Doubly-efficient zkSNARKs without trusted setup

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Abstract. We present a zero-knowledge argument for NP with low communication complexity, low concrete cost for both the prover and the verifier, and no trusted setup, based on standard cryptographic assumptions. Communication is proportional to $d \cdot \log G$ (for d the depth and G the width of the verifying circuit) plus the square root of the witness size. When applied to batched or data-parallel statements, the prover's runtime is linear and the verifier's is sub-linear in the verifying circuit size, both with good constants. In addition, witness-related communication can be reduced, at the cost of increased verifier runtime, by leveraging a new commitment scheme for multilinear polynomials, which may be of independent interest. These properties represent a new point in the tradeoffs among setup, complexity assumptions, proof size, and computational cost.

We apply the Fiat-Shamir heuristic to this argument to produce a zero-knowledge succinct non-interactive argument of knowledge (zkSNARK) in the random oracle model, based on the discrete log assumption, which we call Hyrax. We implement Hyrax and evaluate it against five state-of-the-art baseline systems. Our evaluation shows that, even for modest problem sizes, Hyrax gives smaller proofs than all but the most computationally costly baseline, and that its prover and verifier are each faster than three of the five baselines.

1 Introduction

A zero-knowledge proof convinces a verifier of a statement while revealing nothing but its own validity. Since they were introduced by Goldwasser, Micali, and Rackoff [55], zero-knowledge (ZK) proofs have found applications in domains as diverse as authentication and signature schemes [90, 95], secure encryption [44, 94], and emerging blockchain technologies [12].

A seminal result in the theory of interactive proofs and cryptography is that any problem solvable by an interactive proof (IP) is also solvable by a computational zero-knowledge proof or perfect zero-knowledge argument [8]. This means that, given an interactive proof for any NP-complete problem, one can construct zero-knowledge proofs or arguments for any NP statement. But existing instantiations of this paradigm have large overheads: early techniques [22, 53] require many repetitions to achieve negligible soundness error, and incur polynomial blowups in prover work and communication. More recent work [25, 28, 29, 31, 56, 57, 60] avoids those issues, but generally entails many expensive cryptographic operations.[1]

Several other recent lines of work have sought to avoid these overheads. As detailed in Section 2, however, these works still yield costly protocols or come with significant limitations. In particular, state-of-the-art, general-purpose ZK protocols suffer from one or more of the following problems: (a) they require proof size that is linear or super-linear in the size of the computation verifying an NP witness; (b) they require the prover or verifier to perform work that is super-linear in the time to verify a witness; (c) they require a complex parameter setup to be performed by a trusted party; (d) they rely on non-standard cryptographic assumptions; or (e) they have very high concrete overheads. These issues have limited the use of such general-purpose ZK proof systems in many contexts.

Our goal in this work is to address the limitations of existing general-purpose ZK proofs and arguments. Specifically, we would like to take any computation for verifying an NP statement and turn it into a zero-knowledge proof of the statement's validity. In addition to concrete efficiency, our desiderata are that:

- the proof should be *succinct*, that is, sub-linear in the size of the statement and the witness to the statement's validity;
- the verifier should run in time linear in input plus proof size;
- the prover, given a witness to the statement's validity, should run in time linear in the cost of the NP verification procedure;
- the scheme should not require a trusted setup phase or common reference string; and
- soundness and zero-knowledge should each be either statistical or based on standard cryptographic assumptions. Pragmatically, security in the random oracle model [7] suffices.

Our approach transforms a state-of-the-art interactive proof for arithmetic circuit (AC) satisfiability into a zero-knowledge argument by composing new ideas with existing techniques.

Ben-Or et al. [8] and Cramer and Damgård [40] show how to transform IPs into computationally ZK proofs or perfectly ZK arguments, using cryptographic commitment schemes. At a high level, rather than sending its messages in the clear, the prover sends cryptographic *commitments* corresponding to its messages. These commitments are *binding*, ensuring that the prover cannot cheat by equivocating about its messages. They are also *hiding*, meaning that the verifier cannot learn the committed value and thus ensuring zero-knowledge. Finally, the commitment scheme has a homomorphism property (§3.1) that allows the verifier to check the prover's messages "underneath the commitments."

Accepted wisdom is that such transformations introduce large overheads (e.g., [35, §1.1]). In this paper, we challenge that wisdom by constructing a protocol that meets our desiderata for many cases of interest.

Our starting point is the Giraffe interactive proof [109] with an optimization, adapted from Chiesa et al. [35], that reduces communication complexity (§3.2). We transform this IP into a ZK argument through a straightforward (but careful) application of Cramer-Damgård techniques (§4). This argument uses cryptographic operations (required by the commitment schemes)

[1] Some works avoid these overheads by targeting specific problems with algebraic structure and cryptographic significance, most notably Schnorr-style proofs [95] for languages related to statements about discrete logarithms of group elements.

only for the witness and for the prover's messages, which are sub-linear in the size of the AC. (In contrast, many recent works invoke cryptographic primitives for *each gate* in the verifying circuit [12, 13, 16, 25, 31, 50, 84]; §2.) But the argument is not succinct, and it has high concrete costs, especially for the verifier.

We slash these costs with two key refinements. First, we exploit the IP's structure by tightly integrating the verification procedure with a multi-commitment scheme and a Schnorr-style proof [95] (§5); this reduces communication and computational costs by 3–5× compared to the naive approach. Second, we devise a new witness commitment scheme (§6), yielding a succinct argument and asymptotically reducing the verifier's cost associated with the witness.

Our protocol is public coin; we compile it into *Hyrax*, a zero-knowledge succinct non-interactive argument of knowledge (zkSNARK) [20] in the random oracle model [7], via the Fiat-Shamir heuristic [45] (§7). We evaluate Hyrax and five state-of-the-art baselines (BCCGP-sqrt [25], Bulletproofs [31], Ligero [1], ZKB++ [33], and libSTARK [11]; §8). Even for modest problem sizes, Hyrax gives smaller proofs than all but the most computationally costly baseline, its prover and verifier are each faster than three of the five baselines, and its refinements yield multiple-orders-of-magnitude savings in proof size and verifier runtime.

Contributions. We design, implement, and evaluate Hyrax, a "doubly" (meaning for both prover and verifier) concretely efficient zkSNARK. For input x, witness w, an AC C of width G and depth d, and a design parameter $\iota \geq 2$ that controls a tradeoff between proof length and verifier time:

- Hyrax's proofs are succinct, i.e., sub-linear in $|C|$ and $|w|$: they require $\approx 10d \log G + |w|^{1/\iota}$ group elements;
- its verifier runs in time sub-linear in $|C|$, if C has sufficient parallelism:[2] $O(|x| + d \log G + |w|^{(\iota-1)/\iota})$, with good constants;
- its prover runs in time linear in $|C|$, with good constants, if C has sufficient parallelism (practically, a few tens of parallel instances suffices), and it requires only $O(d \log G + |w|)$ cryptographic operations, also with good constants; and
- it requires no trusted setup, and it is secure under the discrete log assumption in the random oracle model.

We also give a new commitment scheme tailored to multilinear polynomials (§6), which may be of independent interest. This scheme allows the prover to commit to a multilinear polynomial m over $\mathbb{F}$, and later to reveal (a commitment to) $m(r)$ for any r chosen by the verifier. For $\iota \geq 2$, if $|m|$ denotes the number of monomials in m, then the commitment has size $O(|m|^{1/\iota})$, and the time to verify a purported evaluation is $O(|m|^{(\iota-1)/\iota})$.

2 Related work

ZK proofs. Over the past several years there has been significant interest in implementing ZK proof systems. In this section, we discuss those efforts, focusing on the theoretical underpinnings and associated cryptographic assumptions; we compare Hyrax with several of these works empirically in Section 8.

[2] Even without parallelism, the verifier runs in time sub-linear in $|C|$ if C's wiring pattern satisfies a technical "regularity" condition [37, 54] (Thm. 1, §3.2).

Gennaro et al. [50] present a linear probabilistically checkable proof (PCP)[3] and ZK transform that form the basis of many recent zkSNARK implementations [4, 5, 12, 13, 15, 16, 30, 36, 38, 43, 46–48, 69, 81, 84, 110], including systems deployed in applications like ZCash [12, 111]. These implementations build on theoretical work by Ishai et al. [64], Groth [58], Lipmaa [74], and Bitansky et al. [21], as well as implementations and refinements in the non-ZK context [30, 96–98]. Such zkSNARKs give small, constant-sized proofs (hundreds of bytes), and verifier runtime depends only on input size. But ZK systems in this line rely on non-standard, non-falsifiable cryptographic assumptions, require a trusted setup, and have massive prover overhead: runtime is quasi-linear in the verifying circuit size, including a few public key operations per gate, and memory consumption limits the statement sizes these systems can handle in practice [110].

A line of work by Ben-Sasson et al. builds non-interactive ZK arguments from short PCPs, following the seminal work of Kilian [67, 68] and Micali [78], the landmark result of Ben-Sasson and Sudan [17], and recent generalizations of PCPs [9, 14, 92]. The authors reduce the concrete overheads associated with these approaches [10] and implement zero-knowledge scalable transparent arguments of knowledge (zkSTARKs) [11]. zkSTARKs need no trusted setup and no public-key cryptography, but their soundness rests on a non-standard conjecture related to Reed-Solomon codes [11, Appx. B]. Further, zkSTARKs are heavily optimized for statements whose verifying circuits are expressed as a sequence of state-machine transitions; this captures all of NP, but can introduce significant overhead in practice [110]. Both proof size and verifier runtime are logarithmic in circuit size (hundreds of kilobytes and tens of milliseconds, respectively, in practice), and prover runtime is quasi-linear.

Another approach due to Ishai, Kushilevitz, Ostrovsky, and Sahai [65] (IKOS) transforms a secure multi-party computation protocol into a ZK argument. Giacomelli et al. refine this approach and construct ZKBoo [51], a ZK argument system for Boolean circuits with no trusted setup from collision-resistant hashes; ZKB++, by Chase et al. [33], reduces proof size by constant factors. Both schemes are concretely inexpensive for small circuits, but their costs scale linearly with circuit size. Ames et al. [1] further refine the IKOS transform and apply it to a more sophisticated secure computation protocol. Their scheme, Ligero, makes similar security assumptions to ZKBoo but proves an AC C's satisfiability with proof size $\tilde{O}(\sqrt{|C|})$ and prover and verifier work quasi-linear in $|C|$ (where $\tilde{O}$ ignores polylog factors).

Bootle et al. [25] give two ZK arguments for AC satisfiability from the hardness of discrete logarithms, building on the work of Groth [57] and of Bayer and Groth [6]. The first has proof size $O(\sqrt{\mathcal{M}})$ and quasi-linear prover and verifier runtime for an AC with $\mathcal{M}$ multiplications. The second reduces this to $O(\log \mathcal{M})$ at the cost of concretely longer prover and verifier runtimes. Bünz et al. [31] reduce proof size and runtimes in the log scheme by $\approx 3\times$. Bootle et al. [26] give a ZK argument with proof size $O(\sqrt{|C|})$ whose verifier uses $O(|C|)$ *additions* (which are less expensive than multiplications), but the authors state that the constants are large and do not recommend implementing as-is.

[3] The observation that the *quadratic span programs* of GGPR [50] can be viewed as linear PCPs is due to Bitansky et al. [21] and Setty et al. [96].

Most similar to our work, Zhang et al. [112] show how to combine an interactive proof [37, 54, 102] and a verifiable polynomial delegation scheme [66, 83] to construct a succinct, non-ZK interactive argument. A follow-up work [113] (concurrent with and independent from ours) achieves ZK using the same commit-and-prove approach that we use, with several key differences. First, their commitment to the witness w has communication $O(\log |w|)$, but has a trusted setup phase and relies on non-standard, non-falsifiable assumptions. In contrast, our commitment protocol (§6) has no trusted setup and is based on the discrete log assumption, but has communication $O(|w|^{1/\iota})$, $\iota \geq 2$. Second, their argument uses an IP that requires more communication than ours (§3.2). Finally, our method of compiling the IP into a ZK argument uses additional refinements (§5) that reduce costs. Both our IP and our refinements apply to their work; we estimate that they would reduce proof size by $\approx 3\times$ and $\mathcal{V}$ runtime by $\approx 5\times$.

Polynomial commitment schemes were introduced by Kate et al. [66], who gave a construction for univariate polynomials based on pairing assumptions. Several follow-up works [83, 112–114] extend this construction to multivariate polynomials; Libert et al. [70] give a construction based on constant-size assumptions; and Fujisaki et al. [49] give a construction for polynomial evaluation based on the RSA problem that can be immediately adapted to polynomial commitment. None of these schemes meet our desiderata (§1) because of some combination of high cost, trusted setup, and non-standard assumptions. Bootle et al. [26] and Bootle and Groth [27] describe univariate polynomial commitment schemes based on the discrete log assumption; our scheme is closely related to these ideas and extends them to multilinear polynomials. The second of these also presents a general framework for proving simple relations between commitments and field elements; exploring these ideas in our context is future work.

3 Background

3.1 Definitions

We use $\langle A(z_a), B(z_b)\rangle(x)$ to denote the random variable representing the (local) output of machine B when interacting with machine A on common input x, when the random tapes for each machine are uniformly and independently chosen, and A and B has auxiliary inputs z_a and z_b respectively. We use $\mathsf{tr}\langle A(z_a), B(z_b)\rangle(x)$ to denote the random variable representing the entire transcript of the interaction between A and B, and $\mathsf{View}\,(\langle A(z_a), B(z_b)\rangle(x))$ to denote the distribution of the transcript. The symbol $\approx_c$ denotes that two ensembles are computationally indistinguishable.

Arithmetic circuits

Section 3.2 considers the arithmetic circuit (AC) *evaluation* problem. In this problem, one fixes an arithmetic circuit C, consisting of addition and multiplication gates over a finite field $\mathbb{F}$. We assume throughout that C is layered, with all gates having fan-in at most 2 (any arithmetic circuit can be made layered while increasing the number of gates by a factor of at most the circuit depth). C has depth d and input x with length $|x|$. The goal is to evaluate C on input x. In an interactive proof or argument for this problem, the prover sends the claimed outputs y of C on input x, and must prove that $y = C(x)$.

Our end goal in this work is to give efficient protocols for the arithmetic circuit *satisfiability* problem. Let $C(\cdot,\cdot)$ be a layered arithmetic circuit of fan-in two. Given an input x and outputs y, the goal is to determine whether there exists a *witness* w such that $C(x, w) = y$. The corresponding witness relation for this problem is the natural one: $R_{(x,y)} = \{w\colon C(x, w) = y\}$.

Interactive protocols and zero-knowledge

Definition 1 (Interactive arguments and proofs)**.** *A pair of probabilistic interactive machines $\langle \mathcal{P}, \mathcal{V}\rangle$ is called an interactive argument system for a language L if there exists a negligible function η such that the following two conditions hold:*

1. *Completeness: For every $x \in L$ there exists a string w s.t. for every $z \in \{0,1\}^*$, $\Pr[\langle \mathcal{P}(w), \mathcal{V}(z)\rangle(x){=}1] \geq 1 - \eta(|x|)$.*
2. *Soundness: For every $x \notin L$, every interactive PPT $\mathcal{P}^*$, and every $w, z \in \{0,1\}^*$, $\Pr[\langle \mathcal{P}^*(w), \mathcal{V}(z)\rangle(x){=}1] \leq \eta(|x|)$.*

If soundness holds against computationally unbounded cheating provers $\mathcal{P}^*$, then $\langle \mathcal{P}, \mathcal{V}\rangle$ is called an interactive proof (IP).

Definition 2 (Zero-knowledge (ZK))**.** *Let $L \subset \{0,1\}^*$ be a language and for each $x \in L$, let $R_x \subset \{0,1\}^*$ denote a corresponding set of* witnesses *for the fact that $x \in L$. Let R_L denote the corresponding language of valid (input, witness) pairs, i.e., $R_L = \{(x, w)\colon x \in L$ and $w \in R_x\}$. An interactive proof or argument system $\langle \mathcal{P}, \mathcal{V}\rangle$ for L is computational zero-knowledge (CZK) with respect to an auxiliary input if for every PPT interactive machine $\mathcal{V}^*$, there exists a PPT algorithm S, called the* simulator*, running in time polynomial in the length of its first input, such that for every $x \in L$, $w \in R_x$, and $z \in \{0,1\}^*$,*

$$\mathsf{View}\,(\langle P(w), \mathcal{V}^*(z)\rangle(x)) \approx_c S(x, z) \tag{1}$$

when the distinguishing gap is considered as a function of $|x|$. If the statistical distance *between the two distributions is negligible, then the interactive proof or argument system is said to be* statistical *zero-knowledge (SZK). If the simulator is allowed to abort with probability at most $1/2$, but the distribution of its output conditioned on not aborting is* identically *distributed to* $\mathsf{View}\,(\langle P(w), \mathcal{V}^*(z)\rangle(x))$*, then the interactive proof or argument system is called* perfect *zero-knowledge (PZK).*

The left term in Equation (1) denotes the distribution of transcripts after $\mathcal{V}^*$ interacts with $\mathcal{P}$ on common input x; the right term denotes the distribution of simulator S's output on x. For any CZK (resp., SZK or PZK) protocol, Definition 2 requires the simulator to produce a distribution that is computationally (resp., statistically or perfectly) indistinguishable from the distribution of transcripts of the ZK proof or argument system.

Our zero-knowledge arguments also satisfy a proof of knowledge property. Intuitively, this means that in order to produce a convincing proof of a statement, the prover must *know* a witness to the validity of the statement. To define this notion formally, we follow Groth and Ishai [59] who borrow the notion of statistical witness-extended emulation from Lindell [73]:

Definition 3 (Witness-extended emulation [59])**.** *Let L be a language and R_L corresponding language of valid (input, witness) pairs as in Definition 2. An interactive argument system $\langle \mathcal{P}, \mathcal{V}\rangle$*

for L has witness-extended emulation *if for all deterministic polynomial time $\mathcal{P}^*$ there exists an expected polynomial time emulator E such that for all non-uniform polynomial time adversaries A and all $z_{\mathcal{V}} \in \{0,1\}^*$, the following probabilities differ by at most a negligible function in the security parameter λ:*

$$\Pr\left[(x, z_{\mathcal{P}}) \leftarrow A(1^\lambda); t \leftarrow \mathsf{tr}\langle \mathcal{P}^*(z_{\mathcal{P}}), \mathcal{V}(z_{\mathcal{V}})\rangle(x) : A(t) = 1\right]$$

$$\text{and } \Pr\left[\begin{array}{l}(x, z_{\mathcal{P}}) \leftarrow A(1^\lambda); (t, w) \leftarrow E^{\mathcal{P}^*(z_{\mathcal{P}})}(x) : A(t) = 1 \wedge \\ \text{if } t \text{ is an accepting transcript, then } (x, w) \in R_L.\end{array}\right]$$

Here, the oracle called by E permits rewinding the prover to a specific point and resuming with fresh randomness for the verifier from this point onwards.

The protocols of Sections 5 and 6 are *generalized special sound*, which implies witness-extended emulation (Appx. A.6).

Definition 4 (Generalized special soundness)**.** *A $(2\mu + 1)$-move interactive argument $\langle \mathcal{P}, \mathcal{V}\rangle$ is* generalized special sound *if there exists a PPT algorithm* Ex_{GSS} *that extracts a witness except with negligible probability given an $(n_1, \ldots, n_\mu)$-tree of accepting transcripts. This tree comprises n_1 transcripts with fresh randomness in $\mathcal{V}$'s first message; and for each such transcript, n_2 transcripts with fresh randomness in $\mathcal{V}$'s second message; etc., for a total of $\prod_{i=1}^{\mu} n_i$ leaves. The standard notion of special soundness corresponds to $\mu = 1, n_1 = 2$.*

Commitment schemes

Informally, a commitment scheme allows a *sender* to produce a message $C = \mathsf{Com}(m)$ that hides m from a *receiver* but binds the sender to the value m. In particular, when the sender *opens* C and reveals m, the receiver is convinced that this was indeed the sender's original value. We say that $\mathsf{Com}_{\mathrm{pp}}(m; r)$ is a commitment to m with *opening* r with respect to public parameters pp. The sender chooses r at random; to open the commitment, the sender reveals (m, r). We frequently leave the public parameters implicit, and sometimes do the same for the opening, e.g., $\mathsf{Com}(m)$.

Definition 5 (Collection of non-interactive commitments [62])**.** *We say that a tuple of PPT algorithms* $(\mathsf{Gen}, \mathsf{Com})$ *is a* collection of non-interactive commitments *if the following conditions hold:*

- **Computational binding:** *For every (non-uniform) PPT A, there is a negligible function η such that for every $n \in \mathbb{N}$,*

$$\Pr\left[\begin{array}{l}\mathrm{pp} \leftarrow \mathsf{Gen}(1^n) \; ; \\ (m_0, r_0), (m_1, r_1) \leftarrow A(1^n, \mathrm{pp}) \; : \\ m_0 \neq m_1, |m_0| = |m_1| = n, \\ \mathsf{Com}_{\mathrm{pp}}(m_0; r_0) = \mathsf{Com}_{\mathrm{pp}}(m_1; r_1)\end{array}\right] \leq \eta(n)$$

- **Perfect hiding:** *For any* $\mathrm{pp} \in \{0,1\}^*$ *and $m_0, m_1 \in \{0,1\}^*$ where $|m_0| = |m_1|$, the ensembles $\{\mathsf{Com}_{\mathrm{pp}}(m_0)\}_{n\in\mathbb{N}}$ and $\{\mathsf{Com}_{\mathrm{pp}}(m_1)\}_{n\in\mathbb{N}}$ are identically distributed.*

We define only the computational variant of binding and the perfect variant of hiding because the commitment schemes used in our implementation satisfy these properties. The use of such commitment schemes in our context yields a PZK argument. If we instead used perfectly (or statistically) binding, computationally hiding commitments, we would obtain a CZK proof.

Collections of non-interactive commitments can be constructed based on any one-way function [61, 80], but we require a *homomorphism* property (defined below) that these commitments do not provide. (The Pedersen commitment [85], described in Appx. A, provides this property.)

Definition 6 (Additive homomorphism)**.** *Given* $\mathsf{Com}(x; s_x)$ *and* $\mathsf{Com}(y; s_y)$*, there is an operator $\odot$ such that*

$$\mathsf{Com}(x; s_x) \odot \mathsf{Com}(y; s_y) = \mathsf{Com}(x + y; s_x + s_y) \qquad \textit{and}$$

$$\mathsf{Com}(x; s_x)^k \triangleq \mathsf{Com}(x; s_x) \odot \cdots \odot \mathsf{Com}(x; s_x) \qquad \textit{(k times)}$$

In a multi-commitment *scheme, x and y are vectors, and this additive homomorphism is vector-wise.*

3.2 Our starting point: Gir^{++} (Giraffe, with a tweak)

The most efficient known IPs for the AC evaluation problem (§3.1) follow a line of work starting with the breakthrough result of Goldwasser, Kalai, and Rothblum (GKR) [54]. Cormode, Mitzenmacher, and Thaler (CMT) [37] and Vu et al. [107] refine this result, giving $\mathrm{O}(|C| \log |C|)$ prover and $\mathrm{O}(|x| + |y| + d \log |C|)$ verifier runtimes, for AC C with depth d, input x, and output y.

Further refinements are possible in the case where C is *data parallel*, meaning it consists of N identical sub-computations run on different inputs. (We refer to each sub-computation as a *sub-AC* of C, and we assume for simplicity that all layers of the sub-AC have width G, so $|C| = d \cdot N \cdot G$.) Thaler [102] reduced the prover's runtime in the data-parallel case from $\mathrm{O}(|C| \log |C|)$ to $\mathrm{O}(|C| \log G)$. Very recently, Wahby et al. introduced Giraffe [109], which reduces the prover's runtime to $\mathrm{O}(|C| + d \cdot G \cdot \log G)$. Since $|C| = d \cdot N \cdot G$, observe that when $N \geq \log G$, the time reduces to $\mathrm{O}(|C||)$, which is asymptotically optimal. That is, for sufficient data parallelism, the prover's runtime is just a constant factor slower than evaluating the circuit gate-by-gate without providing any proof of correctness.

Our work builds on *Gir*$^{++}$, which reduces Giraffe's communication via an optimization due to Chiesa et al. [35]; our description of Gir^{++} borrows notation from Wahby et al. [109]. Assume for simplicity that N and G are powers of 2, and let $b_N = \log_2 N$ and $b_G = \log_2 G$. Within a layer of C, each gate is labeled with a pair $(i, j) \in \{0,1\}^{b_N} \times \{0,1\}^{b_G}$. Number the layers of C from 0 to d in reverse execution order, so that 0 refers to the output layer, and d refers to the input layer. Each layer i is associated with an evaluator function $V_i \colon \{0,1\}^{b_N} \times \{0,1\}^{b_G} \to \mathbb{F}$ that maps a gate's label to the output of that gate when C is evaluated on input x. For example, $V_0(i, j)$ is the j'th output of the i'th sub-AC, and $V_d(i, j)$ is the jth input to the ith sub-AC.

At a high level, the protocol proceeds in iterations, one for each layer of the circuit. At the start of the protocol, the prover $\mathcal{P}$ sends the claimed outputs y of C (i.e., all the claimed evaluations of V_0). The first iteration of the protocol reduces the claim about V_0 to a claim about V_1, in the sense that it is safe for the verifier $\mathcal{V}$ to believe the former claim as long as $\mathcal{V}$ is convinced of the latter. But $\mathcal{V}$ cannot directly check the claim about V_1, because doing so would require evaluating all of the gates in C other than the outputs themselves. Instead, the second iteration reduces the claim about V_1 to a claim about V_2, and so on, until $\mathcal{P}$ makes a claim about V_d (i.e., the inputs to C), which $\mathcal{V}$ checks itself.

