An Effective Lower Bound on the Number of Orientable Supersingular Elliptic Curves

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Abstract. In this article, we prove a generic lower bound on the number of $\mathfrak{O}$-*orientable* supersingular curves over $\mathbb{F}_{p^2}$, i.e curves that admit an embedding of the quadratic order $\mathfrak{O}$ inside their endomorphism ring. Prior to this work, the only known effective lower-bound is restricted to small discriminants. Our main result targets the case of fundamental discriminants and we derive a generic bound using the expansion properties of the supersingular isogeny graphs.
Our work is motivated by isogeny-based cryptography and the increasing number of protocols based on $\mathfrak{O}$-oriented curves. In particular, our lower bound provides a complexity estimate for the brute-force attack against the new $\mathfrak{O}$-uber isogeny problem introduced by De Feo, Delpech de Saint Guilhem, Fouotsa, Kutas, Leroux, Petit, Silva and Wesolowski in their recent article on the SETA encryption scheme.

1 Introduction

The link between quadratic imaginary orders and elliptic curves have always been of great importance to elliptic curve cryptography, and isogeny-based cryptography is no exception. This connection dates back to the very beginning of the field with the CRS scheme, discovered independently by Couveignes [Cou06] and Rostovtsev and Stolbunov [RS06]. Their original idea is based on isogenies between ordinary curves over finite fields, i.e elliptic curves whose endomorphism ring is isomorphic to a quadratic imaginary order. However, for both security and efficiency reasons, ordinary curves were soon replaced by supersingular curves, i.e curves whose endomorphism ring is isomorphic to a maximal order inside a quaternion algebra. With the CGL hash function [CLG09] and the SIDH key exchange [JDF11] leading the charge, it seemed like the quadratic orders were destined to slowly disappear from the picture. However, they claimed back a share of the spotlight with CSIDH [CLM+18], a revival of CRS in the setting of supersingular curves with the quadratic order obtained by restricting to endomorphisms defined over $\mathbb{F}_p$. In fact, quadratic orders were never really gone as quaternion orders actually contain an infinity of them. Isogeny experts only needed time to understand their place in the rapidly evolving picture of isogeny-based cryptography. In parallel to numerous schemes built upon CSIDH and

its variants ([BKV19,ADFMP20] among others), several papers appeared trying to study the link between isogenies of supersingular curves and quadratic imaginary orders outside of the CSIDH framework. We can mention the OSIDH protocol by Colò and Kohel [CK19] for quadratic orders of smooth discriminant which introduced the terminology of orientations that we use in this paper and the work of Love and Boneh [LB20] on quadratic orders of small discriminant. More recently, Chenu and Smith [CS21] studied the case where the discriminant is a small integer times p. De Quehen et al. have highlighted in [QKL+21] the possibility to use the embedding of a specific quadratic order as a backdoor to break unbalanced variants of SIDH. The SETA encryption scheme [DFFDdSG+21] is built on the same principle. The set of SETA public keys is simply the set of $\mathfrak{O}$-orientable curves for some quadratic order $\mathfrak{O}$ and secret keys are concrete $\mathfrak{O}$-orientations. Additionally, the authors of SETA have introduced the "uber-isogeny assumption" as an attempt to provide a common framework for various security assumptions in isogeny-based cryptography. The formulation of the $\mathfrak{O}$-Uber Isogeny Problem ($\mathfrak{O}$-UIP) is explicitly parametrized by a quadratic imaginary order $\mathfrak{O}$ and it was shown in [DFFDdSG+21] how different variants of the $\mathfrak{O}$-UIP were related to the security of several isogeny-based protocols (including SIDH). Later [Wes21,ACL+22] have studied various algorithmic problems related to $\mathfrak{O}$-orientations and the $\mathfrak{O}$-UIP. Note that the recent attacks [CD22,MM22,Rob22] against SIDH do not apply to the $\mathfrak{O}$-UIP even if those attacks break SETA. It is because the attacks target the encryption mechanism of SETA but do not allow an attacker to perform a key recovery. Thus, the $\mathfrak{O}$-UIP remains an interesting subject of study.

Given the rich history that we have summarized above, it is important to study in more detail the link between quadratic orders and isogenies of supersingular curves. We denote by $\mathcal{E}_{\mathfrak{O}}(p)$ the set of $\mathfrak{O}$-orientable supersingular elliptic curves over $\mathbb{F}_{p^2}$. In this work, we study the cardinal $\#\mathcal{E}_{\mathfrak{O}}(p)$ of this set. The complexity of the brute force algorithm to solve the $\mathfrak{O}$-UIP is linear in $\#\mathcal{E}_{\mathfrak{O}}(p)$. Aiming at the cryptographic applications, we look for an effective bound in the cases where the discriminant of $\mathfrak{O}$ is polynomial in p and both have cryptographic size.

Related works. The number of orientable supersingular curve is related to the number of optimal embeddings of quadratic orders inside maximal orders of the quaternion algebra ramified at p and ∞ and is also linked with the number of representations of integers by ternary quadratic forms. Both quantities have been studied in the literature but not with the same goal. As far as we know, prior to our work, an effective bound is only known for a restricted range of discriminants and is due to Kaneko [Kan89]. In [Voi21, Chapter 30], several formulas are given involving sums of these numbers (such as the Eichler class number formula) from which it seems hard to derive a bound. There are also asymptotic results on number of representations by ternary quadratic forms (see for instance [IK21, Chapter 20]) but they rather target the case where d grows to infinity while p is fixed. Our work also shares some similarities with the trend of work started by Gross and Zagier [ZG85] on singular moduli and later enriched

by Dorman, and Lauter and Viray [Dor87,LV15]. Their results cannot be directly applied to our case because they target simultaneous embeddings of quadratic orders of distinct discriminants while we are going to focus on simultaneous embeddings of quadratic orders with the same discriminant. Nonetheless, some of the techniques developed in these works have inspired part of our analysis.

Contributions. Our main result is Proposition 11, a lower bound on $\#\mathcal{E}_{\mathfrak{O}}(p)$ when $\mathfrak{O}$ is a maximal quadratic order. The proof is based on the study of $K_{\mathfrak{O}}(p)$, the number of quaternion orders obtained from pairs of distinct $\mathfrak{O}$-orientations. With Proposition 14 in Appendix A, we cover the case of non-maximal orders and a lower bound on $\#\mathcal{E}_{\mathfrak{O}}(p)$ can be derived for any quadratic order by combining Propositions 11 and 14.

Asymptotically, our bound becomes trivial when the discriminant grows while the characteristic p stays fix. However, in the case where the discriminant is polynomial in p, our bound proves to be quite tight as we illustrate by using it to verify that the parameters proposed in [DFFDdSG+21] for the SETA encryption scheme reach the claimed level of security.

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The remainder of this paper is organized as follows: Section 2 introduces the necessary notations and mathematical notions. Section 3 is where we present our main result, we treat the case of fundamental discriminants in Section 3.2. We address the case of non-trivial conductor in Appendix A. Finally, in Section 4, we apply our results to a concrete example corresponding to the parameters of SETA.

2 Mathematical background

Notations. Throughout this document, we place ourselves in $B_{p,\infty}$, the definite quaternion algebra ramified at p and ∞ for some prime $p > 3$. We consider supersingular elliptic curves over $\mathbb{F}_{p^2}$ and write N_p for the number of isomorphism classes of such curves over the algebraic closure of $\mathbb{F}_p$.

We fix an imaginary quadratic field $\mathfrak{K}$ of discriminant $-d$ and ring of integers $\mathfrak{O}_{\mathfrak{K}}$. For any $\mathfrak{O} \subset \mathfrak{O}_{\mathfrak{K}}$, we write $f(\mathfrak{O})$ the conductor of $\mathfrak{O}$, i.e the only integer such that $\mathfrak{O} = \mathbb{Z} + f(\mathfrak{O})\mathfrak{O}_{\mathfrak{K}}$ (when it is clear from the context we will simply write f). The class group of $\mathfrak{O}$ is $\mathrm{Cl}(\mathfrak{O})$ and the class number is $h(\mathfrak{O})$. As a convention, we use $\mathcal{O}_*$ for quaternion orders in $B_{p,\infty}$ and $\mathfrak{O}_*$ for quadratic imaginary orders. We write $\mathbb{P}$ for the set of all primes. For any $d \in \mathbb{N}$, we define $\mathbb{P}_d = \{\ell \mid \ \ell \in \mathbb{P} \text{ and } \ell \text{ divides } d\}$.

2.1 Quaternion orders

For $a, b \in \mathbb{Z}^\star$ we denote by $H(a,b) = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + k\mathbb{Q}$ the quaternion algebra over $\mathbb{Q}$ with basis $1, i, j, k$ such that $i^2 = a$, $j^2 = b$ and $k = ij = -ji$. The unique quaternion algebra (up to isomorphism) ramified exactly at p and ∞, is always isomorphic to $H(-q, -p)$ where q is a small integer relatively to p ($q = O(\log(p)^2)$). For instance, when $p \equiv 3 \mod 4$, we can always take $q = 1$. Every quaternion algebra has a canonical involution that sends an element $\alpha = a_1 + a_2 i + a_3 j + a_4 k$ to its conjugate $\overline{\alpha} = a_1 - a_2 i - a_3 j - a_4 k$. We define the *reduced trace* and the *reduced norm* by $tr(\alpha) = \alpha + \overline{\alpha}$ and $n(\alpha) = \alpha\overline{\alpha}$.

Quaternion orders of $B_{p,\infty}$ are lattices of rank 4 inside $B_{p,\infty}$ that are also rings. It can be shown that quaternion orders are integral, i.e that norm and trace of all the elements are in $\mathbb{Z}$. Given a basis $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of $\mathcal{O}$, the *reduced discriminant* (or simply discriminant) of $\mathcal{O}$ is $\mathrm{disc}(\mathcal{O}) = \sqrt{\det(\alpha_i \alpha_j)_{i,j \in [1,4]}}$. For any $\mathcal{O} \subset B_{p,\infty}$, we have $p | \mathrm{disc}(\mathcal{O})$. The discriminant of a suborder $\mathcal{O}' \subset \mathcal{O}$ satisfies $\mathrm{disc}(\mathcal{O}) | \mathrm{disc}(\mathcal{O}')$.

