Shorter Circuit Obfuscation in Challenging Security Models

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Abstract

The study of program obfuscation is seeing great progress in recent years, which is crucially attributed to the introduction of graded encoding schemes by Garg, Gentry and Halevi (Eurocrypt 2013). In such schemes, elements of a ring can be encoded such that the content of the encoding is hidden, but restricted algebraic manipulations, followed by zero-testing, can be performed publicly. This primitive currently underlies all known constructions of general-purpose obfuscators.

However, the security properties of the current candidate graded encoding schemes are not well understood, and new attacks frequently introduced. It is therefore important to assume as little as possible about the security of the graded encoding scheme, and use as conservative security models as possible. This often comes at a cost of reducing the efficiency or the functionality of the obfuscator.

In this work, we present a candidate obfuscator, based on composite-order graded encoding schemes, which obfuscates circuits directly a la Zimmerman (Eurocrypt 2015) and Applebaum-Brakerski (TCC 2015). Our construction requires a graded encoding scheme with only 3 "plaintext slots" (= sub-rings of the underlying ring), which is directly related to the size and complexity of the obfuscated program. We prove that our obfuscator is superior to previous works in two different security models.

1. We prove that our obfuscator is indistinguishability-secure (iO) in the *Unique Representation Generic Graded Encoding* model. Previous works either required a composite-order scheme with polynomially many slots, or were provable in a milder security model. This immediately translates to a polynomial improvement in efficiency, and shows that improved security does not come at the cost of efficiency in this case.
2. Following Badrinarayanan et al. (Eurocrypt 2016), we consider a model where finding any "non-trivial" encoding of zero breaks the security of the encoding scheme. We show that, perhaps surprisingly, secure obfuscation is possible in this model even for some classes of *non-evasive functions* (for example, any class of conjunctions). We define the property required of the function class, formulate an appropriate (generic) security model, and prove that our aforementioned obfuscator is virtual-black-box (VBB) secure in this model.

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1 Introduction

A program obfuscator is a compiler that takes a program as input, and outputs a functionally equivalent program that is hard to reverse engineer. Early works by Hada [Had00] and Barak et al. [BGI+12] provided rigorous definitional treatment of obfuscation, but also showed the impossibility of achieving strong security notions for general circuits. In particular Virtual Black-Box (VBB) security, where interaction with the obfuscated program can be simulated using only black-box access to the obfuscated program, was proven impossible in general.

Constructing secure obfuscators, even heuristically, is a very challenging task. Indeed, until recently, candidate obfuscators were only known to exist for a few simple function classes. The game changer in this field had been the introduction of *graded encoding schemes* (GES) by Garg, Gentry and Halevi [GGH13a] and follow-up constructions by Coron, Lepoint and Tibouchi [CLT13, CLT15]. GES allow to encode ring elements (from some underlying ring) in a way that hides the identity of the ring element, but still allows algebraic manipulation on the encoding (addition and multiplication). Each encoding is associated with a *level*, which is a positive integer (or more generally an integer vector). Addition is only allowed within a level, and in multiplication the level of the output is the sum of the levels of the inputs. A GES allows to test if the contents of an encoding is the zero element, but only at a predetermined "zero-test level", and not beyond. Thus GES allows arithmetic operations of bounded degree.

Garg et al. [GGH+13b] presented a candidate obfuscator for general circuits based on GES. They conjectured, with some supporting evidence, that their obfuscator is a secure *indistinguishability obfuscator* (iO). Indistinguishability obfuscation is a weak security notion and it first glance it may seem useless. However, Sahai and Waters [SW14] showed that iO is actually sufficient for a wide variety of applications. Numerous follow-up works showed how to use iO to construct many desirable cryptographic primitives, thus establishing iO itself as one of the most important cryptographic primitives. The goal of formally establishing the security of obfuscation candidates had since been central in cryptographic research.

Brakerski and Rothblum [BR14b] presented a similar obfuscator candidate, and proved its security in the *generic GES model*. This model addresses adversaries that are restricted to algebraic attacks on the encoding scheme, i.e. generate encodings, perform algebraic manipulations and test for zero, while being oblivious to the representation of the element. This is modeled by representing the encodings using random strings, thus making them completely opaque. The algebraic functionality is provided as oracle. Other candidate obfuscators with generic proofs followed [BGK+14, AGIS14, MSW14]. Pass, Seth and Telang [PST14] replaced the generic model with a strong notion of "uber-assumption".

The constructions mentioned so far were all based on converting the obfuscated program into a branching program, thus having computational cost which scaled with the *formula size* of the program to be obfuscated.[1] This was improved by newer constructions that used *composite order* GES (where the underlying ring is isomorphic to $\mathbb{Z}_N$ for a composite N). In a nutshell, composite order rings allow for "slotted" representation of elements via the Chinese Remainder Theorem, so that each ring element is viewed as a tuple of slots, and algebraic operations are performed slot-wise. In particular, Zimmerman [Zim15] and Applebaum and Brakerski [AB15] presented obfuscators whose overhead relates to the *circuit size* of the program and not its formula size.

[1] An additional "bootstrapping" step established that obfuscating polynomial-size formulae is sufficient in order to obfuscate general circuits.

However, using known candidate GES, the underlying encodings again incorporated overhead that depends on the formula size. Nonetheless, these constructions carry the promise that given more efficient GES candidates, the dependence on the formula size can be completely removed. Proofs in generic models were provided.

Since the generic model restricts the adversary beyond its actual attack capabilities, such proofs should be taken only as evidence in lieu of standard model proofs. In order for the evidence to carry more weight, we should be prudent and use models that pose as few restrictions as possible on the adversary.

For example, [BGK+14, Zim15, AB15] consider a model where one assumes that not only encodings of different elements appear to the adversary as independent uniform strings, but also if the same element is computed in two different ways then it will have two independent-looking representations. This is a fairly strong assumption and in particular one that does not hold in cryptographic multilinear-maps, if such exist [BS02]. It is shown in [AB15] that the suggested obfuscation scheme actually breaks if one is allowed to even test for zero at levels below the zero-test level. They therefore proposed a more robust obfuscator that is secure in the unique representation model of [BR13, BR14a, BR14b], in which each ring element has a unique representation. Unfortunately, this added security came at a cost of reducing efficiency, specifically the number of "input slots" goes up from 2 to $n + 2$ (where n is the input length). This directly translates to an efficiency loss in the construction.[2]

A notable progress in the study of secure obfuscation had been made recently by Gentry, Lewko, Sahai and Waters [GLSW14]. They showed an obfuscator whose security is based on an assumption in the *standard model*. It is yet unclear whether their hardness assumption holds true in known candidate GES (recent attacks [CLR15, MF15] suggest it might not). It should further be noted that this construction again requires a large number of input slots (essentially proportional to the formula size of the obfuscated circuit).[3]

We see that the attempts to come up with a more realistic security model comes at the cost of increasing the number of required slots, and therefore reducing the efficiency. It is not clear whether this trade-off is necessary.

Does a stronger security model come at the cost of efficiency?

In this work, we show that at least in the generic model, one does not need to pay in efficiency to achieve better security.

We proceed to consider an even more conservative security model, one where even finding a non-trivial encoding of the zero element is assumed to obliterate security completely.[4] This model is motivated by new attacks on the security of *all known* proposed GES candidates [GGH13a, CHL+15, CGH+15, HJ15, CLR15, MF15], showing that having access to encodings of the ring's zero element results, in some cases, in a complete security breach. Indeed, current attacks do not work with just *any* non-trivial zero encoding, however they do raise concern that having an adversary access an encoding of zero might be a vulnerability. This concern had been significantly heightened

[2] Miles, Sahai and Weiss [MSW14] suggested constraining the model in a different, orthogonal manner. Their model is less relevant for this work.

[3] They also suggest a construction using a single-slot GES, however the efficiency cost was even greater.

[4] A "trivial" zero is, for example, the result of subtracting an encoding from itself, or of similar computations that nullify based on the syntax of the equation rather than the encoded values.

recently as Miles, Sahai and Zhandry [MSZ16] presented an attack on obfuscators that are based on the [GGH13a] GES candidate. This new attack again makes crucial use of top-level encodings of zero (but does not require "low-level" zero encodings like some prior attacks).

To hedge against these risks, Badrinarayanan, Miles, Sahai and Zhandry [BMSZ15] proposed to avoid zeros completely. Namely, to construct an obfuscator in such a way that the adversary is unable to generate such encodings altogether. However, this seems to defeat the purpose, since zero-testing is the way to extract information out of an encoding for functionality purposes. They get around this barrier in a creative way, by only obfuscating *evasive functions*, where finding an accepting input using oracle access is (unconditionally) hard.[5] Classes of evasive functions have played an important role in the study of obfuscation, since many classes that are desirable to obfuscate are evasive (e.g. various variants of point functions, starting with the work of Canetti [Can97]) and one could hope that they can even be obfuscatable in the strong VBB setting. (See [BBC+14] for more information and the state of the art about evasive functions.) Badrinarayanan et al. show that when their obfuscator is applied to an evasive function, the adversary is unable to find an encoding of zero. The proof here is in the generic model as well. The restriction to evasive functions, however, excludes interesting function classes such as conjunctions [BR13, BVWW16]. We therefore address the following question.

Can we obfuscate non-evasive functions in the zero-sensitive model?

Perhaps surprisingly, we answer this question in the affirmative, and show that our obfuscator (the same as above) is secure in a zero-sensitive model, even for some non-evasive function classes, and in particular for worst-case conjunctions.

1.1 Our Results

A More Efficient Circuit Obfuscator. We present a new direct circuit obfuscator, i.e. one that does not go through branching programs. Our construction is inspired by the "robust obfuscator" RobustObf of [AB15]. However, whereas RobustObf works over a composite order graded encoding scheme with $(n+2)$ message slots, our obfuscator only requires 3 slots. Our obfuscator provides equivalent level of security to RobustObf in the unique representation generic GES model (see details below). This improvement translates directly to a factor n improvement in the size of the encodings, and a $poly(n)$-factor improvement in the computational complexity of generating and evaluating the obfuscated program.[6] We therefore show that at least in the generic model, there is no real efficiency gain to working in a less secure model. We hope that our techniques can be translated to reduce the number of required slots in the non-generic setting as well, in particular in the [GLSW14] scheme.

We prove that the resulting obfuscator is indistinguishability secure in the unique representation graded encoding model. The proof outline is similar in spirit to that of the robust obfuscator of [AB15], while incorporating some proof techniques from [Zim15]. In particular, we rely on the sub-exponential hardness of factoring the order of the underlying ring, in addition to the security of the generic model. In contrast, [AB15] work in a model where the order of the ring is hidden so that factoring it is *information theoretically* hard.

[5] We note that if the [BMSZ15] obfuscator is applied to non evasive functions, and top-level zeros can occur, then the [MSZ16] attack applies. This highlights the significance of completely avoiding zeros.

[6] See e.g. [GLW14, Appendix B] for suggested trade-offs between the number of input slots and the size of the encoding.

We note that one can consider many variants of the generic model: known modulus, unknown modulus and information theoretic hardness, computational hardness. Furthermore, [Zim15] also shows how to prove VBB security at the cost of increasing the size of the obfuscator by additional n^2 encodings. Our improvement can be applied to all of these variants, transforming them to the unique representation model while preserving the number of slots as constant. For the sake of concreteness, we chose to prove in a setting that we found interesting.

The Zero-Sensitive Oracle and All-or-Nothing Functions. We show that the [BMSZ15] approach discussed above can be extended even beyond evasive functions. This may come as a surprise since applying our obfuscator to non-evasive functions gives the adversary access to zero encodings. As a motivating example, consider the class of conjunctions that had been studied in [BR13]. One can think of a conjunction as string-matching with wildcards. Namely, the function is defined by a string $v \in \{0,1,\star\}^n$, and $f_v(x) = 1$ if and only if for all i, either $v[i] = x[i]$ or $v[i] = \star$. Indeed, some distributions on this class of functions are evasive, but what if we want to obfuscate it *in the worst case*?

Naturally, in the worst case there could be an adversary that can find an accepting input (more generally, no function class is evasive in the worst case except the zero function). However, the critical observation is that this does not necessarily hinder security, since given an accepting input, one can *learn the entire function*. In the case of conjunctions this is easy to do by taking an accepting x and flipping each of its bits in turn to see if this bit is a wildcard (and switching it back afterwards). Therefore, if we find an accepting input, we should not expect the obfuscator to hide anything anyway!

We generalize this property and define *All-or-Nothing* (AoN) function classes to be ones where if an adversary finds an accepting input, then it can recover the function in its entirety (a formal definition is provided). We would like to show that indeed such function classes can be securely obfuscated even in a setting where a non-trivial zero encoding implies that the GES is insecure.

In the proof of [BMSZ15] for evasive functions, proving security was split into two tasks: presenting a simulator, and showing that the adversary cannot compute encodings of zero. Our notion of security, however, requires additional definitional treatment, since we would like successful simulation even in the case where an accepting input had been found, and we cannot tell in advance whether such an input will be found or not. We therefore define a new generic model where the GES oracle keeps track of the encodings that the adversary generates, and if one of those is a non-trivial zero, then the adversary gets access to a *decoding oracle* that allows to decode any given encoding to obtain the plaintext. This is how we model the risk in non-trivial zeros.

Finally, we prove that our obfuscator is indeed a secure VBB obfuscator for AoN functions in our new zero-sensitive model. Interestingly, we don't need to use complexity leveraging here and we can prove VBB security without increasing the number of encodings. We view this as evidence that AoN functions may be strictly easier to obfuscate than general functions, and are perhaps a good candidate for VBB obfuscation in the standard model.

What GES Candidate To Use? We stress that our work is completely abstract and not directly related to any specific GES candidate, but naturally it would be more convincing if it could be instantiated with one. To date, the only candidate composite order GES is that of [CLT13,CLT15], and indeed this candidate can be used with our scheme. We stress that the only known attacks on this candidate uses encodings of zero, and there are no known attacks in the zero evading model

(this is also true for the [GGH13a] candidate). In fact, even in the "standard" model, the attacks of [CGH+15,MSZ16] do not seem to apply to our obfuscator when instantiated with [CLT13,CLT15]. However, these attacks suggest that obfuscators such as ours might be vulnerable to future attacks. The goal of finding secure instantiations of composite order candidate GES is a very important one, but orthogonal to the contributions of this paper.

1.2 Our Techniques

Our Obfuscator. Our building block is a graded encoding scheme whose plaintexts are elements in a composite order ring. We denote the encoding of the element a by $[a]$. Encodings can be added, subtracted, multiplied and tested for zero (subject to constraints imposed by the levels, which we will ignore in this outline since they are similar to previous works). We think of a itself as a tuple of elements via the Chinese Remainder Theorem. Each sub-ring is of high cardinality and it is assumed that "isolating" the components of an encoded element is computationally hard (in the generic model this relates to the hardness of factoring the order of the ring). The [AB15] obfuscator (following [BR13, BR14a, BR14b]) adds an additional layer on top of this encoding and rather than encoding $[a]$ itself, it produces a pair of encodings $[r]$ and $[r \cdot a]$, for a random r, i.e. the plaintext value is the ratio between the values in the two encodings. This "rational encoding" plays an important role in both functionality and security. For the purpose of this outline only, we use $[a]^\diamond$ as shorthand notation for the pair $[r], [r \cdot a]$. It can be shown that rational encodings can be added and multiplied, subject to constraints as in previous works.

