A Linear-Time 2-Party Secure Merge Protocol

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Abstract. We present a linear-time, space and communication *data-oblivious* algorithm for securely merging two private, sorted lists into a single sorted, secret-shared list in the *two* party setting. Although merging two sorted lists can be done *insecurely* in linear time, previous *secure* merge algorithms all require super-linear time and communication. A key feature of our construction is a novel method to *obliviously* traverse permuted lists in sorted order. Our algorithm only requires black-box use of the underlying Additively Homomorphic cryptosystem and generic secure computation schemes for comparison and equality testing.

Keywords: Secure Computation · Homomorphic Encryption · Oblivious Protocols.

1 Introduction

Securely merging two sorted lists into a single, globally sorted list with the same asymptotic complexity as in the insecure setting has been a long-standing open problem. It is a fundamental tool in many machine learning and data-processing applications [41,6,56], Oblivious RAM [44,30], and Private Set Intersection (PSI) [34]. A series of works [1,32,31,13] have shown that securely *sorting* a list can be done with the same asymptotic complexity as insecure sorting. On the other hand, for *merging*, a gap remains. In the past, it has been solved with complicated techniques that either run in super-linear time or communication, or make unnatural assumptions.

In the insecure setting, and in the three-party ORAM setting, where there are three servers and a *trusted* client, merging two sorted lists of length n can be done in $O(n)$ time, [10], whereas in the *secure* setting, the best existing 2-party secure merge algorithm requires $O(n \log\log n)$ communication [25].

Our main result is to close this gap. More explicitly, we show

Theorem 1 (Main Theorem). *There exists a 2-party protocol for merging two locally sorted lists in linear-time, space and communication that provides security against semi-honest adversaries. The protocol only requires black-box use of an Additively-Homomorphic cryptosystem and a generic secure computation protocol for comparison and equality-testing on secret shares.*

⋆ Work done while consulting for Stealth Software Technologies, Inc.

Secure 2-party merge protocols arise naturally, since the two participants can each sort their list locally before the protocol begins. Three-party protocols for secure merge are less natural, since there are still only two lists being merged, but these lists are secret-shared amongst the *three* computation parties. If the two lists being merged were initially held in the clear by two parties, then it's unnatural to require a third party to aid in the secure merge procedure. On the other hand, if the two lists were initially secret-shared among two parties (e.g. as the output of a previous 3-party computation) it becomes less natural to assume that they are pre-sorted (since they cannot have been sorted locally).

One application of two-party merge protocols is in Private Set Intersection (PSI). There are many PSI protocols, but most output the intersection *in the clear* (e.g. [27,38,35,20,33,36,21,22,24,50,47,40,51,52,37,12,11,46,16]). In many applications, however, PSI is only a first step in a larger computation, and in these settings the PSI must return *secret shares* of the intersection, rather than the list itself – but these secret-shared PSI protocols (e.g. [17,49,48]) tend to be less efficient than protocols that reveal the intersection in the clear. One of the earliest methods for secret-shared PSI is the *sort-compare* paradigm [34], where the participants sort their joint list, then compare adjacent elements in a linear pass, deleting singletons. The problem with this approach is that the initial sorting step takes $O(n \log n)$ communication. Using our novel linear-time secure merge protocol, the sort-compare paradigm gives a simple, efficient *linear-communication* secret-shared PSI protocol.

Our protocol is inspired by the 3-server ORAM merge protocol of [10], where the two sorted lists are treated as linked lists, then each linked list is shuffled with a collection of "dummy" elements using a linear-time three-party secure shuffle [42]. Thereafter, the trusted client can traverse the shuffled linked lists, comparing one element at a time, as in the standard insecure merge protocol.

There are several obstacles that need to be overcome in order to eliminate the trusted client and one of the servers from the [10] merge protocol. We can use a linear-time 2-party secure shuffle [25] to replace the 3-party shuffle, but updating the pointers in the shuffled lists is challenging without a trusted client.

To overcome this obstacle, we develop a technique for converting values encrypted under the key of one participant into additive secret shares of the same underlying plaintext. (See Section 5.2.) This conversion process is extremely efficient, and only relies on the cryptosystem being additively homomorphic. Moreover, the trusted client in [10] can easily switch from the real to dummy list obliviously once the real list is exhausted; however, this is non-trivial in our 2-party setup since obviously neither party should learn when a real list has been exhausted. We combat this issue by creating a unique, partially circular linked list (Section 5.1, Figure 1) such that the protocol can seamlessly switch from the real to dummy list.

Using this novel linked list construction and ciphertext-to-secret-sharing tool, we give a two party secure merge protocol, where each participant treats their input as a linked list, then allows the other participant to shuffle this linked list (while updating the pointers). The parties then re-share these permuted linked

lists, and compare elements one at a time (using a secure comparison protocol), while the exact sequence of data accesses from each list is independent of the underlying data. See Section 5 for the full construction.

2 Previous Work

2.1 Secure sorting

Merging two sorted lists can be seen as a special case of sorting, and thus any sorting protocol is also a merge protocol. When security is not required, a simple counting argument shows that any comparison-based sorting algorithm requires $O(n \log n)$ comparisons, whereas two sorted lists can be merged using only $O(n)$ comparisons. Although secure merge protocols are a building block for many secure multiparty computations, most applications focus on the more general (and more difficult) problem of secure sorting.