Maximal orders are the orders that admit no proper superorders and, in particular, their discriminant is equal to p. *Eichler* orders are equal to intersection of two maximal orders (not necessarily distinct). Every quaternion order admits the unique decomposition $\mathbb{Z} + f(\mathcal{O})\mathrm{Gor}(\mathcal{O})$ where $f(\mathcal{O}) \in \mathbb{N}$ is the *Brandt Invariant* and $\mathrm{Gor}(\mathcal{O})$ is the *Gorenstein closure*. We can define Gorenstein orders as orders whose Brandt Invariant is 1. As the name suggests, $\mathrm{Gor}(\mathcal{O})$ is always Gorenstein. An order is *Bass* when all its superorders are Gorenstein. Equivalently, Bass orders of $B_{p,\infty}$ are the orders containing a maximal order of a quadratic imaginary field (this was originally the definition of *basic* orders but the two notions were proven equivalent by Chari, Smertnig and Voight in [CSV21]).

We have a chain of proper implication between all those notions

$$\text{maximal} \Rightarrow \text{Eichler} \Rightarrow \text{Bass} \Rightarrow \text{Gorenstein}.$$

We refer the reader to the book of John Voight for more background on quaternion algebras and quaternion orders [Voi21].

In this article, we will make use of *embedding numbers* of Bass orders, i.e the number of distinct maximal orders containing a given order. This problem was studied by Eichler and Brzezinski [BE92,Brz83] and was more recently used in [EHL+20] to estimate the complexity of an algorithm to compute the endomorphism ring of a supersingular curve. For the rest of this section, we fix a Bass order $\mathcal{O}$ of discriminant D. Following [BE92], we denote by $e(\mathcal{O})$ the *embedding number of* $\mathcal{O}$. It turns out that $e(\mathcal{O})$ can be computed efficiently using the local-to-global principle with the formula $e(\mathcal{O}) = \prod_{\ell \in \mathbb{P}} e_\ell(\mathcal{O})$ where $e_\ell(\mathcal{O})$ is the analog of $e(\mathcal{O})$ over the ℓ-adics: $e_\ell(\mathcal{O})$ is the number of maximal orders in $B_{p,\infty} \otimes \mathbb{Q}_\ell$ containing $\mathcal{O}_\ell = \mathcal{O} \otimes \mathbb{Z}_\ell$. An easy preliminary observation is that, $e_\ell(\mathcal{O}) = 1$ when ℓ is coprime with D. Thus, we can rewrite the above formula as $e(\mathcal{O}) = \prod_{\ell \in \mathbb{P}_d} e_\ell(\mathcal{O})$. The value of $e_\ell(\mathcal{O})$ is in fact closely related to the Eichler symbol $(\mathcal{O}/\ell)$, a notion introduced by Eichler in [Eic36]. Let us write k for the

residue field of $\mathbb{Q}_\ell$ and J for the Jacobson radical of $\mathcal{O}_\ell$. Then, we define the Eichler symbol as follows:

$$\left(\frac{\mathcal{O}}{\ell}\right) = \begin{cases} 1 & \text{if } \mathcal{O}_\ell/J \cong k \times k, \\ 0 & \text{if } \mathcal{O}_\ell/J \cong k, \\ -1 & \text{if } \mathcal{O}_\ell/J \text{ is a quadratic extension of } k. \end{cases} \tag{1}$$

The Eichler symbol is a very useful tool to understand the structure of an order $\mathcal{O}$ by the local-global principle. For instance, the order $\mathcal{O}_\ell$ is Eichler if and only if $(\mathcal{O}/\ell) = 1$. The Eichler symbol can be seen as a generalization of the Kronecker symbol, as becomes explicit with the reinterpretation presented in Proposition 1. For any quaternion element α, we write $\Delta(\alpha) = \mathrm{disc}(\mathbb{Z}[\alpha]) = \mathrm{tr}(\alpha)^2 - 4n(\alpha)$.

Proposition 1. *[Voi21]* $\left(\frac{\mathcal{O}}{\ell}\right) = \varepsilon$ *if and only if* $\left(\frac{\Delta(\alpha)}{\ell}\right)$ *takes all the values in* $\{0, \varepsilon\}$ *when* α *ranges over all the elements of* $\mathcal{O}$.

Then, it was shown by Eichler in [Eic36] (see [Brz83] for an account in English of this result) how the value of the Eichler symbol is linked to $e_\ell(\mathcal{O})$. We write $v_\ell(n)$ for the ℓ-adic valuation of a integer n.

Proposition 2. *Let* $\mathcal{O}$ *be a Bass order in* $B_{p,\infty}$ *of discriminant* D *and* $\ell \in \mathbb{P}_D$*:*

$$e_\ell(\mathcal{O}) = \begin{cases} v_\ell(D) + 1 & \text{if } (\mathcal{O}/\ell) = 1, \\ 2 & \text{if } (\mathcal{O}/\ell) = 0 \text{ and } \ell \neq p, \\ 1 & \text{if } (\mathcal{O}/\ell) = -1 \text{ or } ((\mathcal{O}/\ell) = 0 \text{ and } \ell = p)\,. \end{cases}$$

Remark 1. Note that $e_p(\mathcal{O})$ is always equal to 1. This follows from Proposition 2 and the fact that $(\mathcal{O}/p)$ cannot be 1.

2.2 Quadratic orders and oriented supersingular elliptic curves

In this section, we recall the basic definition and properties about orientations of elliptic curves, inspired by Colò and Kohel in [CK19]. The notion of orientation in Definition 1 below corresponds to the one of *primitive orientations with a p-orientation* in [CK19] and it is equivalent under the Deuring correspondence to *optimal embeddings* of quadratic orders inside maximal orders of $B_{p,\infty}$ (see [LB20]). The same notion is referred to as *normalized optimal embeddings* in [Bel08].

Definition 1. *Let* $\mathfrak{K}$ *be a quadratic imaginary field. For any elliptic curve* E*, a* $\mathfrak{K}$*-orientation is a ring homomorphism* $\iota : \mathfrak{K} \hookrightarrow \mathrm{End}(E) \otimes \mathbb{Q}$*. A* $\mathfrak{K}$*-orientation induces an* $\mathfrak{O}$*-orientation if* $\iota(\mathfrak{O}) = \mathrm{End}(E) \cap \iota(\mathfrak{K})$*. In that case, the pair* (E, ι) *is called a* $\mathfrak{O}$*-oriented curve and* E *is an* $\mathfrak{O}$*-orientable curve.*

When $E/\mathbb{F}_{p^2}$ is supersingular, Deuring showed that $\mathrm{End}(E)$ is a maximal order of $B_{p,\infty}$ [Deu41] and so we have $\mathrm{End}(E) \otimes \mathbb{Q} \cong B_{p,\infty}$. We denote by $\mathcal{S}_\mathfrak{O}(p)$ the set of $\mathfrak{O}$-oriented supersingular curves over $\mathbb{F}_{p^2}$ up to isomorphism

and Galois conjugacy. Note that this does not exactly match the definition used in [Onu21,Wes21] where orientations are not considered up to Galois conjugacy. We took this convention because we can state precise results when working up to Galois conjugacy (the Frobenius, which is the only non-trivial element in the Galois group of $\mathbb{F}_{p^2}$, creates somewhat artificial duplicates of a given orientation). The following proposition was shown by Onuki [Onu21, Proposition 3.2] and gives a concrete criterion to determine when $\mathcal{S}_{\mathfrak{O}}(p)$ is not empty.

Proposition 3. *The set $\mathcal{S}_{\mathfrak{O}}(p)$ is not empty if and only if p does not split in $\mathfrak{K}$ and does not divide the conductor of $\mathfrak{O}$.*

The value of $\#\mathcal{S}_{\mathfrak{O}}(p)$ can be easily computed from $p, \mathfrak{O}$. For instance, when p is inert in $\mathfrak{O}$, a consequence of [Onu21, Proposition 3.3, Theorem 3.4] is that $\#\mathcal{S}_{\mathfrak{O}}(p) = h(\mathfrak{O})$. When p ramified in $\mathfrak{O}$, the situation might be more complicated. In general we have that $\#\mathcal{S}_{\mathfrak{O}}(p) \in \{h(\mathfrak{O}), h(\mathfrak{O})/2\}$ by [ACL+22, Proposition 3.3].

In any case, the class group $\mathrm{Cl}(\mathfrak{O})$ acts on $\mathfrak{O}$-orientations through an operation that we write $\mathfrak{a} \star (E, \iota) = (E^{\mathfrak{a}}, \iota^{\mathfrak{a}})$.

Definition 2. *$\mathcal{E}_{\mathfrak{O}}(p)$ is the set of $\mathfrak{O}$-orientable curves (under isomorphism and Galois conjugacy).*

By definition of $\mathcal{E}_{\mathfrak{O}}(p)$, we have the obvious inequality $\#\mathcal{E}_{\mathfrak{O}}(p) \leq \min(\#\mathcal{S}_{\mathfrak{O}}(p), N_p)$.

The recent article [DFFDdSG+21] introduced a new isogeny-based problem: the $\mathfrak{O}$-Uber Isogeny Problem ($\mathfrak{O}$-UIP). We describe below as Problem 1, the $\mathfrak{O}$-UBER variant introduced by Wesolowski in [Wes21]. We assume for Problem 1 that $\mathfrak{O}$ and p satisfy the constraint in Proposition 3.

Problem 1. ($\mathfrak{O}$-UBER) Given $(E, \iota) \in \mathcal{S}_{\mathfrak{O}}(p)$ and $F \in \mathcal{E}_{\mathfrak{O}}(p)$, find $\mathfrak{a} \in \mathrm{Cl}(\mathfrak{O})$, such that $F = E^{\mathfrak{a}}$.

The brute force method to solve Problem 1 consists in trying all ideal classes until a solution is found. The expected complexity of this algorithm is linear in $\#\mathcal{E}_{\mathfrak{O}}(p)$ (and not $\#\mathcal{S}_{\mathfrak{O}}(p)$ since we look for any class connecting E and F and not a specific class).