The starting point of our construction is the "robust obfuscator" from [AB15]. This obfuscator, in turn, is derived from a simpler solution [Zim15, AB15] that applies in a more forgiving generic model. In the "simple obfuscator", for each input bit i, two encodings are given as a part of the obfuscator. These encodings are of the form $[(y_i, b)]^\diamond$, for $b \in \{0, 1\}$, where y_i is a random value that is the same whether $b = 0$ or 1. The weakness of this scheme stems from the ability to subtract the two encodings that correspond to the same i, and cancel out the y_i value to obtain an encoding of the form $[(0, 1)]^\diamond$, which in turn allows to test whether the second slot of a given encoding is zero or not (via multiplying by $[(0, 1)]^\diamond$ and zero-testing). In the less restrictive multiple representation generic model, this attack is prevented by disallowing to test for zero in some situations. However, this cannot be avoided in a model where each element has a unique representation since one can always test for zero by comparing to a known encoding of zero.

The robust obfuscator from [AB15] prevents this problem by adding n additional slots to the encodings, and publishing, for each input bit of the obfuscated function, the values $[w_{i,b}]^\diamond$, for $b \in \{0, 1\}$, where $w_{i,b} = (y_i, b, \rho_{1,b}, \ldots, \rho_{n,b})$. The ρ values are uniform and independent, and therefore subtracting $[w_{i,1}]^\diamond - [w_{i,0}]^\diamond$ here will not cancel out the ρ values. The ρ values should be eliminated in the end of the computation, and this is done by providing additional encodings of a special form $\hat{w}_{i,b} = (\hat{y}_i, \beta_{i,b}, \hat{\rho}_{1,b}, \ldots, \hat{\rho}_{i-1,b}, 0, \hat{\rho}_{i+1,b}, \ldots, \hat{\rho}_{n,b})$. Namely, encodings that zero out the ith ρ value. In the evaluation, the value $\prod_i \hat{w}_{i,x_i}$ is computed and multiplied with the result of the computation so far, thus zeroing out the last n slots. Note that even though the ρ values can be zeroed out, this does not enable the previous attack. This is due to the level constraints that impose structural limitations. In particular, $[\hat{w}_{i,0}]^\diamond$ and $[w_{i,1}]^\diamond$ cannot be used in the same computation, which is in contrast to $[w_{i,0}]^\diamond$ and $[w_{i,1}]^\diamond$ that cannot be prevented from interacting (at a high level, this is because each input bit can be used many times in the circuit, but the $\hat{w}$ values are designed to only be used once).

Our modification to this scheme is quite simple. We observe that the use of n different ρ slots

is only due to the cancellation step via $\hat{w}$, where we need to enforce that an adversary must use a $[\hat{w}_{i,b}]^{\diamond}$ value *for each and every* i. The reason is that this use prevents the dangerous mix-and-match of $[w_{i,0}]^{\diamond}$ and $[w_{i,1}]^{\diamond}$. We notice, however, that since rational encodings can be added and not just multiplied, one could enforce that an $\hat{w}_{i,b}$ is used for every i using a sum rather than a product. We set $\hat{w}_{i,b} = (\hat{y}_i, \beta_{i,b}, \hat{\rho}_i)$, thus reducing the number of sub-rings to only 3. We choose the $\hat{\rho}_i$ values at random, subject to the constraint that $\sum_i \hat{\rho}_i = 0$. This means that in order to zero-out the $\hat{\rho}$ coordinate, an adversary needs to use a $[\hat{w}_{i,b}]^{\diamond}$ element for every i. As before, we must prevent $[\hat{w}_{i,0}]^{\diamond}$ and $[\hat{w}_{i,1}]^{\diamond}$ from interacting, since taking their difference zeros out the $\hat{\rho}$ coordinate and is therefore dangerous, but this is done in the same way as previous works.

Proving Security. As has been shown in a number of previous works, in the generic model, the adversary is limited to applying arithmetic circuits over the encodings received as input, and testing the output for zero. The simulator, therefore, generates a collection of random strings to play the role of the encodings in the obfuscated program, and then to answer queries of the form of an arithmetic circuit, determining whether applying this circuit to the encodings at hand evaluates to zero.[7] The problem is that the simulator needs to do this with only oracle access to the obfuscated circuit. Namely, it does not fully know what is the plaintext in the encoding that it generated.

We use a proof practice that started with [BR13]. They notice that if we use rational encoding as described above, then the polynomial computed by an arithmetic circuit can be decomposed into a sum of terms that we call *semi-monomials*. A semi-monomial is a polynomial of the form $M(\vec{r})Q(\vec{w})$, where $M(\vec{r})$ is a product of "randomizing" variables, and $Q(\vec{w})$ is a polynomial in the "content" variables. Since the randomizer variables are random and independent, the task of testing the polynomial for zero is identical to the task of finding whether there exists a non-zero semi-monomial.

We distinguish between semi-monomials that are "valid", in the sense that they represent a legal evaluation of the circuit on an input, and ones that are "invalid". We show how to test if a semi-monomial is valid or not, and that an invalid semi-monomial cannot zero-out, regardless which circuit had been obfuscated, assuming the hardness of factoring the ring order. We show that "valid" monomials zero-out if and only if the obfuscated circuit accepts their associated input x.

Therefore, our proof strategy is straightforward. We extract semi-monomials from the circuit one after the other (one can show that this can be done). For each semi-monomial, we check whether it is invalid, in which case we can immediately return that the arithmetic circuit computes a non-zero. If the semi-monomial is valid for some input x, we query the obfuscated circuit oracle on x. If it rejects, then the answer is again non-zero, but if it accepts, then the answer is still undetermined and we need to proceed to the next semi-monomial.

This process takes 2^n time in the worst case, since there can be at most 2^n valid semi-monomials. Thus the running time of our iO simulator is exponential in the input length. However, in the case of AoN functions, the situation is much simpler and in fact only *one* semi-monomial needs to be inspected. The reason is that if the extracted x is an accepting input for the circuit, then we don't need to proceed at all, since for AoN functions, we can efficiently learn the code of the circuit, which allows us to continue the simulation trivially by just assigning the right values to the $\vec{w}$ variables. This completes the proof.

[7] It may seem that the simulator needs to do much more than that, but it can be shown that all other functionalities reduce to this problem.

1.3 Paper Organization

Some preliminaries and facts about arithmetic circuits and polynomials appear in Section 2. This section can safely be skipped by a knowledgeable reader. In Section 3 we present our new generic model as well as our new zero-sensitive model, which is a new contribution. Section 4 features the specifics of our obfuscator, and security is proven in Section 5, where we also define the class of AoN functions.

2 Preliminaries

Definition 2.1 (Factoring)**.** *Let λ be a security parameter and let σ be polynomial in the security parameter. The σ-factoring problem is defined as follows. Let $p_1, \ldots, p_\sigma$ be random λ-bit prime numbers, and let $N = \prod_{i=1}^{\sigma} p_i$. The input to the problem is N, and a valid solution is a number K s.t.* $\gcd(N, K) \notin \{1, N\}$.

The σ-factoring hardness assumption is that no polynomial time algorithm can find a valid solution with non-negligible probability. We also consider a sub-exponential variant of the assumption, which states that there exists $\delta > 0$ such that any algorithm running in time 2^{λ^δ} cannot find a valid solution with non-negligible probability.

We note that σ-factoring, for any polynomial σ, is at least as hard as the classical problem of factoring a product of two primes.

2.1 Purely Arithmetic Circuits and Their Structure

Definition 2.2 (Purely arithmetic-circuit)**.** *A* purely arithmetic-circuit *A is a circuit which contains input gates (no fan-in and fan-out > 0), an output gate (fan-in 1, fan-out 0) and operator gates for addition ($+$), subtraction ($-$) and multiplication ($\times$) with fan-in 2 and fan-out > 0. The size of A is the number of gates in A.*

Zero-Testing an Arithmetic Circuit Modulo a Composite. As had been shown in previous works (e.g. [AB15]), a low degree arithmetic circuit can be zero-tested efficiently, so long as all of its factors are large enough. We re-state this property below.

Fact 2.3. *Let $\sigma \in \mathbb{N}$, let $p_1, \ldots, p_\sigma$ be distinct primes and let $N = \prod_{i=1}^{\sigma} p_i$. Then a multivariate polynomial of total degree d has at most d^σ roots over $\mathbb{Z}_P$.*

Moreover, we would like to define an algorithm that given an arithmetic-circuit $A(x_1, \ldots, x_n)$ decides whether or not it computes a multivariate polynomial that is identically zero in $\mathbb{Z}_N[x_1, \ldots, x_n]$. We denote $A \equiv 0$ if A computes the zero-polynomial.

Lemma 2.4. *There exists an PPT algorithm $\mathsf{SZTest}(1^\lambda, A, N)$ such that for all integer λ and every arithmetic-circuit A of total degree d. Then for $N > 2d$ we have:*

$$\Pr[\mathsf{SZTest}(1^\lambda, A, N) = 1 \mid A \equiv 0 \pmod{N}] = 1$$

and:

$$\Pr[\mathsf{SZTest}(1^\lambda, A, N) = 1 \mid A \not\equiv 0 \pmod{N}] \leq 2^{-\lambda}$$

Consider a low-degree multi-variate arithmetic circuit A over some ring with efficient zero-testing, and divide its input variables into two sets $\vec{x}, \vec{y}$. Then the polynomial computed by $A(\vec{x}, \vec{y})$ can be expressed as a sum of semi-monomials of the form $M(\vec{x})Q(\vec{y})$ where $M(\vec{x})$ is a product of variables from $\vec{x}$, and $\vec{y}$ is some polynomial. We denote by $Q_{A,M}(\vec{y})$ the coefficient of $M(\vec{x})$ in the polynomial computed by A (the $Q(\vec{y})$). We omit A from the subscript when the arithmetic-circuit we refer is clear from the context.

The following algorithm takes A as input and outputs A' such that A' is a circuit of size comparable to A, which computes a single term of the terms in A. It follows immediately that the circuit $A - A'$ now computes all of the terms of A except for the one in A'.

Lemma 2.5 (Semi-Monomial Extraction)**.** *Given an arithmetic-circuit $A(\vec{x}, \vec{y})$ as above, there exists a* PPT *algorithm that determines (with all but negligible probability) whether there exists a semi-monomial $M(\vec{x})Q(\vec{y})$ that does not zero-out, that is: $M(\vec{x})Q(\vec{y}) \not\equiv 0$, and outputs an arithmetic-circuit that computes such semi-monomial if exists.*

Proof. We note that we can treat every coordinate in $\vec{x}$ as a formal variable. We denote $\xi \in \vec{x}$ which means that ξ is the formal variable associated with one of the coordinates of $\vec{x}$.

Given A we consider the following algorithm:

1. Use SZTest on A.
2. If the test returned zero:
   (a) Output that A is equivalent to the zero polynomial.
3. For each of the formal variable $\xi \in \vec{x}$:
   (a) Construct an arithmetic circuit A' by hardwiring $\xi = 0$.
   (b) Use SZTest on A'.
   (c) If the test returned non-zero:
      i. Set $A = A'$.
4. Output A.

It is clear that the algorithm is polynomial time, we want to prove the correctness of this algorithm. If $A \equiv 0$, SZTest will return zero with probability 1. But if $A \not\equiv 0$ polynomial, only with negligible probability over the coins of the SZTest we shall get a false-positive result.

We now want to show that the algorithm will output an arithmetic-circuit that computes exactly one semi-monomial.

First we note that the algorithm cannot return an arithmetic-circuit that evaluate the zero-polynomial as the first check will fail and we'll output that A is equivalent to the zero polynomial. Hence if the result is "non-zero", there must be at least one monomial in the output arithmetic-circuit (unless A is equivalent to the zero-polynomial). Assume towards contradiction that the algorithm did result an arithmetic-circuit which equivalent to the zero-polynomial. Then, the test in line 1 had passed, hence A is not equivalent to the zero-polynomial. Since the resulted arithmetic-circuit is equivalent to the zero polynomial, it means that at some iteration of the loop, the polynomial evaluated by A became equivalent to the zero polynomial. But note that this cannot be, because we only replace A when A' is not equivalent to the zero-polynomial.

Now we want to show that with all but negligible probability the resulted arithmetic-circuit will calculate no more than one monomial. Assume towards contradiction that the polynomial evaluated by the resulted arithmetic-circuit contains two different monomials M_1, M_2. Since the monomials are different there exists some variable ξ such that ξ is in one of them but not the other. Without loss of generality assume that ξ is in M_1 but not M_2. Now, in the iteration where ξ was processed, we note that the with all-but-negligible probability SZTest returns that the monomial is non-zero. Hence A will be replaced with an hardwired $\xi = 0$, thus with all-but-negligible probability M_1 won't appear in the polynomial calculated by the resulted arithmetic-circuit. □

We now define the following algorithm that allows us to either check whether two polynomials are equivalent up to multiplication by a constant moduli some number N or it output a proper factor of N. This algorithm will be used in order to determine whether an adversary calculated a "valid" polynomial or not.

Lemma 2.6. *There exists an efficient algorithm* $\mathsf{Div} = \mathsf{Div}^{A,B}(N)$ *as follows. Let* $N \in \mathbb{N}$ *and let* $A(x), B(x)$ *be multivariate polynomials modulo* N*. Consider the polynomial* $C(x, y) = A(y)B(x) - B(y)A(x)$*.*

If $\Pr_x[A(x) = 0] < \epsilon$ *and* $C(x, y) \equiv 0 \pmod N$ *then* $\mathsf{Div}^{A,B}(N)$ *succeeds with probability* $(1 - \epsilon)$ *to either find a scalar* $\alpha \in \mathbb{N}$ *such that* $B(x) = \alpha \cdot A(x)$*, or to find a scalar* β *such that* $1 < \gcd(\beta, N) < N$ *(i.e. a non-trivial factor of* N*).*

Clearly, in the converse case, if there exists α such that $B(x) = \alpha \cdot A(x)$, then $C(x, y) \equiv 0 \pmod N$.

Proof. The algorithm Div samples a random y and computes (via oracle access) $\beta = A(y)$. If $\beta = 0$ then return $\perp$. Otherwise, if β is not a unit modulo N then return β. Otherwise compute $B(y)$ and return $\alpha = \beta^{-1} \cdot B(y)$. Correctness follows immediately. □

2.2 Graded Encoding over Composite Order Groups

This section provides definitions and notation for graded encoding schemes over compositve order groups. It is mostly adopted from [AB15].

2.2.1 General Notation

Partial Order of Natural Valued Vectors. For an integer $\tau \in \mathbb{N}$, we view vectors in $\mathbb{N}^\tau$ as multisets over the universe $[\tau]$. Correspondingly, we define a partial ordering on vectors $\mathbb{N}^\tau$ which corresponds to inclusion. In particular, we say that $\mathbf{v} \leq \mathbf{w}$ if for all $i \in [\tau]$ it holds that $\mathbf{v}[i] \leq \mathbf{w}[i]$. If there exists a coordinate i for which the above does not hold, we say that $\mathbf{v} \not\leq \mathbf{w}$. We note that since our vectors are defined over the naturals, this relation is monotonous: If $\mathbf{v} \leq \mathbf{w}$ then for all $\mathbf{w}' \in \mathbb{N}^\tau$ it also holds that $\mathbf{v} \leq (\mathbf{w} + \mathbf{w}')$, and dually if $\mathbf{v} \not\leq \mathbf{w}$ then for all $\mathbf{v}' \in \mathbb{N}^\tau$ it holds that $(\mathbf{v} + \mathbf{v}') \not\leq \mathbf{w}$.