One route for building a secure sorting protocol is to securely implement a data-oblivious *sorting network* using a generic circuit-based secure multiparty computation (MPC) protocol (e.g. GMW [29], BGW [7] or Garbled Circuits [58,59]). Asymptotically, the best sorting network is the AKS network [1], which requires $O(n \log n)$ comparisons. Although the AKS network is asymptotically optimal, the hidden constants are *extremely* large [2], and so the AKS network has little practical value. In practice, Batcher's bitonic sort [5] which requires $O(n \log^2 n)$ comparisons is much faster and is widely implemented in practice, including in the ABY [23], Obliv-C [60] and EMP [57] compilers. Batcher's sorting network is defined recursively, and thus when using Batcher's network to merge two pre-sorted input lists, all but the final level of the recursion can be omitted. Unfortunately, this does not improve the asymptotic complexity, but it does increase the concrete performance by about a factor of 2.

One problem with implementing traditional sorting algorithms (e.g. quicksort, mergesort, radix sort) using generic secure computation, is that the they are not data-oblivious – even if the comparisons are implemented securely, the *data movement* depends on the underlying values being sorted. The *shuffle-then-sort* paradigm [32,31,13], solves this problem by first *obliviously shuffling* the input lists, then securely executing a traditional sorting algorithm. The initial shuffle ensures that the data movement (which is not hidden by the secure computation) is independent of the underlying data. These techniques yield an asymptotically optimal ($O(n \log n)$) sorting algorithms, that are also efficient in practice.

The efficiency of the shuffle-then-sort paradigm rests on the efficiency of the secure shuffle protocol. In the 3-party setting there are linear-time secure shuffles (based on one-way functions) [42], and in the 2-party there are linear-time secure shuffles (based on additively homomorphic encryption) [28].

Applying the shuffle-then-sort paradigm to the problem of merging immediately yields $O(n \log n)$-communication oblivious merge protocols, but does *not* achieve the $O(n)$-time merging that is possible in the insecure setting. In fact, the $\Omega(n \log n)$ lower bound on comparison-based sorting means that this ap-

proach will never yield a linear-time secure *merge* algorithm – unless we can take advantage of the fact that the initial lists being merged are pre-sorted.

Alternative sorting schemes (e.g. Radix sort) avoid the $\Omega(n \log n)$ lower bounds on comparison-based sorting. These sorting algorithms are efficient, but rely on the RAM model of computation, and their data-dependent access patterns cannot be efficiently implemented in the circuit model. One exception is [4], which uses non-comparison based techniques to beat the $\Omega(n \log(n))$ lower bound, but still remains in the circuit model.

2.2 Secure merging

Secure, multiparty merge protocols have been studied separately from secure sorting protocols, and just as in the insecure case, focusing on the problem of *merging* allows us to circumvent the $\Omega(n \log n)$ lower bound for sorting.

The first secure merge protocol with (asymptotically) less communication than a corresponding secure sort was given in the 3-server ORAM setting (which requires 3-servers and a trusted client), where there is an information-theoretic secure merge protocol with only $O(n)$ communication [10]. In general, any k-server ORAM protocol, the client can be emulated using secure multiparty computation (MPC), thus the protocol of [10] also yields a 3-server secure merge protocol. Unfortunately, using MPC to securely emulate an ORAM client can dramatically hurt performance since the ORAM client may not be "MPC friendly", e.g. the client may have a very large circuit complexity, which leads inefficiencies when emulating the ORAM client under MPC.

The key idea of [10] is to apply "shuffle-then-sort" [32,31,13] to the idea of merging. Essentially, the participants shuffle the two (sorted) linked-lists – updating the pointers to each element's new, shuffled location. Then the participants apply a standard (non-oblivious) merge protocol to traverse these shuffled linked lists (without needing to hide the data movement). These techniques yield a linear-communication secure merge protocol, but the construction of [10] only works in the 3-party ORAM setting, i.e., when there are four parties, three servers and a trusted client.

The "shuffle-then-merge" paradigm is a bit more delicate than the "shuffle-then-sort" paradigm, since the input lists in a merge are pre-sorted, and the merge protocol must process them in this sorted order (even after the oblivious shuffle). To overcome this difficulty, the pre-sorted input lists can be turned into *linked lists*, and the oblivious shuffle can update each item's pointer to point to the permuted position of its successor [10].

In the two-party setting, [25] gives a protocol based on additively homomorphic encryption that securely merges two lists using $O(n \log \log n)$ communication. The key idea of [25] is that since the input lists are pre-sorted, we can divide the entire list into poly-logarithmic sized blocks, and focus on moving these blocks into (nearly) the correct positions. Once the large blocks are in place, the small number of remaining "strays" that are out of place, can be identified and moved efficiently. Although our solution is fundamentally different, like [25], we also rely on a linear-time 2-party shuffle.

Our protocol follows a shuffle-then-merge paradigm that is similar to [10], but in order to adapt this to the two-party setting, we create a new protocol for shuffling linked lists in the two-party setting (which can be seen as an extension of the two-party oblivious shuffle of [28,25]).

3 Overview

3.1 Challenges

In the insecure setting, two parties can merge their locally sorted lists by simply comparing their smallest elements and advancing the list with the smaller element. This operation is linear in the length of the two lists. The core issue in translating this linear-time merge algorithm to a secure version is that advancing a list is not *data-oblivious* – it reveals which list contained the smaller element.