3 The number of $\mathfrak{O}$-orientable supersingular curves.

In this section, we pursue the main goal of this article: finding a generic lower bound on the size of $\mathcal{E}_{\mathfrak{O}}(p)$. Henceforth, we assume that the Legendre symbol $(\mathrm{disc}(\mathfrak{O})/p) \neq 1$ and the conductor $f(\mathfrak{O})$ is coprime with p so we know by Proposition 3 that $\#\mathcal{E}_{\mathfrak{O}}(p) > 0$. We start with Section 3.1, where we introduce useful results from the literature and show a first lower bound when $\mathrm{disc}(\mathfrak{O}) \leq p$. In Section 3.2, we prove our main lower bound in the case where $f(\mathfrak{O}) = 1$. We extend this result to the generic case using the expansion property of the isogeny graphs in Appendix A.

3.1 A first result for small discriminants

The main result of this Section is Proposition 5 that was first proven by Kaneko in [Kan89]. This proposition allows us to derive interesting results on $\mathcal{E}_{\mathfrak{O}}(p)$ with Corollaries 1 and 2 (Corollary 1 being the only effective lower bound on $\#\mathcal{E}_{\mathfrak{O}}(p)$ prior to this work). Proposition 5 is obtained by studying the quaternion order generated by two integral elements in $B_{p,\infty}$. The study of these objects will prove to be very important for our results as well.

The quaternion order generated by two non-commuting elements. Let us take α_1, α_2, two integral elements in $B_{p,\infty}$. We want to look at the order $\mathcal{O}_{1,2} = \langle 1, \alpha_1, \alpha_2, \alpha_1\alpha_2 \rangle$. When α_1 and α_2 are not commuting, $\mathcal{O}_{1,2}$ is a quaternion order, i.e has rank 4 as a $\mathbb{Z}$-module. In Proposition 4, we give the classical formula to compute $\mathrm{disc}(\mathcal{O}_{1,2})$. Proposition 5 is a consequence of this formula and was proven in [Kan89].

Proposition 4. *[Koh96, Chapter 7] Let $\mathfrak{O}_i$ be quadratic orders equal to $\mathbb{Z}[\alpha_i]$ for $i = 1, 2$ such that α_1, α_2 are not commuting. Let $D_i = \mathrm{disc}(\mathfrak{O}_i)$, $t_i = \mathrm{tr}(\alpha_i)$ for $i \in \{1, 2\}$ and $s = \mathrm{tr}(\alpha_1\alpha_2)$, then $\mathrm{disc}(\mathcal{O}_{1,2}) = (D_1D_2 - (t_1t_2 - 2s)^2)/4$.*

Proposition 5. *[Kan89, Theorem 2'] Let $\mathfrak{O}_i$ be quadratic orders equal to $\mathbb{Z}[\alpha_i]$ for $i = 1, 2$ such that α_1, α_2 are not commuting. If $\mathfrak{O}_1, \mathfrak{O}_2$ have respective discriminant $-f_i^2 d$ (where d is a fundamental discriminant) and are contained inside the same quaternion maximal order $\mathcal{O} \subset B_{p,\infty}$, we have that $p \leq f_i f_j d$.*

Remark 2. During the proof for Proposition 5 we showed that $t_1t_2 - 2s \equiv \pm f_1 f_2 d \mod p$. This fact will be useful for what follows in Section 3.2.

Proposition 5 allows us to show interesting properties, including a lower bound on the size of $\mathcal{E}_{\mathfrak{O}}(p)$ (Corollary 1) and a bound on the minimal distance between two $\mathfrak{O}$-oriented curves (Corollary 2).

Corollary 1. *When $|\mathrm{disc}(\mathfrak{O})| < p$, $\#\mathcal{E}_{\mathfrak{O}}(p) = \#\mathcal{S}_{\mathfrak{O}}(p)$.*

Proof. If we assume that $\#\mathcal{E}_{\mathfrak{O}}(p) < \#\mathcal{S}_{\mathfrak{O}}(p)$, then there must be a curve E with two distinct $\mathfrak{O}$-orientations ι_1, ι_2. Under the Deuring correspondence, this implies that there are two distinct quadratic orders $\mathfrak{O}_1, \mathfrak{O}_2$ isomorphic to $\mathfrak{O}$ contained inside the maximal order $\mathcal{O} \cong \mathrm{End}(E)$. By Proposition 5, p must be smaller than d which contradicts our assumption.

Corollary 2. *Let ℓ be a prime different from p. If ℓ is inert in $\mathfrak{O}$ of discriminant d, then the shortest chain of ℓ-isogenies between two curves of $\mathcal{E}_{\mathfrak{O}}(p)$ has degree larger than p/d.*

Proof. (sketch) Let us denote the two curves by E_1, E_2 and by $\mathcal{O}_1, \mathcal{O}_2$, their respective endomorphism rings. Let us take $\varphi : E_1 \to E_2$ the smallest chain of ℓ-isogenies connecting them. Let us write $\theta_i \in \mathcal{O}_i$ such that $\mathfrak{O} \cong \mathbb{Z}[\theta_i]$. Since ℓ is inert in $\mathfrak{O}$, it can be shown that $\alpha_1 = \theta_1$ and $\alpha_2 = \hat{\varphi} \circ \theta_2 \circ \varphi$ are two elements in $\mathcal{O}_1$ that are not commuting (otherwise, at least one of the isogenies composing φ would be commuting with θ which is impossible since $\ell \neq p$ and is inert in $\mathfrak{K}$). Since $\mathrm{disc}(\mathbb{Z}[\alpha_1]) = -d$ and $\mathrm{disc}(\mathbb{Z}[\alpha_2]) = -\deg\varphi^2 d$, we obtain the desired bound by applying Proposition 5.

3.2 The case of $\mathfrak{O}_{\mathfrak{K}}$.

In this section, we focus on the case where $\mathfrak{O} = \mathfrak{O}_{\mathfrak{K}}$ for a quadratic imaginary field $\mathfrak{K}$ of discriminant $-d$. Our main result is Proposition 11.

To improve the reader's understanding, we divide the proof of Proposition 11 into several lemmas and propositions. Next, we give a brief outline and some insights into the generic principle. Our starting point is the observation (already used to prove Corollary 1) that if $\#\mathcal{E}_{\mathfrak{O}}(p) < \#\mathcal{S}_{\mathfrak{O}}(p)$, then there are some curves admitting several $\mathfrak{O}$-orientations. Similarly to Proposition 5, our result is obtained through the analysis of the quaternion orders obtained by combining together the different pairs of orientations. More concretely, we bound the number of these quaternion orders in two very different ways. The first one is a lower bound depending on $\#\mathcal{E}_{\mathfrak{O}}(p)$ and $\#\mathcal{S}_{\mathfrak{O}}(p)$ (Proposition 7) while the second one is an upper-bound (Proposition 10) that involves an explicit quantity that can be computed from d and p. The combination of these two bounds yields Proposition 11.

Here are some notations that we will use throughout this section. For any given $E \in \mathcal{E}_{\mathfrak{O}}(p)$, we write $N_E \geq 1$ for the number of distinct $\mathfrak{O}$-orientations of E. We write $\iota_1, \ldots, \iota_{N_E}$ for these N_E orientations, they induce the existence of endomorphisms $\alpha_1, \ldots, \alpha_{N_E} \in \mathrm{End}(E)$ such that $(\iota_i(\mathfrak{O}))_{1 \leq i \leq N_E} = (\mathbb{Z}[\alpha_i])_{1 \leq i \leq N_E}$. Since $\mathfrak{O}$ is the maximal order of $\mathfrak{K}$, we can assume that α_i is either $\iota_i^{-1}(\sqrt{-q})$ or $\iota_i^{-1}((1+\sqrt{-q})/2)$ where q is the squarefree integer such that $\mathfrak{K} = \mathbb{Q}(\sqrt{-q})$. Let $I^2_{\neq}(N_E)$ be the set of pairs of distinct unordered elements inside $\{1, \ldots, N_E\}$. We define an equivalence relation $\sim_E$ on $I^2_{\neq}(N_E)$ as

$$(i,j) \sim_E (l,m) \text{ iff } \langle 1, \alpha_i, \alpha_j, \alpha_i\alpha_j \rangle = \langle 1, \alpha_l, \alpha_m, \alpha_l\alpha_m \rangle.$$

Definition 3. *The set of equivalence classes under $\sim_E$ is denoted by $\mathcal{K}_E$. We write $K_E = \#\mathcal{K}_E$.*

Intuitively, $\mathcal{K}_E$ is the set of distinct quaternion orders obtained by combining two embeddings of $\mathfrak{O}$ inside $\mathrm{End}(E)$. By the results presented in Section 2.2, we have $\#\mathcal{S}_{\mathfrak{O}}(p) = \sum_{E \in \mathcal{E}_{\mathfrak{O}}(p)} N_E$. The quantity we propose to study is

$$K_{\mathfrak{O}}(p) = \sum_{E \in \mathcal{E}_{\mathfrak{O}}(p)} K_E \tag{2}$$

The link between $\#\mathcal{E}_{\mathfrak{O}}(p)$ and $K_{\mathfrak{O}}(p)$. The number K_E is obviously related to N_E for every curve $E \in \mathcal{E}_{\mathfrak{O}}(p)$. Intuitively, we would like to say that every pair α_i, α_j generates a different quaternion order $\mathcal{O}_{i,j}$ with $1 \leq i < j \leq N_E$ (thus proving that $K_E = N_E(N_E-1)/2$). However, even if this seems to be the case with good probability, it is not true in full generality. The correct statement is given in Proposition 6. Fortunately, Proposition 6 still allows us to derive Corollary 3 that lower-bounds K_E by $CN_E(N_E-1)$ for some constant factor C, which is enough for our purpose.