CRT representation. Let $\sigma \in \mathbb{N}$, let $p_1, \ldots, p_\sigma$ be distinct coprime numbers and let $P = \prod_{i=1}^{\sigma} p_i$. Considering the ring $\mathbb{Z}_P$, the Chinese Remainder Theorem (CRT) asserts that there is an isomorphism $\mathbb{Z}_P \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_\sigma}$ such that if $a \cong (a_1, \ldots, a_\sigma)$ and $b \cong (b_1, \ldots, b_\sigma)$, then $a + b \cong (a_1 + b_1, \ldots, a_\sigma + b_\sigma)$ and $a \cdot b \cong (a_1 \cdot b_1, \ldots, a_\sigma \cdot b_\sigma)$. For a given isomorphism, we will denote by $a[\![i]\!]$ the component $a_i = a \pmod{p_i}$.

2.2.2 Syntax

We begin with the definition of a graded encoding scheme in composite order groups. The definition is adapted from [GGH13a] and follow-up works, but our notation deviates somewhat from that of some previous work.

Definition 2.7 (Graded Encoding Scheme)**.** *Let $\mathcal{R}$ be a ring, and let $\mathbf{v}_{zt} \in \mathbb{N}^\tau$ be an integer vector of dimension $\tau \in \mathbb{N}$. A graded encoding scheme for $\mathcal{R}, \mathbf{v}_{zt}$ is a collection of sets $\{[\alpha]_{\mathbf{v}} \subset \{0,1\}^* : \mathbf{v} \in \mathbb{N}^\tau, \mathbf{v} \leq \mathbf{v}_{zt}, \alpha \in \mathcal{R}\}$ with the following properties:*

1. *For every index $\mathbf{v} \leq \mathbf{v}_{zt}$, the sets $\{[\alpha]_{\mathbf{v}} : \alpha \in \mathcal{R}\}$ are disjoint, and so they are a partition of the indexed set $[\mathcal{R}]_{\mathbf{v}} = \bigcup_{\alpha \in \mathcal{R}} [\alpha]_{\mathbf{v}}$. We slightly abuse notation and often denote $a = [\alpha]_{\mathbf{v}}$ instead of $a \in [\alpha]_{\mathbf{v}}$.*

2. *There are binary operations "$+$" and "$-$" such that for all $\mathbf{v} \in \{0,1\}^\tau$, $\alpha_1, \alpha_2 \in \mathcal{R}$ and for all $u_1 = [\alpha_1]_{\mathbf{v}}$, $u_2 = [\alpha_2]_{\mathbf{v}}$:*

$$u_1 + u_2 = [\alpha_1 + \alpha_2]_{\mathbf{v}} \quad \text{and} \quad u_1 - u_2 = [\alpha_1 - \alpha_2]_{\mathbf{v}} \ ,$$

   where $\alpha_1 + \alpha_2$ and $\alpha_1 - \alpha_2$ are addition and subtraction in $\mathcal{R}$.

3. *There is an associative binary operation "$\times$" such that for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{N}^\tau$ such that $\mathbf{v}_1 + \mathbf{v}_2 \leq \mathbf{v}_{zt}$, for all $\alpha_1, \alpha_2 \in \mathcal{R}$ and for all $u_1 = [\alpha_1]_{\mathbf{v}_1}$, $u_2 = [\alpha_2]_{\mathbf{v}_2}$, it holds that*

$$u_1 \times u_2 = [\alpha_1 \cdot \alpha_2]_{\mathbf{v}_1 + \mathbf{v}_2},$$

   where $\alpha_1 \cdot \alpha_2$ is multiplication in $\mathcal{R}$.

The above definition does not touch upon the computational aspects of graded encoding schemes, which are described below. We note that there is a difference between the definition below and the definitions for the prime order definitions.

Definition 2.8 (Efficient Procedures for Graded Encoding Scheme)**.** *We consider a graded encoding schemes (see above) where the following procedures are efficiently computable.*

- *Composite-Order Instance Generation:* $\mathsf{InstGen}(1^\lambda, 1^\sigma, \mathbf{v}_{zt})$ *outputs the set of parameters params, a description of a Graded Encoding Scheme relative to $\mathbf{v}_{zt}$ and relative to a ring $\mathcal{R}$ such that $\mathcal{R} \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{\sigma}}$, where all p_i are pairwise coprime numbers, i.e. $\mathcal{R} \cong \mathbb{Z}_N$ for $N = \prod p_i$. We assume that each prime factor of $N > 2^\lambda$.*

  In addition, the procedure outputs a subset evparams $\subset$ params that is sufficient for computing addition, multiplication and zero testing, but may be insufficient for sampling, encoding or for randomization.

  We note that for known GES candidates, the running time of the setup procedure (and all other procedures) scales with $\|\mathbf{v}_{zt}\|_1$, and hence for such scheme this procedure may not run in polynomial time. It is conceivable that more efficient instantiations that do not require this additional input will be discovered in the future.

- *Ring Sampler:* $\mathsf{samp}(params)$ *outputs a "level zero encoding" $A \in [a]_{\mathbf{0}}$ for a nearly uniform $a \xleftarrow{\$} \mathcal{R}$.*

- *Sub-Ring Sampler:* $\mathsf{subsamp}(params, i^*)$*, where* $i^* \in [\sigma]$ *outputs a "level zero encoding" in a CRT sub-ring of* $\mathcal{R}$*. Namely, it outputs* $A \in [a]_{\mathbf{0}}$ *for an element* $a \cong (a_1, \ldots, a_\sigma)$*, such that* a_{i^*} *is nearly uniform in* p_{i^*}*, and for all* $i \neq i^*$ *it holds that* $a_i = 0$*. We stress it is very important for the security of our constructions that evparams does not enable such functionality.*

- *Encode and Re-Randomize:* $\mathsf{encRand}(params, \mathbf{v}, A)$ *takes as input an index* $\mathbf{v} \leq \mathbf{v}_{zt}$ *and* $A = [a]_{\mathbf{0}}$*, and outputs an encoding* $B = [a]_{\mathbf{v}}$*, where the distribution of* B *is (statistically close to being) only dependent on* a *and not otherwise dependent on* A*.*

- *Addition and Negation:* $\mathsf{add}(evparams, A_1, A_2)$ *takes* $A_1 = [a_1]_{\mathbf{v}}, A_2 = [a_2]_{\mathbf{v}}$*, and outputs* $B = [a_1 + a_2]_{\mathbf{v}}$*. (If the two operands are not in the same indexed set, then* add *returns* $\perp$*). We often use the notation* $u_1 + u_2$ *to denote this operation when evparams is clear from the context. Similarly,* $\mathsf{negate}(evparams, A_1) = [-a_1]_{\mathbf{v}}$*.*

- *Multiplication:* $\mathsf{mult}(evparams, A_1, A_2)$ *takes* $A_1 = [a_1]_{\mathbf{v}_1}, A_2 = [a_2]_{\mathbf{v}_2}$*. If* $\mathbf{v}_1 + \mathbf{v}_2 \leq \mathbf{v}_{zt}$*, then* mult *outputs* $B = [a_1 \cdot a_2]_{\mathbf{v}_1 + \mathbf{v}_2}$*. Otherwise,* mult *outputs* $\perp$*. We often use the notation* $A_1 \times A_2$ *to denote this operation when evparams is clear from the context.*

- *Zero Test:* $\mathsf{isZero}(evparams, A)$ *outputs 1 if* $A = [0]_{\mathbf{v}_{zt}}$*, and 0 otherwise.*

Noisy encodings. In known candidate constructions, encodings are *noisy* and the noise level increases with addition and multiplication operations, so one has to be careful not to go over a specified noise bound. However, the parameters can be set so as to support $O(\|\mathbf{v}_{\mathrm{zt}}\|_1)$ operations, so long as InstGen is allowed to run in $poly(\|\mathbf{v}_{\mathrm{zt}}\|_1)$ time, as our function interface compels. This will be sufficient for our purposes and we therefore ignore noise management throughout this manuscript.

Remark 2.9. *Given params, we can use* subsamp *to efficiently generate level-0 encodings of related elements, so long as each of their CRT components can be expressed as a polynomial size arithmetic circuit applied to a set of uniformly distributed variables. These variables may not be shared across CRT components, but they can be shared between elements. E.g. in a 2-composite GES, one can generate* $[((a_1 + a_2) \cdot a_3, b_1)]_{\mathbf{0}}$*,* $[(a_3 + a_4, b_2)]_{\mathbf{0}}$*,* $[(a_1 \cdot a_2, b_1 + b_2)]_{\mathbf{0}}$ *(but cannot generate in addition* $[(b_1, a_1)]_{\mathbf{0}}$*. (Note that the product of level zero-encoding results in a level zero encoding.) Combining the above with access to* encRand *allows, given params to encode the aforementioned elements to arbitrary indices* $\mathbf{v} \leq \mathbf{v}_{zt}$*.*

Remark 2.10. *For our application we require that it is intractable to execute* subsamp *using only evparams and without access to params. Our application involves an adversary that is given a set of encodings and evparams. If the adversary is able to perform sub-ring sampling or to modify the level of an encoded element, then our obfuscator will be insecure.*

Concrete instantiations. The candidate constructions of [GGH13a, CLT13] do not support the above functionality out of the box. Specifically, [GGH13a] only allows $\mathcal{R}$ of prime order, whereas [CLT13] does natively support composite order groups, but its security features are unclear if sub-ring sampling is allowed. This issue has been extensively addressed in [GLW14, Appendix B of full version]. In particular the authors there present a variant of [CLT13] that appears to overcome the aforementioned security issues. This variant supports a σ-product ring $\mathcal{R} \cong (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_\sigma})$ where the p_i's are composite numbers with large prime divisors. Note that this is compatible with

our requirements which allow the p_i's to be non-primes. Furthermore, this variant adheres to the constraints we need to impose as per Remark 2.10. Overall, to the best of our knowledge, this candidate is consistent with the requirements of our obfuscator (although we prove security only in a generic model and not under explicit assumptions).

3 The Generic GES Model and Our New Zero-Sensitive Variant

We would like to prove the security of our construction against *generic adversaries*. To this end, we will use the *generic graded encoding scheme* model, adapted from [BR13, BR14a, BR14b, BGK+14], which is analogous to the *generic group model* (see Shoup [Sho97] and Maurer [Mau05]).

There are various flavors of generic models suggested in the literature. In this work, we follow [BR13, AB15] and use the *unique-representation* model, where each element in the underlying ring, at each level, has a unique representation. This is in contrast to the *multiple-representation* model [BGK+14,Zim15,AB15] which (roughly) states that if the same element is being computed via different computational paths, then each path will lead to a different and independent representation of that element. While the latter model makes the task of proving security easier, it is inadequate in some situations, as we described in the introduction. We note that a proof in the unique representation model immediately carries over to the multiple representation model, but not the other way around. We provide a definition of this model in Section 3.1 below.

We then introduce our zero-sensitive model. This model is motivated by recent attacks that leverage non-trivial encodings of zero. In this model we treat a non-trivial encoding of zero at any level as perilous. In particular, once such an encoding had been generated, the GES oracle will no longer keep any secret, and surrender the plaintexts of all encodings to the adversary. As we explained above, we can prove security of all-or-nothing functions in this model. See Section 3.2 for details.

Lastly, in Section 3.3, we define indistiguishability and virtual black-box obfuscation in the presence of our oracles.

3.1 The Ideal GES Oracle

We present the "online" variant of the unique representation model. As shown in previous works, this variant is equivalent to the "offline" variant up to negligible statistical distance. See [BR13, AB15] for more details. We model the GES using an oracle $\mathcal{RG}$ which implements the functionality of a GES in which the representations of elements are uniform and independent random strings.

The Online $\mathcal{RG}$ Oracle. The online $\mathcal{RG}$ oracle is implemented by an *online polynomial time process*, which samples representations for ring elements on-the-fly. Specifically, the oracle will maintain a table of entries of the form $(\mathbf{v}, a, \mathsf{label}_{\mathbf{v},a})$, where $\mathsf{label}_{\mathbf{v},a} \in \{0,1\}^t$ is the representation of $[a]_{\mathbf{v}}$ in $\mathcal{RG}$, and F is either a formal variable or an arithmetic circuit over formal variables. The table is initially empty and is filled as described below.

- Whenever a sampling query is made, $\mathcal{RG}$ generates an element a from $\mathcal{R}$ (or the appropriate sub-ring), and a uniform length t label. It then stores the tuple $(\mathbf{0}, a, \mathsf{label}_{\mathbf{0},a})$ in its table.
- For encoding and arithmetic operations, the oracle takes the input labels and finds appropriate entries in the table that correspond to these labels. If such don't exist then $\perp$ is returned.

Otherwise, the oracle retrieves the appropriate $(\mathbf{v}, a)$ values to perform the operation. It then checks that the level values are appropriate (e.g. $\mathsf{encRand}$ can only be applied to level zero encodings, addition can only take two operands of the same level), and computes the output of the operation. It then performs the computation on the ring elements. Finally, the oracle needs to return an encoding of an element of the form $(\mathbf{v}', a')$. To do this, the oracle checks whether $(\mathbf{v}', a')$ is already in the table, and if so returns the appropriate $\mathsf{label}_{\mathbf{v}',a'}$. Otherwise it samples a new uniform label, and inserts a new entry into the table.. Otherwise it samples a new uniform label, and inserts a new entry into the table.

- Extraction is trivial in our representation, one can just use $\mathsf{label}_{\mathbf{v}',a'}$ as the extracted value for $[a]_{\mathbf{v}}$.
- Zero testing is performed by finding the appropriate entry in the table and checking whether the respective ring element is indeed 0.

3.2 The Zero-Sensitive Generic Model

We propose a new generic model that incorporates the zero-evading requirement of [BMSZ15] into the generic GES model. Whereas our oracle is a modification of the unique representation generic model presented above, similar modifications can be made to other generic models in the literature.

We propose a generic model with an additional *decoding* functionality which will allow the adversary to retrieve the plaintext of any encoding of its choosing, once an encoding of zero had been generated. Some care needs to be taken, since it is easy to produce "syntactic zeros" which are harmless. E.g. subtracting an encoding from itself will produce such a zero encoding, or less trivially, computing an expression of the form $(A + B) * C - (C * A + C * B)$. These expressions will evaluate to zero regardless of the values that are actually encoded in A, B, C and we refer to them as "trivial" or "syntactic" zeros. Such encodings of zero are unavoidable, but they are not dangerous. (Indeed, in known instantiations of GES [GGH13a, CLT13, CLT15], syntactic zeros are always encoded by the all-zero string and thus provide no meaningful information.) We design an oracle that whenever a non-syntactic zero is created (or rather, when it could potentially be created), enables the decoding feature.

We consider the encodings that are generated by the $\mathsf{encRand}$ function as atomic variables, and for every encoding generated by the adversary throughout the computation, we maintain its representation as an algebraic circuit over these variables. Whenever we discover that two syntactically different such arithmetic circuits evaluate to the same value, we enable the encoding feature. Details follow.

The $\mathcal{RG}_{\mathcal{Z}}$ Oracle. The new oracle is based on the functionality of the oracle $\mathcal{RG}$ defined in Section 3.1. It will maintain a table similarly to $\mathcal{RG}$, but in addition each entry in the table will contain an additional value in the form of an arithmetic circuit over the formal variables $X_1, X_2, \ldots$. Elements encoded at level $\mathbf{0}$ will not have a circuit associated with them, but whenever $\mathsf{encRand}$ is executed, the resulting element will be stored in the table together with a new variable X_i. It will also maintain a global binary state decode which is initialized to false.