Protocol 1 A basic, *data-dependent* merge.

Input: Two sorted input lists A, B of lengths n_A and n_B
Output: A sorted output list C of length $n_A + n_B$

```
 1: Initialize i_A = i_B = i_C = 0
 2: while i_C < n_A + n_B do
 3:     if A[i_A] < B[i_B] or i_B ≥ n_B then
 4:         C[i_c] = A[i_A]
 5:         i_A = i_A + 1
 6:     else
 7:         C[i_c] = B[i_B]
 8:         i_B = i_B + 1
 9:     end if
10:     i_C = i_C + 1
11: end while
```

There are two key challenges when trying to adapt the non-oblivious naïve merge protocol (Protocol 1), into an oblivious variant.

1. **Which list is being accessed**: Whether the algorithm reaches Line 4 or Line 7 reveals which *list* is being accessed.
2. **Which location is being accessed**: When the algorithm reaches Line 4 (resp. Line 7), it reveals which element of A's (resp. B's) list is being accessed at iteration i_C.

We also face an additional challenge: we have only *two* participants in the protocol unlike these prior works which had three, either two servers and a trusted client [43] or three servers and trusted client [10].

3.2 Intuition and Construction Overview

Oblivious shuffle with linked list: To address challenge 2, we rely on an oblivious permutation. In the multiparty setting, it is possible to perform efficient (linear-time), oblivious shuffles of secret-shared lists [42]. Similarly, in the two-party scenario, the participants can use additively homomorphic encryption to

obliviously shuffle ciphertexts in linear time [28,25]. These linear-time multiparty shuffles are a key building block of many secure multiparty sorting protocols [32,31,13], and secure merge algorithms [10,25].

By viewing each participant's sorted input as a linked list, then shuffling that list, the parties can decouple the locations being accessed from the iteration of the loop – for example, at Line 4 the protocol would read location $\Pi_A(i_A)$ for some random permutation Π_A, instead of directly reading i_A.

There are some subtleties here, as the parties need to obliviously permute their linked lists, and then obliviously traverse them.

In order to allow the parties to traverse the permuted linked lists in the original (sorted) order, the parties must also update the pointers. Thus if π is a permutation of $[n]$, and the original list is $(v[0], \ldots, v[n-1])$, the parties will create two new arrays

$$w = \left(v\left[\pi^{-1}(0)\right], \ldots, v\left[\pi^{-1}(n-1)\right]\right) \qquad \text{Permuted data}$$
$$t = \left(\pi\left(\pi^{-1}(0)+1\right), \ldots, \pi\left(\pi^{-1}(n-1)+1\right)\right) \qquad \text{Permuted tags}$$

With $t[\pi(n-1)] = \perp$. Thus if $w[i] = v[j]$, then $w[t[i]] = v[j+1]$.

This structure allows the parties to traverse the permuted list, w, by first revealing $\pi(0)$ and then, selectively revealing elements of t, starting with $t[\pi(0)]$, $t[\pi(1)]$, ...

Our goal is for each party to achieve a secret-shared, permutation of their own list permuted (as well as the updated pointers) by the other party. In our construction, the second party acts as a *permuting* party for the first and generates both the permuted list and the corresponding linked list to traverse it. To maintain privacy of the data and obliviousness of the memory accesses, the second party's permutation, and the first party's data must remain private.

Now, if the permuting party holds on to its share of the owner party's list, it is not clear how to obliviously traverse the permutation since the permuting party knows the position of each accessed share, and thus each element.

When there are three participants this can be done information-theoretically, by having each participant generate a permutation and secret-share to the other two participants [10]. In the two party setting, we can use additively homomorphic encryption to (obliviously) permute a private list [28,25], but we cannot use those constructions in a black-box manner, since they do not allow us to create the shared tags needed to traverse the permuted list.

Instead, we recombine the shares at the owner party but to maintain obliviousness, i.e. to hide the data itself so as to not leak the permutation, both parties somehow convert their shares into shares encrypted using the *permuting party's* public key. The owner party can then use the additive homomorphism of the encryption scheme to add the encrypted shares and obtain an encryption of the element under the *other* (permuting) party's public key. Therefore, it cannot decrypt to learn the underlying value (and thus, permutation).

Adding dummies and oblivious pointer advancement: To address challenge 1, we add "dummy" elements to each party's list so that we are able to

advance both lists every iteration of the loop. For simplicity, suppose both parties' lists are of size n. Then, both parties can generate n dummy elements and maintain two separate pointers to keep track of the real and dummy elements respectively. These dummies are interspersed with the real elements to create a list of $2n$ elements. At every iteration, the party with the smaller element advances its real pointer, while the other party advances its dummy pointer. This ensures that an element is consumed from both lists every iteration of the merge.

Finally, we are left with two more operations: (1) comparing encrypted real values efficiently and (2) advancing lists obliviously. We achieve (1) using a trick to convert ciphertexts into secret shares which can be passed to any *state-of-the-art* 2-party comparison protocol [54,18] to avoid executing an expensive decryption circuit jointly; and we accomplish (2) by a clever construction of the linked list. The detailed shuffle and merge protocol is shown in Section 5.