To describe how the reduction from a claim about V_i to a claim about V_{i+1} is performed, we first introduce *multilinear extensions*, *the sum-check protocol*, and *wiring predicates*.

Multilinear extensions. An *extension* of a function $f\colon \{0,1\}^\ell \to \mathbb{F}$ is a ℓ-variate polynomial g over $\mathbb{F}$ such that $g(x) = f(x)$ for all $x \in \{0,1\}^\ell$. Any such function f has a unique multilinear extension (MLE)—a multilinear polynomial—denoted $\tilde{f}$. Given a vector $z \in \mathbb{F}^m$ with $m = 2^\ell$, we will often view z as a function $z\colon \{0,1\}^\ell \to \mathbb{F}$ mapping indices to vector entries, and use $\tilde{z}$ to denote the MLE of z.

The sum-check protocol. Fix an ℓ-variate polynomial g over $\mathbb{F}$, and let $\deg_i(g)$ denote the degree of g in variable i. The sum-check protocol [75] is an interactive proof that allows $\mathcal{P}$ to convince $\mathcal{V}$ of a claim about the value of $\sum_{x\in\{0,1\}^\ell} g(x)$ by reducing it to a claim about the value of $g(r)$, where $r \in \mathbb{F}^\ell$ is a point randomly chosen by $\mathcal{V}$. There are ℓ rounds, and $\mathcal{V}$'s runtime is $O(\sum_{i=1}^{\ell} \deg_i(g))$ plus the cost of evaluating $g(r)$. The mechanics are detailed in Section 4.

Wiring predicates capture the wiring information of the sub-ACs. Define the wiring predicate $\mathrm{add}_i\colon \{0,1\}^{3b_G} \to \{0,1\}$, where $\mathrm{add}_i(g, h_0, h_1)$ returns 1 if (a) within each sub-AC, gate g at layer $i-1$ is an add gate and (b) the left and right inputs of g are, respectively, h_0 and h_1 at layer i (and 0 otherwise). mult_i is defined analogously for multiplication gates. Define the equality predicate $\mathrm{eq} : \{0,1\}^{2b_N} \to \{0,1\}$ as $\mathrm{eq}(a,b) = 1$ iff $a = b$.

Thaler [102, 103] and Wahby et al. [109] show how to express $\tilde{V}_{i-1}$ in terms of $\tilde{V}_i$: for $(q', q) \in \mathbb{F}^{b_N} \times \mathbb{F}^{b_G}$, let $P_{q',q,i}\colon \mathbb{F}^{b_N} \times \mathbb{F}^{b_G} \times \mathbb{F}^{b_G} \to \mathbb{F}$ denote the polynomial

$$
\begin{aligned}
P_{q',q,i}(h', h_L, h_R) = \\
&\tilde{\mathrm{eq}}(q', h') \cdot \big[\tilde{\mathrm{add}}_i(q, h_L, h_R)\left(\tilde{V}_i(h', h_L) + \tilde{V}_i(h', h_R)\right) + \\
&\qquad \tilde{\mathrm{mult}}_i(q, h_L, h_R)\left(\tilde{V}_i(h', h_L) \cdot \tilde{V}_i(h', h_R)\right)\big]
\end{aligned}
$$

Then we have

$$\tilde{V}_{i-1}(q', q) = \sum_{h' \in \{0,1\}^{b_N}} \ \sum_{h_L, h_R \in \{0,1\}^{b_G}} P_{q',q,i}(h', h_L, h_R). \quad (2)$$

Protocol overview

Step 1. At the start of the protocol, $\mathcal{P}$ sends the claimed output y, thereby specifying a function $V_y\colon \{0,1\}^{b_G+b_N} \to \mathbb{F}$ mapping the label of each output gate to the corresponding entry of y. The verifier wishes to check that $V_y = V_0$ (i.e., that the claimed outputs equal the correct outputs of C on input x); to accomplish this, it would be enough to check that $\tilde{V}_y = \tilde{V}_0$. In principle, $\mathcal{V}$ could do that by choosing a random pair $(q', q) \in \mathbb{F}^{b_N} \times \mathbb{F}^{b_G}$ and checking that $\tilde{V}_y(q', q) = \tilde{V}_0(q', q)$; if that check passes, then $\tilde{V}_y = \tilde{V}_0$ with high probability, by the Schwartz-Zippel lemma. On the one hand, $\mathcal{V}$ can and does compute $\tilde{V}_y(q', q)$; this takes $O(NG)$ time [109, §3.3]. But on the other hand, $\mathcal{V}$ cannot compute $\tilde{V}_0(q', q)$ directly—this would require $\mathcal{V}$ to evaluate C.

Step 2 (iterated). Instead, $\mathcal{V}$ outsources evaluation of $\tilde{V}_0(q', q)$ to $\mathcal{P}$, via the sum-check protocol; this is motivated by Equation (2). At the end of the sum-check protocol, $\mathcal{V}$ must evaluate $P_{q',q,1}$ at a random input (r', r_L, r_R), which requires the values $\tilde{V}_1(r', r_L)$ and $\tilde{V}_1(r', r_R)$. $\mathcal{V}$ does not evaluate these points directly; that would be too costly. Instead, $\mathcal{P}$ sends v_0 and v_1, which it claims are the required values. $\mathcal{V}$ uses these to evaluate $P_{q',q,1}$, then checks v_0 and v_1 using a *mini-protocol*, which we describe shortly. At a high level, the mini-protocol transforms $\mathcal{P}$'s claims about v_0, v_1 into a claim about $\tilde{V}_2$. $\mathcal{V}$ checks this claim with a sum-check and mini-protocol invocation, yielding a claim about $\tilde{V}_3$. $\mathcal{P}$ and $\mathcal{V}$ iterate, layer by layer, until $\mathcal{V}$ has a claim about $\tilde{V}_d$.

Final step. $\mathcal{V}$ checks $\mathcal{P}$'s final claim about $\tilde{V}_d$ by evaluating $\tilde{V}_x$ (since $\tilde{V}_d = \tilde{V}_x$); it can do this in $O(NG)$ time [109, §3.3].

Mini-protocols: reducing from $\tilde{V}_i$ to $\tilde{V}_{i+1}$

Gir^{++} differs from Giraffe only in that they use different mini-protocols to reduce $\mathcal{P}$'s claims at the end of one sum-check invocation (i.e., $v_0 = \tilde{V}_i(r', r_L)$ and $v_1 = \tilde{V}_i(r', r_R)$) into the expression that $\mathcal{V}$ and $\mathcal{P}$ use for the next sum-check invocation.

Reducing from two points to one point. This approach is used in Giraffe and prior work [37, 54, 102, 107–109]. $\mathcal{P}$ sends $\mathcal{V}$ the restriction of $\tilde{V}_i$ to the unique line H in $\mathbb{F}^{b_N+b_G}$ passing through the points (r', r_L) and (r', r_R) by specifying the univariate polynomial $f_H(t) = \tilde{V}_i(r', (1-t)\cdot r_L + t\cdot r_R)$, which has degree b_G. $\mathcal{V}$ should believe this claim as long as $f_H(0) = v_0$, $f_H(1) = v_1$, and $f_H(\upsilon) = \tilde{V}_i(r', r_\upsilon)$, where $r_\upsilon = (1-\upsilon)\cdot r_L + \upsilon \cdot r_R$ and υ is chosen by $\mathcal{V}$. By Equation (2), $\mathcal{V}$ can check this latter equality by engaging $\mathcal{P}$ in a sum-check protocol over $P_{r', r_\upsilon, i+1}$.

Alternative: Random linear combination. Each invocation of the prior mini-protocol requires $\mathcal{P}$ to send $b_G + 1$ field elements specifying f_H. The following technique, due to Chiesa et al. [35], eliminates this requirement. Instead, $\mathcal{V}$ checks v_0 and v_1 by checking a random linear combination, via a sum-check invocation over a polynomial we define below.

In more detail, $\mathcal{V}$ samples two field elements μ_0 and μ_1, and sends them to $\mathcal{P}$. Mechanically, $\mathcal{V}$ next checks that

$$\mu_0 \cdot \tilde{V}_i(r', r_L) + \mu_1 \cdot \tilde{V}_i(r', r_R) = \mu_0 \cdot v_0 + \mu_1 \cdot v_1 \quad (3)$$

since, by the Schwartz-Zippel lemma, this implies that $v_0 = \tilde{V}_i(r', r_L)$ and $v_1 = \tilde{V}_i(r', r_R)$ with high probability (formalized in Thm. 1, below). $\mathcal{V}$ checks Equation (3) by exploiting the fact that its LHS can be written as

$$
\begin{aligned}
&\mu_0 \cdot \tilde{V}_i(q', q_L) + \mu_1 \cdot \tilde{V}_i(q', q_R) \\
&\quad = \sum_{h_L, h_R \in \{0,1\}^{b_G}} \ \sum_{h' \in \{0,1\}^{b_N}} \big[\mu_0 \cdot P_{q', q_L, i+1}(h', h_L, h_R) + \\
&\qquad\qquad \mu_1 \cdot P_{q', q_R, i+1}(h', h_L, h_R)\big] \\
&\quad = \sum_{h_L, h_R \in \{0,1\}^{b_G}} \ \sum_{h' \in \{0,1\}^{b_N}} Q_{q', q_L, q_R, \mu_0, \mu_1, i+1}(h', h_L, h_R)
\end{aligned}
$$

where $Q_{q', q_L, q_R, \mu_0, \mu_1, i}\colon \mathbb{F}^{b_N} \times \mathbb{F}^{b_G} \times \mathbb{F}^{b_G} \to \mathbb{F}$ is given by:

$$
\begin{aligned}
Q_{q', q_L, q_R, \mu_0, \mu_1, i}(h', h_L, h_R) &\triangleq \tilde{\mathrm{eq}}(q', h') \cdot \\
&\Big[\left(\mu_0 \cdot \tilde{\mathrm{add}}_i(q_L, h_L, h_R) + \mu_1 \cdot \tilde{\mathrm{add}}_i(q_R, h_L, h_R)\right) \cdot \\
&\qquad \left(\tilde{V}_i(h', h_L) + \tilde{V}_i(h', h_R)\right) \\
&\quad + \left(\mu_0 \cdot \tilde{\mathrm{mult}}_i(q_L, h_L, h_R) + \mu_1 \cdot \tilde{\mathrm{mult}}_i(q_R, h_L, h_R)\right) \cdot \\
&\qquad \left(\tilde{V}_i(h', h_L) \cdot \tilde{V}_i(h', h_R)\right) \Big]
\end{aligned}
$$

This means that $\mathcal{V}$ can check that Equation (3) holds by engaging $\mathcal{P}$ in a sum-check protocol over $Q_{r',r_L,r_R,\mu_0,\mu_1,i+1}$.

Giraffe vs. Gir++ Gir^{++} uses the Alternative above. This reduces communication cost in Gir^{++} compared to Giraffe by a small factor that depends on the amount of data parallelism. We are motivated to reduce communication because communication will translate into proof size and more cryptographic cost (§4).

As an exception, Gir^{++} uses the "reducing from two points to one point" technique after the final sum-check (i.e., the one over $Q_{\ldots,d-1}$); this is to avoid increasing $\mathcal{V}$'s computational costs compared to Giraffe. Recall that in the final step of Gir^{++}, $\mathcal{V}$ checks $\mathcal{P}$'s claim about $\tilde{V}_d$ by evaluating $\tilde{V}_x$ (which is equal to $\tilde{V}_d$). Thus, to check the LHS of Equation (3), $\mathcal{V}$ would require *two* evaluations of $\tilde{V}_x$; the "reducing from two points to one point" technique requires only one. Since evaluating $\tilde{V}_x$ is typically a bottleneck for the verifier [109, §3.3], eliminating the second evaluation is worthwhile even though it slightly increases the size of $\mathcal{P}$'s final message (and thus the proof size; see §4).

We give pseudocode for Gir^{++} in Appendix E. Gir^{++}'s efficiency and security are formalized in the following theorem, which can be proved via a standard analysis [54].

Theorem 1. *The interactive proof Gir^{++} satisfies the following properties when applied to a layered arithmetic circuit C of fan-in two, consisting of N identical sub-computations, each of depth d, with all layers of each sub-computation having width at most G. It has perfect completeness, and soundness error at most* $((1 + 2\log G + 3\log N) \cdot d + \log G)/|\mathbb{F}|$. *After a pre-processing phase taking time* $\mathrm{O}(dG)$*, the verifier runs in time* $\mathrm{O}(|x| + |y| + d\log NG)$*, and the prover runs in time* $\mathrm{O}(|C| + d \cdot G \cdot \log G)$*. If the sub-AC has a regular wiring pattern as defined in [37], then the pre-processing phase is unnecessary.*

4 Compiling Gir^{++} into a ZK argument

In this section, we describe a straightforward application of "commit-and-prove" techniques [8, 40] (§1) to Gir^{++} (§3.2). The result is a public coin, perfect ZK argument "of knowledge" for AC satisfiability (the knowledge property is formalized via witness-extended emulation; §3.1). In Sections 5 and 6, we develop substantial efficiency improvements; in Section 7, we apply the Fiat-Shamir heuristic [45] to make it non-interactive.

Building blocks. This section uses abstract commitments having a homomorphism property (§3.1). We also make black-box use of three sub-protocols, which operate on commitments:

- *proof-of-opening*(C) convinces $\mathcal{V}$ that $\mathcal{P}$ can open C.
- *proof-of-equality*(C_0, C_1) convinces $\mathcal{V}$ that C_0 and C_1 commit to the same value, and that $\mathcal{P}$ can open both.
- *proof-of-product*(C_0, C_1, C_2) convinces $\mathcal{V}$ that C_2 commits to the product of the values committed in C_0 and C_1, and that $\mathcal{P}$ can open all three.

In Appendix A, we give concrete definitions of the above protocols in terms of Pedersen commitments [85].

Protocol overview. This protocol differs from Gir^{++} in three ways. First, it adds an initial step in which $\mathcal{P}$ commits to w such that $C(x, w) = y$. Second, $\mathcal{P}$ replaces all of its messages in Gir^{++} with *commitments* to those messages. Third, $\mathcal{P}$ convinces $\mathcal{V}$ that its committed values pass all of $\mathcal{V}$'s checks in Gir^{++} using the homomorphism property of the commitments and the above sub-protocols. The steps below correspond to the steps of Gir^{++} (§3.2); we describe only how the protocols differ.

Step 0. (This is a new step.) $\mathcal{P}$ sends commitments to each element of $w \in \mathbb{F}^{\ell}$. $\mathcal{P}$ and $\mathcal{V}$ execute proof-of-opening for each.

Step 1. As in Gir^{++}, $\mathcal{V}$ computes $\tilde{V}_y(q', q)$. Afterwards, $\mathcal{V}$ computes $C_0 = \mathsf{Com}(\tilde{V}_y(q', q); 0)$.

Step 2. As in Gir^{++}, this step comprises one sum-check and one mini-protocol per layer of C. We now review the sum-check protocol, and then describe how $\mathcal{P}$ and $\mathcal{V}$ execute the sum-check and mini-protocols "underneath the commitments."

Review of the sum-check protocol. We begin by describing the first layer sum-check protocol in Gir^{++} (others are similar), which reduces $\tilde{V}_y(q', q)$ to a claim about $\tilde{V}_1(\cdot)$. In the first round of the sum-check protocol, $\mathcal{P}$ sends a univariate polynomial $s_1(\cdot)$ of degree 3. $\mathcal{V}$ checks that $s_1(0) + s_1(1) = \tilde{V}_y(q', q)$, and then sends a random field element r_1 to $\mathcal{P}$. In general, in round j of the sum-check protocol, $\mathcal{P}$ sends a univariate polynomial s_j (which is degree 3 in the first b_N rounds and degree 2 in the remaining rounds [102, 109]). $\mathcal{V}$ checks that $s_j(0) + s_j(1) = s_{j-1}(r_{j-1})$, then sends a random field element r_j to $\mathcal{P}$.

We write the vector of all r_j's chosen by $\mathcal{V}$ in the $j_{\text{last}} = b_N + 2b_G$ rounds of the sum-check protocol as $(r_1, \ldots, r_{j_{\text{last}}}) \in \mathbb{F}^{b_N+2b_G}$; let r' denote the first b_N entries of this vector, r_L denote the next b_G entries, and r_R denote the final b_G entries.

In the last round, $\mathcal{P}$ sends v_0 and v_1 (which it claims are equal to $\tilde{V}_1(r', r_L)$ and $\tilde{V}_1(r', r_R)$; §3.2). $\mathcal{V}$ first checks that

$$s_{j_{\text{last}}}(r_{j_{\text{last}}}) = \tilde{\text{eq}}(q', r') \cdot \big[\tilde{\text{add}}_1(q, r_L, r_R) \cdot (v_0 + v_1) + \tilde{\text{mult}}_1(q, r_L, r_R) \cdot v_0 \cdot v_1\big]$$

$\mathcal{V}$ then checks $\mathcal{P}$'s claims about v_0 and v_1 by invoking a mini-protocol (§3.2) and engaging $\mathcal{P}$ in another sum-check at layer 2.

ZK sum-check protocol. In round j of the sum-check, $\mathcal{P}$ commits to $s_j(t) = c_{0,j} + c_{1,j}t + c_{2,j}t^2 + c_{3,j}t^3$, via $\delta_{c_{0,j}} \leftarrow \mathsf{Com}(c_{0,j})$, $\delta_{c_{1,j}} \leftarrow \mathsf{Com}(c_{1,j})$, $\delta_{c_{2,j}} \leftarrow \mathsf{Com}(c_{2,j})$, and $\delta_{c_{3,j}} \leftarrow \mathsf{Com}(c_{3,j})$, and $\mathcal{P}$ and $\mathcal{V}$ execute proof-of-opening for each one. Now $\mathcal{P}$ convinces $\mathcal{V}$ that $s_j(0) + s_j(1) = s_{j-1}(r_{j-1})$. Notice that if $\mathcal{V}$ holds commitments $\mathsf{Com}(s_{j-1}(r_{j-1}))$ and $\mathsf{Com}(s_j(0)+s_j(1))$, $\mathcal{P}$ can use proof-of-equality to convince $\mathcal{V}$ that the above equation holds. Further, $\mathcal{V}$ can use the homomorphism property to compute the required commitments: for $s_j(0)+s_j(1) = 2c_{0,j} + c_{1,j} + c_{2,j} + c_{3,j}$, $\mathcal{V}$ computes $\delta_{c_{0,j}}^2 \odot \delta_{c_{1,j}} \odot \delta_{c_{2,j}} \odot \delta_{c_{3,j}}$. Similarly, for $s_{j-1}(r_{j-1})$ $\mathcal{V}$ computes $\delta_{c_{0,j}} \odot \delta_{c_{1,j}}^{r_j} \odot \delta_{c_{2,j}}^{r_j^2} \odot \delta_{c_{3,j}}^{r_j^3}$.

The first sum-check round ($j = 1$) is an exception to the above: rather than a commitment to s_0, $\mathcal{V}$ holds a commitment to a value that purportedly equals $s_1(0) + s_1(1)$. For the sum-check invocation at layer 1, this value is C_0, which $\mathcal{V}$ computed in Step 1. For subsequent layers, the value is the result of the preceding mini-protocol invocation, which we discuss below.

In the final round j_{last}, $\mathcal{V}$ computes a commitment W to $s_{j_{last}}(r_{j_{\text{last}}})$ as described above. $\mathcal{P}$ sends commitments X, Y, and Z to v_0, v_1, and $v_0 \cdot v_1$, respectively, and uses proof-of-product to convince $\mathcal{V}$ that the committed values satisfy this product relation. Finally, $\mathcal{V}$ computes $\Omega \leftarrow (X \odot Y)^{\tilde{\mathrm{eq}}(q',r')\cdot \tilde{\mathrm{add}}_1(q,r_1,r_2)} \odot Z^{\tilde{\mathrm{eq}}(q',r')\cdot \tilde{\mathrm{mult}}_1(q,r_1,r_2)}$ and $\mathcal{P}$ uses proof-of-equality to convince $\mathcal{V}$ that W and Ω commit to the same value.

ZK mini-protocols. For random-linear-combination, $\mathcal{V}$ computes $\mathsf{Com}(\mu_0 v_0 + \mu_1 v_1) = X^{\mu_0} \odot Y^{\mu_1}$; this is the purported $\mathsf{Com}(s_1(0) + s_1(1))$ for the next sum-check invocation.

To execute reducing-from-two-points-to-one-point, $\mathcal{P}$ commits to the coefficients of f_H and invokes proof-of-opening for each; $\mathcal{V}$ computes commitments to $f_H(0)$ and $f_H(1)$, and $\mathcal{P}$ uses proof-of-equality to show that these commit to the same values as X and Y; and $\mathcal{V}$ samples υ and computes a commitment to $f_H(\upsilon)$, which it uses in the final step.

Final step. $\mathcal{P}$ now convinces $\mathcal{V}$ that $\mathsf{Com}(f_H(\upsilon))$, the result of the final mini-protocol invocation (which is a commitment to $\tilde{V}_d(r', r_\upsilon)$; §3.2), is consistent with x and w.

We let $m = (x, w)$ denote the concatenation of the input x and the witness w; assume for simplicity that $|x| = |w| = 2^\ell$; interpret x, w, and m as functions (§3.2, "Multilinear extensions"); and let $(r_0, \ldots, r_\ell) = (r', r_\upsilon)$. Then by the definitions of $\tilde{m}$, $\tilde{x}$, and $\tilde{w}$,

$$\tilde{m}(r_0, \ldots, r_\ell) = (1 - r_0) \cdot \tilde{x}(r_1, \ldots, r_\ell) + r_0 \cdot \tilde{w}(r_1, \ldots, r_\ell).$$

By analogy to Gir^{++}'s final step, $\mathcal{V}$'s task is to check that $\tilde{V}_d(r', r_\upsilon)$ is equal to $\tilde{m}(r_0, \ldots, r_\ell)$. $\mathcal{V}$ does this by first computing $\mathsf{Com}(\tilde{m}(r_0, \ldots, r_\ell))$ using the commitments to w that $\mathcal{P}$ sent in Step 0 (above), and then engaging $\mathcal{P}$ in proof-of-equality on $\mathsf{Com}(f_H(\upsilon))$ and $\mathsf{Com}(\tilde{m}(r_0, \ldots, r_\ell))$.

To compute $\mathsf{Com}(\tilde{m}(\cdot))$, $\mathcal{V}$ exploits the following expression [37] for the multilinear extension of $w\colon \{0,1\}^\ell \to \mathbb{F}$:

$$\begin{aligned} \tilde{w}(r_1, \ldots, r_\ell) &= \sum_{b \in \{0,1\}^\ell} w(b) \cdot \prod_{k \in \{1,\ldots,\ell\}} \chi_{b_k}(r_k) \\ &= \sum_{b \in \{0,1\}^\ell} w(b) \cdot \chi_b \end{aligned} \tag{4}$$

where $\chi_{b_k}(r_k) = r_k b_k + (1 - r_k)(1 - b_k)$, $\chi_b = \prod_k \chi_{b_k}(r_k)$, and b_k is the (1-indexed) k^{th} bit of b. In more detail, $\mathcal{V}$ first evaluates each χ_b in linear time [107], and then computes

$$F = \bigodot_{b \in \{0,1\}^\ell} \mathsf{Com}(w(b))^{r_0 \cdot \chi_b(r_1, \ldots, r_\ell)}$$

which is $\mathsf{Com}(r_0 \cdot \tilde{w}(r_1, \ldots, r_\ell))$. It then computes, in the clear, $F' = (1 - r_0) \cdot \tilde{x}(r_1, \ldots, r_\ell)$. Finally, $\mathcal{V}$ computes $\mathsf{Com}(\tilde{m}(r_0, \ldots, r_\ell)) = F \odot \mathsf{Com}(F'; 0)$. Invoking proof-of-equality as described above completes the protocol.

The following theorem formalizes the efficiency of the argument of this section. We leave a formal statement of security properties to the final protocol (§7).

Theorem 2. *Let $C(\cdot,\cdot)$ be a layered arithmetic circuit of fan-in two, consisting of N identical sub-computations, each of depth d, with all layers of each sub-computation having width at most G. Assuming the existence of computationally binding, perfectly hiding homomorphic commitment schemes that support proof-of-opening, proof-of-equality, and proof-of-product (Appx. A) with running times upper-bounded by κ, there exists a PZK argument for the NP relation "$\exists w$ such that $C(x, w) = y$." The protocol requires $d \log(G)$ rounds of communication, and has communication complexity $\Theta(|y| + (|w| + d \log G) \cdot \lambda)$, where λ is a security parameter. Given a w such that $C(x, w) = y$, the prover runs in time $\Theta(dNG + G \log G + (|w| + d \log G) \cdot \kappa)$. Verifier runtime is $\Theta(|x| + |y| + dG + (|w| + d \log(NG)) \cdot \kappa)$.*

The above follows from the more general Theorem 3.1 of [8].

5 Reducing the cost of sum-checks

In the PZK argument from Section 4, the prover sends a separate commitment for every message element of Gir^{++} (§3.2), and then independently proves knowledge of how to open each commitment. This leads to long proofs and many expensive cryptographic operations for the verifier.

In this section, we explain how to reduce this communication and the number of cryptographic operations for the verifier by exploiting *multi-commitment* schemes, in which a commitment to a vector of elements has the same size as a commitment to a single element. The Pedersen commitment [85] (Appx. A) supports multi-commitments.

Dot-product proof protocol. Our starting point is an existing protocol for multi-commitments, which we call proof-of-dot-prod. With this protocol, a prover that knows the openings of two commitments, one to a vector $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n$ and one to a scalar $y \in \mathbb{F}$, can prove in zero-knowledge that $y = \langle \vec{a}, \vec{x} \rangle$ for a public $\vec{a} \in \mathbb{F}^n$. The protocol is defined in Appendix A.2.