When the arithmetic functionality of $\mathcal{RG}_{\mathcal{Z}}$ is called, say on operands A_1, A_2 whose table entries are $(\mathbf{v}_1, a_1, A_1, C_1)$, $(\mathbf{v}_1, a_2, A_2, C_2)$, it performs exactly as $\mathcal{RG}$ and computes the values $(\mathbf{v}', a')$ corresponding to the level and value of the result. In addition $\mathcal{RG}_{\mathcal{Z}}$ also defines $C' = C_1 \mathsf{op} C_2$,

where op is the arithmetic operation to be performed (e.g. $C' = C_1 + C_2$ or $C' = C_1 \times C_2$). Then, just like in $\mathcal{RG}$, we search the table to find whether $(\mathbf{v}', a')$ already appears. If it does not, then a new label A' is generated, $(\mathbf{v}', a', A', C')$ is stored in the table, and A' is returned. However, if there already exists $(\mathbf{v}', a', A'', C'')$ in the table, then there is potential for a non-trivial zero in the case where $C' \not\equiv C''$. This equivalence is easy to check (even in polynomial time using Schwartz-Zippel). If the circuits are equivalent: $C' \equiv C''$, then there is no risk, the table entry does not change and A'' is returned. However, if indeed $C' \not\equiv C''$, then the adversary can create a non-trivial zero (since he generated the element a' in two syntactically different ways). Therefore, in this event, $\mathcal{RG}_{\mathcal{Z}}$ sets $\mathsf{decode} = \mathsf{true}$.

As explained above, $\mathcal{RG}_{\mathcal{Z}}$ also provides an additional decoding functionality: $\mathsf{Decode}(A)$. This function, upon receiving an encoding A as input, first checks the decode variable. If $\mathsf{decode} = \mathsf{false}$ then it returns $\perp$. Otherwise, it searches the table for and entry whose label is A, and returns the corresponding "plaintext" value a.

3.2.1 Non-Trivial Zeros in the Unique Representation Model.

Our zero evading model has unique representations, in the sense that the oracle assigns a single string to each ring element. This state of affairs may be confusing, since if there is only one representation for each element (in particular, the zero element), it may seem that the distinction between trivial and non-trivial zeros is meaningless. While this intuition is true in the standard model, in a generic model the $\mathcal{RG}$ oracle can judge whether an encoding of zero is trivial or not even though they are represented by the same string, since it can keep track of the path the adversary took in generating said encoding. In fact, security in our model is *stronger* than in a model that allows multiple representations. Details follow.

We note that unique representation GES (call it uGES for short) is effectively equivalent to multiple representations GES (mGES) in which zero-testing can be performed anywhere below level $\mathbf{v}_{\mathrm{zt}}$ and not just at $\mathbf{v}_{\mathrm{zt}}$ itself. This is because the adversary can always think about the first representation of a specific element as the "real" one. Whenever it sees a new encoding, it can subtract it from all previous ones that it saw in the same level, and test for zero, thus discovering if two different encodings in fact refer to the same element. Therefore, by using uGES we only give the adversary extra power. Another advantage of using uGES is that the extraction procedure becomes trivially defined and does not need additional machinery. One can thus think of our use of uGES as a formalism that allows us to seamlessly handle cases such as mGES with low-level zero-testing (and extraction).

3.3 Obfuscation in the Generic GES Model

These definitions are fairly standard and originate from [BR13]. We start with correctness, which should hold with respect to an arbitrary GES implementation.

Definition 3.1 (Preserving Functionality)**.** *A GES-based obfuscation scheme* $(\mathsf{Obf}, \mathsf{Eval})$ *for* $\mathcal{C}$ *is* functionality preserving *if for every instantiation* $\mathcal{G}$ *of GES, every* $n \in \mathbb{N}$*, every* $C_K \in \mathcal{C}$ *where* $K \in \{0,1\}^{m(n)}$*, and every* $x \in \{0,1\}^n$*, with all but* $\mathsf{negl}(\lambda)$ *probability over the coins of* $\mathsf{Obf}, \mathsf{Eval}$ *and the GES oracle* $\mathcal{G}$ *it holds that:*

$$\mathsf{Eval}^{\mathcal{G}}(1^n, 1^\lambda, \hat{C}, x) = C_K(x), \qquad \text{where } \hat{C} \xleftarrow{\$} \mathsf{Obf}^{\mathcal{G}}(1^n, 1^\lambda, K).$$

We define Indistinguishability Obfuscator with respect to some (possibly inefficient) GES instantiation. Our definition is formulated in terms of unbounded simulation which is equivalent to the more standard indistinguishability-based definition (cf. [BR14b]).

Definition 3.2 (Indistinguishability/VBB Security [BGI+12])**.** *A GES-based obfuscation scheme* $(\mathsf{Obf}, \mathsf{Eval})$ *for* $\mathcal{C}$ *is called an* Indistinguishability Obfuscator *(iO) with respect to some GES instantiation* $\mathcal{G}$ *(which possibly contains a decode function) if for every polynomial size adversary* $\mathcal{A}$*, there exists a (computationally unbounded) simulator* $\mathcal{S}$*, such that for every* $n \in \mathbb{N}$ *and* for every $C_K \in \mathcal{C}$ *where* $K \in \{0,1\}^{m(n)}$*:*

$$\left| \Pr[\mathcal{A}^{\mathcal{G}}(1^\lambda, \hat{C}) = 1] - \Pr[\mathcal{S}^{C_K}(1^{|K|}, 1^n, 1^\lambda) = 1] \right| = \mathsf{negl}(\lambda),$$

where $\hat{C} \xleftarrow{\$} \mathsf{Obf}^{\mathcal{G}}(1^n, 1^\lambda, K)$*. If the simulator can be implemented by polynomial size circuits than the obfuscator is* Virtually Black-Box *(VBB) secure.*

4 Description of Our Obfuscator and Its Correctness

4.1 Setting and Definitions

We define $\mathcal{C} = \{\mathcal{C}_K\}_{K \in \{0,1\}^*}$ to be a family of efficiently computable functions with n-bit inputs, representation size $m = m(n)$ and universal evaluator $\mathcal{U}$. And we let $\hat{\mathcal{U}}$ be the arithmetized version of $\mathcal{U}$. That is, an arithmetic circuits with $\{+, \times\}$ gates such that for any field $\mathbb{F}$ if $(x, K) \in \{0,1\}^{n+m} \subseteq \mathbb{F}^{n+m}$, then $\hat{\mathcal{U}}(x, K) = C_K(x)$. We also denote by $D_{\hat{\mathcal{U}}}$ the degree of the polynomial computed by $\hat{\mathcal{U}}$.

We define the *multiplicity* of input wire i as follows. We consider an enumeration of the wires of $\hat{\mathcal{U}}$ in topological order, such that the first $n + m$ wires refer to the wires of the x, C inputs. For each wire i we define a vector $\mathbf{s}_i \in \mathbb{Z}^{n+m}$ as follows. If $i \leq n + m$, then $\mathbf{s}_i = \mathbf{e}_i$ (the ith indicator vector). For a wire i which is the output wire of a gate whose input wires are j_1, j_2, we define $\mathbf{s}_i = \mathbf{s}_{j_1} + \mathbf{s}_{j_2}$. The *multiplicity* is defined to be $M_i = \mathbf{s}_{\text{out}}[i]$, where "out" is the output wire of $\hat{\mathcal{U}}$.

4.2 The Obfuscator Obf

For all $i \in [n]$, $b \in \{0,1\}$ we define $\mathbf{v}_{i,b} \in \mathbb{Z}^{(n+m+1)\times 4}$ as $\mathbf{v}_{i,b} = \mathbf{e}_i \otimes [b, 1, 1-b, 0]$. We further define $\hat{\mathbf{v}}_{i,b} = \mathbf{e}_i \otimes [(1-b) \cdot M[i], 0, b \cdot M[i], 1]$.

For all $i \in \{n+1, \ldots, n+m\}$ we define $\mathbf{v}_i = e_i \otimes [1, 1, 1, 0]$. We define $\mathbf{v}_0 = \mathbf{e}_{n+m+1} \otimes [1, 1, 1, 0]$ and $\mathbf{v}^* = \mathbf{e}_{n+m+1} \otimes [0, 0, 0, 1]$. Lastly, we define: $\mathbf{v}_{\text{zt}} = (\mathbf{s}_{\text{out}} + \mathbf{e}_{n+m+1}) \otimes [1, 1, 1, 0] + \left(\sum_{i=1}^{n+m} \mathbf{e}_i\right) \otimes [0, 0, 0, 1] + D \cdot \mathbf{v}^* \in \mathbb{Z}^{(n+m+1)\times 4}$, where $D = D_{\hat{\mathcal{U}}} + n$. We note that for all $x \in \{0,1\}^n$ it holds that $\mathbf{v}_{\text{zt}} = \mathbf{v}_0 + \sum_{i=1}^{n} (M[i] \cdot \mathbf{v}_{i,x_i} + \hat{\mathbf{v}}_{i,x_i}) + \sum_{i=n+1}^{n+m} M[i] \cdot \mathbf{v}_i + D \cdot \mathbf{v}^*$. We illustrate the various level vectors in Figure 1.

$$\mathbf{v}_{i,0} = \left[\begin{array}{ccc|c} 0\cdots & 0 & \cdots 0 & 0 \\ 0\cdots & 1 & \cdots 0 & 0 \\ 0\cdots & 1 & \cdots 0 & 0 \\ \hline 0\cdots & 0 & \cdots 0 & 0 \end{array}\right], \quad \mathbf{v}_{i,1} = \left[\begin{array}{ccc|c} 0\cdots & 1 & \cdots 0 & 0 \\ 0\cdots & 1 & \cdots 0 & 0 \\ \hline 0\cdots & 0 & \cdots 0 & 0 \\ \hline 0\cdots & 0 & \cdots 0 & 0 \end{array}\right], \quad \mathbf{v}_{i} = \left[\begin{array}{ccc|c} 0\cdots & 1 & \cdots 0 & 0 \\ 0\cdots & 1 & \cdots 0 & 0 \\ \hline 0\cdots & 0 & \cdots 0 & 0 \\ \hline 0\cdots & 0 & \cdots 0 & 0 \end{array}\right]$$

$$\hat{\mathbf{v}}_{i,0} = \left[\begin{array}{ccc|c} 0\cdots & M[i] & \cdots 0 & 0 \\ 0\cdots & 0 & \cdots 0 & 0 \\ 0\cdots & 0 & \cdots 0 & 0 \\ \hline 0\cdots & 1 & \cdots 0 & 0 \end{array}\right], \quad \hat{\mathbf{v}}_{i,1} = \left[\begin{array}{ccc|c} 0\cdots & 0 & \cdots 0 & 0 \\ 0\cdots & 0 & \cdots 0 & 0 \\ 0\cdots & M[i] & \cdots 0 & 0 \\ \hline 0\cdots & 1 & \cdots 0 & 0 \end{array}\right]$$

$$\mathbf{v}_0 = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right], \quad \mathbf{v}^* = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right]$$

$$\mathbf{v}_{\mathrm{zt}} = \left[\begin{array}{ccc|ccc|c} M[1] & \cdots & M[n] & M[n+1] & \cdots & M[n+m] & 1 \\ M[1] & \cdots & M[n] & M[n+1] & \cdots & M[n+m] & 1 \\ M[1] & \cdots & M[n] & M[n+1] & \cdots & M[n+m] & 1 \\ \hline 1 & \cdots & 1 & 0 & \cdots & 0 & D \end{array}\right]$$

Figure 1: The level vectors for the obfuscator.

The Obfuscator Obf**:**

- **Input:** Circuit identifier $K \in \{0,1\}^m$ where $C_K \in \mathcal{C}$ and a security parameter λ.
- **Output:** Obfuscated program with the same functionality as C_K.
- **Algorithm:**

  1. Instantiate a 3-composite graded encoding scheme
  $$(params, evparams) = \mathrm{InstGen}(1^{\lambda + \log \|\mathbf{v}_{\mathrm{zt}}\|_1}, 1^3, \mathbf{v}_{\mathrm{zt}}).$$
  2. For all $i \in [n]$, compute random encodings $R_{i,b} = [r_{i,b}]_{\mathbf{v}_{i,b}}$ as well as encodings of $Z_{i,b} = [r_{i,b} \cdot w_{i,b}]_{\mathbf{v}_{i,b}+\mathbf{v}^*}$, where $w_{i,b} = (y_i, b, \rho_{i,b})$ and $y_i, \rho_{i,b}$ are uniform.
  3. For all $i \in [i]$, compute random encodings: $\hat{R}_{i,b} = [\hat{r}_{i,b}]_{\hat{\mathbf{v}}_{i,b}}$ as well as encodings of $\hat{Z}_{i,b} = [\hat{r}_{i,b} \cdot \hat{w}_{i,b}]_{\hat{\mathbf{v}}_{i,b}+\mathbf{v}^*}$, where $\hat{w}_i = (\hat{y}_i, \hat{\beta}_i, \hat{\rho}_i)$, where $\hat{y}_i, \hat{\beta}_i, \{\hat{\rho}_i\}_{i \neq n}$ are all uniform but $\hat{\rho}_n = -\sum_{i=1}^{n-1} \hat{\rho}_i$.
  4. For all $i \in \{n+1, \ldots, n+m\}$, compute random encodings $R_i = [r_i]_{\mathbf{v}_i}$ as well as encodings of $Z_i = [r_i \cdot w_i]_{\mathbf{v}_i+\mathbf{v}^*}$, where $w_i = (y_i, K_{i-n}, \rho_i)$, where K_i is the ith bit of the circuit description and y_j, ρ_i are uniform.
  5. Compute random encoding $R_0 = [r_0]_{\mathbf{v}_0}$ and $Z_0 = [r_0 \cdot w_0]_{\mathbf{v}_0 + D\mathbf{v}^*}$, where $w_0 = \left(\sum_{i \in [n]} \hat{w}_i\right) \cdot (y_0, 1, 0)$ and $y_0 = \hat{\mathcal{U}}(y_1, \ldots, y_{n+m})$.

6. The obfuscated program will contain the following:
   - The evaluation parameters $evparams$.
   - For all $i \in [n], b \in \{0,1\}$ the elements $R_{i,b}, Z_{i,b}, \hat{R}_{i,b}, \hat{Z}_{i,b}$.
   - For all $i \in \{n+1, \ldots, n+m\}$ the elements R_i, Z_i.
   - The elements R_0, Z_0.

We denote by $\mathcal{D}_\lambda(n, K)$ the distribution over the encoded ring elements the obfuscator outputs according to the construction.