4 Preliminaries

4.1 Secret sharing

Our protocol makes use of an additive secret sharing scheme, where a secret $x \in \mathcal{G}$ is shared as $(x - r,\ r)$, for some random $r \leftarrow \mathcal{G}$ where $\mathcal{G}$ is the finite group that parameterizes the Group Homomorphic Encryption scheme. In the two-party setting all linear secret-sharing schemes are essentially equivalent [19], so we can focus on this scheme without loss of generality.

As is standard, we use the notation $[\![x]\!]$ to denote a secret sharing of the plaintext x. Using the linearity of the secret sharing scheme, the participants can compute $[\![x + y]\!]$ from $[\![x]\!]$ and $[\![y]\!]$ with *no communication*.

For more complex calculations on shares, we rely on secure multiparty computation (MPC), described below.

4.2 Secure computation

Our protocol makes use of a few simple primitives for processing on secret shares, *comparisons*, *multiplexing* and *equality tests*. These basic primitives are implemented in essentially every secure computation framework, including ABY [23], EMP [57], SCALE-MAMBA [3] and MPyC [53].

We assume that there is an underlying ordering on the elements of $\mathcal{G}$ – this is a necessary assumption since the parties want to *sort* their elements.

Our construction is compatible with both arithmetic and boolean secure computation protocols, although comparisons and equality tests are likely to be more efficient in boolean-circuit-based secure computation protocols.

4.3 Additively homomorphic encryption

Our construction makes use of a semantically secure, additively homomorphic cryptosystem, $(\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Add})$. Our system is compatible with classical additively homomorphic schemes like Paillier [45], or lattice-based schemes that

Comparisons

$$[\![x < y]\!] = \begin{cases} [\![0]\!] & \text{if } x \geq y \\ [\![1]\!] & \text{if } x < y \end{cases}$$

Multiplexing

$$\mathsf{mux}([\![b]\!], [\![x]\!], [\![y]\!]) = \begin{cases} [\![x]\!] & \text{if } b = 0 \\ [\![y]\!] & \text{if } b = 1 \end{cases}$$

Multiplexes are often implemented as a simple multiplication

$$\mathsf{mux}([\![b]\!], [\![x]\!], [\![y]\!]) = [\![x]\!] + [\![b]\!] \cdot ([\![y]\!] - [\![x]\!])$$

Equality tests

$$[\![x = y]\!] = \begin{cases} [\![0]\!] & \text{if } x \neq y \\ [\![1]\!] & \text{if } x = y \end{cases}$$

natively work over $\mathbb{Z}/2\mathbb{Z}$, e.g. BFV [26,9] or CGGI [14,15], both of which are widely supported by current FHE implementations [55]. Note that the security we require for the $\mathsf{Add}(\cdot,\cdot,\cdot)$ is much weaker than full circuit privacy [8], since in our application the summations being computed are known to both parties, and only the *summands* are private.

In order for our final merge protocol to achieve linear communication, the underlying addively homomorphic cryptosystem must have *constant ciphertext expansion*.

4.4 Notation

As there are only two parties, and each party has a unique public key (for the additively homomorphic cryptosystem), when we say "key i" we mean the public key of party i, pk_i.

We denote each party as P_i where $i \in \{0, 1\}$. As all our protocols are two-party protocols (and most are completely symmetric), we take all subscripts modulo 2, thus if P_i is one party, P_{i+1} is the other party.

Several protocols below must be run twice, one time for each party, so we give such protocols an index with respect to which we write the steps within the protocol. For example, $\mathsf{Protocol}_i$ will be called twice, for $i \in \{0, 1\}$ and we use index i within the protocol to identify the parties. Similarly, we use the same index to define the ideal functionality.

We introduce some more notation in Table 1.

Additively Homomorphic Encryption

Semantic security: for all $x, y \in \mathcal{G}$

$$\left\{(pk, c_x) : \begin{array}{l} pk, sk \leftarrow \mathsf{KeyGen}\left(1^\lambda\right) \\ c_x \leftarrow \mathsf{Enc}(pk, x) \end{array}\right\} \approx_c \left\{(pk, c_y) : \begin{array}{l} pk, sk \leftarrow \mathsf{KeyGen}\left(1^\lambda\right) \\ c_y \leftarrow \mathsf{Enc}(pk, y) \end{array}\right\}.$$

Security of Add**:** for all $x, y \in \mathcal{G}$

$$\left\{\begin{array}{c} \\ c, \\ c_x, c_y, \\ pk, sk \\ \\ \end{array} : \begin{array}{l} pk, sk \leftarrow \mathsf{KeyGen}\left(1^\lambda\right) \\ c_x \leftarrow \mathsf{Enc}(pk, x) \\ c_y \leftarrow \mathsf{Enc}(pk, y) \\ r \leftarrow \mathcal{G} \\ c_r \leftarrow \mathsf{Enc}(pk, r) \\ c \leftarrow \mathsf{Add}(pk, c_x, c_r) \end{array}\right\} \approx_c \left\{\begin{array}{c} \\ c, \\ c_x, c_y, \\ pk, sk \\ \\ \end{array} : \begin{array}{l} pk, sk \leftarrow \mathsf{KeyGen}\left(1^\lambda\right) \\ c_x \leftarrow \mathsf{Enc}(pk, x) \\ c_y \leftarrow \mathsf{Enc}(pk, y) \\ r \leftarrow \mathcal{G} \\ c_r \leftarrow \mathsf{Enc}(pk, r) \\ c \leftarrow \mathsf{Add}(pk, c_y, c_r) \end{array}\right\}.$$

Decrypting the sum of two ciphertexts yields nothing about the individual summands.