Squashing $\mathcal{V}$'s checks. To exploit proof-of-dot-prod, we first recall from Section 4 that in each round j of each sum-check invocation in Gir^{++}, $\mathcal{P}$ sends commitments to $c_{0,j}$, $c_{1,j}$, $c_{2,j}$, and (only in the first b_N rounds) $c_{3,j}$. Next, $\mathcal{P}$ proves to $\mathcal{V}$ that $2c_{0,j} + c_{1,j} + c_{2,j} + c_{3,j} = s_{j-1}(r_{j-1})$ (i.e., that $s_j(0) + s_j(1) = s_{j-1}(r_{j-1})$). Finally, $\mathcal{V}$ computes a commitment to $s_j(r_j) = c_{0,j} + c_{1,j} r_j + c_{2,j} r_j^2 + c_{3,j} r_j^3$ for the next round.

Combining the above equations yields $c_{3,j+1} + c_{2,j+1} + c_{1,j+1} + 2c_{0,j+1} - (c_{3,j} r_j^3 + c_{2,j} r_j^2 + c_{1,j} r_j + c_{0,j}) = 0$. $\mathcal{V}$'s final check can likewise be expressed as a linear equation in terms of $v_0, v_1, c_{2,n}, c_{1,n}, c_{0,n}$, and wiring predicate evaluations (§3.2) ($n = b_N + 2b_G$). We can thus write $\mathcal{V}$'s checks during the rounds of the sum-check protocol as the matrix-vector product

$$\begin{bmatrix} M_1 \\ \vdots \\ M_{b_N + 2b_G + 1} \end{bmatrix} \cdot \vec{\pi} = \begin{bmatrix} s_0 \\ 0 \\ \vdots \end{bmatrix} \tag{5}$$

Each M_k is a row in $\mathbb{F}^{4b_N + 6b_G + 3}$ encoding one of $\mathcal{V}$'s checks and $\vec{\pi}$ is a column in $\mathbb{F}^{4b_N + 6b_G + 3}$ comprising $\mathcal{P}$'s messages. ($4b_N + 6b_G + 3$ accounts for b_N rounds with cubic s_j, $2b_G$ rounds with quadratic s_j, and the final values v_0, v_1, and $v_0 v_1$; §4.)

Now we can combine all of the linear equality checks encoded in Equation (5) into a single check, namely, by multiplying each row k by a random coefficient ρ_k and summing the rows.

Lemma 3. *For any $\vec{\pi} \in \mathbb{F}^\ell$, and any matrix $M \in \mathbb{F}^{n+1 \times \ell}$ with rows $M_1, \ldots, M_{n+1}$ for which Eq.* (5) *does not hold, then*

$$\Pr_\rho \left[\left\langle \left(\sum \rho_k \cdot M_k \right), \vec{\pi} \right\rangle = \rho_1 \cdot s_0 \right] \leq 1/|\mathbb{F}|$$

Proof. Observe that $\langle(\sum \rho_k \cdot M_k), \vec{\pi}\rangle$ is a polynomial in $\rho_1, \ldots, \rho_{n+1}$ of total degree 1 (i.e., a linear function in $\rho_1, \ldots, \rho_{n+1}$). Call this linear polynomial ϕ. The coefficients of ϕ are the entries of $M \cdot \vec{\pi}$. Similarly, $\rho_1 \cdot s_0$ is a linear polynomial ψ in $\rho_1, \ldots, \rho_{n+1}$, whose coefficients are the entries of $[s_0, 0, \ldots, 0]$. Note that if Equation (5) does not hold, then ϕ and ψ are distinct polynomials, each of total degree 1. The lemma now follows from the Schwartz-Zippel lemma. □

Putting the pieces together. Lemma 3 implies that, once $\mathcal{P}$ has committed to $\vec{\pi}$, it can use proof-of-dot-prod to convince $\mathcal{V}$ of the sum-check result in one shot. For soundness in Gir^{++}, however, $\mathcal{P}$ must commit to $c_{3,j}, c_{2,j}, c_{1,j}, c_{0,j}$ before the Verifier sends r_j. This means that $\mathcal{P}$ cannot send $\mathsf{Com}(\vec{\pi})$ all at once.

Instead, we observe that $\mathcal{P}$ can send the commitment to $\vec{\pi}$ *incrementally*, using one group element per round of the sum-check. That is, in each round of the sum-check protocol, $\mathcal{P}$ commits to a vector encoding the coefficients of that round's polynomial, and $\mathcal{V}$ responds with its random coin r_j. After $\mathcal{P}$ has committed to all of its messages for the sum-check, $\mathcal{P}$ and $\mathcal{V}$ engage in the protocol of Figure 1, which encodes $\mathcal{V}$'s checks for *all* rounds of the sum-check protocol at once. This protocol replaces $\mathcal{V}$'s checks in Step 2 of the protocol of Section 4.

Lemma 4. *The protocol of Figure 1 is a complete, honest-verifier perfect ZK argument, with generalized special soundness under the discrete log assumption, that its inputs constitute an accepting sum-check relation: on input a commitment C_0, commitments $\{\alpha_j\}$ to polynomials $\{s_j\}$ in a sum-check invocation, rows $\{M_k\}$ of the matrix of Equation* (5)*, and commitments $X = \mathsf{Com}(v_0)$, $Y = \mathsf{Com}(v_1)$, and Z, where $\{r_j\}$ are $\mathcal{V}$'s coins from the sum-check and $n{=}b_N + 2b_G$, the protocol proves that $C_0{=}\mathsf{Com}(s_1(0){+}s_1(1))$; $s_j(0){+}s_j(1){=}s_{j-1}(r_{j-1}), j{\in}\{2, \ldots, n\}$; and $s_n(r_n){=}Q_{\ldots,i}$ evaluated with v_0, v_1 (per §3.2).*

Lemma 4 is proved in Appendix A.4. Relative to Step 2 of Section 4, the protocol of Figure 1 reduces sum-check communication by ≈3×. It also reduces $\mathcal{P}$'s and $\mathcal{V}$'s cryptographic costs by ≈4× and ≈5×, respectively.

proof-of-sum-check($C_0, \{\alpha_j\}, \{M_k\}, X, Y, Z$)

Inputs: $C_0 = \mathsf{Com}(s_0; r_{C_0})$.

$\{\alpha_j\}$ are all of $\mathcal{P}$'s messages from a sum-check invocation: at each round j of the sum-check protocol, $\mathcal{P}$ has sent

$$\alpha_j \leftarrow \mathsf{Com}((c_{3,j}, c_{2,j}, c_{1,j}, c_{0,j}); r_{\alpha_j})$$

$\{M_k\}$ is defined as in Equation (5) and Lemma 3. (These vectors encode $\mathcal{V}$'s random coins $\{r_j\}$ from the sum-check.)

$X = \mathsf{Com}(v_0; r_X)$, $Y = \mathsf{Com}(v_1; r_Y)$, $Z = \mathsf{Com}(v_0 v_1; r_Z)$.

Definitions: $n = b_N + 2b_G$; $\vec{\pi}$ is defined as in Equation (5); $\{\rho_k\}$ are chosen by $\mathcal{V}$ (see below); $\vec{J} = \sum \rho_k \cdot \vec{M}_k$; (J_X, J_Y, J_Z) are the last 3 elements of $\vec{J}$; $\vec{\pi}^*$ and $\vec{J}^*$ are all but the last three elements of $\vec{\pi}$ and $\vec{J}$, respectively.

1. $\mathcal{P}$ and $\mathcal{V}$ execute proof-of-product (§4) on X, Y, and Z.
2. $\mathcal{P}$ picks $r_{\delta_1}, \ldots, r_{\delta_n} \in_R \mathbb{F}$ and $\vec{d} \in_R \mathbb{F}^{4b_N+6b_G}$ where $\vec{d} = (d_{c_{3,1}}, d_{c_{2,1}}, d_{c_{1,1}}, d_{c_{0,1}}, \ldots, d_{c_{0,n-1}}, d_{c_{2,n}}, d_{c_{1,n}}, d_{c_{0,n}})$. $\mathcal{P}$ computes and sends
$$\delta_j \leftarrow \mathsf{Com}((d_{c_{3,j}}, d_{c_{2,j}}, d_{c_{1,j}}, d_{c_{0,j}}); r_{\delta_j}), \quad j \in \{1, \ldots, n\}$$
3. $\mathcal{V}$ chooses and sends $\rho_1, \ldots, \rho_{n+1} \in_R \mathbb{F}$.
4. $\mathcal{P}$ picks $r_C \in_R \mathbb{F}$, then computes and sends
$$C \leftarrow \mathsf{Com}(\langle \vec{J}^*, \vec{d}\rangle; r_C)$$
5. $\mathcal{V}$ chooses and sends challenge $c \in_R \mathbb{F}$.
6. $\mathcal{P}$ computes and sends
$$\begin{aligned} \vec{z} &\leftarrow c \cdot \vec{\pi}^* + \vec{d} \\ z_{\delta_j} &\leftarrow c \cdot r_{\alpha_j} + r_{\delta_j}, \quad j \in \{1, \ldots, n\} \\ z_C &\leftarrow c \cdot (\rho_1 r_{C_0} - J_X r_X - J_Y r_Y - J_Z r_Z) + r_C \end{aligned}$$
7. $\mathcal{V}$ rejects unless the following holds, where we denote $\vec{z} = (z_{c_{3,1}}, z_{c_{2,1}}, z_{c_{1,1}}, z_{c_{0,1}}, \ldots, z_{c_{0,n-1}}, z_{c_{2,n}}, z_{c_{1,n}}, z_{c_{0,n}})$:
$$\mathsf{Com}((z_{c_{3,j}}, z_{c_{2,j}}, z_{c_{1,j}}, z_{c_{0,j}}); z_{\delta_j}) \stackrel{?}{=} \alpha_j^c \odot \delta_j \quad j \in \{1, \ldots, n\}$$
$$(C_0^{\rho_1} \odot X^{-J_X} \odot Y^{-J_Y} \odot Z^{-J_Z})^c \odot C \stackrel{?}{=} \mathsf{Com}(\langle \vec{J}^*, \vec{z}\rangle; z_C)$$

FIGURE 1—This protocol proves the statement derived by applying Lemma 3 to Equation (5), i.e., that the sum-check whose transcript is encoded in the protocol's inputs is accepting. Values corresponding to $c_{3,j}$ are elided for all sum-check rounds j having quadratic s_j.

6 Reducing the cost of the witness

In the protocol of Section 4, $\mathcal{P}$ sends a separate commitment to each element $w_1, \ldots, w_\ell$ of the witness w (§4, "Step 0."). This means that handling a circuit relation with $|w|$ witness elements requires a proof whose size is at least proportional to $|w|$. In this section, we describe a new commitment scheme for multilinear polynomials that reduces witness commitment size (and thus proof size) to sub-linear in $|w|$; it also reduces $\mathcal{V}$'s computation cost to sub-linear in $|w|$ (§6.1). To begin, we require each sub-AC to have separate input and witness elements; we relax this restriction by introducing a *redistribution layer* that allows input and witness sharing among sub-ACs (§6.2).

6.1 A commitment scheme for multilinear polynomials

In Section 4, $\mathcal{V}$'s final step checks that $\mathcal{P}$'s commitments to w are consistent with its other messages by evaluating $\tilde{w}$ (the MLE of w; §3.2, "Multilinear extensions"). Zhang et al. [112] show, in the non-ZK setting, that $\mathcal{V}$ can outsource this evaluation to $\mathcal{P}$. We apply their idea to the ZK setting,[4] reducing communication and saving $\mathcal{V}$ computation, by devising a *polynomial commitment scheme* [66] tailored to multilinear polynomials. Informally, such schemes are hiding and binding (§3.1, Def. 5); they also allow the sender to evaluate a committed polynomial at any point and prove that the evaluation is consistent with the commitment.

Our commitment scheme builds on a matrix commitment idea due to Groth [57] and an inner-product argument due to

[4]In concurrent and independent work, Zhang et al. extend to ZK [113]; see §2.

Bünz et al. [31]. We begin by describing a simplified version of the scheme that gives $O(\sqrt{|w|})$ communication and $\mathcal{V}$ runtime; we then generalize this to O(Sp) communication and O(Ti) $\mathcal{V}$ runtime, $\text{Ti} \geq \sqrt{|w|}, \text{Sp} \cdot \text{Ti} = |w|$. We assume WLOG for notational convenience that $2^\ell = |x| = |w|$.

Square-root commitment scheme. In its final check, $\mathcal{V}$ evaluates $\tilde{w}(r_1, \ldots, r_\ell)$ by computing a commitment to the dot product $\langle(w_0, \ldots, w_{2^\ell-1}), (\chi_0, \ldots, \chi_{2^\ell-1})\rangle$ (Eq. (4), §4). Consider the following strawman protocol for computing this commitment: in Step 0 (§4), $\mathcal{P}$ sends one multi-commitment to w. Later, $\mathcal{P}$ sends a commitment ω, and $\mathcal{P}$ and $\mathcal{V}$ execute proof-of-dot-prod (§5) on $\mathsf{Com}(w)$, ω, and $(\chi_0, \ldots)$. This protocol convinces $\mathcal{V}$ that $\omega = \mathsf{Com}(\tilde{w}(\cdot))$, but does not reduce communication: proof-of-dot-prod requires $\mathcal{P}$ to send $O(|w|)$ messages (Appx. A.2).

To reduce communication, we exploit the structure of the polynomial $\tilde{w}$ and a matrix commitment due to Groth [57]. At a high level, this works as follows (details below). In Step 0, $\mathcal{P}$ encodes w as a matrix T, then sends commitments $\{T_k\}$ to the *rows* of T. Then, in the final step, $\mathcal{P}$ sends a commitment ω that it claims is to $\tilde{w}(r_1, \ldots, r_\ell)$; $\mathcal{V}$ uses $\{T_k\}$ to compute one multi-commitment T'; and $\mathcal{P}$ and $\mathcal{V}$ execute proof-of-dot-prod on T' and ω. In total, communication cost is $O(2^{\ell/2})$.

In more detail: T is the $2^{\ell/2} \times 2^{\ell/2}$ matrix whose column-major order is w, i.e., $T_{i+1,j+1} = w_{i+2^{\ell/2}\cdot j}$. Before defining T' and the proof-of-dot-prod invocation, we define

$$\check{\chi}_b = \prod_{k=1}^{\ell/2} \chi_{b_k}(r_k) \qquad \hat{\chi}_b = \prod_{k=\ell/2+1}^{\ell} \chi_{b_k}(r_k)$$

$$L = (\check{\chi}_0, \check{\chi}_1, \ldots, \check{\chi}_{2^{\ell/2}-1}) \qquad R = (\hat{\chi}_0, \hat{\chi}_{2^{\ell/2}}, \ldots, \hat{\chi}_{2^{\ell/2}\cdot(2^{\ell/2}-1)})$$

To compute T' from commitments $\{T_k\}$ to the rows of T, $\mathcal{V}$ evaluates L (in time $O(2^{\ell/2})$ [109, §3.3]) and uses it to compute

$$T' = \bigodot_{k=0}^{2^{\ell/2}-1} T_{k+1}^{\check{\chi}_k} = \mathsf{Com}(L \cdot T) \tag{6}$$

Finally, $\mathcal{P}$ sends a commitment ω and uses proof-of-dot-prod to convince $\mathcal{V}$ that the dot product of R with the vector committed in T' equals the value committed in ω.

The above proves to $\mathcal{V}$ that $\omega = \mathsf{Com}(\tilde{w}(r_0, \ldots, r_\ell))$, as we now argue. For 1-indexed L and R, we have

$$L_{i+1} \cdot R_{j+1} = \check{\chi}_i \cdot \hat{\chi}_{2^{\ell/2}\cdot j} = \chi_{i+2^{\ell/2}\cdot j}$$

This is true because $\check{\chi}_b$ comprehends the lower $\ell/2$ bits of b, and $\hat{\chi}_b$ the upper $\ell/2$ bits. Then by the definition of T, we have

$$\begin{aligned} L \cdot T \cdot R^{\mathrm{T}} &= \sum_{i=0}^{2^{\ell/2}-1} \sum_{j=0}^{2^{\ell/2}-1} T_{i+1,j+1} \cdot L_{i+1} \cdot R_{j+1} \\ &= \sum_{i=0}^{2^{\ell/2}-1} \sum_{j=0}^{2^{\ell/2}-1} w_{i+2^{\ell/2}\cdot j} \cdot \chi_{i+2^{\ell/2}\cdot j} = \sum_{k=0}^{2^\ell-1} w_k \cdot \chi_k \end{aligned}$$

If $\mathcal{V}$ accepts $\mathcal{P}$'s proof-of-dot-prod on T', ω, and R, then by Equation (6), $\omega = \mathsf{Com}(L \cdot T \cdot R^{\mathrm{T}}) = \mathsf{Com}(\sum_{k=0}^{2^\ell-1} w_k \cdot \chi_k)$, which equals $\mathsf{Com}(\tilde{w}(r_0, \ldots, r_\ell))$ (Eq. (4), §4) as claimed.

In total, communication is $O(2^{\ell/2})$ (for $\{T_k\}$ plus the proof-of-dot-prod invocation), and $\mathcal{V}$'s computational cost is $O(2^{\ell/2})$ (for computing L, R, and T', and executing proof-of-dot-prod).

Reducing the cost of proof-of-dot-prod. In the above protocol, proof-of-dot-prod establishes a lower bound on communication cost. To reduce proof-of-dot-prod's cost, we use an idea due to Bünz et al. [31], who give a dot-product protocol that has cost logarithmic in the length of the vectors. Their protocol works over two committed vectors; we require one that works over one committed and one public vector. In Appendix A.3, we adapt their protocol to the syntax of proof-of-dot-prod; we refer to the result as $\text{proof}_{\log}$-of-dot-prod. Whereas proof-of-dot-prod requires $\mathcal{P}$ to send $4+n$ elements for vectors of length n, $\text{proof}_{\log}$-of-dot-prod requires only $4+2\log n$. In both protocols, $\mathcal{V}$'s computational cost is dominated by a multi-exponentiation [88] of length n.

The full commitment scheme differs from the square-root one in that $\mathcal{P}$ and $\mathcal{V}$ invoke $\text{proof}_{\log}$-of-dot-prod (rather than proof-of-dot-prod) on T', R, and ω. For $T, L, R, \check{\chi}_b, \hat{\chi}_b$ as defined above, $\mathcal{P}$ sends $4+2^{\ell/2}+2\log 2^{\ell/2}$ elements, and $\mathcal{V}$'s runtime is dominated by two multi-exponentiations of length $2^{\ell/2}$, one to compute T' and the other to execute $\text{proof}_{\log}$-of-dot-prod. This gives the same asymptotics as the square-root scheme with $\approx 2\times$ less communication (but with $\approx 3\times$ more computation for $\mathcal{P}$).

More importantly, $\text{proof}_{\log}$-of-dot-prod gives the freedom to reduce communication in exchange for increased $\mathcal{V}$ runtime. For a parameter ι, we redefine T to be the $2^{\ell/\iota} \times 2^{\ell-\ell/\iota}$ matrix whose column-major order is w; redefine $\check{\chi}_b$ to comprehend the lower ℓ/ι bits of b, and $\hat{\chi}_b$ the upper $\ell - \ell/\iota$ bits; and redefine

$$L = (\check{\chi}_0, \check{\chi}_1, \ldots, \check{\chi}_{2^{\ell/\iota}-1}) \quad R = (\hat{\chi}_0, \hat{\chi}_{2^{\ell/\iota}}, \ldots, \hat{\chi}_{2^{\ell/\iota}\cdot(2^{\ell-\ell/\iota}-1)})$$

T has $2^{\ell/\iota}$ rows and T' is a vector of $2^{\ell-\ell/\iota}$ elements, so $\mathcal{P}$ sends $2^{\ell/\iota}$ commitments in Step 0 and $4+\log 2^{\ell-\ell/\iota}$ elements for $\text{proof}_{\log}$-of-dot-prod, which is $O(2^{\ell/\iota})$ in total. Computing T' costs $\mathcal{V}$ one multi-exponentiation of length $2^{\ell/\iota}$, and executing $\text{proof}_{\log}$-of-dot-prod costs one of length $2^{\ell-\ell/\iota}$, which is $O(2^{\ell/\iota}+2^{\ell-\ell/\iota})$ in total. Since this is at least $O(2^{\ell/2})$, $\mathcal{V}$'s runtime is at least $O(\sqrt{|w|})$. We formalize immediately below.

Lemma 5. *Suppose WLOG that $w \in \mathbb{F}^{2^{\iota\cdot\ell'}}$ for $\iota \geq 2$, and that $\mathcal{P}$ commits to w as described above using $2^{\ell'} = |w|^{1/\iota}$ multi-commitments. Then for any $(r_1, \ldots, r_{\iota\cdot\ell'})$, $\mathcal{P}$ can send a commitment ω and argue that it commits to $\tilde{w}(r_1, \ldots, r_{\iota\cdot\ell'})$ in communication $O(|w|^{1/\iota})$, where $\mathcal{V}$ runs in $O(|w|^{(\iota-1)/\iota})$ steps. This is a complete, honest-verifier perfect zero-knowledge argument with generalized special soundness under the discrete log assumption.*

Completeness and ZK follow from the analysis in Appendix A.3. We provide an analysis for generalized special soundness in Appendix A.5. We have described this protocol in terms of the multilinear extension of w, but it generalizes to any multilinear polynomial f using the fact that T comprises the evaluations of f at all binary inputs.

6.2 Sharing witness elements in the data-parallel setting

We have thus far regarded the computation as having one large input and one large witness. When evaluating a data-parallel computation, this means that the sub-ACs' inputs must be disjoint slices of the full input (and similarly for the witness). However, this is not sufficient in many cases of interest.

Consider a case where $\mathcal{P}$ wants to convince $\mathcal{V}$ that it knows leaves of a Merkle tree corresponding to a supplied root. Verifying a witness with M leaves requires $2M-1$ invocations of a hash function. We encode this as a computation with $2M-1$ sub-ACs laid side-by-side, each encoding the hash function.[5] Then, for sub-AC$_b$ processing sub-AC$_a$'s output, $\mathcal{P}$ supplies the purported output to both, and sub-AC$_a$ just *checks* that value and outputs a bit indicating correctness. This is necessary for zero-knowledge: all AC outputs are public, whereas sub-AC$_a$'s output (an intermediate value in the computation) must not be revealed to $\mathcal{V}$.

This arrangement requires sub-ACs to share witness elements—but duplicating entries in the matrix T (§6.1) is not a solution, because $\mathcal{V}$ cannot detect if a cheating $\mathcal{P}$ produces T that gives different values to different sub-ACs. One possibility is a hybrid vector-scalar scheme: $\mathcal{P}$ supplies scalar commitments for each shared witness element and matrix commitment $\{T_k\}$ for the rest. Then, for a scalar commitment δ, $\mathcal{V}$ "injects" the committed value into input index b by multiplying the commitment to $\tilde{V}_d(r', r_v)$ (§4, "Final step") by $\delta^{-r_0 \cdot \chi_b}$.[6] (In contrast, the protocol of Section 6.1 maps each entry of T to a fixed input index.)

This approach works when the number of shared witness elements is small, but it is inefficient when there are many shared elements: each shared element requires a separate commitment and proof-of-opening invocation. For such cases, we enable sharing of witness elements by modifying the arithmetic circuit encoding the NP relation. Specifically, after constructing a data-parallel AC corresponding to the computation, we add one non-data-parallel *redistribution layer* (RDL) whose inputs are the full input and witness, and whose outputs feed the input layers of each sub-AC. Since the RDL is not data parallel, there are no restrictions on how its inputs connect to its outputs, meaning that $\mathcal{V}$ can use it to ensure that the same witness element feeds multiple sub-ACs: the sum-check protocol forces $\mathcal{P}$ to respect the wiring of the RDL, so $\mathcal{P}$ cannot equivocate about w.

Moreover, since the RDL only "re-wires" its inputs, the sum-check invocation corresponding to this layer of the AC can be optimized to require fewer rounds and a simplified final check. Observe that the redistribution layer only requires one-input "pass" gates that copy their input to their output. Thus, following a simplification of the CMT protocol [37, 103], we have that

$$\tilde{V}_{d-1}(q', q) = \sum_{h \in \{0,1\}^{\log(|m|)}} \tilde{\text{pass}}((q', q), h) \cdot \tilde{V}_d(h)$$

where $\tilde{\text{pass}}((q', q), h)$ is the MLE of a wiring predicate (§3.2) that is 1 when the RDL connects from the AC input with index h to input q in sub-AC q', and 0 otherwise. A sum-check over

$$\text{RDL}_{(q',q)}(h) = \tilde{\text{pass}}((q', q), h) \cdot \tilde{V}_d(h)$$

requires $\log(|m|) = \log(|x| + |w|)$ rounds, at the end of which $\mathcal{V}$ evaluates $\text{RDL}_{(q',q)}$ at a random point. This requires $\mathcal{V}$ to evaluate $\tilde{\text{pass}}$, but in contrast to $P_{\ldots,i}$ or $Q_{\ldots,i}$ (§3.2), it only requires *one* evaluation of $\tilde{V}_d$, which $\mathcal{V}$ can check (via the protocol of §6.1) without invoking a mini-protocol (§3.2).

By a standard analysis [37], $\mathcal{P}$'s costs are $O(NG \log |m|)$; $\mathcal{V}$'s primary cost related to the RDL is evaluating $\tilde{\text{pass}}$ at one point, which costs $O(|m| + NG)$ via known techniques [109, §3.3]. We formalize in Theorem 6 (§7).