4.3 Evaluating an Obfuscated Program

We will now describe the evaluation procedure of our obfuscator. As can be seen in the description of our obfuscator, we encode the obfuscated circuit in the w variables, where each variable is encoded relative to an r variable. [AB15] showed that these pair of encodings, the r and the $r \cdot w$ can be manipulated algebraically while keeping the invariant that each value is encoded relative to an r. Which can be achieved by using the following procedure:

Procedure PairOp:

- **Input:** GES evaluation parameters $evparams$, pairs of encodings $\left(R_1 = [r_1]_{\mathbf{v}_1}, Z_1 = [r_1 w_1]_{\mathbf{v}_1 + k\mathbf{v}^*}\right)$, $\left(R_2 = [r_2]_{\mathbf{v}_2}, Z_2 = [r_2 w_2]_{\mathbf{v}_2 + k\mathbf{v}^*}\right)$ and an operation $\mathrm{op} \in \{\times, +, -\}$.
- **Output:** Pair of encodings $\left(R^* = [r_1 r_2]_{\mathbf{v}_1+\mathbf{v}_2}, Z^* = [r_1 r_2 (w_1 \,\mathrm{op}\, w_2)]_{\mathbf{v}_1+\mathbf{v}_2+tk\cdot\mathbf{v}^*}\right)$, where $t = 1$ for $\mathrm{op} \in \{+,-\}$ and $t = 2$ for $\mathrm{op} \in \{\times\}$. If $(\mathbf{v}_1 + \mathbf{v}_2 + tk \cdot \mathbf{v}^*) > \mathbf{v}_{\mathrm{zt}}$, the procedure outputs $\perp$.
- **Algorithm:**
  1. Compute $R^* = R_1 \times R_2$
  2. If $\mathrm{op} = \times$ compute $Z^* = Z_1 \times Z_2$.
  3. If $\mathrm{op} = +$ compute $Z^* = Z_1 \times R_2 + R_1 \times Z_2$.
  4. If $\mathrm{op} = -$ compute $Z^* = Z_1 \times R_2 - R_1 \times Z_2$.

We note that using PairOp iteratively we can evaluate any arithmetic circuit on pairs of encodings and that the multiplicity of $\mathbf{v}^*$ will be exactly the multiplicative degree of the evaluated circuit.

Now, we shall describe the evaluation procedure of our obfuscator.

Procedure Eval:

- **Input:** Obfuscated program as produced by our obfuscator for some identifier K:

$$\mathcal{O} = \left(evparams, \left\{R_{i,b}, Z_{i,b}, \hat{R}_{i,b}, \hat{Z}_{i,b}\right\}_{i\in[n],\, n\in\{0,1\}}, \{R_i, Z_i\}_{i=n+1}^{n+m}, \{R_0, Z_0\}\right),$$

  input $x \in \{0,1\}^n$.
- **Output:** Value $\mathcal{O}(x) \in \{0,1\}$.

- **Algorithm:**

1. We consider the pairs of elements (R_{i,x_i}, Z_{i,x_i}) for $i \in [n]$, and R_i, Z_i for $i = n+1, \ldots, n+m$. We apply the circuit $\hat{\mathcal{U}}$ on these pairs of encodings as described above, to obtain a pair:
$$R_{\mathcal{U}} = [r_{\mathcal{U}}]_{\mathbf{v}_{\mathcal{U}}}\,,\ Z_{\mathcal{U}} = [r_{\mathcal{U}} \cdot w_{\mathcal{U}}]_{\mathbf{v}_{\mathcal{U}}+D_{\hat{\mathcal{U}}}}\,,$$
where $\mathbf{v}_{\mathcal{U}} = \sum_{i=1}^{n} M[i] \cdot \mathbf{v}_{i,x_i} + \sum_{i=n+1}^{n+m} M[i] \cdot \mathbf{v}_i$ and
$$\begin{aligned} w_{\mathcal{U}} &= \hat{\mathcal{U}}(w_{1,x_1}, \ldots, w_{n,x_n}, w_{n+1}, \ldots, w_{n+m}) \\ &= \left(\hat{\mathcal{U}}(y_1, \ldots, y_n, y_{n+1}, \ldots, y_{n+m}), \hat{\mathcal{U}}(x, K), \hat{\mathcal{U}}(\rho_{1,x_i}, \ldots, \rho_{n,x_n}, \rho_{n+1}, \ldots, \rho_{n+m})\right) \\ &= \left(\hat{\mathcal{U}}(\vec{y}), C_K(x), \hat{\mathcal{U}}(\vec{\rho_x})\right). \end{aligned}$$

2. We take the product of the pair of elements $(R_{\mathcal{U}}, Z_{\mathcal{U}})$ with the sum of the pairs $\left(\hat{R}_{i,x_i}, \hat{Z}_{i,x_i}\right)$ to obtain
$$\hat{R}_{\mathcal{U}} = [\hat{r}_{\mathcal{U}}]_{\hat{\mathbf{v}}_{\mathcal{U}}}\,,\ \hat{Z}_{\mathcal{U}} = [\hat{r}_{\mathcal{U}} \cdot \hat{w}_{\mathcal{U}}]_{\hat{\mathbf{v}}_{\mathcal{U}}+D\mathbf{v}^*}\,,$$
where $\hat{w}_{\mathcal{U}} = (\sum_{i=1}^{n} \hat{w}_{i,x_i}) \cdot w_{\mathcal{U}}$, and
$$\hat{\mathbf{v}}_{\mathcal{U}} = \sum_{i=1}^{n} M[i] \cdot \mathbf{v}_{i,x_i} + \sum_{i=n+1}^{n+m} M[i]\,\mathbf{v}_i + \sum_{i=1}^{n} \hat{\mathbf{v}}_{i,x_i} = \mathbf{v}_{\mathrm{zt}} - \mathbf{v}_0$$

3. We subtract the pair $\left(\hat{R}_{\mathcal{U}}, \hat{Z}_{\mathcal{U}}\right)$ from the pair (R_0, Z_0), to obtain:
$$R'' = \left[r''\right]_{\hat{\mathbf{v}}_{\mathcal{U}}+\mathbf{v}_0}\,,\ Z'' = \left[r'' \cdot w''\right]_{\hat{\mathbf{v}}+D\mathbf{v}^*+\mathbf{v}_0}\,,$$
and we notice that indeed $(\hat{\mathbf{v}}_U + D\mathbf{v}^* + \mathbf{v}_0) = \mathbf{v}_{\mathrm{zt}}$ and
$$w'' = w_0 - \left(\sum_{i=1}^{n} \hat{w}_{i,x_i}\right) \cdot \left(\hat{\mathcal{U}}(\vec{y}), C_K(x), \hat{\mathcal{U}}(\vec{\rho_x})\right) = \left(\sum_{i=1}^{n} \hat{w}_{i,x_i}\right) \cdot \left(\hat{\mathcal{U}}(\vec{y}) - \hat{\mathcal{U}}(\vec{y}), 1 - C_K(x), \hat{\mathcal{U}}(\vec{\rho_x})\right)$$
Recalling that $\sum_{i=1}^{n} \hat{\rho}_i = 0$ and thus $\sum_{i=1}^{n} \hat{w}_{i,x_i} = (\alpha, \beta, 0)$, for some α, β, we have that:
$$w'' = (0, \beta\,(1 - C_K(x)), 0)\,.$$

4. Finally, zero testing is applied to Z''. If $\mathsf{isZero}(Z'') = 1$ then output 1, otherwise output 0.

Lemma 4.1 (Correctness)**.** *Considering w'' as above above. Then if $C_K(x) = 0$ then $\Pr_{\mathcal{D}_\lambda(n,K)}[w'' = 0] = negl(\lambda)$, and if $C_K(x) = 1$ then $\Pr_{\mathcal{D}_\lambda(n,K)}[w'' = 0] = 1$.*

Proof. As seen above, $\Pr_{\mathcal{D}_\lambda(n,K)}[w''[\![1]\!] = 0] = 1$ and $\Pr_{\mathcal{D}_\lambda(n,K)}[w''[\![3]\!] = 0] = 1$. It remains to examine the value of $w''[\![2]\!]$. Note that:
$$\begin{aligned} \Pr_{\mathcal{D}_\lambda(n,K)}\left[w'' = 0\right] &= \Pr_{\mathcal{D}_\lambda(n,K)}\left[\left(\sum_{i\in[n]} \hat{\beta}_i\right) \cdot \left(1 - \hat{\mathcal{U}}(x_1, \ldots, x_n, K_1, \ldots, K_m)\right) = 0\right] \\ &= \Pr_{\mathcal{D}_\lambda(n,K)}\left[\left(\sum_{i\in[n]} \hat{\beta}_i\right) \cdot (1 - C_K(x)) = 0\right] \end{aligned}$$

Now, if $C_K(x) = 0$ then the probability is equal to: $\Pr_{\mathcal{D}_\lambda(n,K)}\left[\sum_{i\in[n]} \hat{\beta}_i = 0\right] = \mathsf{negl}(\lambda)$, whereas if $C_K(x) = 1$ then it is obvious that the probability is: $\Pr_{\mathcal{D}_\lambda(n,K)}[0 = 0] = 1$. □

5 Security

This section contains the security proofs for Obf in the standard $\mathcal{RG}$ model (Section 5.2) and for all-or-nothing functions (defined in Section 5.3) in the zero-sensitive $\mathcal{RG}_\mathcal{Z}$ model (Section 5.4). In Section 5.1 below, we present the notion of compatible arithmetic circuits, which will be instrumental in our proofs, and present some useful properties of such circuits.

5.1 Compatible Arithmetic Circuits

We present some tools that will be useful in proving security in the $\mathcal{RG}$ and in the $\mathcal{RG}_\mathcal{Z}$ model. We recall that in both cases, the adversary gets as input a sequence of encodings, and it can perform a sequence of arithmetic operations that respect the levels of the GES. Following [AB15], we say that such an obfuscator is in "canonical form" and further that it defines a class of arithmetic circuits on the encodings that are computable by an adversary.

Definition 5.1 (Obfuscator in Canonical form)**.** *Let λ be the security parameter, $\mathcal{R}_{\lambda,\sigma,\mathbf{v}_{zt}}$ be some ensemble of probability distributions over rings. An obfuscator for the class $\mathcal{C} = \{\mathcal{C}_K\}_{K\in\{0,1\}^*}$ of functions over $\{0,1\}^n$ is in* canonical form *if it can be presented as follows:*

1. *Based on n, the obfuscator deterministically generates $\ell = \ell(n)$ integer-valued vectors $\mathbf{v}_1, \ldots, \mathbf{v}_\ell$, a zero-testing vector $\mathbf{v}_{zt}$ and a ring arity $\sigma \in \mathbb{N}$.*

2. *Based on λ, K, n, the obfuscator defines a joint distribution $\mathcal{D}_\lambda(n, K)$ over ℓ (generic) ring elements $(a_1, \ldots, a_\ell)$.*

3. *Then, the obfuscator initializes the GES which samples $\mathcal{R} \xleftarrow{R} \mathcal{R}_{\lambda,\sigma,\mathbf{v}_{zt}}$ the obfuscator samples the tuple $(a_1, \ldots, a_\ell)$ from $\mathcal{R}$ according to the distribution $\mathcal{D}_\lambda(n, K)$, and outputs the vector of encodings $([a_1]_{\mathbf{v}_1}, \ldots, [a_1]_{\mathbf{v}_\ell})$ together with the evaluation parameters evparams.*

Overall, such a canonical obfuscator can be defined by the length function $\ell = \ell(n)$, the ring arity $\sigma(n)$, the vectors $V_n = (\mathbf{v}_1, \ldots, \mathbf{v}_\ell, \mathbf{v}_{zt})$, the distribution $\mathcal{D}_\lambda(n, K)$, and the ring distribution $\mathcal{R}_{\lambda,\sigma,\mathbf{v}_{zt}}$.

Intuitively, an adversary that tries to break the obfuscated program gets as input a sequence of encodings $([a_1]_{\mathbf{v}_1}, \ldots, [a_1]_{\mathbf{v}_\ell})$ representing the obfuscated program, and has access to arithmetic functionality using the oracle. It can thus evaluate arithmetic circuits over the ring elements $(a_1, \ldots, a_\ell)$. As the definitions of the oracles $\mathcal{RG}, \mathcal{RG}_\mathcal{Z}$ suggests, whether that arithmetic circuit evaluates to zero on the ring elements dictates the behavior of the oracle and thus will be crucial for the simulator. A successful simulator will thus need to be able to take an arithmetic circuit A and determine whether it will evaluate to zero over ring elements of the distribution $\mathcal{D}_\lambda(n, K)$. This distribution is unknown to the simulator since it does not have access to K, only oracle access to $\mathcal{C}_K$. We show below that a lot can be learned about A using only oracle access to $\mathcal{C}_K$. We start by noticing that A has to respect the level restrictions imposed by the GES.

Definition 5.2 (V-compatible circuits)**.** *A purely arithmetic-circuit A is evaluated over the integer valued vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_\ell)$ via the following recursive process. The i-th input gate takes the value $\mathbf{v}_i$, a multiplication gate with input $\mathbf{v}, \mathbf{v}'$ takes the value $\mathbf{v} + \mathbf{v}'$, and an addition (or subtraction) gate with identical inputs $\mathbf{v} = \mathbf{v}'$ takes the value $\mathbf{v}$. If there exists an addition (subtraction) gate with non-identical inputs $\mathbf{v} \neq \mathbf{v}'$ then the circuit is defined to be* syntactically-illegal*. We say that A is compatible with $V = (\mathbf{v}_1, \ldots, \mathbf{v}_\ell, \mathbf{v}_{zt})$ if the computation $A(\mathbf{v}_1, \ldots, \mathbf{v}_\ell)$ is syntactically legal and the level $\mathbf{v}$ of the output gate is lower or equal to the zero-test level $\mathbf{v}_{zt}$, i.e., $\mathbf{v} \leq \mathbf{v}_{zt}$. When $\mathbf{v} = \mathbf{v}_{zt}$ we say that A is strongly compatible with V.*

Concretely for our obfuscator, which is in canonical form, given a low-degree V-compatible arithmetic circuit A on the variables $\vec{r}, \vec{w}$, we would like to find whether $A(\vec{r}, \vec{w}) \equiv 0 \pmod N$ when $\vec{r}, \vec{w}$ are generated as per the obfuscator's description. (We use $\vec{r}$ as shorthand notation for the set of variables $r_{(\cdot)}, \hat{r}_{(\cdot)}$, and $\vec{w}$ as shorthand notation for the set of variables $w_{(\cdot)}, \hat{w}_{(\cdot)}$.) We consider the polynomial computed by $A(\vec{r}, \vec{w})$ as a sum of semi-monomials of the form $M(\vec{r})Q(\vec{w})$, where $M(\vec{r})$ is a monomial in the $\vec{r}$ variables (i.e. a product of a subset of the variables in $\vec{r}$), and Q can be an arbitrary polynomial on variables in $\vec{w}$. It is easy to see that since the $\vec{r}$ variables are all uniform and independent, it suffices to find a single semi-monomial for which the Q component is not congruent to zero in order to conclude that the entire expression computed by A is not congruent to 0 (this is since a low-degree non-zero polynomial will take non-zero value with overwhelming probability).

We would like to use a zero tester for compatible circuits that is unlikely to error throughout the course of the simulation, To this end we note that V-compatible circuits have degree at most $\|\mathbf{v}_{\mathrm{zt}}\|_1$ (the ℓ_1 norm of the $\mathbf{v}_{\mathrm{zt}}$ vector), and by the choice of the order of the ring (N), it holds that $\mathsf{SZTest}(1^{(\lambda+n)}, A, N)$ outputs the correct response with all but $2^{-\lambda-n}$ probability. Therefore, if the number of calls to SZTest is at most $2^n \cdot poly(\lambda)$ then with all but negligible probability, we will have no errors in any of the calls. We will call this procedure $\mathsf{ZeroTest}$ and assume from this point and on that it always outputs the correct value. The following lemma summarizes the properties of $\mathsf{ZeroTest}$.

Lemma 5.3. *Let N be the order of the underlying ring in the construction of* Obf*, and define $\mathsf{ZeroTest}(A) = \mathsf{SZTest}(1^{(\lambda+n)}, A, N)$. Then a simulator that calls $\mathsf{ZeroTest}$ at most $2^n \cdot poly(\lambda)$ times with A which is V-compatible, will always receive a correct output whether $A \equiv 0 \pmod N$ or not, in* all *calls with probability $1 - \mathsf{negl}(\lambda)$.*

In what follows, we analyze the structure of the semi-monomials of A and show that they must take some special forms which dictate certain properties.