Proposition 6. *Let $\mathbb{Z}[\alpha_1], \mathbb{Z}[\alpha_2], \mathbb{Z}[\alpha_3]$ be three distinct embeddings of the quadratic order of discriminant $-d$ (with $d > 10$) inside a maximal quaternion order $\mathcal{O}$. If $d \not\equiv 3 \mod 4$ or $d \not\equiv 0, 1 \mod p$, then $\alpha_3 \notin \mathcal{O}_{1,2} = \langle 1, \alpha_1, \alpha_2, \alpha_1\alpha_2 \rangle$. When $d \equiv 0 \mod p$, either $\alpha_3 \notin \mathcal{O}_{1,2}$ or the trace of $\alpha_1\alpha_2$ is equal to $4n(\alpha_1)$ and α_3 is one of $\pm(\mathrm{tr}(\alpha_1)/2 + \alpha_1 - \alpha_2)$. When $d \equiv 3 \mod 4$ and $d \equiv 1 \mod p$, either $\alpha_3 \notin \mathcal{O}_{1,2}$ or the trace of $\alpha_1\alpha_2$ is $(d-1)/2$ and α_3 is one of $((d-1)/4 + \alpha_2 - \alpha_1\alpha_2), ((d-1)/4 + \alpha_1 - \alpha_1\alpha_2), ((9-d)/4 - \alpha_2 + \alpha_1\alpha_2), ((9-d)/4 - \alpha_1 + \alpha_1\alpha_2)$.*

Proof. Since all orders are isomorphic, we can assume that all α_i have the same trace and norm. Let us write $t = \mathrm{tr}(\alpha_i), n = n(\alpha_i)$ for any $i = 1, 2, 3$. The proof is based on the following claim: any $\alpha_3 \in \mathcal{O}_{1,2}$ corresponds to a solution $x, y, z \in \mathbb{Z}[1/2]$ with $x - y, y - z \in \mathbb{Z}$ to the quadratic equation:

$$q = q(x^2 + y^2) + sxy + z^2(q^2 - s^2/4) \tag{3}$$

for some integers s, q that we will define below. We exclude the trivial solutions $(1, 0, 0)$ and $(0, 1, 0)$.

As a consequence, our proof can be divided in two parts: first, we prove the correspondence between the solutions to Eq. (3) and the $\alpha_3 \in \mathcal{O}_{1,2}$, then we find the solutions of Eq. (3) to identify all the possible α_3.

Proof of the claim. If we assume that $\alpha_3 \in \mathcal{O}_{1,2}$, then there exists $v, x, y, z \in \mathbb{Z}$ such that $\alpha_3 = v + x\alpha_1 + y\alpha_2 + z\alpha_1\alpha_2$. The trace of α_3 implies the equation $t = 2v + t(x + y) + z\mathrm{tr}(\alpha_1\alpha_2)$. Thus, we rewrite $\alpha_3 = t/2 + x(\alpha_1 - t/2) + y(\alpha_2 - t/2) + z(\alpha_1\alpha_2 - \mathrm{tr}(\alpha_1\alpha_2)/2)$.

There are two different cases, depending on the value of $d \mod 4$. If $d \equiv 0 \mod 4$ then $d = 4q$ for some square-free $q \equiv 1 \mod 4$ and so we can assume w.l.o.g. that $t = 0$ and $\alpha_i = \omega_i$ with $\omega_i^2 = -q$. Else $d = q$ for some square-free $q \equiv 3 \mod 4$ and we can take $t = 1$, $\alpha_i = (1 + \omega_i)/2$ with $\omega_i^2 = -q$.

In both cases, let us write $s = \mathrm{tr}(\omega_1\omega_2)$. If $d \equiv 0 \mod 4$, then we obtain the norm equation $q = n(\alpha_3) = q(x^2 + y^2) + sxy + z^2(q^2 - s^2/4)$. When $d \equiv q \equiv 3 \mod 4$, we have $\alpha_3 = (1+\omega_3)/2 = 1/2 + x\omega_1/2 + y\omega_2/2 + (z/4)(1 + \omega_1 + \omega_2 + \omega_1\omega_2 - (1 + s/2))$. Then, we obtain $\omega_3 = (x + z/2)\omega_1 + (y + z/2)\omega_2 + z/2(\omega_1\omega_2 - s/2)$. Writing $x_2 = x + z/2$, $y_2 = y + z/2$, $z_2 = z/2$ and taking the norm, we obtain the equation $q = n(\omega_3) = q(x_2^2 + y_2^2) + sx_2y_2 + z_2^2(q^2 - s^2/4)$.

In conclusion, we need to find the solutions x, y, z in $\mathbb{Z}[1/2]$ and $x - y \in \mathbb{Z}, y - z \in \mathbb{Z}$ to the quadratic equation $q = q(x^2 + y^2) + sxy + z^2(q^2 - s^2/4)$ different from the trivial solutions $(1, 0, 0)$ and $(0, 1, 0)$. The end of the proof is dedicated to the enumeration of all possible solutions.

Finding the solutions of Eq. (3). Throughout this search, we will use heavily the fact that $|s| < 2q$ (which comes from $\mathrm{disc}(\mathbb{Z})[\omega_1\omega_2] = s^2 - 4q^2 < 0$). W.l.o.g. we can assume that $s \geq 0$. We can directly remove the case $x = y = 0$ as it is clearly impossible to find a z such that $q = z^2(q^2 - s^2/4)$.

Our first step is to find the possible values of z. Let us rewrite our equation as $q = q(x^2+y^2)+sxy+z^2(q-s/2)(q+s/2)$. With the bound $(q-s/2)(q+s/2) \geq q/2$ we get that we must have $z \in \{0, \pm 1/2, \pm 1\}$. This implies that $s \equiv 0 \mod 2$ (otherwise $q(x^2+y^2)+sxy+z^2(q-s/2)(q+s/2)$ would not be an integer). With

that additional information, we can actually show that $q^2 - s^2/4 \geq q$. Thus, in fact we must have $z \in \{0, \pm 1/2\}$. We also have $q \geq q(x^2 + y^2) + sxy$.

Now that we have greatly reduced the number of possible z, we can look at the values x, y. We distinguish two cases, depending on the sign of x, y.

Let us assume that $xy \geq 0$. Then, we have $q \geq q(x^2 + y^2) + sxy$. Thus, we must have $x^2 + y^2 \leq 1$. Since we exclude $(x, y) \in \{(0, 0), (0, 1), (1, 0)\}$, the only possibility respecting all our constraints is $(x, y, z) = (1/2, 1/2, \pm 1/2)$. Thus, we obtain $q = q/2 + s/4 + q^2/4 - s^2/16$ which leads to the equation $q^2 - 2q + s(1 - s/4) = 0$. The discriminant of the polynomial $X^2 - 2X + s(1 - s/4)$ is equal to $4 - 4s + s^2 = (s - 2)^2$. The two possible solutions are $s/2$ and $(4 - s)/2$. The first one is impossible by the bound $s < 2q$. Since $s \geq 0$ we obtain $(4 - s)/2 < 2$ and this is incompatible with the bound $d > 10$.

We have seen that we have no solutions to Eq. (3) when $xy \geq 0$. Let us now consider the case $xy < 0$. W.l.o.g. we can assume that $x > 0$ and $y < 0$. Then, we have $q \leq q(x^2 + y^2) - s|xy|$, but the bound $s < 2q$ leads to the inequality $q(x^2 + y^2) - s|xy| > q(x^2 + y^2) - 2q|xy| = q(x + y)^2$. If we want to avoid a contradiction between these two bounds, then we must have $|x + y| < 1$. Since $x - y \in \mathbb{Z}$, the only possibility is $x = -y$.

When $x = -y$, we can rewrite Eq. (3) as

$$q = (q - s/2)(2x^2 + z^2(q + s/2)). \tag{4}$$

Let us study this new equation. Since we have only a few possibilities for z, we can simply see what happens with Eq. (4) for each value of z. Since the equation is in z^2 there are two cases: $z = 0$ and $z = \pm 1/2$.

If $z = 0$, we get $q = x^2(2q - s)$ where $x \in \mathbb{Z}$ and so the only solution is $x = 1$ and $s = q$ since q is square-free. However, looking at the discriminant of $\langle 1, \omega_1, \omega_2, \omega_1\omega_2 \rangle$ with Proposition 4, we get that p must divide q as it divides the discriminant of any maximal order in $B_{p,\infty}$. In that case, we have the solution $(x, y, z) = \pm(1, -1, 0)$.

If $z = \pm 1/2$, the requirement $x - z \in \mathbb{Z}$ implies that we can write $x = x'/2$ with $x' \equiv 1 \mod 2$. Putting all this in Eq. (4), we get $q = (q - s/2)x'^2/2 + 1/4(q - s/2)(q + s/2)$ (we recall that $s' := s/2$ is in $\mathbb{Z}$). It is clear that we must have $q \pm s' \equiv 0 \mod 2$ so let us write $q \pm s' = 2q_\pm$. Our equation becomes $q = q_- + q_+ = q_- x'^2 + q_+ q_-$ which implies that $q_+ \equiv 0 \mod q_-$. Thus, we must have $q_+ = kq_-$ for some $k \in \mathbb{Z}$ and Eq. (4) becomes $q = (k+1)q_- = q_-(x'^2 + kq_-)$ which can only be satisfied if $x' = 1$ and $q_- = 1$. In that case, the only possible solution (up to signs) is $(x, y, z) = (1/2, -1/2, \pm 1/2)$ when $s = 2q - 4$.

In summary, we have showed that our equations have the non-trivial solutions $\pm(1, -1, 0)$ when $d \equiv 0 \mod p$ and $s = q$ or $\pm(1/2, -1/2, \pm 1/2)$ when $d \equiv 3 \mod 4$ and $s = 2q - 4$ and none otherwise. We conclude the proof by computing what are the corresponding values of α_3, so we compute the concrete values $v, x, y, z \in \mathbb{Z}$ such that $\alpha_3 = v + x\alpha_1 + y\alpha_2 + z\alpha_1\alpha_2$ (note that these values x, y, z are not directly the solutions to Eq. (3) when $d = 3 \mod 4$, see the proof of our claim at the beginning of the proof).

For the first solution, selecting the value v to verify the trace equation we get that $\alpha_3 = \pm\mathrm{tr}(\alpha_1) + \alpha_1 - \alpha_2$. It is easily verified that $\mathrm{tr}(\alpha_1\alpha_2) = 4n(\alpha_1\alpha_2)$

when $s = q$. Otherwise, $\alpha_3 = v + x\alpha_1 + y\alpha_2 + z\alpha_1\alpha_2$ can only have a solution when $d \equiv q \equiv 3 \mod 4$ and $\mathrm{tr}(\alpha_1\alpha_2) = 1/2 + 1/4(\mathrm{tr}(\omega_1\omega_2)) = (d-1)/2$. By computing the discriminant of $\mathbb{Z}\langle 1, \alpha_1, \alpha_2, \alpha_1\alpha_2\rangle$ with Proposition 4 when $\mathrm{tr}(\alpha_1) = \mathrm{tr}(\alpha_2) = 1$ and $\mathrm{tr}(\alpha_1\alpha_2) = (q-1)/2$, we see that $\Delta = d - 1$ and so p divides $d - 1$. This proves that $d \equiv 1 \mod p$ is also a necessary condition for our equation to be satisfied.