Correctness: for any $x, y \in \mathcal{G}$, and $c > 0$

$$\Pr\left[\left\{\mathsf{Dec}(sk, c_{x+y}) : \begin{array}{l} pk, sk \leftarrow \mathsf{KeyGen}\left(1^\lambda\right) \\ c_x \leftarrow \mathsf{Enc}(pk, x) \\ c_y \leftarrow \mathsf{Enc}(pk, y) \\ c_{x+y} \leftarrow \mathsf{Add}(pk, c_x, c_y) \end{array}\right\} = x + y\right] > 1 - O\left(\lambda^{-c}\right)$$

$[\![x]\!]$	A secret sharing of the value x
$[\![x]\!]_i$	Party i's secret share of the value x
$\langle\!\langle m \rangle\!\rangle_i$	An encryption of the message m under public key of party i

Table 1. More Notation

5 Construction and protocol definitions

In this section we describe our construction. First, we present a two-party algorithm for creating and shuffling linked lists. Second, we present a technique for converting encryptions (encrypted by one party) into secret shares. Third, we show how to combine these tools into our main construction which is a linear-communication secure merge protocol.

We assume that party i has a key pair (pk_i, sk_i) for an additively homomorphic cryptosystem $(\mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{Add})$.

5.1 Obliviously shuffling input lists

In this section, we describe our novel two-party protocol for padding and shuffling private linked lists. $\mathsf{ShuffleLL}_i$ (Protocol 2). The goal of the $\mathsf{ShuffleLL}_i$ protocol

is for party i to achieve a random permutation of its input list with dummies encrypted under party $(i+1)$'s public key. The protocol takes a parameter, m, defining how many "dummy" elements are created. Although $\mathsf{ShuffleLL}_i$ takes m as a parameter, in our final merge protocol, P_1 should set m equal to the length of its input list. The $\mathsf{ShuffleLL}_i$ protocol realizes the ideal functionality, $\mathcal{F}^i_{\text{shuffle}}$ below.

Ideal Functionality $\mathcal{F}^i_{\text{shuffle}}$

1. *Input:* P_i with sorted list v of size n, and P_{i+1} with permutation $\pi\colon [m+n] \to [m+n]$ for some $m > 0$.
2. Create v' by concatenating m dummy elements to the end of v and shuffle v' using π, $w[j] \leftarrow v'\left[\pi^{-1}(j)\right]$ for $j \in \{0, \ldots, n+m-1\}$.
3. Create linked list t to traverse w, such that if $w[j] = v[k]$, then $w[t[i]] = v[k+1]$.
4. For $j \in \{0, \ldots, n+m-1\}$, *output* $\langle\!\langle w[j] \rangle\!\rangle_{i+1}$, and $\langle\!\langle t[j] \rangle\!\rangle_{i+1}$ to P_i, and $\perp$ to P_{i+1}.
5. *Output* $(\llbracket \pi(n+1) \rrbracket_i, \llbracket \pi(0) \rrbracket_i)$ to P_i, and $(\llbracket \pi(n+1) \rrbracket_{i+1}, \llbracket \pi(0) \rrbracket_{i+1})$ to P_{i+1}.
6. *Output* $\llbracket \pi(n) \rrbracket_i$ to P_i and $\llbracket \pi(n) \rrbracket_{i+1}$ to P_{i+1}.

In the second last step, we output a 2-tuple which are secret shares of the head pointers (positions) of the dummy and real list respectively. In the last step, we output the secret share of the position of a special *end-of-list* dummy element. This special element is used to obliviously switch between the real and dummy list. It is explained in detail in Section 5.1 and 5.3.

Below, we describe the shuffle for party P_0 but in the final protocol they also swap positions and rerun. Assume that P_0 holds a sorted list v of length n, and P_1 generates a random permutation π over $[m+n]$. Then, the protocol proceeds as follows,

1. *Encrypt sorted list:* To hide its real elements (input list), P_0 encrypts each element using its public key pk_0 and sends the list of ciphertexts (in sorted order of the underlying value) to P_1.
2. *Generate shares:* Given a value v', party 1 can create an additive sharing of v' as $(v'-r, r)$ for some random value $r \in \mathcal{G}$. In our setting, however, P_1 does not have the plaintext value, v', but instead has an encryption $\langle\!\langle v' \rangle\!\rangle_0$.
   Using the additively homomorphism, given a ciphertext $\langle\!\langle v' \rangle\!\rangle_0$, party 1 creates the encrypted pair $(\langle\!\langle v'-r \rangle\!\rangle_0, \langle\!\langle r \rangle\!\rangle_1)$. See Line 2.
3. *Concatenate encrypted dummies:* Party P_1 creates a special dummy known as the *end-of-list* element, and $m-1$ random dummy elements. The *end-of-list* element marks the end of both the real and dummy list but also points to the first element of the dummy list. Therefore, the *end-of-list* element along with the dummy elements form a cycle. The *end-of-list* element stores the largest real value in sorted order instead of a random number as its value. P_1 easily constructs the *end-of-list* element encrypted under pk_0 by

just duplicating $\langle\!\langle v\,[n-1] \rangle\!\rangle_0$. Instead of a linked list terminating by pointing to $\perp$, we will have it point to the this *end-of-list* element. The purpose of the special element becomes apparent when either party's real list is exhausted and we must obliviously *switch* to traversing the dummy list while we access the remaining real elements from the other party. (See Section 5.3.)

