $I(\boldsymbol{F}_{\text{opt}})$ is given by $\boldsymbol{F}_{\text{opt}} = \overline{\boldsymbol{V}}_R$.

Proof: Note that maximizing

$$\log_2 \det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\boldsymbol{F}_{\text{opt}}^*\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{F}_{\text{opt}}\right)$$

is equivalent to maximizing

$$\det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\boldsymbol{F}_{\text{opt}}^*\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{F}_{\text{opt}}\right)$$

and thus minimizing

$$\det\left(\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\boldsymbol{F}_{\text{opt}}^*\boldsymbol{H}^*\boldsymbol{H}\boldsymbol{F}_{\text{opt}}\right)^{-1}\right).$$

The latter expression differs from $\det\left(\overline{\text{MSE}}(\boldsymbol{F})\right)$ by a constant scale factor. It therefore follows from Lemma 2 that $\boldsymbol{F}_{\text{opt}} = \overline{\boldsymbol{V}}_R$ maximizes $I(\boldsymbol{F}_{\text{opt}})$. □

Because of the relationship to *MSE-SC*, it is easily seen that $I(\boldsymbol{F}_{\text{opt}}) = I(\boldsymbol{F}_{\text{opt}}\boldsymbol{U})$ for any $\boldsymbol{U} \in \mathcal{U}(M, M)$.

IV. Limited Feedback Precoding: Motivation and Codebook Design

In the preceding section, we derived criteria for selecting the optimal precoding matrix. It is important that the codebook $\mathcal{F}$ is designed specifically for the chosen criterion. To understand the codebook design problem, we first perform a probabilistic characterization of the optimal precoding matrix. We then use this characterization to derive codebooks that maximize average bounds on each of the performance criteria.

A. Probabilistic Characterization of Optimal Precoding Matrix

Let the eigenvalue decomposition of $\boldsymbol{H}^*\boldsymbol{H}$ be given by

$$\boldsymbol{H}^*\boldsymbol{H} = \boldsymbol{V}_R\boldsymbol{\Sigma}^2\boldsymbol{V}_R^* \tag{15}$$

with $\boldsymbol{V}_R$ and $\boldsymbol{\Sigma}$ defined as in (10). In [33], it is shown that, for a MIMO Rayleigh-fading channel, $\boldsymbol{V}_R$, the right singular vector matrix is isotropically distributed on $\mathcal{U}(M_t, M_t)$, the group of unitary matrices. An *isotropically distributed* $M_t \times M$ matrix $\boldsymbol{V}$ is a matrix where $\boldsymbol{\Theta}^*\boldsymbol{V} \overset{d}{=} \boldsymbol{V}$ for all $\boldsymbol{\Theta} \in \mathcal{U}(M_t, M_t)$ with $\overset{d}{=}$ denoting equivalence in distribution [34]. As stated in Section III, the optimal precoder for *MSV-SC, MSE-SC,* and *Capacity-SC* is constructed by simply taking the first M columns of $\boldsymbol{V}_R$. Using the isotropic distribution of $\boldsymbol{V}_R$, it is possible to derive the distribution of $\boldsymbol{F}_{\text{opt}}$.

Lemma 4: For a memoryless, i.i.d. Rayleigh-fading channel $\boldsymbol{H}$, $\boldsymbol{F}_{\text{opt}} = \overline{\boldsymbol{V}}_R$ is isotropically distributed on $\mathcal{U}(M_t, M)$.

Proof: First note that

$$\boldsymbol{F}_{\text{opt}} = \overline{\boldsymbol{V}}_R = \boldsymbol{V}_R\begin{bmatrix}\boldsymbol{I}_M \\ \boldsymbol{0}_{(M_t-M)\times M}\end{bmatrix} \tag{16}$$

where $\boldsymbol{0}_{(M_t-M)\times M}$ is an $(M_t - M) \times M$ matrix of zeros. Since $\boldsymbol{V}_R$ is isotropically distributed

$$\begin{aligned}\boldsymbol{\Theta}^*\boldsymbol{F}_{\text{opt}} &= \boldsymbol{\Theta}^*\boldsymbol{V}_R\begin{bmatrix}\boldsymbol{I}_M \\ \boldsymbol{0}_{(M_t-M)\times M}\end{bmatrix}\\ &\overset{d}{=} \boldsymbol{V}_R\begin{bmatrix}\boldsymbol{I}_M \\ \boldsymbol{0}_{(M_t-M)\times M}\end{bmatrix}\\ &= \boldsymbol{F}_{\text{opt}}.\end{aligned}$$ □

Lemma 4 will allow the effect of the codebook on average distortion to be studied, with distortion to be defined later.

B. Grassmannian Subspace Packing

Before stating design criteria for each of the precoding matrix selection criteria, we present some relevant background about finite sets of matrices in $\mathcal{U}(M_t, M)$. The set $\mathcal{U}(M_t, M)$ defines the complex *Stiefel manifold* [35] of real dimension $2M_tM - M^2$. Each matrix in $\mathcal{U}(M_t, M)$ represents an M-dimensional subspace of $\mathbb{C}^{M_t}$. The set of all M-dimensional subspaces spanned by matrices in $\mathcal{U}(M_t, M)$ is the complex Grassmann manifold, denoted as $\mathcal{G}(M_t, M)$. Thus, if $\boldsymbol{F}_1, \boldsymbol{F}_2 \in \mathcal{U}(M_t, M)$ then the column spaces of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, $\mathcal{P}_{\boldsymbol{F}_1}$ and $\mathcal{P}_{\boldsymbol{F}_2}$, respectively, are contained in $\mathcal{G}(M_t, M)$. Note that the Grassmann manifold can be analyzed using a real or complex Stiefel manifold [35], however, we will only make use of complex subspaces in this correspondence. Our codebook $\mathcal{F}$, which consists of a finite number of matrices chosen from $\mathcal{U}(M_t, M)$, thus represents a set, or packing, of subspaces in the Grassmann manifold. Designing sets of N matrices that maximize the minimum subspace distance (where distance can be chosen in a number of different ways [36]) is known as Grassmannian subspace packing. We will use the interpretation of the precoding codebook $\mathcal{F}$ as a packing of subspaces to simplify notation and analysis.