The other possibilities for α_3 can easily be found by taking $(x + z/2, y + z/2, z/2) = \pm(1/2, -1/2, \pm 1/2)$ and v be such that $\mathrm{tr}(\alpha_3) = 1$.

Corollary 3. $K_E \geq \frac{N_E(N_E-1)}{12}$.

Proof. From $\alpha_l \notin \mathcal{O}_{i,j}$ or $\alpha_m \notin \mathcal{O}_{i,j} \Rightarrow (i,j) \not\sim_E (l,m)$ for any i, j, m, l, we see from Proposition 6 that the cardinality of any equivalence class in $I^2_{\neq}(N_E)$ must be smaller than 6, as there at most four elements $\alpha_{i'}$ contained in $\mathcal{O}_{i,j}$ and we must choose two among the possible i' to get a full quaternion order. This bound combined with $\#I^2_{\neq}(N_E) = N(N-1)/2$ gives the result directly.

The bound obtained in Corollary 3 is the key ingredient to the inequality between $\#\mathcal{E}_{\mathfrak{O}}(p)$, $\#\mathcal{S}_{\mathfrak{O}}(p)$ and $K_{\mathfrak{O}}(p)$ in Proposition 7.

Proposition 7. $K_{\mathfrak{O}}(p) \geq \frac{1}{12}\left(\frac{\#\mathcal{S}_{\mathfrak{O}}(p)^2}{\#\mathcal{E}_{\mathfrak{O}}(p)} - \#\mathcal{S}_{\mathfrak{O}}(p)\right)$

Proof. We have $\#\mathcal{S}_{\mathfrak{O}}(p) = \sum_{E \in \mathcal{E}_{\mathfrak{O}}(p)} N_E$. Using Corollary 3, we get $\sum_{E \in \mathcal{E}_{\mathfrak{O}}(p)} K_E \geq (1/12)\sum_{E \in \mathcal{E}_{\mathfrak{O}}(p)}(N_E^2 - N_E)$. Then, we can use the classical inequality $\sum_{i=1}^n x_i^2 \geq (1/n)(\sum_{i=1}^n x_i)^2$ to get the result.

A generic upper-bound of $K_{\mathfrak{O}}(p)$. If (i,j) is a representative of a class $k \in \mathcal{K}_E$, we define t_k as the value of $\mathrm{tr}(\alpha_i\alpha_j)$ and $\mathcal{O}_k$ as the quaternion order equal to the image of $\langle 1, \alpha_i, \alpha_j, \alpha_i\alpha_j\rangle$ under the isomorphism between $B_{p,\infty}$ and $\mathrm{End}(E) \otimes \mathbb{Q}$ (by definition of $\mathcal{K}_E$, t_k and $\mathcal{O}_k$ are independent of a choice of i, j). The idea is to look at the embedding number of the different orders $\mathcal{O}_k$ for $k \in \mathcal{K}_E$ and $E \in \mathcal{E}_{\mathfrak{O}}(p)$ in order to rewrite $\sum_{E \in \mathcal{E}_{\mathfrak{O}}(p)} K_E$. With the notation from Section 2.1, we write this number $e(\mathcal{O}_k)$ for a given class k and we compute it in Proposition 8. Before proving this result, we need to understand a bit better the structure of the orders $\mathcal{O}_k$. This is the purpose of Lemma 1.

Lemma 1. *Let E be a curve in $\mathcal{E}_{\mathfrak{O}}(p)$ and $k \in \mathcal{K}_E$. The order $\mathcal{O}_k$ is a Bass order.*

Proof. One of the several equivalent definitions of Bass orders inside $B_{p,\infty}$ is that they contain a maximal order inside a commutative subalgebra of $B_{p,\infty}$ [Voi21, Section 24.5]. Since $\mathfrak{O}$ is the maximal order of $\mathfrak{K}$, and the property follows from the definition of $\mathcal{O}_k$.

With the knowledge that the $\mathcal{O}_k$ are Bass orders, we can use Proposition 1 and Proposition 2 to compute $e(\mathcal{O}_k)$.

Proposition 8. *Let $D_k = \mathrm{disc}(\mathcal{O}_k)/p$. The embedding number of $\mathcal{O}_k$ is*

$$e(\mathcal{O}_k) = \prod_{\ell \in \mathbb{P}_{D_k}, (-d/\ell)=1} (v_\ell(D_k)+1) \prod_{\ell \in \mathbb{P}_{D_k}, (-d/\ell)=0, \ell \neq p} 2.$$

Proof. If we show that when ℓ is a prime dividing D_k, $(\mathcal{O}/\ell) = (-d/\ell)$, then the result follows from Proposition 2 and Lemma 1. First, note that when $\ell = p$, the local embedding number $e_\ell(\mathcal{O})$ is always equal to 1 (it is a consequence of Propositions 1 and 2 and the fact that $(-d/p) \neq 1$). Then, it suffices to prove the result for the cases where $\ell \neq p$ is either split or ramified in $\mathfrak{K}$. The two results $(-d/\ell) = 1 \Rightarrow (\mathcal{O}/\ell) = 1$ and $(\mathcal{O}/\ell) = 0 \Rightarrow (-d/\ell) = 0$ are easily implied by Proposition 1. To conclude, it suffices to show $(\mathcal{O}/\ell) = 0 \Leftarrow (-d/\ell) = 0$, as $(\mathcal{O}/\ell) = 1 \Rightarrow (-d/\ell) \in \{0, 1\}$. For that, we will show that ℓ divides the discriminant of every $\alpha \in \mathcal{O}_k$. We recall that there exists α_i, α_j with $\mathfrak{O} \cong \mathbb{Z}[\alpha_i] \cong \mathbb{Z}[\alpha_j]$ and $\mathcal{O}_k = \langle 1, \alpha_i, \alpha_j, \alpha_i\alpha_j \rangle$. By assumption, the property is satisfied for α_i, α_j. We recall the value of $D_k = (d^2 - (\varepsilon - 2t_k)^2)/4p$ where $\varepsilon = \mathrm{tr}(\alpha_i)\mathrm{tr}(\alpha_j)$. From $\ell \mid D_k$ and $\ell \mid d$, we get that ℓ must divide $2t_k - \varepsilon$ which implies that $\ell \mid \Delta(\alpha_i\alpha_j)$. Then, using $\ell \mid d$ and $\ell | (2t_k - \varepsilon)$, we can conclude that $\ell | \Delta(x + y\alpha_i + z\alpha_j + w\alpha_1\alpha_2)$ for any $x, y, z, w \in \mathbb{Z}^4$.

The value of $\mathrm{disc}(\mathcal{O}_k)$ is $(d^2 - (\mathrm{tr}(\alpha_i)\mathrm{tr}(\alpha_j) - 2t_k)^2)/4$ by Proposition 4. It can be shown that $\mathrm{tr}(\alpha_i)\mathrm{tr}(\alpha_j)$ is a constant that is either 0 or 1 depending only on the value of $d \mod 4$. Henceforth, we write this constant ε_d. Inspired by the formulation of Proposition 8, we define the functions

$$D : (t, d, p) \mapsto \frac{(d^2 - (\varepsilon_d - 2t)^2)}{4p}$$

and

$$e : (t, d, p) \mapsto \prod_{\ell \in \mathbb{P}_{D(t,d,p)}, (d/\ell)=1} (v_\ell(D(t, d, p)) + 1) \prod_{\ell \in \mathbb{P}_{D(t,d,p)} (d/\ell)=0, \ell \neq p} 2.$$

Let us define $T_{\mathfrak{O}}(p) = \{t_k | k \in \mathcal{K}_E \text{ for } E \in \mathcal{E}_{\mathfrak{O}}(p)\}$. For each $t \in T_{\mathfrak{O}}(p)$, the values $D(t, d, p)$ and $e(t, d, p)$ are well-defined, when p is prime and d is a fundamental discriminant coprime with p.

Proposition 9. *Let $\mathfrak{O}$ be the maximal quadratic order of discriminant $-d$. Then,*

$$\frac{1}{12} \sum_{t \in T_{\mathfrak{O}}(p)} e(t, d, p) \leq K_{\mathfrak{O}}(p) \leq \sum_{t \in T_{\mathfrak{O}}(p)} e(t, d, p)$$

Proof. By definition, we have that for every class k, there exists $t \in T_{\mathfrak{O}}(p)$ with $t = t_k$ and $e(\mathcal{O}_k) = e(t, d, p)$. Thus, each class k corresponds to at least one embedding of $\mathcal{O}_k$ inside a maximal order and so we must have $K_{\mathfrak{O}}(p) \leq \sum_{t \in T_{\mathfrak{O}}(p)} e(t, d, p)$.

The lower bound of $K_{\mathfrak{O}}(p)$ is more delicate to obtain. For that, we will need to quantify the maximum number of embedding that corresponds to the same

class k. Let us take an element $t \in T_{\mathfrak{O}}(p)$. By definition of $T_{\mathfrak{O}}(p)$, there exists a curve $E \in \mathcal{E}_{\mathfrak{O}}(p)$ and a class $k \in K_E$ with $t_k = t$. By definition of the embedding number, there exist $e(t, d, p)$ distinct maximal orders containing $\mathcal{O}_k$. Each of these maximal orders $\mathcal{O}'$ corresponds to the isomorphism class (up to Galois conjugacy) of supersingular curve E' under the Deuring Correspondence. By definition there also exists a class $k' \in \mathcal{K}_{E'}$ such that $\mathcal{O}_{k'} \cong \mathcal{O}_k$.