4. *Permute ciphertexts and create linked list:* Party P_1 permutes the pair of shares using π by assigning the k^{th} element of the permuted list to the $\pi^{-1}(k)^{\text{th}}$ element of the concatenated list as shown in Line 5. To traverse the permuted list in sorted order, P_1 also generates a linked list such that the i^{th} element is the position of the *next* element in sorted order (see Line 6). We also point the *last* dummy element to the *end-of-list* element. Therefore, the real (resp. dummy) list reaches the *end-of-list* element after n (resp. m) steps. See Figure 1 below which illustrates this construction.

   To hide the linked list from party P_0 (and thus, the underlying permutation), P_1 encrypts each element of the linked list using its public key, pk_1. Finally, it secret shares the position of the first dummy and the first real element as a 2-tuple *head pointer*, *and* the *end-of-list* element.
5. *Recombine shares:* P_1 sends both the shuffled ciphertext pairs and the encrypted linked list to P_0. Party P_0 first decrypts the ciphertexts which were encrypted under its own public key, pk_0 and then *re-encrypts* them using pk_1, P_1's public key. Using the additive property of the encryption scheme, P_0 adds the newly obtained ciphertexts to their corresponding ciphertexts in the pair. Due to the homomorphic property, P_0 obtains an encryption of the sum of the underlying value which is in fact, the original set of real/dummy elements as the pairs were constructed precisely from those values.

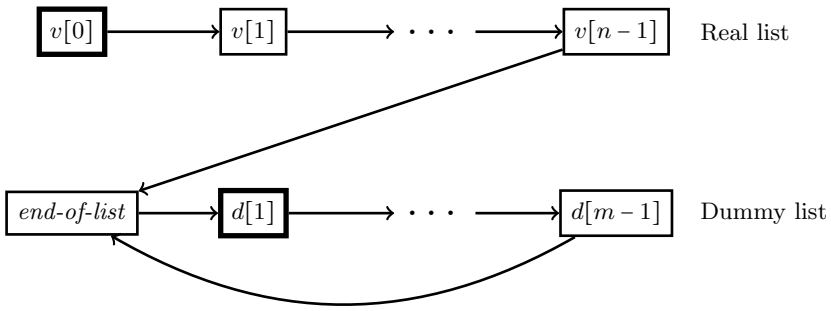

Fig. 1. Construction of the linked list. $d[1]$ and $v[0]$ (as pair of encrypted shares) are the at the head of the dummy and real pointer respectively. Both the last real element, $v[n-1]$ and last dummy element, $d[m-1]$ point to the *end-of-list* element.

Therefore, at the end of Protocol 2, P_0 obtains a permutation (oblivious to itself) of its original list with dummies encrypted under P_1's public key, along with an encrypted linked list to traverse it. Note that the *end-of-list* element is treated as a *dummy* element but stores a *real* value which is crucial in proceeding

obliviously when either party exhausts its real list. We further elaborate on this in Section 5.3.

We prove $\mathsf{ShuffleLL}_i$ securely computes the ideal functionality $\mathcal{F}^i_{\text{shuffle}}$ in Lemma 4.

Protocol 2 $\mathsf{ShuffleLL}_i$: Pad and Permute Linked Lists

Input: Party P_i holds sorted list v of size n; P_{i+1} holds random permutation $\pi : [m+n] \to [m+n]$ for some $m > 0$.

Output: P_i obtains a permutation (under π) of its elements (with m dummies) and linked list, both encrypted using P_{i+1}'s public key.

(index $j \in \{0, \ldots, n+m-1\}$)

1: For $k \in \{0, \ldots, n-1\}$, P_i encrypts $c[k] \leftarrow \langle\!\langle v[k] \rangle\!\rangle_i$, and sends c to P_{i+1}

2: For $k \in \{0, \ldots, n-1\}$, P_{i+1} generates random value $r_k \leftarrow \mathcal{G}$, and creates $c_i[k] \leftarrow (c[k] - \langle\!\langle r_k \rangle\!\rangle_i, \langle\!\langle r_k \rangle\!\rangle_{i+1})$ ▷ 2-tuples of the form $(c_i[k][0], c_i[k][1])$

3: P_{i+1} generates random $r \leftarrow \mathcal{G}$ and sets $c'_i[0] \leftarrow (c[n-1] - \langle\!\langle r \rangle\!\rangle_i, \langle\!\langle r \rangle\!\rangle_{i+1})$ ▷ *end-of-list* element. $c[n-1] - \langle\!\langle r \rangle\!\rangle_i = \langle\!\langle v[n-1] - r \rangle\!\rangle_i$

4: For $k \in \{1, \ldots, m-1\}$, P_{i+1} generates dummies, $d[k] = d_0[k] + d_1[k]$ where $d_0[k], d_1[k] \leftarrow \mathcal{G}$ are random, and creates, $c'_i[k] \leftarrow (\langle\!\langle d_i[k] \rangle\!\rangle_i, \langle\!\langle d_{i+1}[k] \rangle\!\rangle_{i+1})$