7 Hyrax: a zkSNARK based on Gir^{++}

We refer to the honest-verifier PZK argument obtained by applying the refinements of Sections 5 and 6 to the protocol of Section 4 as Hyrax-I; pseudocode is given in Appendix D. Since Hyrax-I is a public-coin protocol, we apply the Fiat-Shamir heuristic [45] to produce a zkSNARK that we call Hyrax whose properties we now formalize:

Theorem 6. *Let $C(\cdot,\cdot)$ be a layered AC of fan-in two, consisting of N identical sub-computations, each having d layers whose width is at most G. Under the discrete log assumption in the random oracle model, for every* Sp, Ti *with* $\text{Ti} \geq \sqrt{|w|}$ *and* $\text{Sp} \cdot \text{Ti} = |w|$, *there exists a perfectly complete, perfect zero-knowledge, non-interactive argument with witness-extended emulation for the NP relation "$\exists w$ such that $C(x, w) = y$." $\mathcal{V}$ runs in time* $O(|x| + |y| + dG + (\text{Ti} + d\log(NG)) \cdot \kappa)$ *for κ a bound on the time to compute a commitment; when using an RDL (§6.2), $\mathcal{V}$ incurs an additional* $O(|x| + |w| + NG)$ *cost. $\mathcal{P}$'s messages have size* $O((\text{Sp} + d\log(NG)) \cdot \lambda)$ *for λ a security parameter.*

We formalize Hyrax-I's properties in Appendix B. Hyrax's security properties follow from these and the properties of the Fiat-Shamir heuristic [7, 45].

We note that it is also possible to compile Hyrax-I into an interactive, malicious-verifier PZK argument in the plain model under the decisional Diffie-Hellman assumption [24] using standard techniques [39, 41].

Implementation. Our implementation of Hyrax is based on Giraffe's code [86, 109]. It uses Pedersen commitments (Appx. A) in an elliptic curve group of order $q_{\mathcal{G}}$ and works with ACs over $\mathbb{F}_{q_{\mathcal{G}}}$. We instantiate the random oracle with SHA-256.

The prover takes as input a high-level description of an AC (in the format produced by Giraffe's C compiler), the public inputs, and an auxiliary executable that generates the witness from the public inputs; the prover's output is a proof. The verifier takes as input the same computation description and public inputs plus the proof, and outputs "accept" or "reject."

We implement Gir^{++}, the techniques of Sections 5 and 6, the random oracle, and proof serialization and deserialization by adding 2800 lines of Python and 300 lines of C to the Giraffe code. We also implemented a library for fast multi-exponentiation comprising 750 lines of C that uses the MIRACL Crypto SDK [79] for elliptic curve operations and selects between Straus's [100] and Pippenger's [19, 88] methods, depending on the problem size. Our library supports Curve25519 [18], M221,

[5]The sub-ACs could instead be arranged sequentially. This would avoid the issues described in this subsection, but would dramatically increase circuit depth, and thus the proof length and associated costs when applying our argument.

[6]In fact, this approach works generally for computations over values to which $\mathcal{V}$ holds a commitment whose opening $\mathcal{P}$ knows. It also applies to committed *vectors*: if $\mathcal{V}$ holds a commitment $\xi = \mathsf{Com}(\vec{x})$, it can inject the committed values into a list of indices $(b_1, \ldots)$ as follows: $\mathcal{P}$ produces a commitment δ and proves to $\mathcal{V}$ that it commits to $\mathsf{Com}(\langle (x_1, \ldots), (\chi_{b_1}, \ldots) \rangle)$ with proof$_{\log}$-of-dot-prod; then $\mathcal{V}$ multiplies $\mathsf{Com}(\tilde{V}_d(r', r_v))$ by δ^{-r_0}. This approach requires more communication and $\mathcal{V}$ computation than the protocol of Section 6.1, because it does not assume any particular structure for $(b_1, \ldots)$.

M191, and M159 [2]. Python code calls this library via CFFI [32]. We produce random group elements by hashing, implemented in 200 lines of Sage [93] adapted from a script by Samuel Neves [2].

We have released full source code [63].

8 Evaluation

In this section we ask:

- How does Hyrax compare to several baseline systems, considering proof size and $\mathcal{V}$ and $\mathcal{P}$ execution time?
- How do Hyrax's refinements (§5–6) improve its costs?
- What is the overall effect of trading greater witness-related $\mathcal{V}$ computation for smaller witness commitments (§6.1)?

A careful comparison of built systems shows that, even for modest problem sizes, Hyrax's proofs are smaller than all but the most computationally costly of the baselines; and that its $\mathcal{V}$ and $\mathcal{P}$ execution times are each faster than three of five baselines. We also find that Hyrax's refinements yield multiple-orders-of-magnitude savings in proof size and $\mathcal{V}$ time, and a small constant savings in $\mathcal{P}$ time. Finally, we find that tuning the witness commitment costs gives much smaller proofs, with little effect on total $\mathcal{V}$ time for a computation using an RDL (§6.2).

8.1 Comparison with prior work

Baselines. We compare Hyrax with five state-of-the-art zero-knowledge argument systems with similar properties, detailed below. We also consider *Hyrax-naive*, which implements the protocol of Section 4 without our refinements (§5–6). We do not compare to systems that require trusted setup (see §2, second paragraph), but we discuss them briefly in Section 8.3.

Like Hyrax (and Hyrax-naive), two of the baselines rely on elliptic curve primitives; but their existing implementations use a different elliptic curve than Hyrax. To evaluate like-for-like, we re-implemented them using the Python scaffolding, C cryptographic library, and elliptic curves that Hyrax uses (§7).

The other three baselines do not use elliptic curves, so some mismatch in implementations is unavoidable. For those systems, we used existing implementations written in C or C++.

• *BCCGP-sqrt* is the square-root-communication argument due to Bootle et al. [25]. We implemented this protocol using Hyrax's libraries, as described above. In addition, this protocol uses polynomial multiplication, for which we used NTL [99]. Finally, we wrote a compiler that converts from Hyrax's AC description format to the required constraint format, with rudimentary optimizations like constant folding and common subexpression elimination. Our implementation comprises 1200 lines of Python and 160 lines of C, which we include in our released code [63].

• *Bulletproofs* is the argument due to Bünz et al. [31] (we also adapted the inner-product argument from this work in §6). We implemented this protocol in 300 lines of Python on top of our BCCGP-sqrt code, which we also include in our release.

• *Ligero* [1]: we report on the authors' C++ implementation.

• *ZKB++* [33]: we report runtime of the C implementation of ZKBoo [51, 115]; per the ZKB++ authors, these systems have similar performance [33, §3.2].[7] We report extrapolated proof sizes (which are linear with AC size) from ZKB++ [33, §3.2.1].

• *libSTARK* [11]: we report on the authors' C++ implementation [72] and SHA-256 primitive [11, Fig. 4], which we adapt to the Merkle tree benchmark (described below).

This implementation supports multi-threading, but we restrict it to a single thread for consistency with the other baselines and to focus on total prover work; we discuss in Section 8.3.

Benchmarks. We evaluate Hyrax, Hyrax-naive, BCCGP-sqrt, and Bulletproofs on all benchmarks below, but Ligero, ZKB++, and libSTARK only on Merkle trees; we discuss in Section 8.3.

• *Matrix factoring* (i.e., matrix multiplication) proves to $\mathcal{V}$ that $\mathcal{P}$ knows two matrices whose product equals the public input. We evaluate on 16×16, 32×32, 64×64, and 128×128 matrices, and for each we vary N, the number of parallel executions.

• *Image scaling* establishes that $\mathcal{V}$'s input, a low-resolution image, is a scaled version of a high-resolution image that $\mathcal{P}$ knows. For scaling, we use Lanczos resampling [105], a standard image transformation in which each output pixel is the result of convolving a two-dimensional windowed sinc function [82] with the input image. We evaluate on 4×, 16×, 64×, and 256× scaling, varying the number of pixels.

This is a data-parallel computation where each sub-AC evaluates one pixel of the low-resolution image, but because of the windowed sinc function, sub-ACs for adjacent pixels must share inputs from the high-resolution image. To accommodate this, we use a redistribution layer (RDL; §6.2).

• *Merkle tree* proves to $\mathcal{V}$ that $\mathcal{P}$ knows an assignment to the leaves of a Merkle tree [77] corresponding to a root that $\mathcal{V}$ provides [23].[8] We use SHA-256 for the hash, varying the number of leaves in the tree; we implement a data-parallel computation in which each sub-computation is one invocation of SHA-256; and we connect outputs at one level of the tree to inputs at the next level using an RDL. For M leaves, the benchmark comprises $2M-1$ sub-computations.

To implement SHA-256 efficiently in an arithmetic circuit, we use an approach from prior work [12] for efficient addition modulo 2^{32}. We describe the approach, and an optimization that may be of independent interest, in Appendix C.

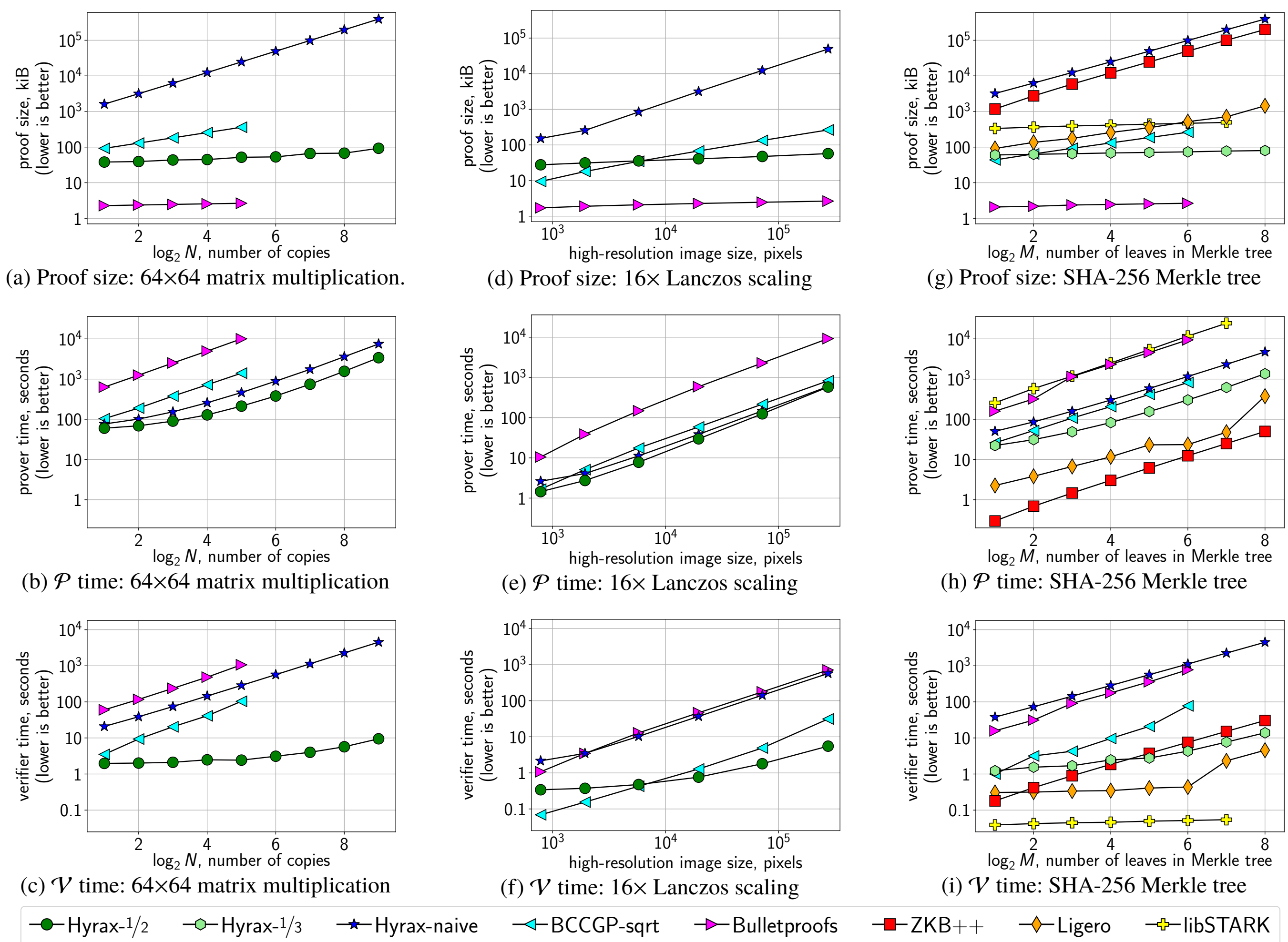
Testbed. We run experiments on Amazon EC2 [3]. For Hyrax, Hyrax-naive, Ligero, and ZKB++, we use c3.4xlarge instances (30 GiB of RAM, 8 Xeon E5-2680v2 cores, 2 threads per core, 2.8 GHz). The BCCGP-sqrt, Bulletproofs, and libSTARK provers are memory intensive ("$\mathcal{P}$ cost," below), so for these we use c3.8xlarge instances (60 GiB of RAM, 16 cores at 2.8 GHz). Only RAM is relevant because we run all tests single threaded.

All testbed machines run Debian GNU/Linux 9 [42]. We run all Python code using PyPy [91], a fast JIT-compiling interpreter.

Security parameters. For Hyrax, Hyrax-naive, BCCGP-sqrt, and Bulletproofs, $\mathcal{G}$ is M191 [2], an elliptic curve over a base field modulo $2^{191}-19$ with a subgroup of order $q_{\mathcal{G}} = 2^{188}+2^{93}+\ldots$, giving ≈90-bit security. We run Ligero, ZKB++, and libSTARK

[7] We run ZKBoo because there is no standalone ZKB++ implementation that can run our benchmarks, only one tailored to the Picnic signature scheme [87].

[8] In related applications (e.g., [111]), $\mathcal{P}$ convinces $\mathcal{V}$ that it knows a *path* from a supplied Merkle root to a leaf. For these systems, a path of length $2M-1$ has essentially the same cost as an M-leaf Merkle tree. We evaluate the full-tree benchmark because it demonstrates a wider range of computation sizes.



(a) Proof size: 64×64 matrix multiplication.
(d) Proof size: 16× Lanczos scaling
(g) Proof size: SHA-256 Merkle tree
(b) $\mathcal{P}$ time: 64×64 matrix multiplication
(e) $\mathcal{P}$ time: 16× Lanczos scaling
(h) $\mathcal{P}$ time: SHA-256 Merkle tree
(c) $\mathcal{V}$ time: 64×64 matrix multiplication
(f) $\mathcal{V}$ time: 16× Lanczos scaling
(i) $\mathcal{V}$ time: SHA-256 Merkle tree

FIGURE 2—Comparison of concrete costs between the baseline systems and Hyrax (§8.1). Hyrax-1/2 is Hyrax where $\mathcal{P}$'s witness commitments have size $|w|^{1/2}$, and likewise Hyrax-1/3 has commitments of size $|w|^{1/3}$ (§6.1). Hyrax-naive is Hyrax without our refinements (§5–6). BCCGP-sqrt [25], Bulletproofs [31], ZKB++ [33], Ligero [1], and libSTARK [11] are prior work. Where the BCCGP-sqrt, Bulletproofs, and libSTARK data are truncated, their provers exceeded available RAM (§8.1). We evaluate ZKB++, Ligero, and libSTARK only on Merkle trees (we discuss in §8.3).

at 2^{-80} soundness error.[9] ACs are over $\mathbb{F}_{q_{\mathcal{G}}}$ and group elements and scalars are 24 bytes, except that Ligero and libSTARK work over smaller fields and ZKB++ works over Boolean circuits.

Method. For each benchmark, we construct a set of arithmetic circuits (and, for image scaling and Merkle trees, RDLs) for a range of computation sizes. We then run each system's prover, feeding the resulting proof into its verifier. We record proof size, and measure time using the high-resolution system clock.

For matrix factoring and image scaling, we set Hyrax's communication and $\mathcal{V}$ runtime to $|w|^{1/2}$ (§6.1). For Merkle trees, we optimize proof size versus $\mathcal{V}$ runtime by setting witness-related communication to $|w|^{1/3}$ and $\mathcal{V}$ runtime to $|w|^{2/3}$; we explore the effect of this setting in Section 8.2.

Results. Figure 2 compares costs for the benchmarks. For matrix factoring and image scaling we show only 64×64 matrices and 16× scaling, respectively; other values give similar results. For an AC $\mathcal{C}$, $\mathcal{M}$ denotes the number of multiplication gates.

Proof size (Figs. 2a, 2d, 2g):

• Hyrax has much larger proofs than Bulletproofs, both asymptotically and concretely.

• Hyrax's proofs are smaller than BCCGP-sqrt's when the cost of the witness commitment dominates the cost of $\mathcal{P}$'s messages in Gir^{++} (i.e., for large enough computations). Specifically, Hyrax's cost tracks $|w|^{1/2}$ ($|w|^{1/3}$ for Merkle trees; Fig. 2g), while BCCGP-sqrt's tracks $\mathcal{M}^{1/2}$. Thus, on matrix factoring (where $|w| \ll \mathcal{M}$) Hyrax has much smaller proofs.

• Hyrax's Merkle tree proofs are asymptotically and concretely smaller than Ligero's: the latter's cost tracks $|\mathcal{C}|^{1/2}$.

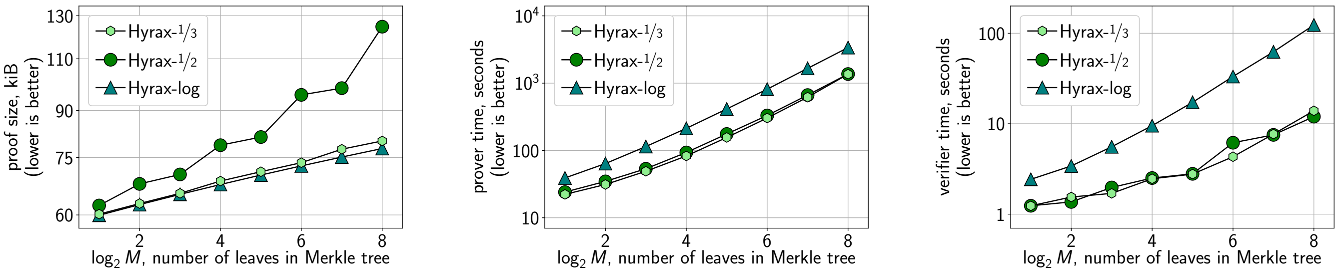
• libSTARK's proof size is asymptotically smaller than Hyrax's, but its proofs are concretely ≈5× larger at these problem sizes.

• ZKB++'s cost is linear in the number of AND gates, and Hyrax-naive's cost tracks $|w|$; both are large.

$\mathcal{P}$ cost (Figs. 2b, 2e, 2h):

• BCCGP-sqrt and Bulletproofs require a number of cryptographic operations proportional to $\mathcal{M}$. Hyrax has lower $\mathcal{P}$ time than these systems because it uses cryptographic operations only

[9] ZKB++, Ligero, and libSTARK give statistical security in the random oracle model, while the other systems make computational assumptions; this makes direct comparison difficult. We have chosen security parameters to give all systems roughly equivalent cost to prove a false statement.



(a) Proof size: SHA-256 Merkle tree (b) $\mathcal{P}$ time: SHA-256 Merkle tree (c) $\mathcal{V}$ time: SHA-256 Merkle tree

FIGURE 3—Proof size and $\mathcal{P}$ and $\mathcal{V}$ runtime for different sizes of $\mathcal{P}$'s witness commitment (§6.1; §8.2). Hyrax-1/2 has commitment size $|w|^{1/2}$, Hyrax-1/3 has commitment size $|w|^{1/3}$, and Hyrax-log has commitment size $\log|w|$. Hyrax-1/2 gives the largest proofs but has the fastest runtimes. Hyrax-log gives the smallest proofs but has the longest runtimes. Hyrax-1/3 gets essentially the best of both for this application.

for $\mathcal{P}$'s messages in Gir^{++} and for w (§4–§6).

- The provers in both BCCGP-sqrt and Bulletproofs ran out of memory for the largest benchmarks (Figs. 2b and 2h) despite having twice as much RAM as Hyrax ("Testbed," above). This is because they operate, roughly speaking, over all wire values in the AC at once. In contrast, Hyrax's $\mathcal{P}$ works layer-by-layer (§3.2).[10]
- Hyrax's $\mathcal{P}$ is more expensive than either ZKB++'s or Ligero's, because those systems do not use any public-key cryptography.
- While Ligero's $\mathcal{P}$ is asymptotically more costly than Hyrax's $\mathcal{P}$, this is not apparent at the problem sizes we consider.
- libSTARK's $\mathcal{P}$ is 12–40× more expensive than Hyrax's for these problem sizes. It is also memory intensive: for the largest problem, it exceeded available RAM despite having twice as much as Hyrax ("Testbed," above).
- Hyrax's refinements compared to Hyrax-naive (§5–6) yield a constant factor lower $\mathcal{P}$ cost, at most ≈3×.

$\mathcal{V}$ *time* (Fig. 2c, 2f, 2i):

- For matrix factoring, Hyrax's $\mathcal{V}$ bottleneck is sum-check invocations for small N, and $\tilde{V}_y$ evaluation for large N (§3.2). The RDL (§6.2) dominates $\mathcal{V}$'s costs in the other two benchmarks.
- Hyrax's $\mathcal{V}$ cost is lower than BCCGP-sqrt for large enough problems: the latter requires $O(\mathcal{M})$ field operations.
- Hyrax's $\mathcal{V}$ cost is much less than Bulletproofs's: the latter requires a multi-exponentiation of length $2\mathcal{M}$ (which can be computed using $O(\mathcal{M}/\log\mathcal{M})$ cryptographic operations [88]).
- ZKB++ has verification cost linear in the problem size, so Hyrax wins on large enough problems.
- Ligero's $\mathcal{V}$ amortizes its bottleneck computation over repeated SHA-256 instances [1, §5.4], so over this range of problem sizes it has sublinear scaling and concretely fast verification time.
- libSTARK's $\mathcal{V}$ has the best asymptotics among all systems and extremely low concrete costs.
- Hyrax-naive requires cryptographic operations proportional to $|w|$; Hyrax's refinements give more than 100× savings.

8.2 Effect of trading $\mathcal{V}$ runtime for smaller proofs

Method. We run the Merkle tree benchmark using the same setup as in Section 8.1, except that we vary the size of $\mathcal{P}$'s witness commitment (§6.1). We experiment with commitments of size $\log|w|$, $|w|^{1/3}$, and $|w|^{1/2}$. $\mathcal{V}$'s witness-related work at these three settings is $O(|w|)$, $O(|w|^{2/3})$, and $O(|w|^{1/2})$, respectively.

Results. Figure 3 shows proof size and runtime for the specified commitment sizes. For Hyrax-1/2, proof sizes are large but $\mathcal{P}$ and $\mathcal{V}$ runtimes are small; Hyrax-log is the opposite. Hyrax-1/3 has similar runtimes to Hyrax-1/2: $\mathcal{P}$'s costs are dominated by Gir^{++}, $\mathcal{V}$'s by the RDL (§6.2). Meanwhile, its proof sizes are not much larger than Hyrax-log, because the Gir^{++}-related proof costs are the same in both cases, and because the constants hidden in the asymptotic notation mean that the log and cube-root protocols have similar concrete costs at these problem sizes. In other words, Hyrax-1/3 gets very nearly the best of both worlds.

8.3 Discussion

Our results show that Hyrax is competitive with the baselines, and that the refinements of Sections 5 and 6 give substantial improvements. Hyrax gives smaller proofs than all but Bulletproofs, which pays for its smaller proofs with very high computational costs. Meanwhile, for problem sizes of practical interest, only Ligero is faster for both $\mathcal{P}$ and $\mathcal{V}$; ZKB++ has faster $\mathcal{P}$ but often slower $\mathcal{V}$; libSTARK has faster $\mathcal{V}$ but much slower $\mathcal{P}$; and all three systems produce larger proofs than Hyrax.

On the other hand, there are several limitations to this analysis. First, because Gir^{++} is geared to data-parallel computations (§3.2; Thm. 1), Hyrax is competitive with prior work primarily when computations contain sufficient parallelism or are amenable to batching; this is evident in the way Hyrax's performance relative to the baselines improves as parallelism increases in Figure 2. While an RDL (§6.2) lets Hyrax take advantage of parallelism within one computation (as it did in the Merkle tree and image scaling benchmarks), not all applications fit these paradigms. Moreover, the RDL is asymptotically and concretely costly for $\mathcal{V}$; eliminating this bottleneck is future work.

Second, we compare ZKB++, Ligero, and libSTARK only on the SHA-256 Merkle tree benchmark. This makes sense for ZKB++ because it is geared to Boolean circuits, where SHA-256 is a natural benchmark; similarly, Ligero's primary evaluation is on SHA-256 [1, §6]. For libSTARK, however, a hash function that is more efficient in $\mathbb{F}_{2^{64}}$ would improve performance [11, Fig. 4]; future work is to compare Hyrax and all baselines on Merkle trees using hash functions tailored to each system (e.g., [11, Appx. E; 15, §5.2; 30, §3.2]). Furthermore, Ligero and libSTARK can in principle work over large fields, but the current

[10]It is probably possible to engineer the BCCGP-sqrt and Bulletproofs provers to reduce memory requirements, e.g., by streaming from disk. We attempted a standard approach—paging memory to an array of fast SSDs—but this caused thrashing and dramatically worsened runtimes.

implementations do not [11, 106], so we could not evaluate on matrix factoring or image scaling; future work is to do so.[11]

Third, our comparison does not consider multi-threaded performance because, to our knowledge, libSTARK is the only baseline with a multi-threaded implementation [11, 72]. Prior work [104, 108, 109] suggests that Gir^{++} is highly parallelizable; exploring this in Hyrax is future work.