A normalized invariant measure μ is induced on $\mathcal{G}(M_t, M)$ by the Haar measure in $\mathcal{U}(M_t, M)$. This measure allows the computation of volumes within $\mathcal{G}(M_t, M)$. Subspaces within the Grassmann manifold can be related by their distance from each other [36]–[38]. A number of different distances can be defined [36], [39], but we will only make use of three. The *chordal distance* between the two subspaces $\mathcal{P}_{\boldsymbol{F}_1}$ and $\mathcal{P}_{\boldsymbol{F}_2}$ is

$$d_{\text{chord}}(\boldsymbol{F}_1, \boldsymbol{F}_2) = \frac{1}{\sqrt{2}} \|\boldsymbol{F}_1\boldsymbol{F}_1^* - \boldsymbol{F}_2\boldsymbol{F}_2^*\|_F = \sqrt{M - \sum_{i=1}^{M} \lambda_i^2\{\boldsymbol{F}_1^*\boldsymbol{F}_2\}}.$$

The projection two-norm distance between two subspaces $\mathcal{P}_{\boldsymbol{F}_1}$ and $\mathcal{P}_{\boldsymbol{F}_2}$ is

$$d_{\text{proj}}(\boldsymbol{F}_1, \boldsymbol{F}_2) = \|\boldsymbol{F}_1\boldsymbol{F}_1^* - \boldsymbol{F}_2\boldsymbol{F}_2^*\|_2 = \sqrt{1 - \lambda_{\min}^2\{\boldsymbol{F}_1^*\boldsymbol{F}_2\}}.$$

The *Fubini–Study distance* between two subspaces $\mathcal{P}_{\boldsymbol{F}_1}$ and $\mathcal{P}_{\boldsymbol{F}_2}$ is

$$d_{FS}(\boldsymbol{F}_1, \boldsymbol{F}_2) = \arccos |\det(\boldsymbol{F}_1^*\boldsymbol{F}_2)|.$$

Each of these distances corresponds to different ideas of distance between subspaces. The chordal distance generalizes the distance between points on the unit sphere through an isometric embedding from $\mathcal{G}(M_t, M)$ to the unit sphere [37]. Maximizing this distance corresponds to minimizing the sum of the eigenvalues of $\boldsymbol{F}_2^*\boldsymbol{F}_1\boldsymbol{F}_1^*\boldsymbol{F}_2$ or, similarly, $\|\boldsymbol{F}_1^*\boldsymbol{F}_2\|_F^2$. The projection two-norm distance is maximized by minimizing the smallest singular value of $\boldsymbol{F}_1^*\boldsymbol{F}_2$, while the Fubini–Study distance is maximized by minimizing the product of the singular values of $\boldsymbol{F}_1^*\boldsymbol{F}_2$. Note that

$$\|\boldsymbol{F}_1^*\boldsymbol{F}_2\|_F^2 \geq M\lambda_{\min}^2\{\boldsymbol{F}_1^*\boldsymbol{F}_2\} \geq M\,|\det(\boldsymbol{F}_1^*\boldsymbol{F}_2)|^2 \tag{17}$$

thus,

$$d_{\text{chord}}(\boldsymbol{F}_1, \boldsymbol{F}_2) \leq \sqrt{M} d_{\text{proj}}(\boldsymbol{F}_1, \boldsymbol{F}_2) \leq \sqrt{M}\sin(d_{FS}(\boldsymbol{F}_1, \boldsymbol{F}_2)). \tag{18}$$

Let $\mathcal{S} = \{\mathcal{P}_{\boldsymbol{F}_1}, \mathcal{P}_{\boldsymbol{F}_2}, \ldots, \mathcal{P}_{\boldsymbol{F}_N}\}$ be the packing of column spaces of the codebook matrices where $\mathcal{P}_{\boldsymbol{F}_i}$ is the column space of $\boldsymbol{F}_i$. Similarly to binary error correcting codes [36], a packing can be characterized by its minimum distance

$$\delta = \min_{1 \leq i < j \leq N} d(\boldsymbol{F}_i, \boldsymbol{F}_j)$$

where $d(\cdot,\cdot)$ is a distance function on $\mathcal{G}(M_t, M)$.

Consider the open ball in $\mathcal{G}(M_t, M)$ of radius $\gamma/2$ defined as

$$\mathcal{B}_{\boldsymbol{F}_i}(\gamma/2) = \{\mathcal{P}_{\boldsymbol{U}} \in \mathcal{G}(M_t, M) \mid d(\boldsymbol{U}, \boldsymbol{F}_i) < \gamma/2\}.$$

This metric ball can be defined with respect to any of the distance functions on $\mathcal{G}(M_t, M)$. Note that if $d_{\text{chord}}(\boldsymbol{F}_1, \boldsymbol{F}_2) < \sqrt{1-\rho^2}$, with $0 \leq \rho \leq 1$, then we are guaranteed that $d_{\text{proj}}(\boldsymbol{F}_1, \boldsymbol{F}_2) < \sqrt{1-\rho^2}$ and $d_{FS}(\boldsymbol{F}_1, \boldsymbol{F}_2) < \arccos(\rho^M)$. This follows by restricting the largest $M-1$ singular values of $\boldsymbol{F}_1^*\boldsymbol{F}_2$ to be unity in order to find a lower bound on the minimum singular value. This observation yields

$$\mathcal{B}_{\boldsymbol{F}_i}^{\text{chord}}(\delta_{\text{proj}}/2) \subseteq \mathcal{B}_{\boldsymbol{F}_i}^{\text{proj}}(\delta_{\text{proj}}/2) \tag{19}$$

and

$$\mathcal{B}_{\boldsymbol{F}_i}^{\text{chord}}\left(\sqrt{1-\cos^{2/M}(\delta_{FS}/2)}\right) \subseteq \mathcal{B}_{\boldsymbol{F}_i}^{FS}(\delta_{FS}/2) \tag{20}$$

where the superscript indicates the distance used.