We will provide an upper bound on the number of these $e(t, d, p)$ classes that are equal. Up to composition with the relevant isomorphisms, we can assume that all the orders $\mathcal{O}_{k'}$ are actually equal (and not simply isomorphic). Let us take $\mathcal{O}^1 \neq \mathcal{O}^2$, maximal orders with $\mathcal{O}_k \subset \mathcal{O}^i$ for $i = 1, 2$ and assume that these two embeddings of $\mathcal{O}_k$ lead to the same class k' and curve E'. We must have $\mathcal{O}^1 \cong \mathcal{O}^2 \cong \mathrm{End}(E)'$, so let us write $\sigma_i : \mathcal{O}^i \to \mathrm{End}(E)$ the corresponding isomorphisms. By definition of our equivalence relation, we must have $\sigma_1(\mathcal{O}_k) = \sigma_2(\mathcal{O}_k)$, which means that $\mathcal{O}_k$ is stable under the isomorphism $\sigma_1^{-1} \circ \sigma_2 : \mathcal{O}^2 \to \mathcal{O}^1$. This shows that our problem reduces to counting the number of isomorphisms that leave stable $\mathcal{O}_k$ but are not the identity. For that, it suffices to count the number of possible images of the two elements α_i, α_j such that $k = (i, j)$. Proposition 6 tells us that there are at most two other elements of same norm and trace as α_i, α_j. Thus, we have $3 \times 4 = 12$ possible image pair for α_i, α_j, and so we can conclude that each class k corresponds to at most 12 distinct embeddings of $\mathcal{O}_k$ inside distinct maximal orders. This proves the result.

With Proposition 9, we have all the necessary ingredients to prove our generic upper-bound of $K_{\mathfrak{O}}(p)$. We introduce in Definition 4, a final notation to simplify the formulation of Proposition 10.

Definition 4. *The function $\tau : \mathbb{N} \to \mathbb{N}$ is defined as $\tau(N) = \prod_{\ell \in \mathbb{P}_N} (v_\ell(N) + 1)$, and it computes the number of divisors of N.*

Proposition 10. $K_{\mathfrak{O}}(p) \leq \left\lceil \frac{d+1}{4p} \right\rceil \max_{0 \leq N \leq d^2/(4p)} \tau(N)$.

Proof. It is clear from the definition of the functions τ, D, e that $\tau(D(d, t, p)) \geq e(d, t, p)$. Since $0 \leq D(t, d, p) \leq d^2/(4p)$ we get that

$$\sum_{t \in T_{\mathfrak{O}}(p)} e(t, d, p) \leq \#T_{\mathfrak{O}}(p) \max_{0 \leq N \leq d^2/(4p)} \tau(N).$$

Next, we prove that $\#T_{\mathfrak{O}} \leq \lceil (d+1)/4p \rceil$. If $t \in T_{\mathfrak{O}}$, we must have that $D(d, t, p) = \mathrm{disc}(\mathcal{O}_k)/p \in \mathbb{N}^\star$ for some class k. The condition on the discriminant yields $d^2 - (\varepsilon_d - 2t)^2 \equiv 0 \mod 4p$ and $d^2 > (\varepsilon_d - 2t)^2$. When $t > 0$, we have $|2t| - \varepsilon_d > 0$ and so get the bound $(|d| + \varepsilon_d)/2 > |t|$. There are two possible values of $t \mod 4p$ and combining that with the $0 < |t| < (|d| + \varepsilon_d)/2$ we obtain at most $\lfloor (d+1)/(4p) \rfloor$ possible values. Adding $t = 0$, we obtain the desired bound. The proof is concluded by Proposition 9.

We obtain a generic lower bound on $\#\mathcal{E}_{\mathfrak{O}}(p)$ in Proposition 11. It is a combination of Proposition 7 and Proposition 10.

Proposition 11. *$\#\mathcal{E}_{\mathfrak{O}}(p) \geq \frac{AB}{A+B} \geq \frac{1}{2}\min(A,B)$ where $A = \#\mathcal{S}_{\mathfrak{O}}(p)$ and $B = \frac{\#\mathcal{S}_{\mathfrak{O}}(p)^2}{3(4p+d+1)} \frac{p}{\max\limits_{0\leq N\leq d^2/4p} \tau(N)}$.*

Before proving Proposition 11, we prove Lemma 2 that will be useful.

Lemma 2. *For every 3 values $x, A, B > 0$ such that $x \geq \lambda A$ or $x \geq (1-\lambda)B$ for every $0 < \lambda < 1$. Then, $x \geq \frac{AB}{A+B} \geq \frac{1}{2}\min(A,B)$.*

Proof. We are going to start with the intermediary result that $x \geq \lambda A$ or $x \geq (1-\lambda)B$ for every $0 < \lambda < 1$ imply that $x \geq \max_{0<\lambda<1} \min(\lambda A, (1-\lambda)B)$. The function $\lambda \mapsto \min(\lambda A, (1-\lambda)B)$ is increasing on $]0, \lambda_m]$ and decreasing on $[\lambda_m, 1[$ for the value λ_m such that $\lambda_m A = (1-\lambda_m)B$. Thus, we get $\lambda_m = B/(A+B)$ and $\max_{0<\lambda<1} \min(\lambda A, (1-\lambda)B) = \lambda_m A = AB/(A+B)$.

To conclude it is easy to verify that

$$\frac{AB}{A+B} \geq \frac{1}{2}\min(A,B).$$

Proof. (Proposition 11) We will apply Lemma 2 with $x = \#\mathcal{E}_{\mathfrak{O}}(p)$ and the values A, B as in Proposition 11. Thus, we need to prove that either $\#\mathcal{E}_{\mathfrak{O}}(p) \geq \lambda A$ or $\#\mathcal{E}_{\mathfrak{O}}(p) \geq (1-\lambda)B$ for any $0 < \lambda < 1$.

Note that when $\#\mathcal{E}_{\mathfrak{O}}(p) < \lambda \#\mathcal{S}_{\mathfrak{O}}(p)$ for some $\lambda \leq 1$, we must have $K_{\mathfrak{O}}(p) > 0$ because there is at least one curve with two distinct orientations. Thus, Proposition 7 proves that

$$\#\mathcal{E}_{\mathfrak{O}}(p) \geq \frac{\#\mathcal{S}_{\mathfrak{O}}(p)^2 - \#\mathcal{S}_{\mathfrak{O}}(p)\#\mathcal{E}_{\mathfrak{O}}(p)}{12K_{\mathfrak{O}}(p)}.$$

For any $\lambda \in [0,1]$, if $\#\mathcal{E}_{\mathfrak{O}}(p) < \lambda \#\mathcal{S}_{\mathfrak{O}}(p)$, we have that

$$\#\mathcal{E}_{\mathfrak{O}}(p) > (1-\lambda)\frac{\#\mathcal{S}_{\mathfrak{O}}(p)^2}{12K_{\mathfrak{O}}(p)}.$$

The proof is concluded by Proposition 10 and $\lceil (d+1)/4p \rceil \leq 1 + (d+1)/4p$.

Remark 3. Our bound becomes less and less tight when the size of d grows in comparison to p. Asymptotically, we have

$$\lim_{d\to\infty} \frac{\#\mathcal{S}_{\mathfrak{O}}(p)^2}{3(4p+d+1)} \frac{p}{\max\limits_{0\leq N\leq d^2/4p} \tau(N)} = 0$$

which is very far from the expected $\mathcal{E}_{\mathfrak{O}}(p) = N_p$ when $d \to \infty$. However, when the value of d is polynomial in p, classical analysis on the τ function detailed below shows that our bound will never be trivial even as p grows to infinity. This is typically the case needed for isogeny-based cryptography as illustrated by our numerical application in Section 4 for a prime $p \approx 2^{400}$ and a discriminant d satisfying $p < d < p^2$.

Remark 4. Note that we can derive a family of upper-bounds on the class number $h(\mathfrak{O})$ from Proposition 11. Indeed, since we have the trivial bound $p/12+1 \geq N_p \geq \#\mathcal{E}_{\mathfrak{O}}(p)$, in the cases where $A > B$ (which will happen when d is much bigger than p as explained in Remark 3), we obtain $p/12+1 \geq B/2$. When we replace $\#\mathcal{S}_{\mathfrak{O}}(p)$ by the correct value $ch(\mathfrak{O})$ (with $c \in \{1/2, 1\}$) in the formula for B we can obtain

$$h(\mathfrak{O})^2 < \frac{(p+12)(4p+d+1)}{2c^2 p} \max_{0 \leq N \leq d^2/4p} \tau(N) \tag{5}$$

Intuitively, the best bounds should be obtained when $d \approx p$. The estimates we provide on $\tau(N)$ below do not allow us to conclude that this would lead to an improvement on the state of the art upper bounds on class numbers.

Remark 5. When p divides d, it might be possible to get better bounds. For instance, when d/p is a prime smaller than $p/4$, a lower bound was proven in [EHL+20, Theorem 3.9], using the fact that a curve in $\mathcal{E}_{\mathfrak{O}}(p)$ must be d/p-isogenous to its Galois conjugate E^p. Another possibility, is to exploit the fact that when $d \equiv 0 \mod p$, the element $\omega_1\omega_2/p$ is integral (see the proof of Proposition 6 for the definition of ω_1, ω_2) and so we may be able to consider superorders of the $\mathcal{O}_k$ (which might give a better bound since the discriminants are smaller). While this idea seems promising, it does not appear trivial to obtain the analog of Proposition 6 and this is why we left the study of this special case open for future work.

Upper bound on the number of divisors function. The number of divisors function τ is well-studied and generic upper-bounds can be found in the literature. Since Wigert [Wig07], we know that $\tau(N) = O(N^{\varepsilon})$ for any $\varepsilon > 0$. In 1983, Nicolas and Robin showed that

$$\tau(N) \leq 2^{\eta_1 \frac{\log(N)}{\log\log(N)}}, \text{ for any } N \geq 3 \tag{6}$$

where $\eta_1 = 1.53793986\ldots$

More recently De Konick and Letendre [DKL18] proved several new upper-bounds involving $\omega(N)$ the number of distinct prime factors of N. In particular they showed that for any composite $n \geq 2$

$$\tau(N) \leq \left(1 + \eta_3 \frac{\log(N)}{\omega(N)\log(\omega(N))}\right)^{\omega(N)} \tag{7}$$

where $\eta_3 = 1,1999953\ldots$

When $\omega(N) \geq 74$ they even prove that

$$\tau(N) \leq \left(1 + \frac{\log(N)}{\omega(N)\log(\omega(N))}\right)^{\omega(N)} \tag{8}$$

We will use this last bound in the numerical application we propose in Section 4.