Finally, our comparison does not consider argument systems like libsnark [16, 71] that require trusted setup and non-standard, non-falsifiable assumptions (§2, paragraph 2); Hyrax's goal is to avoid these requirements. Ignoring this, Hyrax's proofs are bigger: libsnark's proofs are a constant ≈300-bytes, independent of the AC C. Hyrax's $\mathcal{P}$ cost is concretely and asymptotically smaller: libsnark has a logarithmic overhead in $|C|$, and it requires cryptographic operations per AC gate, while Hyrax's $\mathcal{P}$ is essentially linear in computation size and requires cryptographic operations only for $\mathcal{P}$'s Gir^{++} messages and for w (§4–§6). For $\mathcal{V}$, libsnark's offline setup is very expensive [110, §5.4], and it must be performed by $\mathcal{V}$ or someone $\mathcal{V}$ trusts; but libsnark's online $\mathcal{V}$ costs are essentially always cheaper than Hyrax's (and roughly comparable to libSTARK's, in practice).

9 Conclusion

We have described a succinct zero-knowledge argument for NP with no trusted setup and low concrete cost for both the prover and the verifier, based on standard cryptographic assumptions. This scheme is practical because it tightly integrates three components: a state-of-the-art interactive proof (IP), which we tweak to reduce communication complexity; a highly optimized transformation from IPs to zero-knowledge arguments following the approach of Ben-Or et al. [8] and Cramer and Damgård [40]; and a new cryptographic commitment scheme tailored to multilinear polynomials that adapts prior work [31, 57] to allow a sender to commit to a $\log G$-variate multilinear polynomial and later to open it at one point, with $O(G^{1/\iota})$ total communication and $O(G^{(\iota-1)/\iota})$ receiver runtime for any $\iota \geq 2$. A careful comparison with prior work shows that our argument system is competitive on both proof size and computational costs. Key future work is to further reduce proof size without increasing verifier runtime.

More broadly, ours and other recent work [112–114] suggest that the applicability of the GKR interactive proof [54] has been underestimated. In particular, GKR seemingly requires deterministic arithmetic circuits, and saves work for the verifier (relative to computing the circuit) only when those circuits have low depth. Zhang et al. sidestep these issues, extending GKR to non-deterministic, low-depth computations [112] and more recently to arbitrary RAM programs [114], in both cases saving work asymptotically for the verifier. But even those enhanced protocols fall short of state-of-the-art work-saving zkSNARKs [4, 5, 12, 13, 15, 16, 30, 36, 38, 43, 46–48, 69, 81, 84, 110], because they fail to address zero-knowledge applications. This work (and concurrent work by Zhang et al. [113]) closes that gap—and, in our view, attests to the power and versatility of the GKR protocol.

[11]We note that, since Hyrax's proof size is primarily due to witness size $|w|$ rather than arithmetic circuit size $|C|$, we expect it to outperform Ligero on applications like matrix factoring where $|w| \ll |C|$.

We have released Hyrax's source code and our BCCGP-sqrt and Bulletproofs implementations as open-source software [63].

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PEDERSEN COMMITMENT SCHEME

Definitions: Let $\mathcal{G}$ be a (multiplicative) cyclic group of prime order $q_{\mathcal{G}}$ with group operation $\odot$ and inverse $\oslash$. $\mathcal{V}$ publishes generators $g, h \in \mathcal{G}$.

$\mathsf{Com}(m)$**:** $\mathcal{P}$ picks $s \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}$ and sends $g^m \odot h^s$.

$\mathsf{Open}(\alpha)$**:** $\mathcal{P}$ sends (m, s). $\mathcal{V}$ checks $\alpha \stackrel{?}{=} g^m \odot h^s$.

Multi-commitments: For commitments to vectors, $\mathcal{V}$ publishes generators $g_1, \ldots, g_n, h \in G$ and $\mathcal{P}$ sends

$$\mathsf{Com}((m_1, \ldots, m_n)) = h^s \odot \bigodot_i g_i^{m_i}$$

FIGURE 4—The Pedersen commitment scheme.

A Instantiations of commitment schemes

In this section, we review the Pedersen commitment scheme [85] (Fig. 4) and related protocols.

Theorem 7 ([85])**.** *The Pedersen commitment scheme is a non-interactive commitment scheme assuming the hardness of the discrete logarithm problem in $\mathcal{G}$.*

Knowledge of opening. Schnorr [95] shows how $\mathcal{P}$ can give a ZK proof that it knows an x, r such that $C_0 = \mathsf{Com}(x; r)$.

proof-of-opening(C_0)

Inputs: $C_0 = g^x \odot h^r$. $\mathcal{P}$ knows x and r.

1. $\mathcal{P}$ picks $t_1, t_2 \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}$ and sends $\alpha \leftarrow g^{t_1} \odot h^{t_2}$.
2. $\mathcal{V}$ sends a challenge $c \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}$.
3. $\mathcal{P}$ sends $z_1 \leftarrow xc + t_1$ and $z_2 \leftarrow rc + t_2$.
4. $\mathcal{V}$ checks that $g^{z_1} \odot h^{z_2} \stackrel{?}{=} C_0^c \odot \alpha$.

Theorem 8 ([95])**.** proof-of-opening *is complete, honest-verifier perfect ZK, and special sound under the discrete log assumption.*

Commitment to the same value. Using similar ideas, $\mathcal{P}$ can show in ZK that $C_1 = \mathsf{Com}(v_1; s_1)$ and $C_2 = \mathsf{Com}(v_2; s_2)$ are commitments to the same value, i.e., $v_1 = v_2$. Given $C_u = \mathsf{Com}(u; s_u)$ and a value v, $\mathcal{P}$ can also convince $\mathcal{V}$ that $u = v$.

proof-of-equality(C_1, C_2)

Inputs: $C_1 = g^{v_1} \odot h^{s_1}$ and $C_2 = g^{v_2} \odot h^{s_2}$.
$\mathcal{P}$ knows $v_1 = v_2$, s_1, and s_2.

1. $\mathcal{P}$ picks $r \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}$ and sends $\alpha \leftarrow h^r$.
2. $\mathcal{V}$ sends a challenge $c \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}$.
3. $\mathcal{P}$ sends $z \leftarrow c \cdot (s_1 - s_2) + r$
4. $\mathcal{V}$ checks that $h^z \stackrel{?}{=} (c_1 \oslash c_2)^c \odot \alpha$.

Theorem 9 (Folklore)**.** proof-of-equality *is complete, honest-verifier perfect zero-knowledge, and special sound under the discrete log assumption.*

A.1 Proving a product relationship

Figure 5 gives a protocol in which $\mathcal{P}$ convinces $\mathcal{V}$ that it has openings to three Pedersen commitments having a product relationship. This is folklore; for example, we know that Maurer [76] describes a very similar protocol.

proof-of-product(X, Y, Z)

Inputs: $X = g^x \odot h^{r_X}$, $Y = g^y \odot h^{r_Y}$, and $Z = g^{x \cdot y} \odot h^{r_Z}$.
$\mathcal{P}$ knows x,y,r_X,r_Y, and r_Z.

1. $\mathcal{P}$ picks $b_1, \ldots, b_5 \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}$ and sends
$$\alpha \leftarrow g^{b_1} \odot h^{b_2} \quad \beta \leftarrow g^{b_3} \odot h^{b_4} \quad \delta \leftarrow X^{b_3} \odot h^{b_5}$$
2. $\mathcal{V}$ sends a challenge $c \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}$
3. $\mathcal{P}$ sends
$$z_1 \leftarrow b_1 + c \cdot x \quad z_2 \leftarrow b_2 + c \cdot r_x \quad z_3 \leftarrow b_3 + c \cdot y$$
$$z_4 \leftarrow b_4 + c \cdot r_y \quad z_5 \leftarrow b_5 + c(r_z - r_x y)$$
4. $\mathcal{V}$ checks that
$$\alpha \odot X^c \stackrel{?}{=} g^{z_1} \odot h^{z_2} \tag{7}$$
$$\beta \odot Y^c \stackrel{?}{=} g^{z_3} \odot h^{z_4} \tag{8}$$
$$\delta \odot Z^c \stackrel{?}{=} X^{z_3} \odot h^{z_5} \tag{9}$$

FIGURE 5—ZK proof of knowledge for a product relationship (§A.1).

Theorem 10. *Given commitments X, Y, and Z,* proof-of-product *proves that Z is a commitment to the product of the values committed in X and Y. This protocol is complete, honest-verifier perfect zero-knowledge, and special sound under the discrete log assumption.*

Proof. **Completeness.** It is easy to check that if the prover sends all values as prescribed, then the first two equations hold. Checking that the third equation holds is a straightforward (if slightly tedious) calculation:

$$\begin{aligned}
\delta \odot Z^c &= X^{b_3} \odot h^{b_5} \odot (g^{xy} \odot h^{r_z})^c = X^{b_3} \odot g^{xyc} \odot h^{b_5 + r_z \cdot c} \\
&= X^{b_3} \odot \left((g^x \odot h^{r_x})^{yc} \odot h^{-(r_x \odot yc)} \right) \odot h^{b_5 + r_z \cdot c} \\
&= X^{b_3} \odot X^{yc} \odot h^{b_5 + c(r_z - r_x y)} \\
&= X^{z_3} \odot h^{z_5}.
\end{aligned}$$

Special soundness. To show that the protocol is special sound, for a given theorem statement (X, Y, Z), let (α, β, δ) be a first message, and let $(c, z_1, \ldots, z_5)$ and $(c', z_1', \ldots, z_5')$ be two transcripts such that the verification equations above hold, $c \neq c'$.

Define the variables

$$x = \frac{z_1 - z_1'}{c - c'} \qquad r_x = \frac{z_2 - z_2'}{c - c'}$$
$$y = \frac{z_3 - z_3'}{c - c'} \qquad r_y = \frac{z_4 - z_4'}{c - c'} \qquad w = \frac{z_5 - z_5'}{c - c'}$$

We first show that the values (x, r_x), (y, r_y), are proper openings of the commitments to X and Y.

Recall the following two equations hold:

$$\alpha \odot X^c = g^{z_1} \odot h^{z_2} \qquad \alpha \odot X^{c'} = g^{z_1'} \odot h^{z_2'}$$

It follows by dividing the two equations, that

$$X^{c-c'} = g^{z_1-z_1'} \odot h^{z_2-z_2'}$$

and moreover that

$$X = g^{(z_1-z_1')/(c-c')} \odot h^{(z_2-z_2')/(c-c')} = g^x \odot h^{r_x} \tag{10}$$

which shows that (x, r_x) is indeed an opening for commitment X. The same follows for Y. Finally, since

$$\delta \odot Z^c = X^{z_3} \odot h^{z_5} \qquad \delta \odot Z^{c'} = X^{z_3'} \odot h^{z_5'}$$

it follows again by dividing that

$$Z = X^{(z_3-z_3')/(c-c')} \odot h^{(z_5-z_5')/(c-c')} = X^y \odot h^w$$

Substituting from (10), we have

$$Z = (g^x \odot h^{r_x})^y \odot h^w = g^{xy} \odot h^{r_x y + w}$$

which shows that Z can be opened as the product $(xy, r_x y + w)$.

Perfect ZK. Here, we show an honest-verifier simulation. Using Fiat-Shamir [45] or Cramer-Damgård [39] techniques, one can compile the protocol into a malicious-verifier ZK protocol.

- The simulator $S(X, Y, Z, c)$, on input X, Y, Z and a random challenge c does the following:
  Sample $z_1, \ldots, z_5 \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}$, then compute:

$$\alpha \leftarrow g^{z_1} \odot h^{z_2} \oslash X^c \qquad \beta \leftarrow g^{z_3} \odot h^{z_4} \oslash Y^c$$
$$\delta \leftarrow X^{z_3} \odot h^{z_5} \oslash Z^c$$

- Then the simulator outputs the transcript

$$(\alpha, \beta, \delta), c, z_1, z_2, z_3, z_4, z_5$$

By inspection, the transcript satisfies equations (7)–(9) above.

It remains to show that the distribution over transcripts produced by S and those produced by the honest prover and honest verifier are identical. To show this, we show a one-to-one mapping between every transcript produced by the honest prover and a transcript produced by the Simulator. Fix a theorem statement (X, Y, Z) having the correct product relation, a witness, and a challenge c. The Prover's random coins consist of the 5 values b_1, b_2, b_3, b_4, b_5. Given the fixed values, each 5-tuple uniquely defines (α, β, δ) and the values z_1, z_2, z_3, z_4, z_5. This 5-tuple is chosen uniformly over $\mathbb{Z}_{q_{\mathcal{G}}}^5$. Likewise, the simulator S uses random coins $z_1', z_2', z_3', z_4', z_5'$, again chosen uniformly over the same probability space. When S picks the random coins $z_1 = b_1 + c \cdot x$,..., then S produces exactly the same transcript $(\alpha, \beta, \delta), c, (z_1, z_2, z_3, z_4, z_5)$. Thus, for every possible transcript, both the honest prover and simulator produce that transcript with the same probability. $\square$

proof-of-dot-prod$(\xi, \tau, \vec{a})$

Inputs: Commitments $\xi = \mathsf{Com}(\vec{x}; r_\xi)$, $\tau = \mathsf{Com}(y; r_\tau)$, and a vector $\vec{a}$, where $\vec{x}, \vec{a} \in \mathbb{Z}_{q_{\mathcal{G}}}^n$ and $y = \langle \vec{x}, \vec{a} \rangle \in \mathbb{Z}_{q_{\mathcal{G}}}$.
$\mathcal{P}$ knows $\vec{x}$, r_ξ, y, and r_τ.

1. $\mathcal{P}$ samples the vector $\vec{d} \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}^n$ and the values $r_\beta, r_\delta \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}$ and sends
$$\delta \leftarrow \mathsf{Com}(\vec{d};\ r_\delta) = h^{r_\delta} \odot \bigodot_i g_i^{d_i} \tag{11}$$
$$\beta \leftarrow \mathsf{Com}(\langle \vec{a}, \vec{d} \rangle; r_\beta) = g^{\langle \vec{a}, \vec{d} \rangle} \odot h^{r_\beta} \tag{12}$$
2. $\mathcal{V}$ sends a challenge $c \xleftarrow{\$} \{1, \ldots, q_{\mathcal{G}}\}$.
3. $\mathcal{P}$ sends
$$\vec{z} \leftarrow c \cdot \vec{x} + \vec{d}, \quad z_\delta \leftarrow c \cdot r_\xi + r_\delta, \quad z_\beta \leftarrow c \cdot r_\tau + r_\beta$$
4. $\mathcal{V}$ checks that
$$\xi^c \odot \delta \stackrel{?}{=} \mathsf{Com}(\vec{z}; z_\delta) = h^{z_\delta} \odot \bigodot_i g_i^{z_i} \tag{13}$$
$$\tau^c \odot \beta \stackrel{?}{=} \mathsf{Com}(\langle \vec{z}, \vec{a} \rangle; z_\beta) = g^{\langle \vec{z}, \vec{a} \rangle} \odot h^{z_\beta} \tag{14}$$

FIGURE 6—ZK vector dot-product proof (§A.2).

A.2 Proving a dot-product relationship

In the protocol of Figure 6, $\mathcal{P}$ convinces $\mathcal{V}$ that it has openings to one multi-commitment $\xi = \mathsf{Com}(\vec{x}; r_\xi)$ and one scalar commitment $\tau = \mathsf{Com}(y; r_\tau)$ such that, for a supplied vector $\vec{a}$ it holds that $y = \langle \vec{x}, \vec{a} \rangle$. Intuitively, this protocol works because

$$\langle \vec{z}, \vec{a} \rangle = \langle c\vec{x} + \vec{d}, \vec{a} \rangle = c\langle \vec{x}, \vec{a} \rangle + \langle \vec{d}, \vec{a} \rangle = cy + \langle \vec{d}, \vec{a} \rangle$$

The above identity is verified in the exponent in Equation (14).

Theorem 11. *The protocol of Figure 6 is complete, honest-verifier perfect zero-knowledge, and special sound under the discrete log assumption.*

Proof. **Completeness.** If both the prover and the verifier follow the protocol correctly both checks will succeed because

$$\begin{aligned}
\xi^c \odot \delta &= \left(h^{r_\xi} \odot \bigodot_i g_i^{x_i} \right)^c \odot h^{r_\delta} \odot \bigodot_i g_i^{d_i} \\
&= h^{c \cdot r_\xi + r_\delta} \odot \bigodot_i g_i^{c \cdot x_i + d_i} = h^{z_\delta} \odot \bigodot_i g_i^{z_i} \\
\tau^c \odot \beta &= (h^{r_\tau} \odot g^y)^c \odot g^{\vec{a} \cdot \vec{d}} \odot h^{r_\beta} = h^{c \cdot r_\tau + r_\beta} \odot g^{cy + \langle \vec{a}, \vec{d} \rangle} \\
&= h^{z_\beta} \odot g^{c \cdot \langle \vec{x}, \vec{a} \rangle + \langle \vec{d}, \vec{a} \rangle} = h^{z_\beta} \odot g^{\vec{z} \cdot \vec{a}}
\end{aligned}$$

Special soundness. For a given theorem instance $(\xi, \tau, \vec{a})$, let (δ, β) be a first message, and let $(c, \vec{z}, z_\delta, z_\beta)$ and $(c', \vec{z}', z_\delta', z_\beta')$ be two valid transcripts. Since both transcripts satisfy both of $\mathcal{V}$'s checks, we have

$$\xi^c \odot \delta = h^{z_\delta} \odot \bigodot_i g_i^{z_i}$$

$$\xi^{c'} \odot \delta = h^{z'_\delta} \odot \bigodot_i g_i^{z'_i}$$

$$\tau^c \odot \beta = h^{z_\beta} \odot g^{\langle \vec{z}, \vec{a} \rangle}$$

$$\tau^{c'} \odot \beta = h^{z'_\beta} \odot g^{\langle \vec{z}', \vec{a} \rangle}$$

Dividing the top by the bottom in each pair yields

$$\xi = h^{(z_\delta - z'_\delta)/(c-c')} \odot \bigodot_i g_i^{(z_i - z'_i)/(c-c')}$$

$$\tau = h^{z_\beta - z'_\beta/(c-c')} \odot g^{(\vec{z}-\vec{z}')\cdot\vec{a}/(c-c')}$$

which implies that $\vec{x} = (\vec{z}-\vec{z}')/(c-c')$, that $r_\xi = (z_\delta - z'_\delta)/(c-c')$, and that $r_\tau = (z_\beta - z'_\beta)/(c-c')$.

Honest-verifier perfect zero-knowledge. (Analogous to the ZK proof for protocol for proving product of commitment.) The zero-knowledge property follows from standard reverse-ordering techniques. In particular, the simulator first picks c, then $\vec{z}, z_\delta, z_\beta$, and finally computes an appropriate first message δ, β which satisfies check equations (13) and (14).

The simulator S, on input the theorem instance $(\xi, \tau, \vec{a})$ and a challenge c does the following:

- Sample $\vec{z}' \xleftarrow{\$} \mathbb{Z}^n_{q_\mathcal{G}}$ and $z'_\delta, z'_\beta \xleftarrow{\$} \mathbb{Z}_{q_\mathcal{G}}$
- Produce values

$$\delta \leftarrow \left(h^{z'_\delta} \odot \bigodot_i g_i^{z'_i} \right) \oslash \xi^c \tag{15}$$

$$\beta \leftarrow \left(g^{\langle \vec{z}', \vec{a} \rangle} \odot h^{z'_\beta} \right) \oslash \tau^c \tag{16}$$

- Output the transcript $(\delta, \beta), c, (\vec{z}, z_\delta, z_\beta)$.

By inspection, one can certify that the transcript passes both checks of the verifier and the verifier will accept it.

It remains to show that the distribution over transcripts produced by S and those produced by the honest verifier and honest prover are identical. To show this, we show a one to one mapping between every transcript produced by the honest prover and a trancript produced by the simulator. We fix a theorem statement $(\xi, \tau, \vec{a})$, a witness $w = (\vec{x}, r_\xi, r_\tau)$ and a challenge c. The prover's random coins consist of the $(n+2)$-tuple $(\vec{d}, r_\delta, r_\beta)$. Given the fixed statement, witness, and challenge, each $(n+2)$-tuple uniquely determines (δ, β) and the values $\vec{z}, z_\delta, z_\beta$. This $(n+2)$-tuple is chosen uniformly over $\mathbb{Z}^{n+2}_{q_\mathcal{G}}$.

Likewise, the simulator S uses an $(n+2)$-tuple $\vec{z}', z'_\delta, z'_\beta$ chosen uniformly at random over the probability space. We show a one-to-one mapping between the prover's coins and the simulator's output. In particular, when

$$M(\vec{z}') = \vec{z}' - c \cdot \vec{x} \quad M(z'_\delta) = z'_\delta - c \cdot r_\xi \quad M(z'_\beta) = z'_\beta - c \cdot r_\tau$$

By inspection, this mapping is one-to-one, and when the prover runs with coins $M(\vec{z}', z'_\delta, z'_\beta)$, it produces the same transcript as the simulator. Thus, the output distribution of S is identical to that of the prover on this instance. This property holds for all instances, and any challenge c, which concludes the proof. □

$\mathsf{proof_{log}\text{-}of\text{-}dot\text{-}prod}(\xi, \tau, \vec{a})$

Inputs: $\xi = \mathsf{Com}_{\vec{g}}(\vec{x}; r_\xi) = h^{r_\xi} \odot \bigodot_{i=1}^n g_i^{x_i}$,
$\tau = \mathsf{Com}(y; r_\tau) = g^y \odot h^{r_\tau}$. $\vec{x}, \vec{a} \in \mathbb{Z}^n_{q_\mathcal{G}}$, $y, r_\xi, r_\tau \in \mathbb{Z}_{q_\mathcal{G}}$.

$\mathcal{P}$ knows $\vec{x}$, y, r_ξ, and r_τ.

1. Let $\Upsilon = \xi \odot \tau = h^{r_\Upsilon} \odot g^y \odot \bigodot_{i=1}^n g_i^{x_i}$ where $r_\Upsilon = r_\tau + r_\xi$. $(\hat{\Upsilon}, \hat{a}, \hat{g}) \leftarrow \mathsf{bullet\text{-}reduce}(\Upsilon, \vec{a}, \vec{g})$ (see Fig. 8).

At this point, $n = 1$ and $\hat{\Upsilon} = \hat{g}^{\hat{x}} \odot g^{\hat{y}} \odot h^{\hat{r}_\Upsilon}$ where $\hat{y} = \hat{x} \cdot \hat{a}$.

2. $\mathcal{P}$ samples $d, r_\delta, r_\beta \xleftarrow{\$} \{1, \ldots, q_\mathcal{G}\}$ and sends

$$\delta \leftarrow \mathsf{Com}_{\hat{g}}(d; r_\delta) = \hat{g}^d \odot h^{r_\delta}$$

$$\beta \leftarrow \mathsf{Com}_g(d; r_\beta) = g^d \odot h^{r_\beta}$$

3. $\mathcal{V}$ chooses and sends $c \xleftarrow{\$} \{1, \ldots, q_\mathcal{G}\}$.
4. $\mathcal{P}$ sends $z_1 \leftarrow d + c \cdot \hat{y}$ and $z_2 \leftarrow \hat{a}\left(c \cdot \hat{r}_\Upsilon + r_\beta\right) + r_\delta$.
5. $\mathcal{V}$ checks that

$$\left(\hat{\Upsilon}^c \odot \beta\right)^{\hat{a}} \odot \delta \stackrel{?}{=} \left(\hat{g} \odot g^{\hat{a}}\right)^{z_1} \odot h^{z_2} \tag{17}$$

FIGURE 7—Protocol for dot-product relation based on Bulletproofs [31]. $\mathsf{Com}_{\vec{g}}$ indicates a multi-commitment over generators $\vec{g}$.

A.3 Dot-product argument from Bulletproofs

The dot-product argument of Appendix A.2 has communication $4+n$ elements for a vector of length n. By adapting the Bulletproof recursive reduction of Bünz et al. [31], we reduce this to $4+2\log n$. Figures 7 and 8 detail this protocol.

As in Appendix A.2, we have $\vec{x}$, $\vec{a}$, and $y = \langle \vec{x}, \vec{a} \rangle$, where $n = |\vec{x}| = |\vec{a}|$. Given $\vec{a}$ and $\Upsilon = \mathsf{Com}_{\vec{g}}(\vec{x}) \odot \mathsf{Com}(y)$, each recursive call to bullet-reduce produces $\vec{a}'$ and $\Upsilon' = \mathsf{Com}_{\vec{g}'}(\vec{x}') \odot \mathsf{Com}(y')$ such that $y' = \langle \vec{x}', \vec{a}' \rangle$.

After $\log n$ such recursive calls, we are left with a scalar $\hat{a}$ and a commitment $\hat{\Upsilon} = \hat{g}^{\hat{x}} g^{\hat{y}} h^{\hat{r}_\Upsilon}$. $\mathcal{P}$ can now use a Schnorr proof to convince $\mathcal{V}$ that $\hat{y} = \hat{x} \cdot \hat{a}$. Expanding Equation (17) (Fig. 7),

$$\begin{aligned}
\left(\hat{\Upsilon}^c \odot \beta\right)^{\hat{a}} \odot \delta &= \left(\hat{g}^{c\cdot\hat{x}} \odot g^{c\cdot\hat{y}} \odot h^{c\cdot\hat{r}_\Upsilon} \odot g^d \odot h^{r_\beta}\right)^{\vec{a}} \odot \delta \\
&= \left(\hat{g}^{c\cdot\hat{x}} \odot g^{c\cdot\hat{y}+d} \odot h^{c\cdot\hat{r}_\Upsilon + r_\beta}\right)^{\vec{a}} \odot \hat{g}^d \odot h^{r_\delta} \\
&= \hat{g}^{c\cdot\hat{x}\cdot\hat{a}+d} \odot g^{\hat{a}(c\cdot\hat{y}+d)} \odot h^{\hat{a}(c\cdot r_\Upsilon + r_\beta)+r_\delta} \\
&= \hat{g}^{c\cdot\hat{y}+d} \odot g^{\hat{a}(c\cdot\hat{y}+d)} \odot h^{z_2} \\
&= \left(\hat{g} \odot g^{\hat{a}}\right)^{z_1} \odot h^{z_2}
\end{aligned}$$

In total, $\mathcal{P}$ sends $2\log n$ elements during the bullet-reduce calls and 4 elements for the final Schnorr proof. Adapting suggestions by Poelstra [89], $\mathcal{V}$'s work computing $\hat{g}$ can be reduced to one multi-exponentiation of length n and one field inversion, and computing $\hat{\Upsilon}$ costs one multi-exponentiation of length $1+2\log n$.

bullet-reduce$(\Upsilon, \vec{a}, \vec{g})$

Inputs: $\Upsilon = h^{r_\Upsilon} \odot g^y \odot \bigodot_{i=1}^n g_i^{x_i}$, $\vec{x}, \vec{a} \in \mathbb{Z}_{q_\mathcal{G}}^n$, $y, r_\Upsilon \in \mathbb{Z}_{q_\mathcal{G}}$.