The *density* of a subspace packing with respect to a distance γ ($\gamma \leq \delta$) is

$$\Delta(\gamma) = \mu\left(\bigcup_{i=1}^{N} \mathcal{B}_{\boldsymbol{F}_i}(\gamma/2)\right) = \sum_{i=1}^{N} \mu\left(\mathcal{B}_{\boldsymbol{F}_i}(\gamma/2)\right)$$

where $\mathcal{B}_{\boldsymbol{F}_i}(\gamma/2)$ can be defined with respect to any distance function on the Grassmann manifold. The density of a packing is a measure of how well the codebook matrices "cover" $\mathcal{G}(M_t, M)$. The density allows the probability of the isotropically distributed $\overline{\boldsymbol{V}}_R$ falling in one of the set $\mathcal{B}_{\boldsymbol{F}_i}(\gamma/2)$, with $\gamma \leq \delta$, to be expressed as

$$Pr\left(\overline{\boldsymbol{V}}_R \in \bigcup_{i=1}^{N} \mathcal{B}_{\boldsymbol{F}_i}(\gamma/2)\right) = \Delta(\gamma). \tag{21}$$

Furthermore, (19) and (20) yield

$$\Delta_{\text{chord}}(\delta_{\text{proj}}) \leq \Delta_{\text{proj}}(\delta_{\text{proj}}) \tag{22}$$

and

$$\Delta_{\text{chord}}\left(2\sqrt{1-\cos^{2/M}(\delta_{FS}/2)}\right) \leq \Delta_{FS}(\delta_{FS}) \tag{23}$$

where the subscript indicates the distance used. Notice that the factor of 2 in (23) follows from the fact that the Fubini–Study minimum distance is halved inside of the cosine function. For large M_t it has been shown in [36] that

$$\Delta_{\text{chord}}(\delta) \approx N\left(\frac{\delta}{2\sqrt{M}}\right)^{2M_tM + o(M_t)}. \tag{24}$$

C. Codebook Design Criteria

We now derive the codebook design criteria for each specific selection criterion using the distribution of the optimal unquantized precoding matrix derived in Section IV-A and the Grassmannian subspace packing results in Section IV-B.

ML-SC, MSV-SC, and MSE-SC (With Trace Cost Function): We will show that the *ML-SC, MSV-SC,* and *MSE-SC* (with the trace cost function) selection criteria all relate to maximizing the average minimum singular value of the effective channel matrix $\boldsymbol{HF}$. We will use this result to derive a codebook design criterion that relates to subspace packing on the Grassmann manifold using the projection two-norm distance.

Using (3), we can bound

$$d_{\min,R} \geq \sqrt{\frac{\mathcal{E}_s}{M}}\left(\min_{\boldsymbol{s}_1,\boldsymbol{s}_2 \in \mathcal{W}^M : \boldsymbol{s}_1 \neq \boldsymbol{s}_2} \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|_2\right)\lambda_{\min}\{\boldsymbol{HF}\}. \tag{25}$$

Thus, maximizing the lower bound on $d_{\min,R}$ is equivalent to maximizing $\lambda_{\min}\{\boldsymbol{HF}\}$. Thus this bound shows *ML-SC* requires maximizing $\lambda_{\min}\{\boldsymbol{HF}\}$.

MSE-SC using the trace cost function chooses $\boldsymbol{F} \in \mathcal{F}$ that maximizes

$$\mathrm{tr}\left(\frac{\mathcal{E}_s}{M}\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{HF}\right)^{-1}\right).$$

For high SNR, this can be approximated by $N_0 \,\mathrm{tr}\left((\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{HF})^{-1}\right)$, and we can bound

$$\min_{\boldsymbol{F}_i \in \mathcal{F}} N_0 \,\mathrm{tr}\left((\boldsymbol{F}_i^*\boldsymbol{H}^*\boldsymbol{HF}_i)^{-1}\right) \leq \min_{\boldsymbol{F}_i \in \mathcal{F}} \frac{MN_0}{\lambda^2_{\min}\{\boldsymbol{HF}_i\}}. \quad (26)$$

The bound in (26) uses the fact that the maximum eigenvalue of $(\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{HF})^{-1}$ is the inverse of the minimum eigenvalue of $\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{HF}$. Minimizing the bound approximately minimizes the trace of the MSE matrix. Therefore, maximizing the $\lambda_{\min}\{\boldsymbol{HF}\}$ is an approximate method for minimizing the trace of the MSE matrix.

Based on (25) and (26), we can relate the selection of the optimal codeword in a given codebook for the *ML-SC* and *MSE-SC* (using the trace cost function) cases to selection of the optimal codeword based on *MSV-SC*. To define a notion of an optimal codebook, we need a distortion measure with which to measure the average distortion. To design codebooks for the *ML-SC*, *MSE-SC*, and *MSV-SC* case we will use the error difference

$$\lambda^2_{\min}\{\boldsymbol{HF}_{\mathrm{opt}}\} - \lambda^2_{\min}\{\boldsymbol{HF}_i\}$$

which is nonnegative for any choice of $\boldsymbol{F}_i \in \mathcal{F}$. Thus, we will choose our codebook to minimize the average distortion

$$E_{\boldsymbol{H}}\left[\lambda^2_{\min}\{\boldsymbol{HF}_{\mathrm{opt}}\} - \max_{\boldsymbol{F}_i \in \mathcal{F}} \lambda^2_{\min}\{\boldsymbol{HF}_i\}\right] = E_{\boldsymbol{H}}\left[\lambda^2_M\{\boldsymbol{H}\} - \max_{\boldsymbol{F}_i \in \mathcal{F}} \lambda^2_{\min}\{\boldsymbol{HF}_i\}\right]. \quad (27)$$

Evaluating the expectation exactly in (27) is difficult; therefore, we will minimize an upper bound on the average distortion.