Claim 5.4. *Let N be the order of the underlying ring in the construction of* Obf*, and let A be a V-compatible circuit such that $A \not\equiv 0 \pmod N$. Then:*

$$\Pr_{\vec{r},\vec{w}}[A(\vec{r}, \vec{w}) = 0 \pmod N] \leq 2^{-\lambda}$$

where the elements of $\vec{r}, \vec{w}$ are randomly chosen from N

Definition 5.5. *Let $P(X_1, \cdots, X_n)$ be a polynomial. We say that P is X_i-free if all monomials that contain X_i take zero value in P's coefficient vector. We extend this notation to monomials and say that P is $(\prod X_i^{d_i})$-free if all monomials that are divisible by $(\prod X_i^{d_i})$ take zero value in P's coefficient vector. For a set of monomials $\{M_1, \ldots, M_k\}$ we say that P is $\{M_1, \ldots, M_k\}$-free if it is M_j-free for all $j = 1, \ldots, k$.*

The Structure of the Q Component. We assert the following structural claims on Q which hold due to A's being V-compatible. These properties hold here for the same reason as in [AB15] but we add the proofs for the sake of completeness.

Lemma 5.6. *There exists a constant a and a w_0-free polynomial $Q'(w)$ such that*

$$Q(\vec{w}) = a \cdot w_0 - Q'(\vec{w})$$

Proof. First, we note that the level associated with w_0 prevents it from being multiplied with any of the other $\vec{w}$ variables. The reason is that w_0 is of level $\geq (D+n)\mathbf{v}^*$, and the other $\vec{w}$ variables are encoded at level $\geq \mathbf{v}^*$. Thus, the product of w_0 with any other $w_{(\cdot)}$ variable will result an element in a level $\geq (D+n+1)\mathbf{v}^* \not\leq \mathbf{v}_{\text{zt}}$ and contradiction follows. □

Lemma 5.7. *For all $i \in [n]$, the polynomial Q (and therefore also Q' from Lemma 5.6) is $\hat{w}_i^2$-free.*

Proof. Assume towards contradiction that this is not the case. Therefore, there exists an i and $b_1, b_2 \in \{0,1\}$ such that $\hat{r}_{i,b_1} \cdot \hat{r}_{i,b_2} \mid M$. Thus, P is of level at least: $\hat{\mathbf{v}}_{i,b_1} + \hat{\mathbf{v}}_{i,b_2} \not\leq \mathbf{v}_{\text{zt}}$, and contradiction follows. □

Lemma 5.8. *If $Q'(\vec{w}) = \left(\sum_{i\in[n]} \hat{w}_i\right) \cdot Q''(\vec{w})$ for some Q'', then there exists $\vec{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$ such that Q'' is $\{w_{i,1-x_i}\}_{i\in[n]}$-free.*

And by Lemma 5.7, Q'' is also $\{\hat{w}_i\}_{i\in[n]}$-free.

Proof. For the sake of convenience, we denote: $\hat{w}_x = \sum_{i\in[n]} \hat{w}_{i,x_i}$. Note that the level of $\hat{w}_x$ is: $\hat{\mathbf{v}}_x = \sum_{i=1}^{n} \hat{\mathbf{v}}_{i,x_i}$.

Assume towards contradiction that this is not the case. Namely that there exists $i \in [n]$ such that $Q''(w)$ is neither $w_{i,0}$-free nor $w_{i,1}$-free. This means that $r_{i,0} \cdot r_{i,1} \mid M(r)$. However it also holds that $\hat{r}_{i,b} \mid M(r)$ for some $b \in \{0,1\}$. The latter is since $\hat{w}_x \mid Q'$ and thus Q is not $\hat{w}_{i,x_i}$-free. However, for all $b \in \{0,1\}$, $\mathbf{v}_{i,0} + \mathbf{v}_{i,1} + \hat{\mathbf{v}}_{i,b} \not\leq \mathbf{v}_{\text{zt}}$ and contradiction follows. □

Classification of Semi-Monomials. We distinguish between three main classes of semi-monomials.

- **Invalid I:** It holds that $\left(\sum_{i\in[n]} \hat{w}_i\right) \nmid Q'(\vec{w})$
- **Invalid II:** It holds that $\left(\sum_{i\in[n]} \hat{w}_i\right) \mid Q'(\vec{w})$, namely there exists $Q''(\vec{w})$ which is $\{w_0, \hat{w}_1, \ldots, \hat{w}_n\}$-free and:
$$Q(\vec{w}) = aw_0 - \left(\sum_{i\in[n]} \hat{w}_i\right) \cdot Q''(w).$$
However,
$$Q''(\vec{w}) \neq a \cdot \hat{\mathcal{U}}\left(w_{1,x_1}, \ldots, w_{n,x_n}, w_{n+1}, \ldots, w_{n+m}\right)$$
for any $\vec{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$.
- **Valid:** There exists $\vec{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$ such that:
$$Q(\vec{w}) = a \cdot \left(w_0 - \left(\sum_{i\in[n]} \hat{w}_i\right) \cdot \hat{\mathcal{U}}\left(w_{1,x_1}, \ldots, w_{n,x_n}, w_{n+1}, \ldots, w_{n+m}\right)\right).$$

We define the canonical valid semi-monomial for input $\vec{x}$ as follows. We let

$$Q^*_{\vec{x}}(\vec{w}) = w_0 - \left(\sum_{i\in[n]} \hat{w}_i\right) \cdot \hat{\mathcal{U}}\left(w_{1,x_1}, \ldots, w_{n,x_n}, w_{n+1}, \ldots, w_{n+m}\right) \ ,$$

and denote $A^*_{\vec{x}}(\vec{r},\vec{w}) = M^*_{\vec{x}}(\vec{r})Q^*_{\vec{x}}(\vec{w})$, where $M^*_{\vec{x}}(\vec{r})$ the monomial in $\vec{r}$ that was yielded by the truthful evaluation of the obfuscated program.

Validity Testing. We show that the validity of a semi-monomial, represented by an arithmetic circuit, can be tested efficiently.

Lemma 5.9 (Validity Check)**.** *There exists an algorithm* $\mathsf{isValid}^{\mathsf{ZeroTest}}(1^\lambda, A, N)$ *that on an input A which is a V-compatible arithmetic circuit that computes a single semi-monomial, and a modulus N, either returns a non-trivial factor of N or answers whether A is a valid semi-monomial and if so outputs it's corresponding input.*

Proof. First we note that we can extract a unique input-configuration from the semi-monomial A computes. We do that by simply by iterating over $r_{i,b}$, hardwiring them to zero and getting a new arithmetic circuit A'. We then check using $\mathsf{ZeroTest}$ on the new arithmetic-circuit whether $A' \equiv 0$ or not. If the $A' \equiv 0$ when hardwiring the $r_{i,b}$ it means that it is present in the semi-monomial, therefore we say that the input-configuration of A incorporates (i,b). We say that the input-configuration is inconsistent if it incorporates both $(i,0)$ and $(i,1)$ for some i, and in such case we output that the semi-monomial is "invalid". We also output that the monomial is "invalid" if it doesn't incorporate neither $(i,0)$ nor $(i,1)$ for some i.

Now, we have the input-configuration of A, denote it by $\vec{x}$, and we made sure that A contains exactly one semi-monomial. We construct $A^*_{\vec{x}}$, the valid semi-monomial associated with the input-configuration $\vec{x}$. We use the algorithm Div in lemma 2.6 on the A and $A^*_{\vec{x}}$, storing the result in α. We check if α is a proper factor of N by checking $\gcd(\alpha, N)$ if the result is different than 1 we output it. For last, run $\mathsf{ZeroTest}$ on the circuit defined by $\alpha \cdot A - A^*_{\vec{x}}$ and output "valid" and the value of $\vec{x}$ if the result was "zero" and "invalid" otherwise.

Correctness holds since $A^*_{\vec{x}}$ is a valid semi-monomial, it means that $A^*_{\vec{x}} \not\equiv 0$ and thus $\Pr_\xi[A^*_{\vec{x}}(\xi) = 0] = \mathsf{negl}(\lambda)$.If Div output a proper factor of N then the gcd will return an answer $\neq 1$, thus we will return a factor as required. If not, than it returns a factor s.t. $\alpha \cdot A = A^*_{\vec{x}}$ or determine that A is indeed invalid. □

Invalid Semi-Monomials Should not Zero-Out. We show that an invalid semi-monomial cannot evaluate to zero with more than negligible probability modulo N, unless it is easy to factor N.

Lemma 5.10. *Let A be a V-compatible circuit that computes an invalid semi-monomial. If for some non-negligible ϵ:*

$$\Pr_{\vec{r},\vec{w}\leftarrow\mathcal{D}_\lambda(n,K)}[A(\vec{r},\vec{w}) = 0] > \epsilon \ ,$$

then there exists an efficient algorithm that outputs a non-trivial factor of N in polynomial time.

Proof. Let A be an invalid semi-monomial, let M, Q be such that $A(r,w) = M(r)Q(w)$ and let $Q'(w)$ be derived from $Q(w)$ as above.

If A is Invalid-I, then $\sum_{i=1}^{n} \hat{w}_{i,x_i}$ does not divide $Q'(w)$. Define the distribution $\mathcal{D}_1$ on r, w such that $w_0 = 0$, $\hat{w}_{i,0} = \hat{w}_{i,1} = \hat{w}_i$ (chosen uniformly) for all $1 \leq i \leq n-1$, $\hat{w}_{n,0} = \hat{w}_{n,1} = -\sum_{i=1}^{n-1} \hat{w}_i \pmod N$ the rest of the formal variables are uniformly random. We will prove that $\Pr_{\mathcal{D}_1}[A(r,w) = 0 \pmod N] = \mathsf{negl}(\lambda)$ and $\Pr_{\mathcal{D}_1}[A(r,w) = 0 \pmod{p_3}] > \epsilon$, which allows us to find a factor for N efficiently.

To show that $\Pr_{\mathcal{D}_1}[A(r,w) = 0 \pmod{p_3}] > \epsilon$, we observe that $\mathcal{D}_1$ is equivalent to $\mathcal{D}_\lambda(n, K)$ modulo p_3. Thus by the assumption we have $\Pr_{\mathcal{D}_1}[A(r,w) = 0 \pmod{p_3}] > \epsilon$.

To show that $\Pr_{\mathcal{D}_1}[A(r,w) = 0 \pmod N] = \mathsf{negl}(\lambda)$ we notice that Q' is of the form:

$$Q'(\vec{w}) = a(\vec{w}) + \hat{w}_n \cdot b(\vec{w})$$

where a and b are both $\hat{w}_n$-free because we know that Q' is $\hat{w}_n^2$-free by Lemma 5.7. We define a new polynomial $\tilde{Q}$ by substituting $\hat{w}_n = -\sum_{i=0}^{n-1} \hat{w}_i$ and we get that:

$$\tilde{Q}(\vec{w}) = a(\vec{w}) - \left(\sum_{i=0}^{n-1} \hat{w}_i\right) b(\vec{w}).$$

If Q' evaluates to zero with noticeable probability over $\mathcal{D}_1$ so must $\tilde{Q}$. But note that $\tilde{Q}$ is a multivariate polynomial with one less variables and we substitute uniformly random values. Hence, from Claim 5.4, if we get 0 with noticeable probability it has to be equivalent to the zero polynomial. Thus:

$$a(\vec{w}) = \left(\sum_{i=0}^{n-1} \hat{w}_i\right) b(\vec{w}).$$

So, looking again at Q' we get:

$$Q'(\vec{w}) = \left(\sum_{i=0}^{n-1} \hat{w}_i\right) b(\vec{w}) + \hat{w}_n \cdot b(\vec{w})$$

Hence $\sum_{i=1}^{n} \hat{w}_{i,x_i}$ divides Q' which contradicts the assumption.

On the other hand, if A is Invalid-II then we let $Q''(\vec{w})$ be as above and by assumption

$$\forall \vec{x} \in \{0,1\}^n \;\; Q''(\vec{w}) \not\equiv a \cdot \hat{\mathcal{U}}\left(w_{1,x_1}, \ldots, w_{n,x_n}, w_{n+1}, \ldots, w_{n+m}\right).$$

Define the distribution $\mathcal{D}_2$ on r, w such that all the variables are chosen uniformly only that: $\hat{w}_{i,0} = \hat{w}_{i,1} = \hat{w}_i$ for all $1 \leq i \leq n$ and $w_{i,0} = w_{i,1} = w_i$ for all $1 \leq i \leq n$ and we define: $w_0 = \left(\sum_{i=0}^{n-1} \hat{w}_i\right) \hat{\mathcal{U}}(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{n+m})$. We will prove that $\Pr_{\mathcal{D}_2}[A(r,w) = 0 \pmod N] = \mathsf{negl}(\lambda)$ and $\Pr_{\mathcal{D}_2}[A(r,w) = 0 \pmod{p_1}] > \epsilon$, which allows us to find a factor for N efficiently.

To show that $\Pr_{\mathcal{D}_2}[A(r,w) = 0 \pmod{p_1}] > \epsilon$, we observe that $\mathcal{D}_2$ is equivalent to $\mathcal{D}_\lambda(n, K)$ modulo p_1. Thus by the assumption we have $\Pr_{\mathcal{D}_2}[A(r,w) = 0 \pmod{p_1}] > \epsilon$.

To show that $\Pr_{\mathcal{D}_2}[A(r,w) = 0 \pmod N] = \mathsf{negl}(\lambda)$ we notice that by Lemma 5.8, there exists a polynomial $\hat{Q}$ such that:

$$Q''(w) = \hat{Q}\left(w_{1,x_1}, \ldots, w_{n,x_n}, w_{n+1}, \ldots, w_{n+m}\right).$$

Because we are going to take the probability over the distribution $\mathcal{D}_2$, and in that distribution $w_{i,0} = w_{i,1} = w_i$ for $i \leq i \leq n$ we can also write:

$$Q''(w) = \hat{Q}\left(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{n+m}\right).$$

We now note that:

$$\begin{aligned}
\Pr_{\mathcal{D}_2}\left[Q(w) = 0\right] &\leq \Pr_{\mathcal{D}_2}\left[\sum_{i\in[n]} \hat{w}_i = 0\right] + \Pr_{\mathcal{D}_2}\left[a \cdot w_0 - Q''(\vec{w}) = 0\right] \\
&\leq \Pr_{\mathcal{D}_2}\left[a \cdot w_0 - Q''(\vec{w}) = 0\right] + \mathsf{negl}(\lambda) \\
&= \Pr_{\mathcal{D}_2}\left[a \cdot w_0 - \hat{Q}(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{n+m}) = 0\right] + \mathsf{negl}(\lambda) \\
&= \Pr_{\mathcal{D}_2}\left[a \cdot \hat{\mathcal{U}}(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{n+m}) - \hat{Q}(w_1, \ldots, w_n, w_{n+1}, \ldots, w_{n+m}) = 0\right] + \mathsf{negl}(\lambda)
\end{aligned}$$

However, because A is Invalid-II we have that: $\left(a \cdot \hat{\mathcal{U}}(\cdot) - Q''(\cdot)\right) \not\equiv 0$, and therefore the probability of zero is negligible because all of the variables in the polynomial that is written inside the last probability are uniformly random, hence from Claim 5.4 we get the bound we wish for. □

Rejecting Semi-Monomials Should not Zero-Out. Moreover, we show that an valid semi-monomial cannot evaluate to zero with more than negligible probability modulo N if the input corresponding to this monomial is a rejecting input, unless it is easy to factor N.