4 A numerical application to the parameters of SETA

In this section, we are going to use Proposition 11 to provide a lower bound on the complexity of the brute-force algorithm to solve the $\mathfrak{O}$-UBER problem where $\mathfrak{O}$ is the quadratic order used in the SETA encryption scheme. This will give a lower bound on the hardness of SETA key recovery (using brute-force) and answer the interrogations left open in [DFFDdSG+21, Section 5.3] on the concrete hardness of the uber isogeny problem. More precisely, with our new result, we are able to prove, under a few reasonable assumptions, that there exists a fitting choice of quadratic order $\mathfrak{O}$ such that the best brute force key recovery attack is hard enough for the claimed security level in [DFFDdSG+21].

The SETA parameters. The set of SETA keys is $\mathcal{E}_{\mathfrak{O}}(p)$ where $\mathfrak{O} = \mathbb{Z}[\sqrt{-n}]$ where n is a solution of the quadratic equation $z^2 + nD^2 = N^2$ with $-n$ being a quadratic non-residue modulo all the prime divisors of D and p, where D, N and p are the three main parameters of SETA.

the authors of the SETA paper [DFFDdSG+21] provided an implementation of their protocol with given values for p, D, N. The characteristic p is a 400 bits primes equal to $2 \cdot 8426067021^{12} - 1$, and the two other parameters are:

$$\begin{aligned} D &= 43^{12} \cdot 84719^{11}, \\ N &= 3^{21} \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 73 \cdot 257^{12} \cdot 313 \cdot 1009 \cdot 2857 \cdot 3733 \cdot 5519 \cdot 6961 \\ &\quad \cdot 53113 \cdot 499957 \cdot 763369 \cdot 2101657 \cdot 2616791 \cdot 7045009 \cdot 11959093 \\ &\quad \cdot 17499277 \cdot 20157451 \cdot 33475999 \cdot 39617833 \cdot 45932333. \end{aligned}$$

The concrete value of n was not given in [DFFDdSG+21] but several solutions can be found quite easily. Below, we computed one such solution where n is easy to factor so that we could compute the conductor. For instance, we found the value:

$$\begin{aligned} n = 113 \cdot 337 \cdot 43913 \cdot 69522129914593554713466657355275000660185257908972492522431413808553767205401453148081325894556965991428307754649539266\\ 033342875068026023370667830770225304 57. \end{aligned}$$

Since, n is square free and equal to $1 \mod 4$, the order $\mathfrak{O} = \mathbb{Z}[\sqrt{-n}]$ is the ring of integers of $\mathfrak{K} = \mathbb{Q}(\sqrt{-n})$. Since p is inert in $\mathfrak{O}$, we have $\#\mathcal{S}_{\mathfrak{O}}(p) = h(\mathfrak{O})$. Under GRH, Littlewood [Lit28] proved the inequality

$$h(\mathfrak{O}) > (\frac{\pi}{12e^{\gamma}} + o(1))\frac{\sqrt{4n}}{\log\log(4n)}$$

where γ is the Euler-Mascheroni constant with $e^{\gamma}/\pi \approx 0.56693$. To derive the concrete lower-bound in Corollary 4 from Proposition 11, we apply the classical lower bound on $h(\mathfrak{O})$ stated above and the upper-bound in Eq. (6) on the size of τ.

Corollary 4. *Let the values p, n be as above and $\mathfrak{O} = \mathbb{Z}[\sqrt{-n}]$. Assuming GRH, the size of $\mathcal{E}_{\mathfrak{O}}(p)$ is bigger than 2^{269}.*

Proof. Proposition 11 tells us that $\#\mathcal{E}_{\mathfrak{O}}(p)$ is bigger than $(1/2)\min(A, B)$ where:

$$A = h(\mathfrak{O}) \text{ and } B = \frac{h(\mathfrak{O})^2}{3(4p + 4n + 1)} \frac{p}{\max\limits_{0 \leq N \leq 4n^2/p} \tau(N)}.$$

Assuming that our n is big enough for it to hold, we are going to use $A = h(\mathfrak{O}) > \frac{\pi}{24e^{\gamma}} \frac{\sqrt{4n}}{\log\log(4n)} > 2^{270}$. To get a lower bound on B, it remains to get an upper bound on $\max_{0 \leq N \leq 4n^2/p} \tau(N)$. We can compute this bound manually using Eq. (8). Indeed, it can be easily verified that $\omega(4n^2/p) \geq 98 \geq 74$ and so we can compute

$$\left(1 + \frac{\log(4n^2/p)}{k \log(k)}\right)^k \text{ for all } 1 \leq k \leq 98.$$

As expected, the maximum is obtained for $k = 98$ and we have that

$$\max_{0 \leq N \leq 4n^2/p} \tau(N) < 2^{105}.$$

Thus, using the bound on $h(\mathfrak{O})$, we get that $B > 2^{279}$. So, under GRH and the assumption that n is big enough so that our simplification of the Littlewood bound hold, we get that

$$\#\mathcal{E}_{\mathbb{Z}[\sqrt{-n}]}(p) > 2^{269}.$$

5 Conclusion and open problems.

We have given a new generic lower bound on the size of $\mathcal{E}_{\mathfrak{O}}(p)$ and proven that our bound was useful in practice by applying it to a concrete example taken from isogeny-based cryptography. While our bound seems satisfying for the example we took, its behavior is a bit counter-intuitive as it tends to 0 when the discriminant of $\mathfrak{O}$ tends to infinity and p is fixed. It is an interesting question to see if this asymptotic behavior is intrinsic to our method or if we could derive a better bound. We have also observed that studying the case where p divides the discriminant of $\mathfrak{O}$ could lead to interesting improvements. Finally, it could be interesting to see if we could adapt our analysis to the case of embeddings of quadratic orders of distinct discriminants inside the same quaternion order and see how the resulting bound would compare with the results from the singular moduli literature (the ones from [LV15] for instance).

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A The case of non-maximal orders

In this section, we provide an upper-bound on $\#\mathcal{E}_{\mathfrak{O}}(p)$ when $\mathfrak{O}$ is not maximal. Most of the ideas exposed below are not fundamentally new, and have been considered for instance by Lubotzky, Phillips and Sarnak [LPS86], but we expose them here for completeness as simple statements matching our needs are rather hard to extract.

For the rest of this section, let us take $\mathfrak{O} = \mathbb{Z} + f\mathfrak{O}_0$ where $f > 1$ is coprime with p and $\mathfrak{O}_0$ is a maximal quadratic order. The fundamental property underlying our result in Proposition 14 is summarized in Proposition 12. This is a consequence of [LB20, Lemma 5.4, Corollary 5.5] showed by Love and Boneh (they talk about optimal embeddings inside maximal orders rather than orientations of elliptic curves in their paper but the two notions are the same under the Deuring Correspondence) and the standard characterization of *horizontal, ascending and descending* isogenies (see [CK19,Onu21] for instance).

Proposition 12. *Let $\mathfrak{O}$ be a quadratic order of conductor f and discriminant d such that $\mathcal{E}_{\mathfrak{O}}(p)$ is not empty. Let $\ell \neq p$ be a prime number. For every curve $E \in \mathcal{E}_{\mathfrak{O}}(p)$, among the $\ell+1$ curves ℓ-isogenous to E, there are $\ell - (d/\ell)$ curves contained in $\mathcal{E}_{\mathbb{Z}+\ell\mathfrak{O}}$. If ℓ is coprime with f, the $1 + (d/\ell)$ remaining curves are in $\mathcal{E}_{\mathfrak{O}}(p)$ and if $\ell \mid f$, then the final curve is contained in $\mathcal{E}_{\mathfrak{O}'}(p)$ where $\mathfrak{O}'$ is the quadratic order of discriminant d/ℓ^2 such that $\mathfrak{O} = \mathbb{Z} + \ell\mathfrak{O}'$.*

Note that by ℓ-isogenous we mean connected by an ℓ-isogeny (that must be cyclic since ℓ is prime). From Proposition 12, we see that we can tie the generic case to the case of fundamental discriminant using the expanding properties of the isogeny graphs. For simplicity, we assume henceforth that $f = \ell^e$ for some prime ℓ and $e \in \mathbb{N}$ and $\mathfrak{O}_0$ is any maximal quadratic order. By Proposition 12, each coprime factor of the conductor can be treated independently. The exact bound in Proposition 14 depends on whether ℓ is split, ramified or inert in $\mathfrak{K}$ but the three cases can be treated in a similar manner. As a warm-up, we start with Proposition 13, to give a lower bound on the number of curves that are f-isogenous to a curve in an arbitrary subset $\mathcal{E}_0$ of the isomorphism classes of supersingular curves over $\mathbb{F}_{p^2}$.

For any prime ℓ coprime with p, the graph of cyclic ℓ-isogenies on isomorphism classes of supersingular curves is Ramanujan with degree of regularity equal to $\ell+1$. For ℓ^e-isogenies, we obtain an almost-Ramanujan graph with degree of regularity equal to $\lambda_1(\ell^e) = \ell^e(1+1/\ell)$. We write $A(\ell^e)$ for the adjacency matri of the graph of cyclic ℓ^e-isogenies on supersingular curves. The matrices $A(\ell^r)$ for $r \in \mathbb{N}$ are related to the Brandt matrices $B(\ell^r)$ for $r \in \mathbb{N}$ under the relation $A(\ell^r) = B(\ell^r) - B(\ell^{r-2})$ and $B(\ell) = A(\ell)$ and $B(1) = A(1) = I$. The Brandt matrices $B(m)$ correspond to the action of the Hecke operator T_m on the space of modular forms of weight 2 on $\Gamma_0(N)$ when m and N are coprime. The graph associated to the $B(\ell^r)$ are $\sum_{i=0}^{r} \ell^i$-regular. The matrices $A(\ell^r)$ and $B(\ell^r)$ are real symmetric positive and have N_p ordered real eigenvalues. The biggest eigenvalue is always equal to the degree of regularity k of the associated graph and the corresponding eigenspace is generated by the vector $(1/\sqrt{N_p})_{1 \leq i \leq N_p}$ of

norm 1. The expansion of the graph can be measured by the size of the second eigenvalue. The graph is said to be *Ramanujan* when this value is smaller than $2\sqrt{k-1}$. A consequence of the Riemann hypothesis for function fields, proven by Deligne (see [Kat76] for instance) is that the second eigenvector of $B(\ell^r)$ is smaller than $(r+1)\sqrt{\ell^r}$. When $r=1$, this proves that the graph of ℓ-isogenies is Ramanujan. For $r>1$, the bound is not good enough to prove the same thing. This is why the other graphs are to be almost-Ramanujan. We will use $A(\ell^r)=B(\ell^r)-B(\ell^{r-2})$ to deduce results on the expansion of the graph of ℓ^e isogenies.