Using the singular value representation used in Section III and the properties of Grassmannian subspace packing,

$$\begin{aligned} E_{\boldsymbol{H}}\left[\max_{\boldsymbol{F}_i \in \mathcal{F}} \lambda^2_{\min}\{\boldsymbol{HF}_i\}\right] &= E_{\boldsymbol{H}}\left[\max_{\boldsymbol{F}_i \in \mathcal{F}} \lambda^2_{\min}\{\boldsymbol{\Sigma V}_R^*\boldsymbol{F}_i\}\right] \\ &\geq E_{\boldsymbol{H}}\left[\max_{\boldsymbol{F}_i \in \mathcal{F}} \lambda^2_{\min}\{\overline{\boldsymbol{\Sigma}}\overline{\boldsymbol{V}}_R^*\boldsymbol{F}_i\}\right] \\ &\geq E_{\boldsymbol{H}}\left[\lambda^2_M\{\boldsymbol{H}\}\right] E_{\boldsymbol{H}}\left[\max_{\boldsymbol{F}_i \in \mathcal{F}} \lambda^2_{\min}\{\overline{\boldsymbol{V}}_R^*\boldsymbol{F}_i\}\right] \end{aligned} \quad (28)$$

where $\overline{\boldsymbol{\Sigma}}$ is the matrix constructed from the first M columns of $\boldsymbol{\Sigma}$. The result in (28) follows from the fact that singular values and singular vectors of complex normal matrices are independent [33], [35]. Due to the results in (27) and (28) we obtain

$$\begin{aligned} &E_{\boldsymbol{H}}\left[\lambda^2_{\min}\{\boldsymbol{HF}_{\mathrm{opt}}\} - \max_{\boldsymbol{F}_i \in \mathcal{F}} \lambda^2_{\min}\{\boldsymbol{HF}_i\}\right] \\ &\leq E_{\boldsymbol{H}}\left[\lambda^2_M\{\boldsymbol{H}\}\right] E_{\boldsymbol{H}}\left[\left(1 - \max_{\boldsymbol{F}_i \in \mathcal{F}} \lambda^2_{\min}\{\overline{\boldsymbol{V}}_R^*\boldsymbol{F}_i\}\right)\right] \\ &\leq E_{\boldsymbol{H}}\left[\lambda^2_M\{\boldsymbol{H}\}\right] \\ &\quad \cdot \left(\frac{\delta^2_{\mathrm{proj}}}{4}\Delta_{\mathrm{proj}}(\delta_{\mathrm{proj}}) + (1 - \Delta_{\mathrm{proj}}(\delta_{\mathrm{proj}}))\right) \quad (29) \\ &\lessapprox E_{\boldsymbol{H}}\left[\lambda^2_M\{\boldsymbol{H}\}\right] \\ &\quad \cdot \left(1 + N\left(\frac{\delta_{\mathrm{proj}}}{2\sqrt{M}}\right)^{2M_tM+o(M_t)}\left(\frac{\delta^2_{\mathrm{proj}}}{4} - 1\right)\right). \quad (30) \end{aligned}$$

The bound in (29) is a result of partitioning the possible outcomes into two cases: i) the subspace of $\overline{\boldsymbol{V}}_R$ falls within a codeword metric ball of radius δ_{proj} and ii) the subspace of $\overline{\boldsymbol{V}}_R$ does not fall within a codeword metric ball. The codewords fall within a metric ball with probability $\Delta_{\mathrm{proj}}(\delta_{\mathrm{proj}})$ and must have distance less than $\delta_{\mathrm{proj}}/2$ from some codeword when they fall within a metric ball. Substituting the density bound in (22) and the approximation in (24) results in (30). Differentiation and making the assumption that $2M_tM + o(M_t) > 2/3$, gives the following design criterion. We always have that $\delta_{\mathrm{proj}} < 1$ so the function is a decreasing function of δ_{proj}.

Codebook Design Criterion: A codebook $\mathcal{F}$ for a system using *ML-SC*, *MSV-SC*, or *MSE-SC* (using the trace cost function) to select $\boldsymbol{F}$ from $\mathcal{F}$ should be designed by maximizing the minimum projection two-norm distance between any pair of codeword matrix column spaces.

MSE-SC (With Determinant Cost Function) and Capacity-SC: Similarly to the above, we will show that the *MSE-SC* (with the determinant cost function) and *Capacit-SC* selection criteria require maximizing the average system *Capacity-SC*. This result will be used to derive a codebook design criterion that relates to subspace packing on the Grassmann manifold using the Fubini–Study distance.