5: P_{i+1} permutes, $c^\pi_i[j] \leftarrow (c_i \| c'_i)\left[\pi^{-1}(j)\right]$

6: P_{i+1} creates linked list, $t'[\pi(j)] \leftarrow \pi(j+1)$ with $t'[\pi(n+m-1)] = \pi(n)$ ▷ point the *last* dummy to the *end-of-list* element

7: P_{i+1} encrypts $t_i[j] \leftarrow \langle\!\langle t'[j] \rangle\!\rangle_{i+1}$

8: P_{i+1} secret shares $\mathfrak{p}_i = (\pi(n+1), \pi(0))$ ▷ head pointers tuple

9: P_{i+1} secret shares $\mathfrak{e}_i = \pi(n)$ ▷ end-of-list element

10: P_{i+1} sends c^π_i, and t_i to P_i

11: P_i recombines $c^\pi[j] \leftarrow c^\pi_i[j][1] + \langle\!\langle \mathsf{Dec}(sk_0, c^\pi_i[j][0]) \rangle\!\rangle_{i+1}$

5.2 Converting ciphertexts to secret shares

In this section, we give an efficient 2-party protocol for converting ciphertexts from an additively homomorphic cryptosystem into secret shares of the same underlying value. A similar idea was used implicitly for creating "blinded permutations" [28].

In principle, a general-purpose MPC protocol can always be used to convert ciphertexts to secret shares by evaluating the decryption circuit for the encryption scheme within the MPC, but, in general, this is extremely inefficient. $\mathsf{EncToSS}_i$ (Protocol 3) gives an extremely efficient two-party protocol for achieving the same result when the underlying cryptosystem is additively homomorphic. $\mathsf{EncToSS}_i$ realizes the ideal functionality, $\mathcal{F}^i_{\text{decrypt}}$ defined below.

Ideal Functionality $\mathcal{F}^i_{\text{decrypt}}$

1. *Input:* P_i with ciphertext, $\langle\!\langle v \rangle\!\rangle_{i+1}$.
2. *Output* secret shares of value v: $[\![v]\!]_i$ to P_i, and $[\![v]\!]_{i+1}$ to P_{i+1}.

In our setting, party i holds a ciphertext $c = \langle\!\langle v \rangle\!\rangle_{i+1}$ of a private value, v, encrypted under party $(i+1)$'s key. At the end of the protocol, the parties hold additive secret shares of the underlying value v, and neither party learns anything about v.

We prove that $\mathsf{EncToSS}_i$ securely computes $\mathcal{F}^i_{\text{decrypt}}$ in Lemma 3.

Protocol 3 $\mathsf{EncToSS}_i$: Convert Ciphertext to Secret Share

Input: Party P_i inputs ciphertext, $c = \langle\!\langle v \rangle\!\rangle_{i+1}$ (encrypted using pk_{i+1}).
Output: Returns secret sharing of the underlying plaintext, v.

P_i generates random value, $r_i \leftarrow \mathcal{G}$
P_i encrypts $\langle\!\langle r_i \rangle\!\rangle_{i+1}$
P_i uses the additive homomorphism to compute $\langle\!\langle v + r_i \rangle\!\rangle_{i+1}$
P_i sends $c' = \langle\!\langle v + r_i \rangle\!\rangle_{i+1}$ to P_{i+1}
P_{i+1} decrypts $v' \leftarrow \mathsf{Dec}(sk_{i+1}, c')$
P_{i+1} shares v'
P_i sets $[\![v'']\!]_i = [\![v']\!]_i - r_i$
return $[\![v'']\!]$

5.3 Securely merging obliviously shuffled lists

We are finally ready to *securely* merge the two parties' lists. Our Merge protocol realizes the ideal functionality, $\mathcal{F}_{\text{merge}}$ defined below.

Ideal Functionality $\mathcal{F}_{\text{merge}}$

1. *Input:* For $i \in \{0,1\}$, P_i with list v_i of size n_i.
2. $\mathcal{F}_{\text{merge}}$ merges the two lists v_1 and v_2 such that the resultant list, v is sorted.
3. *Output* secret shares of each element of v, $[\![v\,[j]]\!]_0$ to P_0, and $[\![v\,[j]]\!]_1$ to P_1, for $j \in \{0, \ldots, n_0 + n_1\}$.

Suppose party P_i holds list v_i of size n_i. The protocol proceeds as described below.

1. *Obliviously shuffle padded list with linked list:* First, both parties call $\mathsf{ShuffleLL}_i$ (for $i \in \{0,1\}$ (as described in Protocol 2) to obtain an encrypted, permuted version of their input list padded with dummies (including the *end-of-list* element). $\mathsf{ShuffleLL}_i$ also outputs an encrypted linked list that party i later uses to traverse their list without leaking the accessed positions to party $i+1$ (who knows the permutation).

2. *Access elements from shuffled list:* The parties maintain a secret-shared bit for each party, $[\![b_i]\!]$, and $b_i = 1$ at iterations where P_i needs to access a real element, and $b_i = 0$ at iterations where P_i needs to access a dummy element. In the first step, both parties access their first real element, in all subsequent steps $b_0 \neq b_1$ since only one party advances its real list.[4] The bit, b_i, allows the parties to select and update the appropriate values obliviously using the mux operation (e.g. Protocol 5 line 9).