Proposition 13. *Let $e\in\mathbb{N}$ and $\ell\in\mathbb{P}$ different from p and let $\mathcal{E}_0$ be a non-empty subset of isomorphism classes of supersingular elliptic curves. Let us write $C_0=\#\mathcal{E}_0$ and write $\mathcal{E}_{\ell^e}$ for the set of isomorphism classes of curves ℓ^e-isogenous to a curve of $\mathcal{E}_0$. We have the bound*

$$\#\mathcal{E}_{\ell^e}\geq\frac{N_P}{1+(N_P/C_0-1)\frac{(e+1)^2\ell^e+(e-1)^2\ell^{e-2}}{\lambda_1(\ell^e)^2}}$$

Proof. Assume that we have an ordering $E_1,\dots,E_{N_p}$ of all supersingular elliptic curves (up to isomorphism). Let us write $X=(x_i)_{1\leq i\leq N_p}\in\mathbb{R}^{N_p}$, the vector such that $x_i=1$ if $E_i\in\mathcal{E}_0$ and 0 otherwise. If we write $Y=A(\ell^e)X$, then $\#\mathcal{E}_{\ell^e}$ is equal to the number of non-zero entries of Y. A very classical bound tells us that

$$\#\mathcal{E}_{\ell^e}\geq\frac{\|Y\|_1^2}{\|Y\|_2^2}.$$

We see easily that $\|Y\|_1=\lambda_1(\ell^e)C_0$. To upper-bound $\|Y\|_2^2$, we are going to use the Ramanujan property. Let us write $f_1,\dots,f_{N_p}$ the orthonormal basis of eigenvectors of $A(\ell^e)$. We already explained that $f_1=(1/\sqrt{N_p})_{1\leq i\leq N_p}$. We can write $X=z_1f_1+X'$ where X' is orthogonal to f_1. The coefficient z_1 is equal to $\langle X,f_1\rangle=C_0/\sqrt{N_p}$. We have $\|A(\ell^e)X\|_2^2=\lambda_1(\ell^e)^2z_1^2+\|A(\ell^e)X'\|^2$. We have the equality $A(\ell^e)=B(\ell^e)-B(\ell^{e-2})$. The vector X' lies in a space of dimension N_p-1 where all the eigenvalues of $B(\ell^r)$ have absolute value smaller than $(r+1)\sqrt{\ell^r}$ for every $r\geq 1$. We have $\|A(\ell^e)X'\|_2^2\leq\|B(\ell^e)X'\|_2^2+\|B(\ell^{e-2})X'\|_2^2$ by the triangular inequality. Thus, $\|A(\ell^e)X\|_2^2\leq\lambda_1(f)^2z_1^2+((e+1)^2\ell^e+(e-1)^2\ell^{e-2})\|X'\|_2^2$. We can easily compute that $\|X'\|_2^2=\|X\|_2^2-z_1^2$. Thus we obtain $\|Y\|_2^2\leq\lambda_1(\ell^e)^2C_0^2/N_p+((e+1)^2\ell^e+(e-1)^2\ell^{e-2})(C_0-C_0^2/N_p)$. Thus, we obtain

$$\#\mathcal{E}_{\ell^e}\geq\frac{\lambda_1(\ell^e)^2C_0^2}{\frac{\lambda_1(\ell^e)^2C_0^2}{N_p}+((e+1)^2\ell^e+(e-1)^2\ell^{e-2})(C_0-C_0^2/N_p)}$$

$$\#\mathcal{E}_{\ell^e}\geq\frac{N_P}{1+(N_P/C_0-1)\frac{(e+1)^2\ell^e+(e-1)^2\ell^{e-2}}{\lambda_1(\ell^e)^2}}.$$

Proposition 14. *Let $\mathfrak{O}=\mathbb{Z}+\ell^e\mathfrak{O}_0$ for some $\ell\in\mathbb{N}$ coprime with p and a maximal quadratic order $\mathfrak{O}_0$. Let us write $C_0=\#\mathcal{E}_{\mathfrak{O}_0}$.*

If ℓ is inert in $\mathfrak{K}$:

$$\#\mathcal{E}_{\mathfrak{O}}(p) \geq \frac{N_P}{1+(N_P/C_0-1)\frac{(e+1)^2\ell^e+(e-1)^2\ell^{e-2}}{(\ell^e+\ell^{e-1})^2}},$$

else if ℓ is ramified:

$$\#\mathcal{E}_{\mathfrak{O}}(p) \geq \frac{N_P}{1+(N_P/C_0-1)\frac{((e+1)^2\ell^e+e^2\ell^{e-1})}{\ell^{2e}}},$$

else, ℓ is split and:

$$\#\mathcal{E}_{\mathfrak{O}}(p) \geq \frac{N_P}{1+(N_P/C_0-1)\frac{\sum_{j=0}^{e}(2^{\lfloor (1+j)/2\rfloor}-2^{\lfloor (1+j-2)/2\rfloor})^2(e-j+1)^2\ell^{e-j+1}}{(\ell^e-\ell^{e-1})^2}}.$$

Proof. The case ℓ inert is a simple combination of Proposition 12 and Proposition 13 since the set $\mathcal{E}_{\mathbb{Z}+\ell\mathfrak{O}}(p)$ is exactly the set of curves ℓ-isogenous to curves in $\mathcal{E}_{\mathfrak{O}}(p)$. When ℓ is ramified or split, the situation is slightly more complicated. In fact, in both cases, the result is obtained by rewriting the reasoning used in the proof of Proposition 13. Indeed, we are going to show that $\#\mathcal{E}_{\mathbb{Z}+\ell^e\mathfrak{O}_0}$ is equal to the number of non-zero entries of a vector Y computed as MX where X is defined as in the proof of Proposition 13 and M is a linear combination of the $A(\ell^i)$ for $i \in [0, e]$. For Proposition 13 (and ℓ inert) we can simply take $M = A(\ell^e)$. When ℓ is not inert, we need to remove some of the ℓ^e isogenies and this is why we have a more complicated expression for M.
For what remains, let us assume that the labelling of the N_P-isomorphism classes of supersingular curves is such that $E_1, \ldots, E_{C_0}$ are the C_0 curves contained in $E \in \mathcal{E}_{\mathfrak{O}_0}(p)$. X_i is the vector of $\mathbb{N}^{N_p}$ such that $(X_i)_j = 1$ if $j = i$ and 0 otherwise and $X = \sum_{i=1}^{C_0} X_i$.
When ℓ is ramified, Proposition 12 implies that there exists a permutation σ of $[1, C_0]$ such that E_i and $E_{\sigma(i)}$ are ℓ-isogenous and σ^2 is the identity. To get the curves of $\mathcal{E}_{\mathfrak{O}}(p)$, we need to exclude all the ℓ^e-isogenies that can be written as $\phi \circ \varphi_i$ where φ_i is the ℓ-isogeny between E_i and $E_{\sigma(i)}$. $A(\ell^e)X_i$ gives all the curves that are ℓ^e-isogenous to E_i. To remove the ones that are obtained through the wrong isogenies we can subtract by $A(\ell^{e-1})X_{\sigma(i)}$ but with that we have also subtracted the curves that are ℓ^{e-2}-isogenous to E_i. So we need to compensate by adding $A(\ell^{e-2})X_i$ and iterating this reasoning, we end up with the $\sum_{j=0}^{e}(-1)^j A(\ell^{e-j})X_{\sigma^j(i)}$. Thus, we get $M = \sum_{j=0}^{e}(-1)^j A(\ell^{e-j})$ after summing this formula for all $i \in [1, C_0]$. The lower bound on the number of zeroes of $Y = MX$ is given by $\|Y\|_1^2/\|Y\|_2^2$. A simple counting gives that $\|Y\|_1^2 = (\sum_{j=0}^{e}(-1)^j\lambda_1(\ell^{e_i}))^2 = \ell^{2e}$. To lower bound $\|Y\|_2^2$, we see that M has the same eigenvector f_1 (see the notations of the proof of Proposition 13) for the eigenvalue ℓ^{2e}. Thus, we can decompose $X = z_1 f_1 + X'$ with $z_1 = C_0/\sqrt{N_p}$. Once again we replace each $A(\ell^r)$ by $B(\ell^r) - B(\ell^{r-2})$ and $B(\ell) = A(\ell)$ and $B(1) = A(1)$. Interestingly, a lot of terms cancel out in M and we end with $B(\ell^e) - B(\ell^{e-1})$. As in the proof of Proposition 13, we conclude with the bound

on the eigenvalues of the $B(\ell^r)$ and the triangular inequality. This is how we get $\|MX'\|_2^2 \leq ((e+1)^2\ell^e + e^2\ell^{e-1})\|X'\|_2^2$. The proof is concluded in a similar way to Proposition 13.
When ℓ is split, we have by Proposition 12 that there exists two permutations σ_l, σ_r such that E_i is ℓ-isogenous to $E_{\sigma_l(i)}$ and $E_{\sigma_r(i)}$ and $\sigma_l \circ \sigma_r = \sigma_r \circ \sigma_l$ is the identity. A similar reasoning proves that we can take

$$M = \sum_{j=0}^{e} (-1)^j 2^{\lfloor (1+j)/2 \rfloor} A(\ell^{e-j}).$$

Once again, we use our relation between A and B matrices to get a sum on the $B(\ell^{e-j})$. We have $\ell^e(1-1/\ell) = \sum_{j=0}^{e}(-1)^j 2^{\lfloor (1+j)/2 \rfloor}\lambda_1(\ell^{e-j})$ and the final result follows from the same ideas as before.