Selecting $\boldsymbol{F} \in \mathcal{F}$ using *MSE-SC* with the determinant cost function requires solving for $\boldsymbol{F}$ that minimizes

$$\det\left(\frac{\mathcal{E}_s}{M}\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{HF}\right)^{-1}\right) = \left(\frac{\mathcal{E}_s}{M}\right)^M \left(\det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{HF}\right)\right)^{-1}.$$

This is equivalent to solving for the $\boldsymbol{F}$ that maximizes

$$\tilde{I}(\boldsymbol{F}) \doteq \det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\boldsymbol{F}^*\boldsymbol{H}^*\boldsymbol{HF}\right)$$

the same expression maximized in *Capacity-SC*. Using the singular value decomposition (SVD) representation of $\boldsymbol{H}$

$$\begin{aligned} \tilde{I}(\boldsymbol{F}) &= \det\left(\boldsymbol{F}^*\boldsymbol{V}_R\left(\boldsymbol{I}_{M_t} + \frac{\mathcal{E}_s}{MN_0}\boldsymbol{\Sigma}^T\boldsymbol{\Sigma}\right)\boldsymbol{V}_R^*\boldsymbol{F}\right) \\ &\geq \left|\det\left(\overline{\boldsymbol{V}}_R^*\boldsymbol{F}\right)\right|^2 \det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\overline{\boldsymbol{\Sigma}}^T\overline{\boldsymbol{\Sigma}}\right). \end{aligned}$$

To define a notion of an optimal codebook, we need a distortion measure to measure the average performance loss in this case. Since $\boldsymbol{F}_{\mathrm{opt}}$ that maximizes the mutual information over $\mathcal{U}(M_t, M)$ gives $\tilde{I}(\boldsymbol{F}_{\mathrm{opt}}) = \det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\overline{\boldsymbol{\Sigma}}^T\overline{\boldsymbol{\Sigma}}\right)$, we will use the error difference

$$\tilde{I}(\boldsymbol{F}_{\mathrm{opt}}) - \left|\det\left(\overline{\boldsymbol{V}}_R^*\boldsymbol{F}\right)\right|^2 \det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\overline{\boldsymbol{\Sigma}}^T\overline{\boldsymbol{\Sigma}}\right)$$

which is nonnegative for any choice of $\boldsymbol{F}_i \in \mathcal{F}$. Thus, we will choose our codebook to minimize the average distortion

$$\begin{aligned} &E_{\boldsymbol{H}}\left[\det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\overline{\boldsymbol{\Sigma}}^T\overline{\boldsymbol{\Sigma}}\right) - \max_{\boldsymbol{F}_i \in \mathcal{F}}\left|\det\left(\overline{\boldsymbol{V}}_R^*\boldsymbol{F}_i\right)\right|^2 \det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\overline{\boldsymbol{\Sigma}}^T\overline{\boldsymbol{\Sigma}}\right)\right] \quad (31) \\ &= E_{\boldsymbol{H}}\left[\det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\overline{\boldsymbol{\Sigma}}^T\overline{\boldsymbol{\Sigma}}\right)\right] \\ &\quad \cdot \left(1 - E_{\boldsymbol{H}}\left[\max_{\boldsymbol{F}_i \in \mathcal{F}}\left|\det\left(\overline{\boldsymbol{V}}_R^*\boldsymbol{F}_i\right)\right|^2\right]\right) \quad (32) \end{aligned}$$

where (32) follows from the independence of $\boldsymbol{\Sigma}$ and $\boldsymbol{V}_R$ [33], [35].

The distortion cost function can be bounded as

$$\begin{aligned} &E_{\boldsymbol{H}}\left[\det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\overline{\boldsymbol{\Sigma}}^T\overline{\boldsymbol{\Sigma}}\right)\right] \\ &\quad\cdot\left(1 - E_{\boldsymbol{H}}\left[\max_{\boldsymbol{F}_i\in\mathcal{F}}\left|\det\left(\overline{\boldsymbol{V}}_R^*\boldsymbol{F}_i\right)\right|^2\right]\right) \\ &\le E_{\boldsymbol{H}}\left[\det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\overline{\boldsymbol{\Sigma}}^T\overline{\boldsymbol{\Sigma}}\right)\right] \\ &\quad\cdot\left(1-\cos^2\left(\delta_{FS}/2\right)\Delta_{FS}(\delta_{FS})\right) \qquad (33) \\ &\lessapprox E_{\boldsymbol{H}}\left[\det\left(\boldsymbol{I}_M + \frac{\mathcal{E}_s}{MN_0}\overline{\boldsymbol{\Sigma}}^T\overline{\boldsymbol{\Sigma}}\right)\right]\left(1 - N\cos^2\left(\delta_{FS}/2\right)\right. \\ &\quad\left.\cdot\left(\sqrt{\frac{1-\cos^{2/M}(\delta_{FS}/2)}{M}}\right)^{2M_tM+o(M_t)}\right). \qquad (34) \end{aligned}$$

The result in (33) follows from the facts that the subspace of $\overline{\boldsymbol{V}}_R$ lies within a codeword metric ball with probability $\Delta_{FS}(\delta_{FS})$ and that all subspaces within the metric balls have distance less than $\delta_{FS}/2$. Using the density bound in (23) and the approximation in (24) yields (34). Differentiating this bound, and assuming that $M_t + o(M_t)/(2M) \ge (2^{1/M}-1)$, tells us that we want to maximize δ_{FS} in order to minimize the distortion cost function.We can now state the following.

Codebook Design Criterion: A codebook $\mathcal{F}$ for a system using *MSE-SC* with the determinant cost function or *Capacity-SC* to select $\boldsymbol{F}$ from $\mathcal{F}$ should be designed by maximizing the minimum Fubini–Study distance between any pair of codeword matrix column spaces.

Discussion: In summary, thinking of the codebook $\mathcal{F}$ as a packing of M-dimensional subspaces rather than a set of $M_t \times M$ matrices allows us to bound the distortion for each of the selection criteria proposed in Section III. The distortion bound for *ML-SC, MSV-SC,* and *MSE-SC* with the trace cost function is minimized by maximizing the minimum projection two-norm distance between any pair of codebook subspaces. The *Capacity-SC* and *MSE-SC* with the determinant cost function bound is minimized by maximizing the minimum Fubini–Study distance between any pair of codebook subspaces. Thus, the codebook design is equivalent to subspace packing in the Grassmann manifold.