$\mathcal{P}$ knows $\vec{x}$,y, and r_Υ.

Define $\vec{x}_1 = (x_1, \ldots, x_{n/2})$, $\vec{x}_2 = (x_{1+n/2}, \ldots, x_n)$ and similarly for $\vec{a}_1$, $\vec{a}_2$, $\vec{g}_1$, and $\vec{g}_2$; and define

$$(g_1, g_2 \ldots)^k \circ (g_{1+n/2}, g_{2+n/2} \ldots)^\ell = (g_1^k \odot g_{1+n/2}^\ell, g_2^k \odot g_{2+n/2}^\ell, \ldots)$$

1. If $n = 1$, return (Υ, a_1, g_1).
2. $\mathcal{P}$ samples $r_{\Upsilon_{-1}}, r_{\Upsilon_1} \xleftarrow{\$} \{1, \ldots, q_\mathcal{G}\}$ and sends
$$\Upsilon_{-1} \leftarrow h^{r_{\Upsilon_{-1}}} \odot g^{\langle \vec{x}_1, \vec{a}_2 \rangle} \odot \bigodot_{i=1}^{n/2} g_{i+n/2}^{x_i}$$
$$\Upsilon_1 \leftarrow h^{r_{\Upsilon_1}} \odot g^{\langle \vec{x}_2, \vec{a}_1 \rangle} \odot \bigodot_{i=1}^{n/2} g_i^{x_{i+n/2}}$$
3. $\mathcal{V}$ chooses and sends $c \xleftarrow{\$} \{1, \ldots, q_\mathcal{G}\}$.
4. $\mathcal{P}$ and $\mathcal{V}$ both compute
$$\Upsilon' \leftarrow \Upsilon_{-1}^{c^2} \odot \Upsilon \odot \Upsilon_1^{c^{-2}}$$
$$\vec{a}' \leftarrow c^{-1} \cdot \vec{a}_1 + c \cdot \vec{a}_2$$
$$\vec{g}' \leftarrow \vec{g}_1^{\,c^{-1}} \circ \vec{g}_2^{\,c} = (g_1, \ldots, g_{n/2})^{c^{-1}} \circ (g_{1+n/2}, \ldots, g_n)^c$$
$\mathcal{P}$ computes
$$\vec{x}' \leftarrow c \cdot \vec{x}_1 + c^{-1} \cdot \vec{x}_2$$
$$y' \leftarrow c^2 \cdot \langle \vec{x}_1, \vec{a}_2 \rangle + y + c^{-2} \cdot \langle \vec{x}_2, \vec{a}_1 \rangle$$
$$r'_\Upsilon \leftarrow r_{\Upsilon_{-1}} \cdot c^2 + r_\Upsilon + r_{\Upsilon_1} \cdot c^{-2}$$
5. Return bullet-reduce$(\Upsilon', \vec{a}', \vec{g}')$.

If $y = \langle \vec{x}, \vec{a} \rangle$, then $y' = \langle \vec{x}', \vec{a}' \rangle$, and $\mathcal{P}$ knows $\vec{x}'$, y', r'_Υ.

FIGURE 8—Reduction step for the protocol of Figure 7.

Lemma 12. *The protocol of Figures 7–8 is complete, honest-verifier perfect ZK, and generalized special sound under the discrete log assumption.*

Completeness follows from the derivation of Equation (17) above and the completeness of bullet-reduce [31, Thm. 2, Appx. A], and ZK follows from standard reverse-ordering techniques. Generalized special soundness follows from the properties of Schnorr protocols and an argument similar to the proof of [31, Thm. 2, Appx. A]. In total, the extractor requires $n+2$ transcripts.

A.4 Proof of Lemma 4

Proof. **Completeness.** If the theorem statement

$$(C_0, (\alpha_1, \ldots \alpha_n), (M_1, \ldots, M_{n+1}), X, Y, Z)$$

holds, the prover knows openings of the commitments $C_0, (\alpha_1, \ldots \alpha_n), X, Y, Z$ and he sends all values as prescribed, it follows from the correctness of proof-of-product that X, Y, Z will pass all the checks of that subprocess. Moreover, for all k it holds that

$$\begin{aligned}
&\mathsf{Com}((z_{c_{3,k}}, z_{c_{2,k}}, z_{c_{1,k}}, z_{c_{0,k}}); z_{\delta_k}) \\
&\quad = \mathsf{Com}((c \cdot c_{3,k} + d_{c_{3,k}}, c \cdot c_{2,k} + d_{c_{2,k}}, \\
&\qquad\qquad c \cdot c_{1,k} + d_{c_{1,k}}, c \cdot c_{0,k} + d_{c_{0,k}}); c \cdot r_{\alpha_k} + r_{\delta_k}) \\
&\quad = \left(\mathsf{Com}((c_{3,k}, c_{2,k}, c_{1,k}, c_{0,k}); r_{\alpha_k})\right)^c \odot \\
&\qquad \mathsf{Com}((d_{c_{3,k}}, d_{c_{2,k}}, d_{c_{1,k}}, d_{c_{0,k}}); r_{\delta_k}) \\
&\quad = \alpha_k^c \odot \delta_k
\end{aligned}$$

It also holds that

$$\begin{aligned}
&(C_0^{\rho_1} \odot X^{-J_X} \odot Y^{-J_Z} \odot Z^{-J_Z})^c \odot C \\
&\quad = \mathsf{Com}(c(\rho_1 \cdot s_0 - J_X \cdot x - J_Y \cdot y - J_Z \cdot z) + \langle \vec{J}^*, \vec{d} \rangle; \\
&\qquad\qquad c(\rho_1 \cdot r_{C_0} - J_x \cdot r_X - J_Y \cdot r_Y - J_Z \cdot r_Z) + r_C) \\
&\quad = \mathsf{Com}(c \cdot \langle \vec{J}^*, \vec{\pi}^* \rangle + \langle \vec{J}^*, d \rangle; z_C) \\
&\quad = \mathsf{Com}(\langle \vec{J}^*, \vec{z} \rangle; z_C)
\end{aligned}$$

Generalized special soundness. For theorem instance $(C_0, (\alpha_1, \ldots \alpha_n), (M_1, \ldots, M_{n+1}), X, Y, Z)$, the transcripts:

$$\left(\mathrm{Tr}_{\mathrm{prod}}, (\delta_1, \ldots, \delta_n), (\rho_1, \ldots \rho_{n+1}), C, c, (\vec{z}, \{z_{\delta_k}\}, z_C)\right)$$
$$\left(\mathrm{Tr}_{\mathrm{prod}}, (\delta_1, \ldots, \delta_n), (\rho_1, \ldots \rho_{n+1}), C, c', (\vec{z}', \{z'_{\delta_k}\}, z'_C)\right)$$
$$\left(\mathrm{Tr}'_{\mathrm{prod}}, \ldots\right)$$

are sufficient to extract a witness for the statement except with negligible probability, where $\mathrm{Tr}_{\mathrm{prod}}$ and $\mathrm{Tr}'_{\mathrm{prod}}$ are transcripts for proof-of-product. In this context, by witness we mean openings to the prover's messages that satisfy the checks that the verifier of Gir^{++} does during the corresponding invocation of the sum-check protocol.

The extractor proceeds as follows:

1. Exploiting the first condition checked by the verifier in Figure 1 (Step 7), extract openings for $\{\alpha_k\}$ via
$$\begin{aligned}\alpha_k^{c-c'} = \mathsf{Com}\big(&(z_{c_{3,k}} - z'_{c_{3,k}}, z_{c_{2,k}} - z'_{c_{2,k}}, \\ &z_{c_{1,k}} - z'_{c_{1,k}}, z_{c_{0,k}} - z'_{c_{0,k}}); z_{\delta_k} - z_{\delta'_k}\big)\end{aligned}$$
i.e., $r_{\alpha_k} = \frac{z_{\delta_k} - z_{\delta'_k}}{c-c'}$ and $c_{j,k} = \frac{z_{c_{j,k}} - z'_{c_{j,k}}}{c-c'}$, $j \in \{0, 1, 2, 3\}$.
2. Use the extractor for proof-of-product and $\mathrm{Tr}_{\mathrm{prod}}, \mathrm{Tr}'_{\mathrm{prod}}$ to extract openings $\hat{x}, r_X, \hat{y}, r_Y, \hat{z}, r_Z$ of X, Y, Z.
3. Use the openings from the previous step to extract an opening for C_0. Specifically, exploiting the second condition checked by the verifier in Figure 1 (Step 7), we have:
$$(c - c')(s_0 - J_X \cdot \hat{x} - J_Y \cdot \hat{y} - J_Z \cdot \hat{z}) = \langle \vec{J}^*, \vec{z} - \vec{z}' \rangle,$$
from where we can solve for s_0.
4. Check that Equality (5) holds for the extracted values. If not, reject and output $\perp$.

Note that the extractor aborts only when for the extracted witness, $\langle (\sum \rho_k \cdot M_k), \vec{\pi} \rangle = \rho_1 \cdot s_0$ but Equality (5) does not hold.

From Lemma 3 the probability that this happens is at most $1/|\mathbb{F}|$, which is negligible when the field is of superpolynomial size. Also, note that we are exploiting the fact that proof-of-product ensures that $\hat{z} = \hat{x} \cdot \hat{y}$, since the verifier's checks in the sum-check protocol require this.

Honest-verifier perfect zero-knowledge. The simulator, on input statement $(C_0, (\alpha_1, \dots \alpha_n), (M_1, \dots, M_{n+1}), X, Y, Z)$ and messages from $\mathcal{V}$, will work as follows. First it uses the simulator for proof-of-product to produce a valid transcript $\mathrm{Tr}_{\mathrm{prod}}$ for proof-of-product (X, Y, Z). Then, it follows the below steps to simulate the rest of the interaction between $\mathcal{P}$ and $\mathcal{V}$:

1. Pick random $\vec{z}$ and values $z_{\delta_1}, \dots, z_{\delta_n}, z_C$.
2. For all k, set $\delta_k = \mathsf{Com}((z_{c_{3,k}}, z_{c_{2,k}}, z_{c_{1,k}}, z_{c_{0,k}}); z_{\delta_k}) \oslash \alpha_k^c$ and set $C = \mathsf{Com}(\langle J^*, \vec{z} \rangle; z_C) \oslash (C_0^{\rho_1} \odot X^{-J_X} \odot Y^{-J_Z} \odot Z^{-J_Z})^c$.
3. Output the transcript:

$$\left(\mathrm{Tr}_{\mathrm{prod}}, (\delta_1, \dots, \delta_n), (\rho_1, \dots \rho_{n+1}), C, c, (\vec{z}, \{z_{\delta_i}\}, z_C)\right)$$

By construction this passes $\mathcal{V}$'s checks. It remains to show that the distribution of transcripts produced by S and those produced by the honest prover and honest verifier are identical. To show this, we show a one-to-one mapping between every transcript produced by the honest prover and a transcript produced by the simulator. We fix a theorem statement, a witness, and $\mathcal{V}$'s challenges. Since $\mathrm{Tr}_{\mathrm{prod}}$ is produced by the simulator for proof-of-product its distribution is identical to the distribution of the messages of an honest prover.

Now, we analyze the rest of the transcript. $\mathcal{P}$'s random coins comprise the $(5b_N + 8b_G + 1)$-tuple $(\vec{d}, r_{\delta_1}, \dots, r_{\delta_n}, r_C)$. Given the fixed statement, witness, and challenges, each such tuple uniquely determines $(\delta_1, \dots, \delta_n)$, C and the values $\vec{z}, z_{\delta_1}, \dots, z_{\delta_n}, z_C$. This $(5b_N + 8b_G + 1)$-tuple is chosen uniformly over $\mathbb{Z}_{q_\mathcal{G}}^{5b_N+8b_G+1}$.

Likewise, the simulator S uses a $(5b_N + 8b_G + 1)$-tuple $(\vec{z}, z_{\delta_1}, \dots, z_{\delta_n}, z_C)$ chosen uniformly at random over the probability space. We show a one-to-one mapping between the prover's coins and the simulator's output. We define this mapping as:

$$\begin{aligned} M(\vec{z}) &= \vec{z} - c \cdot \pi^* \\ M(z_{\delta_k}) &= z_{\delta_k} - c \cdot r_{\alpha_k}, \quad k \in \{1, \dots, n\} \\ M(z_C) &= z_C - c \cdot (\rho_1 \cdot r_{C_0} - J_X r_X - J_Y r_Y - J_Z r_Z) \end{aligned}$$

By inspection, this mapping is one-to-one, and when the prover runs with coins $\left(M(\vec{z}), M(z_{\delta_1}), \dots, M(z_{\delta_n}), M(z_C)\right)$, it produces the same transcript as the simulator. Thus, the output distribution of S is identical to that of the prover on this instance. This holds for all instances and any challenge c, which concludes the proof. □

A.5 Proof of Lemma 5

We now show how the matrix T can be extracted from the prover in the protocol of Section 6.1. That protocol first compresses T into one vector commitment T', and then uses $\mathrm{proof}_{\log}$-of-dot-prod (Appx. A.3) to show the dot-product relation for this vector. By the properties of $\mathrm{proof}_{\log}$-of-dot-prod, $2^{\ell-\ell/\iota} + 2$ transcripts with the same first message can be used to extract the underlying witness, which is the vector:

$$\left(\sum_{i=0}^{2^{\ell/\iota}-1} w_i \check{\chi}_i, \ \sum_{i=0}^{2^{\ell/\iota}-1} w_{i+2^{\ell/\iota}} \check{\chi}_i, \ \dots, \ \sum_{i=0}^{2^{\ell/\iota}-1} w_{2^\ell - 2^{\ell/\iota}+i} \check{\chi}_i \right)$$

In other words, each term is a weighted sum of one column of T, with weights given by the $\check{\chi}_i$ values. For $\iota \ge 2$, given $2^{\ell/\iota} \cdot \left(2^{\ell-\ell/\iota} + 2\right) \le 2^{\ell+1} = 2\,|w|$ transcripts with linearly independent $(\check{\chi}_0, \dots, \check{\chi}_{2^{\ell/\iota}-1})$ (which happens except with negligible probability for randomly selected r_k), an extractor can solve the resulting system of linear equations for the values $w_0, \dots, w_{2^\ell-1}$.

A.6 Witness-extended emulation from perfect generalized special soundness

We employ a lemma from Bootle et al. [25, Lem. 1] which shows that generalized special soundness implies witness-extended emulation. Bootle et al. analyze the case where the extractor always succeeds, but their argument establishes the statement below directly.

Lemma 13 (Forking lemma [25])**.** *Let (P, V) be a $(2\mu+1)$-move, public coin interactive protocol. Let $\mathrm{Ex}_{\mathrm{GSS}}$ be a witness extraction algorithm that extracts a witness from an $(n_1, \dots, n_\mu)$-tree of accepting transcripts (Def. 4, §3.1) in probabilistic polynomial time with at most negligible failure probability. Assume that $\prod_{i=\mu}^{n} n_i$ is bounded above by a polynomial in the security parameter λ. Then (P, V) has witness-extended emulation.*

B Proofs of Hyrax-I's properties

Theorem 14. *Hyrax-I is complete: if the prover knows a witness w s.t. $y = C(x, w)$ and follows the prescribed steps in the pseudocode (Appx. D), the verifier will accept.*

Proof. The completeness of Hyrax-I follows from the completeness of Gir^{++} and the completeness of all the ZK sub-protocols used to establish that the verifier's checks hold. □

Theorem 15. *Hyrax-I is honest-verifier perfect ZK.*

Proof. On input a theorem instance (C, x, y), and a set of messages $(q_0', q_{0,0}, q_1', q_{1,0}, q_{1,1}, \dots)$ of the verifier, the simulator S proceeds as follows:

1. For $i = 1, \dots, \sqrt{|w|}$, set $T_i = \mathsf{Com}(0; r_{T_i})$ for random r_{T_i}.
2. Compute $\tilde{V}_y(q_0', q_{0,0})$ and set $a_0 = \mathsf{Com}(\mathcal{V}_y(q_0', q_{0,0}); 0)$
3. For each layer $i = 1, \dots, d$, for all sum-check rounds j, pick $r_{i,j}$ uniformly at random and set $\alpha_{i,j} = \mathsf{Com}((0,0,0,0); r_{i,j})$. At the end of the sum-check, pick $r_{X_i}, r_{Y_i}, r_{Z_i}$ uniformly at random from $\mathbb{F}$ and set $X_i = \mathsf{Com}(0; r_{X_i})$, $Y_i = \mathsf{Com}(0; r_{Y_i})$ and $Z_i = \mathsf{Com}(0; r_{Z_i})$. Use the simulator of proof-of-sum-check (Fig. 1, §5; Appx. A.4) to get a valid transcript Tr_i for the theorem statement $\left(a_{i-1}, (\alpha_{i,1}, \dots \alpha_{i,n}), (M_1, \dots, M_{n+1}), X_i, Y_i, Z_i\right)$. Pick random $\mu_{i,0}, \mu_{i,1}$ from $\mathbb{F}$ and set $a_i = X_i^{\mu_{i,0}} \odot Y_i^{\mu_{i,1}}$ (Line 19, Fig. 9).
4. For all $j = 0, \dots, b_G$ pick r_{H_j} uniformly at random from $\mathbb{F}$ and set $\mathsf{Com}(H_j) = \mathsf{Com}(0; r_{H_j})$

5. Run the simulator of proof-of-opening for the values $\mathsf{Com}(H_0), \ldots, \mathsf{Com}(H_{b_G})$ and get back the accepting transcripts $\mathrm{Tr}_{H_0}, \ldots, \mathrm{Tr}_{H_{b_G}}$
6. Run the simulator of proof-of-equality for the input statement $(\mathsf{Com}(H_0), X)$ and get back the accepting transcript $\mathrm{Tr}_{\mathrm{same},X}$
7. Run the simulator of proof-of-equality for the input statement $(\mathsf{Com}(H_{b_G}) \odot \ldots \odot \mathsf{Com}(H_0), Y)$ and get back the accepting transcript $\mathrm{Tr}_{\mathrm{same},Y}$
8. Compute q_d, ζ and T' as in Lines 24–26 of Figure 10
9. Run the simulator of $\mathrm{proof}_{\log}$-of-dot-prod (Appx. A.3) for the input statement $(T', \zeta \oslash g^{(1-q_d[0])\tilde{V}_x(q_d[1,\ldots,b_N+b_G-1])}, R)$ and get back the accepting transcript $\mathrm{Tr}_{\mathrm{dotProd}}$
10. Output the transcript:
$(T_1, \ldots, T_{\sqrt{|w|}}, q_0', q_{0,0}, q_{1,0}, \alpha_{1,0}, \ldots, X_1, Y_1, Z_1, \mathrm{Tr}_1, \ldots,$
$\mathrm{Tr}_d, \mathsf{Com}(H_0), \ldots, \mathrm{Tr}_{H_{b_G}}, \mathrm{Tr}_{\mathrm{same},X}, \mathrm{Tr}_{\mathrm{same},Y}, \tau, \mathrm{Tr}_{\mathrm{dotProd}})$

The above transcript is accepting: The verifier can only reject during one of the ZK subroutines $\mathrm{proof}_{\log}$-of-dot-prod, proof-of-opening, etc., but the simulators of these routines produce accepting transcripts for each of the input theorem statements.

By a standard hybrid argument, the distribution of transcripts produced by S and those produced by the honest prover and honest verifier are identical. We fix a theorem statement (C, x, y), a witness w and verifier's challenges $q_0', q_{0,0}, \ldots$. Recall $\mathsf{View}(\langle \mathcal{P}(w), \mathcal{V}^*(z)\rangle(x))$ denotes the distribution of transcripts that the prover produces when interacting with the honest verifier. In the following we define experiments $\mathsf{Exp}_1, \mathsf{Exp}_2, \ldots$ which produce a distribution of transcripts, and show that each is identical to the previous.

1. Let Exp_1 be the experiment that outputs a transcript in which the prover behaves as the honest prover except that in Line 28 of Figure 10, instead of executing the protocol $\mathrm{proof}_{\log}$-of-dot-prod honestly, the protocol's simulator is used to produce the transcript for the same theorem. From Theorem 11, since this simulator produces transcripts that are identically distributed to those that an honest prover produces, the distributions Exp_1 and $\mathsf{View}(\langle \mathcal{P}(w), \mathcal{V}^*(z)\rangle(x))$ are identical.
2. Let Exp_2 be the experiment in which the prover behaves as the prover in Exp_1 except that in Line 3 of Figure 10, a commitment to 0 is used. By Theorem 7 which establishes that the Pedersen commitment scheme is perfectly hiding, each of these commitments to zero is identically distributed to commitment used in Exp_1. As a result, the distributions Exp_2 and Exp_1 are identically distributed.
3. Let Exp_3 be the experiment in which the prover behaves as the prover in Exp_2 except that in each of the Lines 21 and 22 of Figure 10, the simulator of the protocol proof-of-equality is used to generate the transcript instead of executing the protocol. By Theorem 9 this simulator produces transcripts for each of these relations that are identical to those that an honest prover would produce.
4. Let Exp_4 be the experiment in which the prover behaves as the prover in Exp_3 except that in the b_G iterations of Line 20 of Figure 10, the simulator of the protocol proof-of-opening is used to generate a transcript instead of executing it honestly. From Theorem 8, this simulator produces transcripts that are identically distributed to those that an honest prover produces. So, the distributions H_4 and H_3 are identical.
5. Let Exp_5 be the experiment in which the prover behaves as the prover in Exp_4 except that in Line 18 of Figure 10, commitments to 0 are used. By Theorem 7 which establishes that the Pedersen commitment scheme is perfectly hiding, the distribution Exp_5 is identical to the distribution Exp_4
6. For all sum-check iterations $i = d, \ldots, 1$:
   - Let $\mathsf{Exp}_{3(d-i)+6}$ be the experiment in which the prover behaves as the prover of the previous experiment $\mathsf{Exp}_{3(d-i)+5}$ except that in Line 54 of figure 11, the simulator for the protocol proof-of-sum-check is used to generate the transcript. From Lemma 4 this simulator produces transcripts that are identically distributed to those that an honest prover produces. So, the distributions $\mathsf{Exp}_{3(d-i)+6}$ and $\mathsf{Exp}_{3(d-i)+5}$ are identical.
   - Let $\mathsf{Exp}_{3(d-i)+7}$ be the experiment in which the prover behaves as the prover of the previous experiment $Exp_{3(d-i)+6}$ except that in Line 52 of figure 11 produces the commitments X_i, Y_i, Z_i by committing to 0. By Theorem 7 which establishes that the Pedersen commitment scheme is perfectly hiding, the distribution $\mathsf{Exp}_{3(d-i)+7}$ is identical to the distribution $\mathsf{Exp}_{3(d-i)+6}$.
   - Let $\mathsf{Exp}_{3(d-i)+8}$ be the experiment in which the prover behaves as the prover of the previous experiment $\mathsf{Exp}_{3(d-i)+8}$ except that for all sum-check rounds $j = 1 = n, \ldots, 1$, in Lines 19 and 47 of figure 11 the commitments α_i are produced by committing to the zero vector. By Theorem 7 which establishes that the Pedersen commitment scheme is perfectly hiding, the distribution $\mathsf{Exp}_{3(d-i)+8}$ is identical to the distribution $\mathsf{Exp}_{3(d-i)+7}$

Thus, $\mathsf{View}(\langle \mathcal{P}(w), \mathcal{V}^*(z)\rangle(x))$ is identically distributed to $\mathsf{Exp}_{3(d-1)+8}$. The experiment $\mathsf{Exp}_{3(d-1)+8}$ is identical to running the Simulator S on the instance (C, x, y) and verifier's challenges $q_0', q_{0,0}, \ldots$. As a result, $\mathsf{Exp}_{3(d-1)+8}$ corresponds to the distribution of transcripts that S produces which completes the proof. □

Theorem 16. *Under the discrete log assumption, Hyrax-I has witness-extended emulation.*

Proof. We construct an extractor E that simultaneously outputs a transcript and a witness for a given theorem x with roughly the same probability that prover $\mathcal{P}^*$ causes the honest verifier to accept theorem (C, x, y). At a high level, this extractor E uses $\mathcal{P}^*$ as an oracle, and runs the extractor from Lemma 5 in order to find a witness w. It checks the validity of the witness by verifying that $C(x, w) = y$ and outputs w or else aborts by outputting $\perp$. We show that when the extractor succeeds in recovering w, then (except with negligible probability) it indeed outputs w and thus satisfies witness-extended emulation. The last condition follows by the soundness of the Gir^{++} protocol: a $\mathcal{P}^*$ that succeeds at convincing the verifier to accept with some non-negligible probability ϵ, but which causes E to abort, can be used to break the soundness of Gir^{++}. We use a proof technique borrowed from the elegant analysis of various parallel repetition theorems where the same style of soundness reduction (albeit more complicated) is required.