Lemma 5.11. *Given an $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ and a semi-monomial that evaluate such that:*

$$Q(\vec{w}) = a \cdot \left(w_0 - \left(\sum_{i\in[n]} \hat{w}_i\right) \cdot \hat{\mathcal{U}}\left(w_{1,x_1}, \ldots, w_{n,x_n}, w_{n+1}, \ldots, w_{n+m}\right)\right)$$

and $C_K(x) = 0$. Then if $\Pr_{\mathcal{D}_\lambda(n,K)}\left[Q = 0\right] > \epsilon$ for a non-negligible ϵ there exists an efficient algorithm that outputs a non-trivial factor of N in polynomial time.

Proof. Let x, Q be as in the lemma statement. By definition, $\Pr_{\mathcal{D}_\lambda(n,K)}\left[Q[\![1]\!] = 0\right] = 1$ and $\Pr_{\mathcal{D}_\lambda(n,K)}\left[Q[\![3]\!] = 0\right] = 1$. It remains to examine the value of $Q(\vec{w})[\![2]\!] = Q[\![2]\!](w[\![2]\!])$. Note that:

$$\begin{aligned}
\Pr_{\mathcal{D}_\lambda(n,K)}\left[Q(\vec{w}) = 0\right] &= \Pr_{\mathcal{D}_\lambda(n,K)}\left[Q[\![2]\!](\vec{w}[\![2]\!]) = 0\right] \\
&= \Pr_{\mathcal{D}_\lambda(n,K)}\left[a[\![2]\!] \cdot \left(\sum_{i\in[n]} \hat{\beta}_i\right) \cdot \left(1 - \hat{\mathcal{U}}\left(x_1, \ldots, x_n, K_1, \ldots, K_m\right)\right) = 0\right] \\
&= \Pr_{\mathcal{D}_\lambda(n,K)}\left[a[\![2]\!] \cdot \left(\sum_{i\in[n]} \hat{\beta}_i\right) \cdot \left(1 - C_K(x)\right) = 0\right] \\
&= \Pr_{\mathcal{D}_\lambda(n,K)}\left[a[\![2]\!] = 0\right] + \Pr\left[\left(\sum_{i\in[n]} \hat{\beta}_i\right) \cdot \left(1 - C_K(x)\right)\right]
\end{aligned}$$

Note that since $C_K(x) = 0$ then the probability of the second term is equal to: $\Pr_{\mathcal{D}_\lambda(K)}\left[\sum_{i\in[n]} \hat{\beta}_i = 0\right] = \mathsf{negl}(\lambda)$.Hence, if $\Pr_{\mathcal{D}_\lambda(n,K)}[Q(\vec{w}) = 0] > \epsilon$ it means that $\Pr_{\mathcal{D}_\lambda(n,K)}[a[\![2]\!] = 0] > \epsilon - \mathsf{negl}(\lambda)$. In that case we construct $Q^*_{\vec{x}}$, the valid evaluation of x (without the a factor), and use Div from Lemma 2.6. Which either output a or a factor of n. But since $a[\![2]\!] = 0$ it means that $p_2 \mid a$, thus either way we've found a factor of N with probability $> \epsilon - \mathsf{negl}(\lambda)$. □

5.2 Indistinguishability Obfuscation Security in the Classic Generic Model

Theorem 5.12. *The obfuscator* Obf *is an indistinguishability obfuscator in the generic GES model relative to the oracle* $\mathcal{RG}$*, under the sub-exponential hardness of factoring (Definition 2.1), so long as* $\lambda \geq n^{1/\delta}$*.*

Proof. In order to prove security, we construct a simulator that will simulate the view of the adversary with only oracle access to C_K. Since we only want to prove indistinguishability obfuscation, we can allow the simulator to run in time $2^n \cdot poly(\lambda)$. Thus it can read the entire truth table of C_K.

Initialization: The simulator generate a number N from the same distribution as the ring order in the GES. The simulator also creates a table $\mathcal{L}$. For each encoding the obfuscator outputs, $\mathcal{S}$ will create a row in the table associating random label string with the formal variable represented by the encoding and the appropriate level of the encoding. $\mathcal{S}$, just like the obfuscator, will output a list of label strings for each of the obfuscated encodings and give them to the adversary.

$\mathcal{S}.\mathsf{Add}(\mathrm{enc}_1, \mathrm{enc}_2), \mathcal{S}.\mathsf{Mult}(\mathrm{enc}_1, \mathrm{enc}_2), \mathcal{S}.\mathsf{Negate}(\mathrm{enc})$**:** Given an arithmetic operation (Add, Mult, Negate), the simulator will construct an arithmetic-circuit $A_{\mathrm{res}} = A_{\mathsf{enc}_1}$ op A_{enc_2} (where A_{enc_1} and A_{enc_2} are the arithmetic-circuits associated with enc_1 and enc_2 respectively) and check if it is equivalent to one of the other elements in the table with the same level. It can easily be done by subtracting A_{res} from the arithmetic-circuit in the table and using isZero procedure. If they are equivalent, the simulator will response with the same label. Otherwise, the simulator will create a new row in $\mathcal{L}$ containing a new label, A_{res} and the new level. Outputs the label to the adversary.

$\mathcal{S}.\mathsf{isZero}(\mathrm{enc})$**:** Upon an isZero request, the simulator checks the table $\mathcal{L}$ for the encoding element enc. If it is not contained in the table, the simulator will output $\perp$. Otherwise, we denote by A the arithmetic-circuit associated with enc.

From Lemma 5.13 we can determine whether A evaluates to zero or not with overwhelming probability. This process runs in time $2^n poly(\lambda)$, which is allowed in the context of iO.

□

Lemma 5.13. *Let* $\lambda \geq n^{1/\delta}$*, there exists an algorithm* iOZero^{C_K} *that given a* V*-compatible arithmetic-circuit* A *outputs a bit* $\beta \in \{0,1\}$ *such that with all-but-negligible probability:*

$$\left| \Pr_{\vec{r},\vec{w}\leftarrow\mathcal{D}_\lambda(n,K)}[A(\vec{r},\vec{w}) = 0] - \beta \right| \leq \mathsf{negl}(\lambda)$$

under the sub-exponential assumption of factoring (Definition 2.1).

Note that this implies that with all-but-negligible probability A either evaluates to zero almost everywhere or not to zero almost everywhere under $\mathcal{D}_\lambda(n, K)$.

Proof. Consider the following algorithm:

1. Create a new empty arithmetic-circuit S.
2. For every $\vec{x} \in \{0,1\}^n$:
   (a) Generate a new arithmetic-circuit $A_{\vec{x}}$ which is created by taking A and hardwiring $r_{i,1-x_i}, \hat{r}_{i,1-x_i}$ to zero.
   (b) Check "minimality" of $A_{\vec{x}}$: meaning that $A_{\vec{x}}$ does not contain more than one semi-monomial and that all of $r_{i,x_i}, \hat{r}_{i,x_i}, r_i, r_0$ are used in the semi-monomial represented by $A_{\vec{x}}$. We do that by a process similar to the one in the semi-monomial extraction algorithm (Lemma 2.5). We simply try to hardwire more rs to zero and if we can hardwire another r without getting an arithmetic-circuit that is equivalent to the zero-circuit (denote this arithmetic-circuit by A') it means that all the semi-monomials $M'(\vec{r})Q'(\vec{w})$ in A' (that must also be contained in $A_{\vec{x}}$) maintain that either r_0 isn't present in $M'(\vec{r})$, there exists an $1 \leq i \leq n$ such that either r_{i,x_i} or $\hat{r}_{i,x_i}$ not present in $M'(\vec{r})$ or there exists an $n+1 \leq i \leq n+m$ such that r_i not present in $M'(\vec{r})$. By using the semi-monomial extraction procedure (Lemma 2.5) on A' we can extract such monomial, by doing so we've found an invalid semi-monomial in $A_{\vec{x}}$ and therefore in A so we output that A evaluates to "non-zero".
   (c) Now we know that $A_{\vec{x}}$ contains only one semi-monomial. Run the validity check (Lemma 5.9) on $A_{\vec{x}}$. If it is invalid output that A evaluates to "non-zero".
   (d) Query the C_K-oracle on the input $\vec{x}$ if it returned 0 output "non-zero".
   (e) Replace S with $S + A_{\vec{x}}$.
3. Check whether S and A are equivalent by using $\mathsf{ZeroTest}$ on the arithmetic-circuit $S - A$. If they are equivalent output "zero". Otherwise, using semi-monomial extraction on $S - A$ will yield an invalid semi-monomial, so we can output "non-zero".

Correctness: We first note that this algorithm runs in time $2^n poly(n, \lambda)$, therefore under the sub-exponential assumption on factoring, this algorithm cannot factor N. Hence, the validity check cannot return a proper factor of N with more than negligible probability and therefore will return whether the semi-monomials are valid or not and by Lemma 5.10 we get that all invalid semi-monomials evaluate to zero only with negligible probability. Hence if the algorithm finds an invalid semi-monomial during the evaluation, we've found a semi-monomial which is not congruent to zero, thus A evaluates to non-zero with overwhelming probability, so we can output "non-zero" and quit the process.

If we encounter a valid semi-monomial such that the C_K-oracle on the input $\vec{x}$ returned 0 then according to Lemma 5.11, since this algorithm cannot break factoring then with overwhelming probability this semi-monomial does not evaluate to zero, and again it means that with overwhelming probability A evaluates to non-zero.

It remains to check the case in which all semi-monomials in the loop were valid. In this case we check whether S and A are equivalent. If they are, it means that A consists only of the

semi-monomials we've checked, and we know that each and every one of them evaluates to zero (otherwise, we would have stopped the process beforehand) and therefore A evaluates to zero. But if A and S are not equivalent, that means that there is another semi-monomial in A that was not covered in the loop. This semi-monomial has to be invalid because we covered all possible valid semi-monomials. Therefore it will evaluate to zero only with negligible probability because of Lemma 5.10 and as a result so does A. □

5.3 All-or-Nothing (AoN) Functions

We define a category of "all or nothing" functions. These are functions such that are either evasive or perfectly learnable, namely, finding an accepting input for a function in the class implies that the code of the function can be retrieved. This class is an extension of the class of evasive functions. For simplicity we provide the definition in the standalone setting, but it can be extended to the auxiliary input setting as well.

Definition 5.14. *An ensemble of functions $\mathcal{C} = \{\mathcal{C}_n\}$ is* AoN *if for any* PPT *algorithm $\mathcal{A}$, there exists a* PPT *algorithm $\mathcal{B}$ such that for all $C \in \mathcal{C}_n$,*

$$\Pr_r \left[\left(C\left(\mathcal{A}^C\left(1^n; r\right)\right) = 1\right) \wedge \left(\mathcal{B}^C(1^n; r) \neq C\right)\right] = \mathsf{negl}(\lambda) \ ,$$

that is $\mathcal{A}, \mathcal{B}$ use the same random tape r.

We can also define an average-case analogue:

Definition 5.15. *An ensemble of functions $\mathcal{C} = \{\mathcal{C}_n\}$ together with distributions $\{\mathcal{D}_n\}$ over $\mathcal{C}$ is average-case* AoN *if for any* PPT *algorithm $\mathcal{A}$, there exists a* PPT *algorithm $\mathcal{B}$ such that:*

$$\Pr_{r,\, C \leftarrow \mathcal{D}_n} \left[\left(C\left(\mathcal{A}^C\left(1^n; r\right)\right) = 1\right) \wedge \left(\mathcal{B}^C(1^n; r) \neq C\right)\right] = \mathsf{negl}(\lambda) \ ,$$

that is $\mathcal{A}, \mathcal{B}$ use the same random tape r.

Note that we ask that $\mathcal{B}$ outputs the exact code of C, given only black box access. Therefore, AoN function classes which are not evasive need to have programs with unique representations. This indeed holds for classes such as conjunctions.

5.4 Zero-Sensitive Security for All-or-Nothing Functions

The following theorem states the VBB security of Obf for any class of AoN functions. We note that while we provide a proof for worst-case AoN, the average case setting follows by a similar proof (note that there could exist function classes that are average case AoN but not worst case AoN).

Theorem 5.16. *Assuming factoring is hard then if $\mathcal{C}$ is a family of* AoN *functions, then* Obf *is VBB-secure with respect to the oracle $\mathcal{RG}_{\mathcal{Z}}$.*

Proof. In order to prove VBB security, we want to define an efficient simulator $\mathcal{S}$ that will simulate the view of the adversary using only an oracle access to C_K.

Similarly to the definition of the $\mathcal{RG}_{\mathcal{Z}}$ oracle in Section 3.2, the simulator $\mathcal{S}$ will need to act differently when a non-trivial encoding of zero is encountered (that is, simulate the performance of $\mathcal{RG}_{\mathcal{Z}}$ when decode = true). The simulator will maintain a variable decode that upon initialization

will be set to false and only when we encounter a non-trivial encoding of zero it will be set to true. As long as decode = false, we use the hardness of factorization in order to show that finding non-trivial zero using invalid monomials is unlikely, therefore up to the point where such encoding is found, the simulator will not use the factorization of the ring at all. The factorization will only be used afterwards in order to continue the simulation after decode was set to true.

Initialization: The simulator generate a number N which it knows how to factor into three factors p_1, p_2, p_3 (which have $\gcd(p_i, p_j) = 1$ for $i \neq j$, but does not have to be primes). In similar with the $\mathcal{RG}_{\mathcal{Z}}$ oracle, the simulator will also create a table $\mathcal{L}$. For each encoding the obfuscator outputs, $\mathcal{S}$ will create a row in the table associating random label string with the formal variable represented by the encoding and the appropriate level of the encoding. $\mathcal{S}$, just like the obfuscator, will output a list of label strings for each of the obfuscated encodings and give them to the adversary. The only difference between the simulator and the oracle here is that the ring element is not stored in the table at this point.

$\mathcal{S}.\mathsf{Add}(\mathrm{enc}_1, \mathrm{enc}_2), \mathcal{S}.\mathsf{Mult}(\mathrm{enc}_1, \mathrm{enc}_2), \mathcal{S}.\mathsf{Negate}(\mathrm{enc})$**:** Given an arithmetic operation (Add, Mult, Negate), the simulator will construct an arithmetic-circuit $A_{\mathrm{res}} = A_{\mathsf{enc}_1}$ op A_{enc_2} (where A_{enc_1} and A_{enc_2} are the arithmetic-circuits associated with enc_1 and enc_2 respectively) and check if it is equivalent to one of the other elements in the table with the same level. It can easily be done by subtracting A_{res} from the arithmetic-circuit in the table and using isZero procedure. If they are equivalent, the simulator will response with the same label. Otherwise, the simulator will create a new row in $\mathcal{L}$ containing a new label, A_{res} and the new level. Outputs the label to the adversary.

$\mathcal{S}.\mathsf{isZero}(\mathrm{enc})$**:** The isZero algorithm works differently when decode is set to true or false.

The case where decode = false**:** The simulator will check if enc is in $\mathcal{L}$. If not it will output $\perp$, otherwise the simulator use the following algorithm:

1. Use the AoNZero algorithm from Lemma 5.18 on the arithmetic-circuit associated with enc in order to determine whether it evaluates to zero or in order to find an accepting input. If the algorithm output a decision regarding the evaluation of the arithmetic-circuit output it.
2. Otherwise, we note that the adversary together with the simulator up to this point is an efficient algorithm that finds an accepting input. From the definition of the function class (Definition 5.14) we can use the $\mathcal{B}$ algorithm associated with this combined algorithm in order to find the code of the obfuscated circuit C.
3. Generate values to all the formal variables given in the initialization step using the known factorization of the ring. And store the values for future use.