Observe that both design criteria make assumptions on the relation between M_t, M, and the $o(M_t)$ term. Numerical experiments have shown that for most $M_t > 2$ the assumptions are satisfied. When $M = 1$, it is also known that the $o(M_t)$ term is always -2 and (24) is an exact expression [20].

Finding good packings in the Grassmann manifold for arbitrary M_t, M, and N, and thus finding good codebooks, is difficult (see, for example, [37], [38], [40]). The problem is exasperated by the use of the projection two-norm and Fubini–Study distances instead of the more common chordal distance [36], [39]. For instance, in the simplest case of $M = 1$ where the Rankin lower bound on line packing correlation [38] can be employed, packings that achieve equality with the lower bound are often impossible to design. One simple method for designing good packings with arbitrary distance functions is to use the noncoherent constellation designs from [41]. We have found that the algorithms for constellation design in [41] yield codebooks with large minimum distances and can be easily modified to work with any distance function on the Grassmann manifold.

Note that this work generalizes the analysis in [20]. When $M = 1$, both the projection two-norm and Fubini–Study distance criteria minimize $\max_{1\le i<j\le N}\left|\boldsymbol{f}_i^*\boldsymbol{f}_j\right|$ where the codebook is given by $\mathcal{F} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_N\}$. This would be expected because the probability of error and capacity cost functions are both functions of the effective channel gain $\|\boldsymbol{H}\boldsymbol{f}\|_2^2$ when $M = 1$.

V. Simulations

Monte Carlo simulations were performed to illustrate the performance of Grassmannian precoders. The codebooks were designed using the criteria proposed in Section IV-C. To optimize the criteria, the codebooks were designed using the noncoherent constellations in [41]. Example codebooks can be downloaded at [42]. For each of the precoding systems using M_t transmit antennas and M substreams, we also plotted the $M \times M_r$ spatial multiplexing results with both ZF and ML decoding. In addition, we simulated the unquantized MMSE precoding using the trace cost function and both the sum power and maximum singular value constraints [3] and maximum minimum singular value antenna selection [12]. As well, we use the notation $M_t \times M_r$ to denote an M_t transmit and M_r receive antenna system.

A. Comparisons Between Receiver Architectures

This simulation used binary phase-shift keying (BPSK) modulation and two substreams on a 4×2 wireless system. The results are shown in Fig. 2. We simulated 6–bit limited feedback precoding using *Capacity-SC, MSV-SC,* and *ML-SC*. ZF decoding and precoding using *Capacity-SC* provided more than a 4 dB performance gain at a probability of symbol vector error of 2×10^{-3} over unprecoded decoding using ML decoding. Precoding using *MSV-SC* provided a 0.5 dB gain over *Capacity-SC*. Unquantized MMSE precoding with the sum power constraint performs approximately 1.5 dB better than limited feedback precoding using *MSV-SC*. As expected, ML decoding combined with *ML-SC* provided a large performance gain and outperformed unquantized MMSE precoding at a probability of symbol vector error of 10^{-3} by around 2.5 dB.

B. Feedback Allows Lower Complexity Decoding

We simulated three substream precoding on a 6×3 wireless system in this experiment using 16-QAM with results shown in Fig. 3. We used limited feedback precoding with 6 bits of feedback. Limited feedback precoding with *Capacity-SC* and with *MSE-SC* (using the determinant cost function) performed approximately 0.25 dB better than antenna selection. Limited feedback precoding using *MSV-SC* and *MSE-SC* with the trace cost function both performed approximately the same. They both provide around a 0.25 dB improvement over *Capacity-SC* and perform within 1 dB of unquantized optimal precoding using ZF decoding. Note that all selection criteria outperformed unprecoded spatial multiplexing using an ML receiver. This shows though the power of precoding: near-ML or better than ML performance with low-complexity receivers at the expense of feedback.

C. Comparison With Direct Channel Quantization

The purpose of this experiment is to demonstrate the problems associated with directly quantizing the matrix channel $\boldsymbol{H}$. The results are presented in Fig. 4. This experiment considered a 4×2 wireless system using two substrems and 16-QAM. Directly quantizing the channel with 16 bits of feedback performs approximately 4.7 dB worse than a 6-bit limited feedback precoder at a probability of symbol vector error of 10^{-2}. The limited feedback precoder obtains performance approximately identical to that of the unquantized MMSE precoder with the maximum singular value power constraint.

VI. Conclusion and Future Work

We proposed a system for precoding when only a limited feedback channel is available to convey channel state information. The essential component of this system is a codebook of precoding matrices that is known to both the transmitter and receiver. The codebook is designed offline based on the distribution of the channel. Observations of the channel at the receiver are used to determine the optimal precoder in the codebook and this index is conveyed to the transmitter.