To simplify the presentation, we assume that the Hyrax-I verifier runs all zero-knowledge proofs at the very end of the protocol (even though the pseudocode describes these checks as occurring over the course of the interaction); we refer to this ZK challenge step as Step ($*$). By moving all sigma protocols to the end in Hyrax-I, and by standard AND composition of sigma protocols [34], all ZK proofs can use the same verifier challenge $c \in \mathbb{F}$ and the combined proof enjoys the same honest-verifier zero-knowledge and special soundness properties (we think of Step (*) as consisting of one giant ZK proof instead of many separate ZK proofs). Similarly, the Gir^{++} verifier defers all checks to the end of the interaction. This change to Gir^{++} does not affect soundness.

On input theorem statement (C, x, y), the extractor $E^{\mathcal{P}^*}(C, x, y)$ works as follows:

1. Run the honest Hyrax-I verifier $\mathcal{V}$ on instance (C, x, y) with prover $\mathcal{P}^*(C, x, y)$ and record the partial transcript Tr$'$ until Step ($*$) when the zero-knowledge proof is given.
2. By Lemma 13, there exist witness-extended emulators $E_w^{\mathcal{P}^*}$ and $E_O^{\mathcal{P}^*}$ for the protocols of Lemma 5 and Theorem 8, respectively. Run $(\mathrm{Tr}''_e, w_e) \leftarrow E_w^{\mathcal{P}^*}(C, x, y)$ to extract the witness w_e. Run $(\mathrm{Tr}''_{H_i}, (H_i, r_{H_i})) \leftarrow E_O^{\mathcal{P}^*}$ to extract openings for each $\mathsf{Com}(H_i)$ (Fig. 9, Line 26), and use these openings to compute the opening (m_ζ, r_ζ) of ζ (Fig. 9, Line 32).
3. Run the rest of the ZK sub-protocols of Step (*) and record the transcript $\mathrm{Tr}''_{\mathrm{rest}}$. Let $\mathrm{Tr}'' = \mathrm{Tr}''_e || \mathrm{Tr}''_{H_0} || \mathrm{Tr}''_{H_1} || \ldots || \mathrm{Tr}''_{\mathrm{rest}}$, and let $\mathrm{Tr} = \mathrm{Tr}' || \mathrm{Tr}''$.
4. If m_ζ is not consistent with (x, w_e) or if Tr is rejecting, output $(\mathrm{Tr}, 0)$ and halt. Otherwise, if $C(x, w_e) = y$, then output (Tr, w_e). Otherwise, output $\perp$.

Fix a polynomial-time adversary A and $\mathcal{P}^*$. Without loss of generality, assume that $A, \mathcal{P}^*$ are deterministic (i.e., their best random tapes are hard-wired). Thus, the instance and state given by $(u_\lambda{=}(C, x, y), s_\lambda) \leftarrow A(1^\lambda)$ are just indexed on the security parameter. We must show that E runs in expected polynomial time (which follows by inspection of each step) and that

$$\Pr\left[\mathrm{Tr} \leftarrow \mathrm{tr}\langle \mathcal{P}^*(s_\lambda), \mathcal{V}\rangle(u_\lambda) : A(\mathrm{Tr}) = 1\right] \tag{18}$$

and

$$\Pr\left[\begin{array}{l}(\mathrm{Tr}, w) \leftarrow E^{\mathcal{P}^*(s_\lambda)}(u_\lambda) : A(\mathrm{Tr}) = 1 \wedge \\ \text{if Tr is an accepting transcript, } C(x, w) = y.\end{array}\right] \tag{19}$$

differ by a negligible amount in the security parameter.

When E outputs a pair, it outputs the transcript Tr produced by an honest verifier interacting with $\mathcal{P}^*$, and thus the probability that $A(\mathrm{Tr}) = 1$ when E does not abort is identical in (19) and (18). As we show below in Claim 17, the probability that E aborts is a negligible function $\eta(\cdot)$, and thus, it follows that $\Pr\left[(\mathrm{Tr}, w) \leftarrow E^{\mathcal{P}^*(u,s)}(u) : A(\mathrm{Tr}) = 1\right] \geq (1 - \eta(\cdot)) \cdot$ Eq. (18). Furthermore, when E does not abort and the transcript is accepting, then because of Step 4, E outputs a pair (Tr, w) such that $C(x, w) = y$. Thus, the second condition of Eq. (19) also holds, and equations (18) and (19) differ by at most a negligible function, which completes the proof of the theorem.

E only aborts when $\mathcal{P}^*$ succeeds in creating an accepting transcript, extractor E_w succeeds in extracting a witness w_e, but $C(x, w_e) \neq y$. Denote this event as badwit.

Claim 17. *For every $\mathcal{P}^*$, there exists a $\hat{\mathcal{P}}$ that runs in expected time* $\mathrm{poly}(|u_\lambda| + \mathrm{time}(\mathcal{P}^*))$ *such that if* $\Pr[\mathsf{badwit}] = \epsilon(\lambda)$*, then*

$$\Pr[\langle \hat{\mathcal{P}}(s_\lambda), \mathcal{V}_g\rangle(u_\lambda) = 1] > \epsilon(\lambda)^2 - \eta(\lambda)$$

where $\mathcal{V}_g$ is the Gir^{++} verifier and η is a negligible function.

By the soundness of Gir^{++} from Theorem 1, it follows that

$$\epsilon(\lambda)^2 - \eta(\lambda) < \mathrm{O}(|u_\lambda| / |\mathbb{F}|)$$

and by rearranging, that $\epsilon(\lambda) < \sqrt{\mathrm{O}(|u_\lambda| / |\mathbb{F}|) + \eta(\lambda)}$ which shows that ϵ is a negligible function because $|\mathbb{F}|$ is assumed exponential in the security parameter.

The idea of the construction of $\hat{\mathcal{P}}$. Both the Gir^{++} verifier's ($\mathcal{V}_g$) challenge messages and the Hyrax-I verifier's ($\mathcal{V}$) challenge messages are $(\mu_{0,0}, \mu_{0,1}, r_{1,1}, r_{1,2}, \ldots, r_{d,b_G}, \tau)$, and in addition, the Hyrax-I verifier makes a single challenge c in Step ($*$). For simplicity, we denote these challenges $\vec{r} = (r_1, \ldots, r_\ell) \in \mathbb{F}^\ell$ and $c \in \mathbb{F}$ and assume that $|\mathbb{F}| > 2^\lambda$.

The only difference between $\mathcal{V}_g$ and $\mathcal{V}$ is that the latter performs its checks "under the commitments" by verifying a ZK proof instead of directly checking the provers responses $(m_1, \ldots, m_\ell)$ after every step. Thus the job of $\hat{\mathcal{P}}$ is to forward the messages "under the commitments" from $\mathcal{P}^*$ to $\mathcal{V}_g$.

In particular, $\hat{\mathcal{P}}$ needs to sample an accepting transcript between $\mathcal{V}$ and $\mathcal{P}^*$, and then rewind the challenge message of the ZK proof to extract the witness w_e from the session and initiate a session of Gir^{++} with $\mathcal{V}_g$ on the theorem statement $C(x, w_e) = y$. As it receives the challenges from $\mathcal{V}_g$ for the sum-check iterations, it forwards the same challenge to $\mathcal{P}^*$, awaits a response commitment, and then extracts the value from the commitment by completing the execution and extracting a witness from the ZK proof at Step ($*$) by using the (generalized) special soundness of each sub-protocol along with Lemma 13.

Bounding the probability that $\mathcal{P}^*$ aborts. A natural concern is handling provers $\mathcal{P}^*$ that abort the Hyrax-I protocol during $\hat{\mathcal{P}}$'s attempts to find a completion of a partial transcript. Intuitively, $\mathcal{P}^*$ succeeds on ϵ-fraction of all $\mathcal{V}$ challenges, and thus sufficient sampling should lead to a success with high probability.

More formally, consider a j-move partial execution between $\mathcal{P}^*$ and $\mathcal{V}$ in which $\mathcal{V}$'s challenges are $\vec{r}_j = (r_1, \ldots, r_j)$. A *continuation* of $\vec{r}_j$ is a set of challenges $(r_{j+1}, \ldots, r_\ell, c)$, and a *good continuation* is one in which P^* responses to $(r_1, \ldots, r_\ell, c)$ causes V to accept. Define the set G_λ to be $G_\lambda = \{\vec{r} = (r_1, \ldots, r_\ell)\}$ the set of ℓ-move challenges such that each prefix $\vec{r}_j$ of $\vec{r} \in \mathcal{G}_\lambda$, has an ϵ/ℓ-fraction of good continuations. These are the "heavy" challenges for which $\mathcal{P}^*$ produces accepting transcripts that facilitate extraction via special-soundness. Let $\nu_\lambda = \Pr_{\vec{r} \xleftarrow{\$} \mathbb{F}^\ell}[\vec{r} \in G_\lambda]$. Next we lower bound ν_λ.

Lemma 18. $\nu_\lambda \geq \epsilon/8$ *for* $\ell \geq 3, \ell \in \mathbb{N}$.

Proof. Consider the tree of transcripts with leaves $(r_1, \ldots, r_\ell) \in \mathbb{F}^\ell$ representing the portion of an execution of Hyrax-I right before the challenge c for the ZK proof is given. Level j of this tree contains the internal nodes $\vec{r}_j = (r_1, \ldots, r_j) \in \mathbb{F}^j$ (so there are $|\mathbb{F}|^j$ nodes at layer j of the tree); we are interested in analyzing *good* continuations over these leaves. Let $\rho_{\vec{r}_j}$ be the fraction of good continuations for prefix $\vec{r}_j$. Suppose that over all nodes at level i, an ϵ_i fraction of continuations are good. Define the heavy set $\mathcal{H}_i$ as those prefixes for which at least an ϵ_i/ℓ-fraction of continuations succeed.

Claim 19. *If an ϵ_i fraction of continuations of all nodes $\vec{r}_i$ at level i succeed, then at least an $\epsilon_i(1 - 1/\ell)$ fraction of continuations of nodes in $\mathcal{H}_i$ at level i succeed. That is, if $\sum_{\vec{r}_i \in \mathbb{F}^i} \rho_{\vec{r}_i} > \epsilon_i$, then $\sum_{\mathcal{H}_i} \rho_{\vec{r}_i} \geq \epsilon_i(1 - \frac{1}{\ell-1})$.*

Proof. Partition the sum

$$|\mathbb{F}|^{(\ell-i)} \sum_{\vec{r}_i \in \mathbb{F}^i} \rho_{\vec{r}_i} = |\mathbb{F}|^{(\ell-i)} \left(\sum_{\mathcal{H}_i} \rho_{\vec{r}_i} + \sum_{\overline{\mathcal{H}_i}} \rho_{\vec{r}_i} \right) \geq \epsilon_i \cdot |\mathbb{F}|^\ell \quad (20)$$

Using the fact that $|\mathcal{H}_i| + |\overline{\mathcal{H}_i}| = |\mathbb{F}|^i$, we bound $|\overline{\mathcal{H}_i}|$ by overestimating its weight as follows:

$$\begin{aligned}
|\mathbb{F}|^{(\ell-i)} \left(|\mathcal{H}_i| \cdot 1 + \left|\overline{\mathcal{H}_i}\right| \left(\frac{\epsilon_i}{\ell}\right) \right) &\geq \epsilon_i \cdot |\mathbb{F}|^\ell \\
|\mathbb{F}|^{(\ell-i)} \left((|\mathbb{F}|^i - \left|\overline{\mathcal{H}_i}\right|) + \left|\overline{\mathcal{H}_i}\right| \left(\frac{\epsilon_i}{\ell}\right) \right) &\geq \epsilon_i \cdot |\mathbb{F}|^\ell \\
|\mathbb{F}|^{(\ell-i)} \left(|\mathbb{F}|^i + \left|\overline{\mathcal{H}_i}\right| \left(-1 + \left(\frac{\epsilon_i}{\ell}\right)\right) \right) &\geq \epsilon_i \cdot |\mathbb{F}|^\ell \\
|\mathbb{F}|^{(\ell-i)} \left|\overline{\mathcal{H}_i}\right| \left(-1 + \left(\frac{\epsilon_i}{\ell}\right)\right) &\geq (-1 + \epsilon_i) \cdot |\mathbb{F}|^\ell \\
\left|\overline{\mathcal{H}_i}\right| &\leq \frac{(1 - \epsilon_i)}{\left(1 - \left(\frac{\epsilon_i}{\ell}\right)\right)} \cdot |\mathbb{F}|^i \quad (21)
\end{aligned}$$

Substituting (21) into (20)

$$\begin{aligned}
\sum_{\mathcal{H}_i} \rho_{\vec{r}_i} &\geq \epsilon_i \cdot |\mathbb{F}|^i - \sum_{\overline{\mathcal{H}_i}} \rho_{\vec{r}_i} \\
&\geq \epsilon_i \cdot |\mathbb{F}|^i - \frac{(1 - \epsilon_i)}{\left(1 - \left(\frac{\epsilon_i}{\ell}\right)\right)} \cdot |\mathbb{F}|^i \cdot \frac{\epsilon_i}{\ell} \\
&= \epsilon_i \cdot |\mathbb{F}|^i \left[1 - \frac{(1 - \epsilon_i)}{\ell - \epsilon_i}\right] \\
&\geq \epsilon_i \cdot |\mathbb{F}|^i \left[1 - \frac{1}{\ell - 1}\right]
\end{aligned}$$

□

Thus, at level i, the probability mass over the "heavy" prefixes remains roughly the same. At level ℓ in the tree (i.e., full challenges), we have by assumption that an ϵ fraction succeeds, and thus by the calculation above, at least $(1 - 1/(\ell-1))$ fraction are heavy leaves for which ϵ/ℓ-fraction of the continuations (over the ZK challenge c) succeed. Now consider the parents of these heavy leaves at level $\ell - 1$. At least an $\epsilon_{\ell-1} = \epsilon(1 - 1/(\ell-1))$ fraction of nodes at this level have children which are heavy. Applying Claim 19, it follows that an $\epsilon_{\ell-1} \cdot (1 - 1/(\ell-1))$ fraction of the nodes at this level are heavy. By induction, we have that at the top-level of the tree, at least a fraction of all challenges

$$\epsilon \left[1 - \frac{1}{\ell - 1}\right]^\ell \geq \epsilon/8.$$

are heavy, because $(1 - 1/(x-1))^x \geq 1/8$ for $x \geq 3, x \in \mathbb{N}$. □

We now consider the task of sampling an accepting transcript that allows extracting w starting from a partial transcript with challenges $\vec{r}_j = (r_1, \ldots, r_j)$. Define procedure $\text{SAM}^{\mathcal{P}^*}_{\lambda,u}(\vec{r}_j)$ as:

1. For up to t attempts:

(a) Sample $(r'_{j+1}, \ldots, r'_\ell) \leftarrow \mathbb{F}^{\ell-j}$

(b) Run $\mathcal{V}$ with $\mathcal{P}^*$ (from its current state) using challenges $(r'_{j+1}, \ldots, r'_\ell)$ until $\hat{\mathcal{P}}$ generates the first message of the ZK proof, and then sample t transcripts with randomly sampled ZK challenges $c \leftarrow \mathbb{F}$. Succeed if $2|u_\lambda|$ transcripts accept, and return the accepting transcripts.

Since $|u_\lambda| > |w|$, $2|u_\lambda|$ transcripts are sufficient to extract w (Appx. A.5). Further, SAM always runs in time polynomial in time($\mathcal{P}^*$) and $2|u_\lambda|$ (which are both polynomial in the instance size) and t, which we set below. More importantly, for $j \in \{1, \ldots, \ell\}$, conditioned on $\vec{r}_j$ being the prefix of some $\vec{r} \in G_\lambda$, then SAM succeeds with high probability. In particular, for SAM to fail, all t attempts at sampling $(r_{j+1}, \ldots, r_\ell)$ must fail to yield a candidate $\vec{r} \in G_\lambda$ and more than $t - 2|u_\lambda|$ attempts to find good continuations of each such candidate fail. For the first case, because the samples are independent, the failure probability is less than $(1 - \epsilon/8)^t$. For the second case, failure occurs when there are fewer than $2|u_\lambda|$ accepting transcripts. Conditioned on the prefix being in G_λ, an ϵ/ℓ fraction of continuations result in accepting transcripts, so sampling $2|u_\lambda|$ accepting transcripts requires an expected $2|u_\lambda| \cdot \ell/\epsilon$ attempts. Setting $t \geq 20|u_\lambda| \cdot \ell/\epsilon \cdot \lambda$ makes the first case negligible by inspection and the second case negligible by a Chernoff bound.

Details of $\hat{\mathcal{P}}$. For theorem (C, x, y), $\hat{\mathcal{P}}$ does the following:

1. Sample $\text{Tr}_{\text{full}} \leftarrow \langle \mathcal{P}^*, \mathcal{V}(C, x, y) \rangle$. If Tr_{full} is not accepting, then abort.
2. Rewind $\mathcal{P}^*$ to Step $(*)$ and sample accepting transcripts until there are enough to extract the witness w_e and an opening (m_ζ, r_ζ) of ζ via the perfect generalized special soundness of the ZK step (Step $(*)$) and Lemma 13. Send w_e to Gir^{++} verifier $\mathcal{V}_g$ (Line 3 of Figure 12), thereby making the claim that $C(x, w_e) = y$.

   Rewind $\mathcal{P}^*$ to after it sends its first commitment message (Line 3, Fig. 14). From this point, $\hat{\mathcal{P}}$ plays man in the middle between $\mathcal{V}_g$ and $\mathcal{P}^*$ as follows.

3. Repeat to determine each message that the Gir^{++} prover $\hat{\mathcal{P}}$ sends to the Gir^{++} verifier $\mathcal{V}_g$:

(a) For message j, await the random challenge r_j from $\mathcal{V}_g$, forward r_j to $\mathcal{P}^*$. Run SAM to generate enough accepting continuations and ZK transcripts to extract prover messages $(m^j_1, \ldots, m^j_\ell)$ for every committed message in the protocol.

(b) For any $k < j$, if there exists a pair $m_k^k \neq m_k^j$, (i.e., the extracted value of message k differs from a previous extraction of message k), then abort.

(c) Forward the extracted message m_j^j to $\mathcal{V}_g$.

During its execution, $\hat{\mathcal{P}}$ samples fresh random challenges on behalf of the Hyrax-I verifier, and it receives challenge messages from $\mathcal{V}_g$. Let coll denote the event that two such samples are equal (over the entire execution including all sampling of continuations). Because both $\hat{\mathcal{P}}$ and $\mathcal{V}_g$ choose samples randomly and uniformly from $\mathbb{F}$, and because the expected number of samples chosen is $O(\text{poly}(d \log NG))$ over all rewinds, it follows that $\Pr[\mathsf{coll}] \leq \eta_1(\lambda)$ for a negligible function η_1. For the rest of this argument, we condition on the event $\overline{\mathsf{coll}}$, which implies that every extraction of a witness succeeds by perfect generalized special soundness and Lemma 13.

$\hat{\mathcal{P}}$ may abort in Step 3b. Denote this event by bind; when this event occurs, $\hat{\mathcal{P}}$ can be modified to output a commitment α_j and two different openings of α_j. This event occurs with negligible probability η_2 by the binding of the Pedersen commitment, and we also condition the rest of the analysis on the event $\overline{\mathsf{bind}}$.

We can view verifier $\mathcal{V}_g$ as sampling all of its challenges $\vec{r} \in \mathbb{F}^\ell$ at the start of the protocol (but only sending them one by one). Let event good occur when $\mathcal{V}_g$'s sampled challenge $\vec{r} \in G_\lambda$. By Lemma 18, this event occurs with probability $\epsilon/8$. Even conditioned on good, the invocations of $\textsc{sam}$ may fail, via the union bound, with negligible probability at most $\eta_3 \geq e^{-\lambda} \cdot \text{poly}(d, \log(NG))$.

The first step of $\hat{\mathcal{P}}$ succeeds with probability ϵ. When this occurs, the second step runs an expected $2\,|u_\lambda|\,/\epsilon$ times to recover enough transcripts to extract the witness. Conditioned on the four events above, $\hat{\mathcal{P}}$ always succeeds in convincing $\mathcal{V}_g$, thus:

$$\begin{aligned}\Pr[\langle\hat{\mathcal{P}}(s_\lambda), V_g\rangle(u_\lambda) = 1] &\geq \epsilon \cdot \epsilon/8 \cdot (1 - \eta_1(\lambda)) \\ &\quad \cdot (1 - \eta_2(\lambda)) \cdot (1 - \eta_3(\lambda)) \\ &\geq \epsilon^2/8 - \eta_4(\lambda)\end{aligned}$$

where η_4 is a negligible function and η_1, η_2, and η_3 correspond to the events bind, coll and sam.

To compute the running time, there are three cases: (a) when step 1 fails (with probability $1 - \epsilon$), (b) when step 1 succeeds and events good, $\overline{\mathsf{bind}}$, and $\overline{\mathsf{coll}}$ all occur, (c) when step 1 succeeds and $\overline{\mathsf{good}}$ occurs. In (a), $\hat{\mathcal{P}}$ runs one execution of $\hat{\mathcal{P}}$, in (b), $\hat{\mathcal{P}}$ makes an expected polynomial number of executions of $\hat{\mathcal{P}}$, and in (c), we use the loop bound t to ensure that the runtime remains polynomial.

Finally, the loop bound t in $\textsc{sam}$ relies on ϵ. We can either estimate this probability after Step 1 using an analysis technique from Goldreich and Kahan [52], or we can be given this value as advice for this security parameter. Furthermore, we can use a standard technique of aborting the reduction after 2^λ steps to catch the unexpected steps that may occur due to sampling errors. □

C Randomized checks "inside the AC"

We describe a simple technique for running randomized tests "inside the AC." We use this technique improve the cost of modulo-2^{32} addition in an AC for SHA-256, but it is applicable to many randomized checks. Zhang et al. use a related approach to verify set intersections in vSQL [112].

Our implementation of SHA-256 uses a prior technique [12] to compute $h = f + g \mod 2^{32}$ in an AC over a large prime field. Specifically, $\mathcal{P}$ supplies witness elements $\{b_i\}$ that are purportedly the binary digits of $f + g$, i.e., $f + g = \sum_i 2^i b_i$. The AC computes and outputs the value $f + g - \sum_i 2^i b_i$, which $\mathcal{V}$ checks is equal to zero. The AC also encodes "bit tests" that ensure $b_i \in \{0, 1\}$, namely, it computes and outputs $\{b_i \cdot (1 - b_i)\}$, which $\mathcal{V}$ also checks are zero. Finally, the AC uses $\{b_i\}$ to compute h, i.e., $h = \sum_{i=0}^{31} 2^i b_i = f + g \mod 2^{32}$.

In our implementation, the SHA-256 AC entails more than 7 000 bit tests. Because Gir^{++} requires layered ACs (§3.2), the results of these tests must be passed layer-by-layer to the output, meaning that they cost ≈7000d gates in total. As a result, bit tests nearly double the size of the AC.

To avoid this, we observe that it is safe for $\mathcal{V}$ to believe that all bit tests have passed if $\sum_i r_i \cdot b_i \cdot (1 - b_i) = 0$ for $r_i \in_R \mathbb{F}$—as long as $\mathcal{P}$ *does not know* $\{r_i\}$ when it chooses $\{b_i\}$. To ensure this, we introduce a small modification to Hyrax-I. Specifically, $\mathcal{V}$ first encodes the above check into the AC. Then, during protocol execution, after $\mathcal{V}$ sends x, $\mathcal{P}$ commits to the witness (including $\{b_i\}$; §6); now $\mathcal{V}$ chooses and sends $\{r_i\}$, after which $\mathcal{P}$ returns y and the protocol continues as normal. $\mathcal{V}$ rejects if the output corresponding to the above sum is not 0. Informally, since $\mathcal{P}$ commits to $\{b_i\}$ before $\mathcal{V}$ chooses $\{r_i\}$, it cannot "fool" the test.

Completeness follows by inspection; soundness follows from the Schwartz-Zippel lemma. This protocol is public coin, so it can be made non-interactive with the Fiat-Shamir heuristic. Finally, we note that in the data-parallel setting $\mathcal{V}$ can supply a single $\{r_i\}$ for all N sub-ACs (because $\mathcal{P}$ commits to the entire witness at once).

D Hyrax-I pseudocode

In this section, we provide pseudocode for Hyrax-I. Figure 9 details $\mathcal{V}$'s work; Figures 10 and 11 detail $\mathcal{P}$'s. Our presentation borrows from Wahby et al. [109].

E Gir^{++} specification

In this section, we provide pseudocode for Gir^{++}. Figures 12 and 13 detail $\mathcal{V}$'s work; Figures 14 and 15 detail $\mathcal{P}$'s. Our presentation borrows from Wahby et al. [109].