   We note that when we choose random variables, we could have broken consistency with previous queries, as it could have been that using these values previous isZero calls would have response with true. But note that such inconsistency can only occur with negligible probability.
4. Set decode = true and run $\mathcal{S}.\mathsf{isZero}(\mathrm{enc})$ again.

Remark 5.17. *The* isZero *algorithm this case can only return that the value is indeed "zero" if the encoded element is a trivial zero. In any other case we either output "non-zero" or we change to the case where* decode = true*.*

The case where decode = true**:** In this case, the simulator has already assigned values to each of the formal variables in the table $\mathcal{L}$, and therefore it can easily evaluate the result of the arithmetic-circuit associated with enc and reply to the isZero accordingly.

$\mathcal{S}$.Decode(enc)**:** If decode = false simply return $\perp$ as this is what the simulator will do. We note that in every arithmetic operation that the adversary does, we initiate isZero on all the elements at the same level. Therefore, if the adversary succeeded in finding a non-trivial zero or received the same element in two different ways, the simulator will change decode to be true. Thus in that case, the simulator has already assigned values to all the formal variables used in the arithmetic-circuit associated with enc. By substituting those variables into this arithmetic-circuit results the decoded value of enc which we can output to the adversary.

Correctness: We want to show the correctness of the $\mathcal{S}$.isZero procedure in both cases. We note that if decode = true, the simulator already knows the function evaluated and it have assignments to all the formal variables that are in use, therefore, it is clear that substituting this values in the arithmetic-circuit associated with the encoding the adversary wish to zero test will yield a correct answer.

On the other hand, while decode = false, the correctness is immediate from the correctness of Lemma 5.18 together with the definition of the AoN class and the hardness of factoring. But using the hardness of factoring is delicate since $\mathcal{S}$ knows factors of N, therefore we cannot simply solve factoring using the simulator, because in order to construct the simulator those factors are needed to be known in advanced. We note that once decode = true the hardness of factoring doesn't play a role in the correctness of the simulator.

Because we only care when decode = false, we can construct a new simulator $\mathcal{S}_1$ that will abort when decode = true. It is clear that if AoNZero in $\mathcal{S}$ broke factoring while decode = false so it must during $\mathcal{S}_1$. Now, we introduce the simulator $\mathcal{S}_2$ which is similar to $\mathcal{S}_1$ only that $\mathcal{S}_2$ does not know any proper factors of N. We notice that those factors are only being used when we set decode = true, and since $\mathcal{S}_1$ aborts when decode is set to true the behavior of $\mathcal{S}_1$ and $\mathcal{S}_2$ is the same, and therefore the behavior of $\mathcal{S}_2$ and $\mathcal{S}$ is the same as long as decode = false.

Now, we want to bound the probability that AoNZero, when being used in the $\mathcal{S}$ during the time decode = true, will output a factor or fail (which occurs only in negligible probability as explained in Lemma 5.18). We note that in simulator $\mathcal{S}_2$ the probability to either of those event is negligible since factoring is hard. Thus, because the behavior of $\mathcal{S}$ and $\mathcal{S}_2$ is the same as long as decode = false, the probability will have to be negligible in $\mathcal{S}$.

□

Lemma 5.18. *Let C is from a family of* AoN *functions. There exists an algorithm* $\mathsf{AoNZero}^C$ *that when given an arithmetic circuit A either determines whether it evaluates to zero, outputs an accepting input for C or output a non-trivial factor of N.*

Proof. Consider the following algorithm:

1. Run ZeroTest on A, if the output of the ZeroTest was "zero" - output "zero" as the circuit is with overwhelming probability computes the zero-polynomial.

2. Use the Semi-Monomial Extraction algorithm from lemma 2.5 in order to extract an arbitrary semi-monomial from the associated arithmetic-circuit.

3. Check whether the evaluated semi-monomial is valid by using the validity check algorithm in lemma 5.9. If the semi-monomial is invalid, use the algorithm defined by Lemma 5.10 in order to try and exact a factor of N if manage to find a factor output it otherwise output "nonzero". If the the validity check returned a proper factor of N, output it. For last, if the semi-monomial was valid, denote the input of semi-monomial by x.

4. Query the C-oracle for $C(x)$, if $C(x) = 0$, output "non-zero". As the resulted encoding cannot zero-out in this case with non-negligible probability.

5. Otherwise, output the x.

Correctness: If A is equivalent to the zero-circuit, the first line of the algorithm uses ZeroTest that with probability 1 will output "zero", hence we indeed return the right answer.

If A is not equivalent to the zero-circuit, then by using the semi-monomial extraction algorithm (Lemma 2.5) in the second step of the algorithm we get a semi-monomial.

We note that since factoring is hard, validity check will not return a proper factor with overwhelming probability. Therefore it will output whether the semi-monomial is valid or not. If this semi-monomial is valid then the validity check must accept and also output the corresponding input, denote it by $\vec{x}$. In step 4 we query the oracle for the value $C(\vec{x})$, if it returns that the $C(\vec{x}) = 1$ then we have successfully found an accepting input, and we output it. On the other hand, if oracle response was that $C(\vec{x}) = 0$, then according to Lemma 5.11, since factoring is hard this semi-monomial cannot zero-out with noticeable probability. Thus from Claim 5.4 we know that A only returns 0 with negligible probability as it contains a monomial that does not cancels out.

Now, if A is an invalid semi-monomial and if the encoding is indeed encoding of zero then from Lemma 5.10 we can construct an algorithm that will output a factor of N. By amplification it will fail only in negligible probability. If it fails, that mean that with overwhelming probability A will not zero-out. Therefore, from Claim 5.4 we get that A will not zero out with overwhelming probability. □

References

[AB15] Benny Applebaum and Zvika Brakerski. Obfuscating circuits via composite-order graded encoding. In *Theory of Cryptography - 12th Theory of Cryptography Conference, TCC 2015, Warsaw, Poland, March 23-25, 2015, Proceedings, Part II*, pages 528–556, 2015.

[AGIS14] Prabhanjan Vijendra Ananth, Divya Gupta, Yuval Ishai, and Amit Sahai. Optimizing obfuscation: Avoiding barrington's theorem. In *Proceedings of the 2014 ACM SIGSAC Conference on Computer and Communications Security, Scottsdale, AZ, USA, November 3-7, 2014*, pages 646–658, 2014.

[BBC+14] Boaz Barak, Nir Bitansky, Ran Canetti, Yael Tauman Kalai, Omer Paneth, and Amit Sahai. Obfuscation for evasive functions. In Lindell [Lin14], pages 26–51.

[BGI+12] Boaz Barak, Oded Goldreich, Russell Impagliazzo, Steven Rudich, Amit Sahai, Salil P. Vadhan, and Ke Yang. On the (im)possibility of obfuscating programs. *J. ACM*, 59(2):6, 2012. Preliminary version in CRYPTO 2001.

[BGK+14] Boaz Barak, Sanjam Garg, Yael Tauman Kalai, Omer Paneth, and Amit Sahai. Protecting obfuscation against algebraic attacks. In *Advances in Cryptology - EUROCRYPT 2014*, pages 221–238, 2014.

[BMSZ15] Saikrishna Badrinarayanan, Eric Miles, Amit Sahai, and Mark Zhandry. Post-zeroizing obfuscation: new mathematical tools, and the case of evasive circuits. *IACR Cryptology ePrint Archive*, 2015:167, 2015. To appear in Eurocrypt 2016.

[BR13] Zvika Brakerski and Guy N. Rothblum. Obfuscating conjunctions. In Ran Canetti and Juan A. Garay, editors, *Advances in Cryptology - CRYPTO 2013 - 33rd Annual Cryptology Conference, Santa Barbara, CA, USA, August 18-22, 2013. Proceedings, Part II*, volume 8043 of *Lecture Notes in Computer Science*, pages 416–434. Springer, 2013.

[BR14a] Zvika Brakerski and Guy N. Rothblum. Black-box obfuscation for d-cnfs. In Moni Naor, editor, *Innovations in Theoretical Computer Science, ITCS'14, Princeton, NJ, USA, January 12-14, 2014*, pages 235–250. ACM, 2014.

[BR14b] Zvika Brakerski and Guy N. Rothblum. Virtual black-box obfuscation for all circuits via generic graded encoding. In Lindell [Lin14], pages 1–25.

[BS02] Dan Boneh and Alice Silverberg. Applications of multilinear forms to cryptography. *IACR Cryptology ePrint Archive*, 2002:80, 2002.

[BVWW16] Zvika Brakerski, Vinod Vaikuntanathan, Hoeteck Wee, and Daniel Wichs. Obfuscating conjunctions under entropic ring LWE. In Madhu Sudan, editor, *Proceedings of the 2016 ACM Conference on Innovations in Theoretical Computer Science, Cambridge, MA, USA, January 14-16, 2016*, pages 147–156. ACM, 2016.

[Can97] Ran Canetti. Towards realizing random oracles: Hash functions that hide all partial information. In *Advances in Cryptology - CRYPTO '97, 17th Annual International Cryptology Conference, Santa Barbara, California, USA, August 17-21, 1997, Proceedings*, pages 455–469, 1997.

[CGH+15] Jean-Sébastien Coron, Craig Gentry, Shai Halevi, Tancrède Lepoint, Hemanta K. Maji, Eric Miles, Mariana Raykova, Amit Sahai, and Mehdi Tibouchi. Zeroizing without low-level zeroes: New MMAP attacks and their limitations. In Rosario Gennaro and Matthew Robshaw, editors, *Advances in Cryptology - CRYPTO 2015 - 35th Annual Cryptology Conference, Santa Barbara, CA, USA, August 16-20, 2015, Proceedings, Part I*, volume 9215 of *Lecture Notes in Computer Science*, pages 247–266. Springer, 2015.

[CHL+15] Jung Hee Cheon, Kyoohyung Han, Changmin Lee, Hansol Ryu, and Damien Stehlé. Cryptanalysis of the multilinear map over the integers. In Elisabeth Oswald and Marc Fischlin, editors, *Advances in Cryptology - EUROCRYPT 2015 - 34th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Sofia, Bulgaria, April 26-30, 2015, Proceedings, Part I*, volume 9056 of *Lecture Notes in Computer Science*, pages 3–12. Springer, 2015.

[CLR15] Jung Hee Cheon, Changmin Lee, and Hansol Ryu. Cryptanalysis of the new clt multilinear maps. Cryptology ePrint Archive, Report 2015/934, 2015. `http://eprint.iacr.org/`.

[CLT13] Jean-Sébastien Coron, Tancrède Lepoint, and Mehdi Tibouchi. Practical multilinear maps over the integers. In Ran Canetti and Juan A. Garay, editors, *Advances in Cryptology - CRYPTO 2013 - 33rd Annual Cryptology Conference, Santa Barbara, CA, USA, August 18-22, 2013. Proceedings, Part I*, volume 8042 of *Lecture Notes in Computer Science*, pages 476–493. Springer, 2013.

[CLT15] Jean-Sébastien Coron, Tancrède Lepoint, and Mehdi Tibouchi. New multilinear maps over the integers. *IACR Cryptology ePrint Archive*, 2015:162, 2015.

[GGH13a] Sanjam Garg, Craig Gentry, and Shai Halevi. Candidate multilinear maps from ideal lattices. In *Advances in Cryptology - EUROCRYPT 2013, 32nd Annual International Conference on the Theory and Applications of Cryptographic Techniques, Athens, Greece, May 26-30, 2013. Proceedings*, pages 1–17, 2013.

[GGH+13b] Sanjam Garg, Craig Gentry, Shai Halevi, Mariana Raykova, Amit Sahai, and Brent Waters. Candidate indistinguishability obfuscation and functional encryption for all circuits. In *54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, 2013, Berkeley, CA, USA*, pages 40–49, 2013.

[GLSW14] Craig Gentry, Allison Lewko, Amit Sahai, and Brent Waters. Indistinguishability obfuscation from the multilinear subgroup elimination assumption. Cryptology ePrint Archive, Report 2014/309, 2014. To appear in FOCS 2015.

[GLW14] Craig Gentry, Allison B. Lewko, and Brent Waters. Witness encryption from instance independent assumptions. In *Advances in Cryptology - CRYPTO 2014 - 34th Annual Cryptology Conference, Santa Barbara, CA, USA, August 17-21, 2014, Proceedings, Part I*, pages 426–443, 2014.

[Had00] Satoshi Hada. Zero-knowledge and code obfuscation. In Tatsuaki Okamoto, editor, *Advances in Cryptology - ASIACRYPT 2000, 6th International Conference on the Theory and Application of Cryptology and Information Security, Kyoto, Japan, December 3-7, 2000, Proceedings*, volume 1976 of *Lecture Notes in Computer Science*, pages 443–457. Springer, 2000.

[HJ15] Yupu Hu and Huiwen Jia. Cryptanalysis of ggh map. Cryptology ePrint Archive, Report 2015/301, 2015. `http://eprint.iacr.org/`.

[Lin14] Yehuda Lindell, editor. *Theory of Cryptography - 11th Theory of Cryptography Conference, TCC 2014, San Diego, CA, USA, February 24-26, 2014. Proceedings*, volume 8349 of *Lecture Notes in Computer Science*. Springer, 2014.

[Mau05] Ueli . Maurer. Abstract models of computation in cryptography. In Nigel P. Smart, editor, *IMA Int. Conf.*, volume 3796 of *Lecture Notes in Computer Science*, pages 1–12. Springer, 2005.

[MF15] Brice Minaud and Pierre-Alain Fouque. Cryptanalysis of the new multilinear map over the integers. Cryptology ePrint Archive, Report 2015/941, 2015. `http://eprint.iacr.org/`.

[MSW14] Eric Miles, Amit Sahai, and Mor Weiss. Protecting obfuscation against arithmetic attacks. *IACR Cryptology ePrint Archive*, 2014:878, 2014.

[MSZ16] Eric Miles, Amit Sahai, and Mark Zhandry. Annihilation attacks for multilinear maps: Cryptanalysis of indistinguishability obfuscation over ggh13. Cryptology ePrint Archive, Report 2016/147, 2016. `http://eprint.iacr.org/`.

[PST14] Rafael Pass, Karn Seth, and Sidharth Telang. Indistinguishability obfuscation from semantically-secure multilinear encodings. In *Advances in Cryptology - CRYPTO 2014 - 34th Annual Cryptology Conference, Santa Barbara, CA, USA, August 17-21, 2014, Proceedings, Part I*, pages 500–517, 2014.

[Sho97] Victor Shoup. Lower bounds for discrete logarithms and related problems. In *EUROCRYPT*, pages 256–266, 1997.

[SW14] Amit Sahai and Brent Waters. How to use indistinguishability obfuscation: deniable encryption, and more. In David B. Shmoys, editor, *Symposium on Theory of Computing, STOC 2014, New York, NY, USA, May 31 - June 03, 2014*, pages 475–484. ACM, 2014.

[Zim15] Joe Zimmerman. How to obfuscate programs directly. In Elisabeth Oswald and Marc Fischlin, editors, *Advances in Cryptology - EUROCRYPT 2015 - 34th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Sofia, Bulgaria, April 26-30, 2015, Proceedings, Part II*, volume 9057 of *Lecture Notes in Computer Science*, pages 439–467. Springer, 2015.