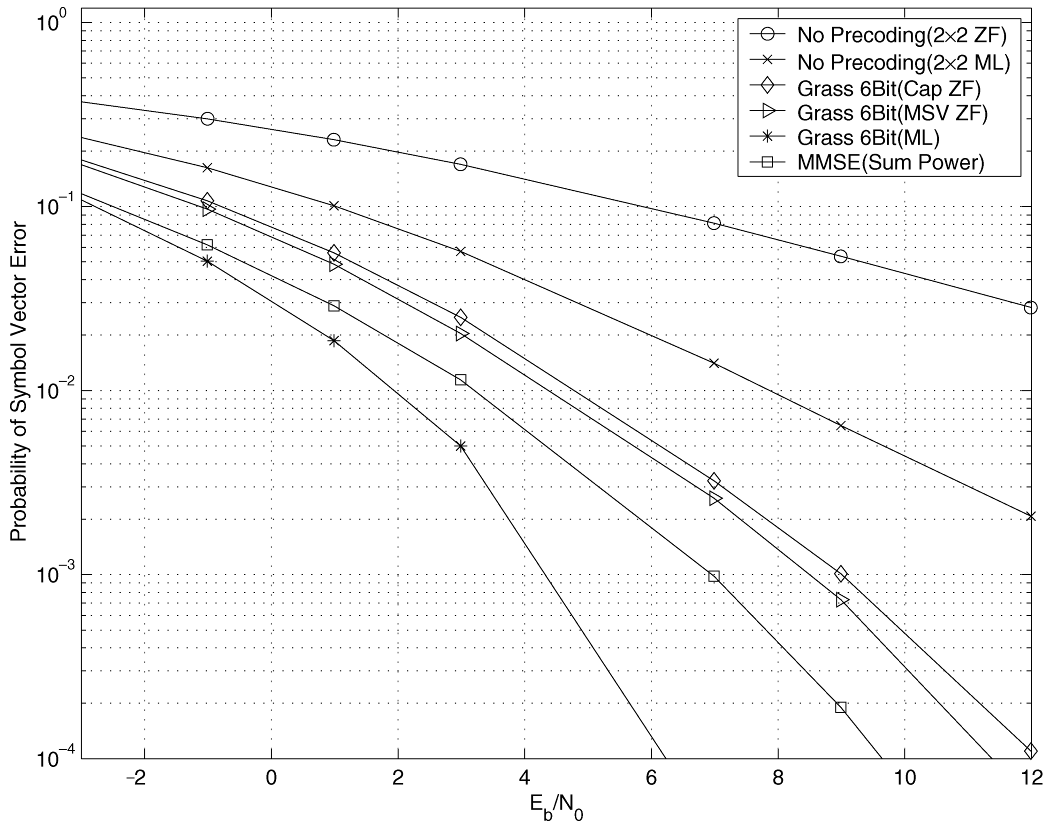

Fig. 2. Probability of symbol vector error comparison of various precoding schemes for a two substream 4×2 system using BPSK.

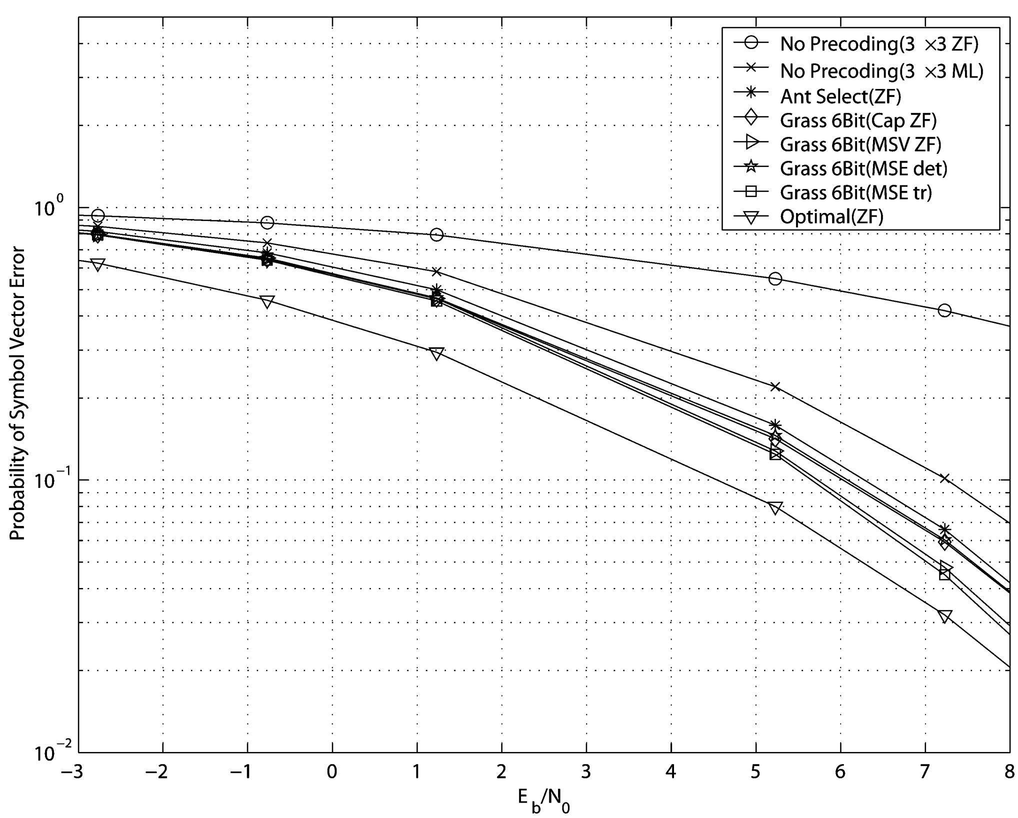

Fig. 3. Probability of symbol vector error comparison of various precoding schemes for a three substream 6×3 system using 16-QAM.