1: **function** HYRAX-VERIFY(ArithCircuit c, input x, output y, parameter ι)
2: // Receive commitments to the rows of the matrix T
3: $(T_0, \ldots, T_{|w|^{1/\iota}}) \leftarrow$ ReceiveFromProver() // see Line 3 of Figure 10
4: $b_N \leftarrow \log N, b_G \leftarrow \log G$
5: $(q'_0, q_{0,L}) \xleftarrow{R} \mathbb{F}^{b_N} \times \mathbb{F}^{b_G}$
6: $\mu_{0,0} \leftarrow 1, \mu_{0,1} \leftarrow 0, q_{0,R} \leftarrow q_{0,L}$
7: $a_0 \leftarrow \mathsf{Com}(\tilde{V}_y(q'_0, q_{0,0}); 0)$
8: SendToProver($(q'_0, q_{0,0})$) // see Line 5 of Figure 10
9: $d \leftarrow c$.depth
10:
11: **for** i=1,...,d **do**
12: $(X, Y, r', r_L, r_R) \leftarrow$ ZK-SUMCHECKV($i, a_{i-1}, q'_{i-1}, q_{i-1,L}, q_{i-1,R}$)
13: // $X = \mathsf{Com}(v_0), Y = \mathsf{Com}(v_1)$
14:
15: **if** i<d **then**
16: // Pick the next random $\mu_{i,0}, \mu_{i,1}$ and
17: // compute random linear combination (§3.2)
18: $\mu_{i,0}, \mu_{i,1} \xleftarrow{R} \mathbb{F}$
19: $a_i \leftarrow X^{\mu_{i,0}} \odot Y^{\mu_{i,1}}$
20: $(q'_i, q_{i,L}, q_{i,R}) \leftarrow (r', r_L, r_R)$
21: SendToProver($\mu_{i,0}, \mu_{i,1}$) // see Line 14 of Figure 10
22:
23: // For the final check, reduce from two points to one point (§3.2)
24: $(\mathsf{Com}(H_0), \ldots, \mathsf{Com}(H_{b_G})) \leftarrow$ ReceiveFromProver() // see Line 18 of Figure 10
25: **for** $i = 0, \ldots, b_G$ **do**
26: proof-of-opening ($\mathsf{Com}(H_i)$)
27: proof-of-equality ($\mathsf{Com}(H_0), X$)
28: proof-of-equality ($\mathsf{Com}(H_{b_G}) \odot \ldots \odot \mathsf{Com}(H_0), Y$)
29: $\tau \xleftarrow{R} \mathbb{F}$
30: SendToProver(τ) // see Line 23 of Figure 10
31: $q_d \leftarrow (r', (1-\tau) \cdot r_L + \tau \cdot r_R)$
32: $\zeta = \mathsf{Com}(H_{b_G})^{\tau^{\log G}} \odot \mathsf{Com}(H_{b_G-1})^{\tau^{\log G-1}} \odot \ldots \odot \mathsf{Com}(H_0)$
33: $T' \leftarrow \bigodot_{i=0}^{|w|^{1/\iota}-1} T_i^{\check{\chi}_i}$ // $\check{\chi}_b$ is defined in Section 6
34: $R \leftarrow (\hat{\chi}_0, \hat{\chi}_{|w|^{1/\iota}}, \ldots, \hat{\chi}_{|w|^{1/\iota}\cdot(|w|^{1/\iota}-1)})$ // $\hat{\chi}_b$ is defined in Section 6
35: proof$_{\log}$-of-dot-prod ($T'^{q_d[0]}, \zeta \oslash g^{(1-q_d[0])\tilde{V}_x(q_d[1,\ldots,b_N+b_G-1])}, R$)
36: **return** accept
37:
38: **function** ZK-SUMCHECKV(layer i,$a_{i-1}, q'_{i-1}, q_{i-1,L}, q_{i-1,R}$)
39: $(r', r_L, r_R) \xleftarrow{R} \mathbb{F}^{\log N} \times \mathbb{F}^{\log G} \times \mathbb{F}^{\log G}$
40: $r \leftarrow (r', r_L, r_R)$
41: **for** $j = 1, \ldots, \log N + 2\log G$ **do**
42: $\alpha_j \leftarrow$ ReceiveFromProver() // α_j is $\mathsf{Com}(s_j)$; see Lines 19,47 of Figure 11
43: SendToProver($r[j]$) // see Lines 20,48 of Figure 11
44: $(X, Y, Z) \leftarrow$ ReceiveFromProver() // see Line 52 of Figure 11
45: // $X = \mathsf{Com}(v_0), Y = \mathsf{Com}(v_1), Z = \mathsf{Com}(v_0 v_1)$
46: // $\mathcal{V}$ computes $\{M_j\}$ as defined in Equation (5)
47: proof-of-sum-check ($a_{i-1}, \{\alpha_j\}, \{M_j\}, X, Y, Z$) // see Figure 1
48: **return** ($\mathsf{Com}(v_0), \mathsf{Com}(v_1), r', r_L, r_R$)

FIGURE 9—Pseudocode for $\mathcal{V}$ in Hyrax-I (§7). $\mathcal{P}$'s work is described in Figures 10 and 11. For notational convenience, we assume $|x| = |w|$, as in Section 6.1.

1: **function** HYRAX-PROVE(ArithCircuit c, input x, witness w, parameter ι)
2: // Commit to the rows of T via commitments $T_1, \ldots, T_{|w|^{1/k}}$
3: SendToVerifier($T_0, \ldots, T_{|w|^{1/k}-1}$) // see Line 3 of Figure 9
4: $b_N \leftarrow \log N, b_G \leftarrow \log G$
5: $(q'_0, q_{0,L}) \leftarrow$ ReceiveFromVerifier() // see Line 8 of Figure 9
6: $\mu_{0,0} \leftarrow 1, \mu_{0,1} \leftarrow 0, q_{0,R} \leftarrow q_{0,L}$
7: $a_0 \leftarrow \mathsf{Com}(\tilde{V}_y(q'_0, q_{0,L}); 0)$
8: $d \leftarrow c$.depth
9:
10: **for** i=1,...,d **do**
11: $(X, Y, q'_i, q_{i,L}, q_{i,R}) \leftarrow$ ZK-SUMCHECKP($c, i, a_{i-1}, \mu_{i-1,0}, \mu_{i-1,1},$
12: $q'_{i-1}, q_{i-1,L}, q_{i-1,R}$)
13: **if** i<d **then**
14: $(\mu_{i,0}, \mu_{i,1}) \leftarrow$ ReceiveFromVerifier() // see Line 21 of Figure 9
15: $a_i \leftarrow X^{\mu_{i,0}} \odot Y^{\mu_{i,1}}$
16:
17: // Compute Coefficients of the degree b_G polynomial H: $H_0, \ldots, H_{\log G}$
18: SendToVerifier($\mathsf{Com}(H_0), \ldots, \mathsf{Com}(H_{b_G})$) // see Line 21 of Figure 9
19: **for** $i = 0, \ldots, b_G$ **do**
20: proof-of-opening ($\mathsf{Com}(H_i)$)
21: proof-of-equality ($\mathsf{Com}(H_0), X$)
22: proof-of-equality ($\mathsf{Com}(H_{b_G}) \odot \ldots \odot \mathsf{Com}(H_0), Y$)
23: $\tau \leftarrow$ ReceiveFromVerifier() // see Line 30 of Figure 9
24: $q_d \leftarrow (q'_d, (1-\tau) \cdot q_{d,L} + \tau \cdot q_{d,R})$
25: $\zeta = \mathsf{Com}(H_{b_G})^{\tau^{\log G}} \odot \mathsf{Com}(H_{b_G-1})^{\tau^{\log G-1}} \odot \ldots \odot \mathsf{Com}(H_0)$
26: $T' \leftarrow \bigodot_{i=0}^{|w|^{1/k}-1} T_i^{\check{\chi}_i}$ // $\hat{\chi}_b$ is defined in Section 6
27: $R \leftarrow (\hat{\chi}_0, \hat{\chi}_{|w|^{1/\iota}}, \ldots, \hat{\chi}_{|w|^{1/\iota}\cdot(|w|^{1/\iota}-1)})$ // $\hat{\chi}_b$ is defined in Section 6
28: proof$_{\log}$-of-dot-prod ($T'^{q_d[0]}, \zeta \oslash g^{(1-q_d[0])\tilde{V}_x(q_d[1,\ldots,b_N+b_G-1])}, R$)

FIGURE 10—Pseudocode for $\mathcal{P}$ in Hyrax-I (§7). The ZK-SumCheckP subroutine is defined in Figure 11. $\mathcal{V}$'s work is described in Figure 9. For notational convenience, we assume $|x| = |w|$, as in Section 6.1.

function ZK-SumcheckP(ArithCircuit c, layer i, a_{i-1},)
$\mu_{i-1,0}, \mu_{i-1,1}, q'_{i-1}, q_{i-1,L}, q_{i-1,R}$
for $j = 1, \ldots, b_N$ **do**
// In these rounds, prover sends commitment to degree-3 polynomial s_j
for all $\sigma \in \{0,1\}^{b_N - j}$ and $g \in \{0,1\}^{b_G}$ and $k \in \{-1, 0, 1, 2\}$ **do**
$s \leftarrow (g,\ g_L,\ g_R)$ // g_L, g_R are labels of g's layer-i inputs in sub-circuit.
termP $\leftarrow \widetilde{\text{eq}}(q'_{i-1}, r'[1], \ldots, r'[j-1],\ k,\ \sigma[1], \ldots, \sigma[b_N - j]) \cdot$
$(\mu_{i-1,0} \cdot \chi_g(q_{i-1,L}) + \mu_{i-1,1} \cdot \chi_g(q_{i-1,R}))$
termL $\leftarrow \tilde{V}_i\,(r'[1], \ldots, r'[j-1],\ k,\ \sigma[1], \ldots, \sigma[b_N - j],\ g_L)$
termR $\leftarrow \tilde{V}_i\,(r'[1], \ldots, r'[j-1],\ k,\ \sigma[1], \ldots, \sigma[b_N - j],\ g_R)$

if g is an add gate **then** $s_j[\sigma, g][k] \leftarrow$ termP $\cdot$ (termL + termR)
else if g is a mult gate **then** $s_j[\sigma, g][k] \leftarrow$ termP $\cdot$ termL $\cdot$ termR

for $k \in \{-1, 0, 1, 2\}$ **do**
$s_j[k] \leftarrow \sum_{\sigma \in \{0,1\}^{b_N - j}} \sum_{g \in \{0,1\}^{b_G}} s_j[\sigma, g][k]$

// Compute coefficients of s_j and create a multi-commitment (§5)
SendToVerifier($\mathsf{Com}(s_j)$) // see Line 42 of Figure 9
$r'[j] \leftarrow$ ReceiveFromVerifier() // see Line 43 of Figure 9

$r' \leftarrow (r'[1], \ldots, r'[b_N])$ // notation

for $j = 1, \ldots, 2b_G$ **do**
// In these rounds, prover sends commitment to degree-2 polynomial $s_{b_N + j}$.
for all gates $g \in \{0,1\}^{b_G}$ and $k \in \{-1, 0, 1\}$ **do**
$s \leftarrow (g,\ g_L,\ g_R)$ // g_L, g_R are labels of g's layer-i inputs in subcircuit
$u_{k,0} \leftarrow (q_{i-1,L}[1], \ldots, q_{i-1,L}[b_G],\ r[1], \ldots, r[j-1],\ k)$
$u_{k,1} \leftarrow (q_{i-1,R}[1], \ldots, q_{i-1,R}[b_G],\ r[1], \ldots, r[j-1],\ k)$
termP $\leftarrow \widetilde{\text{eq}}(q'_{i-1}, r') \cdot (\mu_{i,0} \cdot \prod_{\ell=1}^{b_G + j} \chi_{s[\ell]}(u_{k,0}[\ell]) +$
$\mu_{i,1} \cdot \prod_{\ell=1}^{b_G + j} \chi_{s[\ell]}(u_{k,1}[\ell]))$

if $j \le b_G$ **then**
termL $\leftarrow \tilde{V}_i\,(r', r[1], \ldots, r[j-1],\ k,\ g_L[j{+}1], \ldots, g_L[b_G])$
termR $\leftarrow \tilde{V}_i\,(r', g_R)$
else // $b_G < j \le 2b_G$
termL $\leftarrow \tilde{V}_i(r', r[1], \ldots, r[b_G])$
termR $\leftarrow \tilde{V}_i\,(r', r[b_G{+}1], \ldots, r[j{-}1],\ k,\ g_R[j{-}b_G{+}1], \ldots, g_R[b_G])$

if g is an add gate **then** $s_{b_N + j}[g][k] \leftarrow$ termP $\cdot$ (termL + termR)
else if g is a mult gate **then** $s_{b_N + j}[g][k] \leftarrow$ termP $\cdot$ termL $\cdot$ termR

for $k \in \{-1, 0, 1\}$ **do**
$s_{b_N + j}[k] \leftarrow \sum_{g \in \{0,1\}^{b_G}} s_{b_N + j}[g][k]$

// Compute coefficients of $s_{b_N + j}$ and create a multi-commitment (§5)
SendToVerifier($\mathsf{Com}(s_{b_N + j})$) // see Line 42 of Figure 9
$r[j] \leftarrow$ ReceiveFromVerifier() // see Line 43 of Figure 9

$r_0 \leftarrow (r[1], \ldots, r[b_G])$ $r_1 \leftarrow (r[b_G + 1], \ldots, r[2b_G])$ // notation
$v_0 \leftarrow \tilde{V}_i(r', r_0)$ $v_1 \leftarrow \tilde{V}_i(r', r_1)$ // $X = \mathsf{Com}(v_0)$, $Y = \mathsf{Com}(v_1)$, $Z = \mathsf{Com}(v_0 v_1)$
SendToVerifier(X, Y, Z) // see Line 44 of Figure 9
// $\mathcal{P}$ computes $\{M_k\}$ as defined in Equation (5).
proof-of-sum-check (a_{i-1}, $\{\mathsf{Com}(s_j)\}$, $\{M_k\}$, X, Y, Z) // see Figure 1

return ($\mathsf{Com}(v_0)$, $\mathsf{Com}(v_1)$, r', r_0, r_1)

Figure 11—$\mathcal{P}$'s side of the zero-knowledge sum-check protocol in Hyrax-I (§7).

function Verify(ArithCircuit c, input x, output y)
// $\mathcal{P}$ sends witness w to $\mathcal{V}$ in the clear at start of protocol.
$w \leftarrow$ ReceiveFromProver() // see Line 3 of Figure 14
$(q'_0,\ q_0) \xleftarrow{R} \mathbb{F}^{\log N} \times \mathbb{F}^{\log G}$
$a_0 \leftarrow \tilde{V}_y(q'_0, q_0)$ // $\tilde{V}_y$ is the multilin. ext. of the output y

// The first iteration of sum-check only involves a claim about a single
// evaluation of $\tilde{V}_0$, rather than a linear combination of two evaluations;
// we encode this as a trivial linear combination.
$\mu_{0,0} \leftarrow 1, \mu_{0,1} \leftarrow 0, q_{0,L} \leftarrow q_0, q_{0,R} \leftarrow q_0$

SendToProver(q'_0, q_0) // see Line 4 of Figure 14
$d \leftarrow$ c.depth

for $i = 1, \ldots, d$ **do**
// Reduce $\mu_{i-1,0} \cdot \tilde{V}_{i-1}(q'_{i-1}, q_{i-1,L}) + \mu_{i-1,1} \cdot \tilde{V}_{i-1}(q'_{i-1}, q_{i-1,R}) \stackrel{?}{=} a_{i-1}$ to
// $Q_i(r', r_L, r_R) \stackrel{?}{=} e$, where Q_i is given in §3.2
$(e, r', r_L, r_R) \leftarrow$ SumCheckV($i,\ a_{i-1}$)

if $i < d$ **then**
$(q'_i,\ q_{i,L}, q_{i,R}) \leftarrow (r',\ r_L, r_R)$
$v_0, v_1 \leftarrow$ ReceiveFromProver() // see Line 53 of Figure 15
termA1 $\leftarrow \mu_{i-1,0} \cdot \tilde{\text{add}}_i(q_{i-1,L}, r_L, r_R)$
termA2 $\leftarrow \mu_{i-1,1} \cdot \tilde{\text{add}}_i(q_{i-1,R}, r_L, r_R)$
termM1 $\leftarrow \mu_{i-1,0} \cdot \tilde{\text{mult}}_i(q_{i-1,L}, r_L, r_R)$
termM2 $\leftarrow \mu_{i-1,1} \cdot \tilde{\text{mult}}_i(q_{i-1,R}, r_L, r_R)$

if $e \neq \widetilde{\text{eq}}(q'_{i-1},\ r') \cdot \big[(\text{termA1} + \text{termA2}) \cdot (v_0 + v_1)$
$+ (\text{termM1} + \text{termM2}) \cdot v_0 \cdot v_1\big]$ **then**
return reject

// Reduce the two v_0, v_1 questions to a random linear combination thereof
$\mu_{i,0}, \mu_{i,1} \xleftarrow{R} \mathbb{F}$
$a_i \leftarrow \mu_{i,0} \cdot v_0 + \mu_{i,1} \cdot v_1$

SendToProver($\mu_{i,0}, \mu_{i,1}$) // see Line 12 of Figure 14

// For the final layer, $\mathcal{P}$ and $\mathcal{V}$ reduce two points to one point (§3.2)
$(H_0, \ldots, H_{b_G}) \leftarrow$ ReceiveFromProver() // see Line 56 of Figure 15
$\tau \xleftarrow{R} \mathbb{F}$
$q'_d \leftarrow r'$
$q_d \leftarrow (1 - \tau) \cdot r_L + \tau \cdot r_R$
$a_d \leftarrow H_{b_G} \cdot \tau^{\log G} + H_{b_G - 1} \cdot \tau^{\log G - 1} + \ldots + H_0$

// $\tilde{V}_d(\cdot)$ is the multilinear extension of (x, w) where x is input and w is witness
if $\tilde{V}_d(q'_d,\ q_d) \neq a_d$ **then**
return reject
return accept

Figure 12—$\mathcal{V}$'s side of our non-zero-knowledge interactive proof Gir^{++}. Gir^{++} is equivalent to the interactive proof in Giraffe, but incorporates a technique of Chiesa et al. [35] in place of GKR's "reducing from two points to one point" step. $\mathcal{V}$'s side of the sum-check protocol and $\mathcal{P}$'s work are described in Figures 13, 14, and 15.

function SumCheckV(layer i, a_{i-1})
  $e \leftarrow a_{i-1}$
  $(r', r_L, r_R) \xleftarrow{R} \mathbb{F}^{b_N} \times \mathbb{F}^{b_G} \times \mathbb{F}^{b_G}$
  $r \leftarrow (r', r_L, r_R)$
  for $j = 1, 2, \ldots, (b_N + 2b_G)$ **do**
    // F_j is a degree-2 or degree-3 polynomial
    $F_j \leftarrow$ ReceiveFromProver()   // see Lines 18,47 of Figure 15
    if $F_j(0) + F_j(1) \neq e$ **then**
      return reject
    SendToProver($r[j]$)   // see Lines 19,48 of Figure 15
    $e \leftarrow F_j(r[j])$
  return (e, r', r_L, r_R)

Figure 13—$\mathcal{V}$'s side of the sum-check protocol within Gir^{++}. This protocol reduces the claim that a_i equals the sum $\sum_{n,h_L,h_R} Q_i(n, h_L, h_R)$ (this sum equals $\mu_{i-1,L} \cdot \tilde{V}_{i-1}(q'_{i-1}, q_{i-1,L}) + \mu_{i-1,R} \cdot \tilde{V}_{i-1}(q'_{i-1}, q_{i-1,R})$, per §3.2, "Random linear combination") to the claim $e = Q_i(r', r_L, r_R)$.

function Prove(ArithCircuit c, input x, output y)
  // Let w be a witness such that $c(x, w) = y$
  SendToVerifier(w)   // see Line 3 of Figure 12.
  $(q'_0, q_0) \leftarrow$ ReceiveFromVerifier()   // see Line 12 of Figure 12
  $\mu_{0,0} \leftarrow 1$, $\mu_{0,1} \leftarrow 0$, $q_{0,L} \leftarrow q_0$, $q_{0,R} \leftarrow q_0$.
  $d \leftarrow$ c.depth
  // each circuit layer induces one sum-check invocation
  for $i = 1, \ldots, d$ **do**
    $(q'_i, q_{i,L}, q_{i,R}) \leftarrow$ SumCheckP(c, i, $q'_{i-1}, \mu_{i-1,0}, q_{i-1,L}, \mu_{i-1,1}, q_{i-1,R}$)
    if $i < d$ **then**
      $(\mu_{i,0}, \mu_{i,1}) \leftarrow$ ReceiveFromVerifier()   // see Line 36 of Figure 12

Figure 14—Pseudocode for $\mathcal{P}$ in Gir^{++}. SumCheckP is defined in Figure 15.

function SumCheckP(ArithCircuit c, layer i, q'_{i-1}, $\mu_{i-1,0}$, $q_{i-1,0}$, $\mu_{i-1,1}$, $q_{i-1,1}$)
  for $j = 1, \ldots, b_N$ **do**
    // Prover sends degree-3 polynomial F_j. Does this by computing $F_j(-1), F_j(0), F_j(1), F_j(2)$ and then interpolating.

    for all $\sigma \in \{0,1\}^{b_N - j}$ and $g \in \{0,1\}^{b_G}$ and $k \in \{-1, 0, 1, 2\}$ **do**
      $s \leftarrow (g,\, g_L,\, g_R)$ // g_L, g_R are labels of g's layer-i inputs in sub-circuit.

      termP $\leftarrow \widetilde{\text{eq}}(q'_{i-1}, r'[1], \ldots, r'[j-1],\, k,\, \sigma[1], \ldots, \sigma[b_N - j]) \cdot \big(\mu_{i-1,0} \cdot \chi_g(q_{i-1,0}) + \mu_{i-1,1} \cdot \chi_g(q_{i-1,1})\big)$
      termL $\leftarrow \tilde{V}_i\,(r'[1], \ldots, r'[j-1],\, k,\, \sigma[1], \ldots, \sigma[b_N - j],\, g_L)$
      termR $\leftarrow \tilde{V}_i\,(r'[1], \ldots, r'[j-1],\, k,\, \sigma[1], \ldots, \sigma[b_N - j],\, g_R)$

      if g is an add gate **then** $F_j[\sigma, g][k] \leftarrow$ termP $\cdot$ (termL + termR)
      else if g is a mult gate **then** $F_j[\sigma, g][k] \leftarrow$ termP $\cdot$ termL $\cdot$ termR

    for $k \in \{-1, 0, 1, 2\}$ **do**
      $F_j[k] \leftarrow \sum_{\sigma \in \{0,1\}^{b_N - j}} \sum_{g \in \{0,1\}^{b_G}} F_j[\sigma, g][k]$
    // Use Lagrange interpolation to compute coefficients of F_j and send them to $\mathcal{V}$
    SendToVerifier(F_j, 3) // see Line 10 of Figure 13
    $r'[j] \leftarrow$ ReceiveFromVerifier() // see Line 15 of Figure 13

  $r' \leftarrow (r'[1], \ldots, r'[b_N])$ // notation

  for $j = 1, \ldots, 2b_G$ **do**
    // In these rounds, prover sends degree-2 polynomial $F_{b_N + j}$.
    for all gates $g \in \{0,1\}^{b_G}$ and $k \in \{-1, 0, 1\}$ **do**

      $s \leftarrow (g,\, g_L,\, g_R)$ // g_L, g_R are labels of g's layer-i inputs in subcircuit
      $u_{k,0} \leftarrow (q_{i-1,0}[1], \ldots, q_{i-1,0}[b_G],\, r[1], \ldots, r[j-1],\, k)$
      $u_{k,1} \leftarrow (q_{i-1,1}[1], \ldots, q_{i-1,1}[b_G],\, r[1], \ldots, r[j-1],\, k)$
      termP $\leftarrow \widetilde{\text{eq}}(q'_{i-1}, r') \cdot \Big(\mu_{i,0} \cdot \prod_{\ell=1}^{b_G + j} \chi_{s[\ell]}(u_{k,0}[\ell]) + \mu_{i,1} \cdot \prod_{\ell=1}^{b_G + j} \chi_{s[\ell]}(u_{k,1}[\ell])\Big)$

      if $j \le b_G$ **then**
        termL $\leftarrow \tilde{V}_i\,(r', r[1], \ldots, r[j-1],\, k,\, g_L[j+1], \ldots, g_L[b_G])$
        termR $\leftarrow \tilde{V}_i\,(r', g_R)$
      else // $b_G < j \le 2b_G$
        termL $\leftarrow \tilde{V}_i(r', r[1], \ldots, r[b_G])$
        termR $\leftarrow \tilde{V}_i\,(r', r[b_G+1], \ldots, r[j-1],\, k,\, g_R[j-b_G+1], \ldots, g_R[b_G])$

      if g is an add gate **then**
        $F_{b_N+j}[g][k] \leftarrow$ termP $\cdot$ (termL + termR)
      else if g is a mult gate **then**
        $F_{b_N+j}[g][k] \leftarrow$ termP $\cdot$ termL $\cdot$ termR

    for $k \in \{-1, 0, 1\}$ **do**
      $F_{b_N+j}[k] \leftarrow \sum_{g \in \{0,1\}^{b_G}} F_{b_N+j}[g][k]$
    // Use Lagrange interpolation to compute coefficients of F_{b_N+j} and send them to verifier
    SendToVerifier(F_{b_N+j}, 2) // see Line 10 of Figure 13
    $r[j] \leftarrow$ ReceiveFromVerifier() // see Line 15 of Figure 13

  $r_0 \leftarrow (r[1], \ldots, r[b_G])$ $r_1 \leftarrow (r[b_G+1], \ldots, r[2b_G])$ // notation

  if $i < c$.depth **then**
    SendToVerifier($\tilde{V}_i(r', r_0), \tilde{V}_i(r', r_1)$) // see Line 22 of Figure 12
  else // in the last sum-check invocation, $\mathcal{P}$ and $\mathcal{V}$ reduce two points to one point (§3.2)
    // First, compute coefficients of $H(\cdot)$, i.e., $\tilde{V}_d$ restricted to the line passing through (r', r_0) and (r', r_1).
    SendToVerifier($(H_0, \ldots, H_{b_G})$) // see Line 39 of Figure 12

  return (r', r_0, r_1)

Figure 15—$\mathcal{P}$ pseudocode in our non-zero-knowledge interactive proof Gir^{++} for the layer-i sum-check invocation.