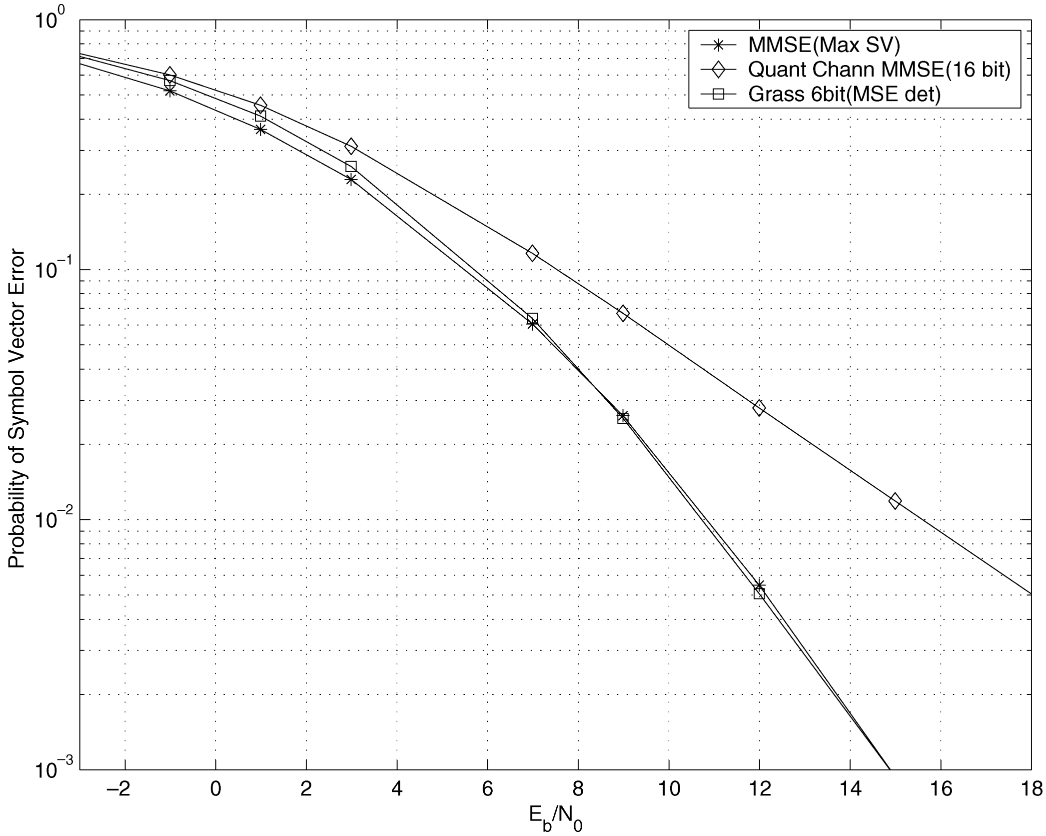

Fig. 4. Probability of symbol vector error comparison for direct channel quantization, unquantized MMSE, and limited feedback precoding on a two substream 4×2 system using 16-QAM.

In this correspondence, we presented solutions to the problems of precoder selection and codebook construction. We presented criteria for choosing the optimal precoder from the codebook according to different performance criteria and found that they were related directly to the unquantized codebook design criteria in [3] and the antenna selection criteria in [12]. We defined a notion of average distortion and used this to propose a codebook design criterion that minimizes a bound on the average distortion. We showed that the proposed design essentially relates to the problem of packing subspaces in the Grassmann manifold using the projection two-norm and the Fubini–Study distances. This problem of spacing subspaces is a famous problem in applied mathematics known as Grassmannian subspace packing [36], [37].

One important point of future work that is not addressed in this correspondence is the effect of delay and errors in the feedback channel. This will lead to a degradation of the bit-error rate performance compared with ideal channels. This analysis is currently being studied in the IEEE 802.16E standards body.

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Quadratic Forms on Complex Random Matrices and Multiple-Antenna Systems

T. Ratnarajah, *Associate Member, IEEE,* and Rémi Vaillancourt

***Abstract*—In this correspondence, the densities of quadratic forms on complex random matrices and their joint eigenvalue densities are derived for applications to information theory. These densities are represented by complex hypergeometric functions of matrix arguments, which can be expressed in terms of complex zonal polynomials. The derived densities are used to evaluate the two most important information-theoretic measures, the so-called ergodic channel capacity and capacity versus outage of multiple-input multiple-output (MIMO) spatially correlated Rayleigh-distributed wireless communication channels. We also derive the probability density function of the mutual information between transmitted and received complex signals of MIMO systems with a *finite* number of transmit and receive antennas. Numerical results show how channel correlation degrades the capacity of MIMO communication systems.**

***Index Terms*—Capacity versus outage, complex random matrices, ergodic channel capacity, quadratic form on complex random matrices, Rayleigh-distributed MIMO channels, zonal polynomials.**

I. Introduction

Let an $n \times m$ $(n \geq m)$ complex Gaussian (or normal) random matrix $\boldsymbol{X}$ be distributed as $\boldsymbol{X} \sim \mathcal{C}N(0, \Sigma_1 \otimes \Sigma_2)$ with mean $\mathcal{E}\{\boldsymbol{X}\} = 0$ and covariance $\operatorname{cov}\{\boldsymbol{X}\} = \Sigma_1 \otimes \Sigma_2$, where $\Sigma_1 \in \mathbb{C}^{n\times n}$ and $\Sigma_2 \in \mathbb{C}^{m\times m}$ are positive-definite Hermitian matrices. Here we read the symbol "$\sim$" as "is distributed as," $\mathcal{C}N$ denotes the complex normal distribution, and $\otimes$ denotes the Kronecker product. The quadratic form $\boldsymbol{S}$ on $\boldsymbol{X}$ associated with the positive-definite Hermitian matrix A is defined by

$$\boldsymbol{S} = \boldsymbol{X}^H A \boldsymbol{X}.$$

Here, we study the distribution of $\boldsymbol{S}$, denoted by $\mathcal{C}Q_{n,m}(A, \Sigma_1, \Sigma_2)$, and its application to information theory. We also derive the joint eigenvalue densities of $\boldsymbol{S}$, which are represented by complex zonal polynomials. Complex zonal polynomials are symmetric polynomials in the eigenvalues of a Hermitian matrix, see [17], and they enable us to represent the derived densities as infinite series. The distributions

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T. Ratnarajah was with the Department of Mathematics and Statistics, University of Ottawa, Ottawa ON K1N 6N5, Canada. He is now with ECIT, Queen's University of Belfast, Belfast BT7 9DT, Northern Ireland, U.K. (e-mail: T.Ratnarajah@ieee.org).

R. Vaillancourt is with the Department of Mathematics and Statistics, University of Ottawa, Ottawa ON K1N 6N5, Canada (e-mail: remi@mathstat.uottawa.ca).

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