   At every step in the protocol, the parties also maintain a secret sharing of the last observed real value in P_i's list, cur_i. In any iteration where a dummy element must be consumed from party i's list, we use b_i to obliviously select cur_i over the dummy value, effectively discarding it in place of the actual real value to be compared. See Line 14 of Protocol 5.
3. *Compare real values:* Using b_i, we obtain the real values at the head of each real list. To find the smaller element, we use a generic comparison protocol (Section 4.2) which returns a (secret-shared) bit equal to 1 if party 0's real value was smaller than party 1's. Therefore, we set b_0 to the result of the comparison protocol (line 15) and $b_1 \leftarrow 1 - b_0$ (line 16) allowing us to appropriately update the head pointer for the next step.
4. *Update head pointer:* Now, we advance one party's real list and the other party's dummy list as follows. First, we find the next position from the encrypted linked list using $\mathsf{EncToSS}_i$. Then, we update the appropriate entry of the head pointer using bit, b_i (line 1). If $b_i = 1$, then this means that P_i's real value was smaller and we must advance the real (resp. dummy) pointer to obtain the next real (resp. dummy) value from P_i's (resp. P_{i+1}'s) list. Protocol 4 details how the head pointer is advanced. Lemma 2 shows that every memory location in the shuffled list is accessed exactly once, which makes the overall access pattern independent of the underlying data.
5. *Switching from an exhausted list:* When either party exhausts their real list, we must somehow *notify* the protocol and secret-share the remaining values of the other real list.

   We keep track of when a real list is exhausted by checking when the real pointer reaches the *end-of-list* element. We do so securely using a generic equality testing MPC protocol as described in Section 4.2. We maintain another secret-shared bit, fin initialized to 0, which acts like a boolean flag and is inverted as soon as either real pointer reaches its corresponding *end-of-list* element. See line 10 of Protocol 5.

   Without loss of generality, suppose that party 0 exhausted its real list first. This implies that $b_0 = 1$ (and $b_1 = 0$) from the previous iteration, and the real pointer has been advanced to store the position of the *end-of-list* element. Recall that the underlying value of the *end-of-list* element is exactly the same as the largest real value, i.e., the most recent element that party 0 accessed in the previous iteration. So on Line 14, val_0 will equal the *end-of-list* element i.e., the largest real value of party 0, and val_1 will equal cur_1, the most recent

[4] Since $b_0 = \neg b_1$ at every iteration after the first, we could increase efficiency by storing only a single bit, but the exposition is simpler if we forego this minor optimization.

real value from party 1 that has not been advanced and secret-shared yet. Therefore, essentially, we will perform the same comparison as the previous iteration and conclude that val_0 is smaller. However, val_0 is a duplicate of the most recent real value that was secret-shared in the previous iteration. This is where we use the fin bit to "reverse" the bits so that we instead select val_1 as the next real value, and advance the real pointer of party 1 (and dummy pointer of party 0) as required since we're only left with real values from party 1's list. As val_0 is smaller than every remaining real value in party 1's list, every comparison hereafter will always return $b_0 = 1$ which we always *invert* hereafter using fin. We prove fin remains 1 once set in Lemma 1, thus proving the correctness of the algorithm. In summary, performing these *dummy* comparisons allows the protocol to remain oblivious by still accessing elements from the permuted list, and using the fin bit allows the the protocol to *correctly* compute the merge.

Lastly, notice that if party 1 exhausts it real list first, then by construction, party 0's dummy pointer will reach the *end-of-list* element as we consume one dummy for each real element after the first one and thus, cycle back from the last dummy element to the *end-of-list* element. And since party 1 just exhausted its real list, we know $b_0 = 0$ and $b_1 = 1$. So, pos_0 is equal to the position of the dummy pointer, i.e., the position of the *end-of-list* element. Therefore, in either case (whether party 0 or 1 exhausts a real list), pos_0 will always equal the position of the *end-of-list* element and it is sufficient to only test pos_0 for setting fin (line 10).

Protocol 4 $\mathsf{UpdateHead}_i$: Update Head Pointer to Linked Lists

Input: Bit, $[\![b]\!]$; Head pointer tuple, $[\![\mathfrak{p}]\!]$; linked list, t held by party P_i.

Output: Head pointer tuple updated with the next real or dummy position from t according to bit, b.

1: $[\![pos]\!] \leftarrow \mathsf{mux}([\![b]\!], [\![\mathfrak{p}[0]]\!], [\![\mathfrak{p}[1]]\!])$
2: $\mathsf{Reveal}_i(pos)$ ▷ The revealed pos is an index in the shuffled list
3: $[\![next]\!] \leftarrow \mathsf{EncToSS}_i(t[pos])$
4: $[\![\mathfrak{p}_{new}[1]]\!] \leftarrow \mathsf{mux}([\![b]\!], [\![\mathfrak{p}[1]]\!], [\![next]\!])$
5: $[\![\mathfrak{p}_{new}[0]]\!] \leftarrow \mathsf{mux}([\![b]\!], [\![next]\!], [\![\mathfrak{p}[0]]\!])$
6: **return** $[\![\mathfrak{p}_{new}]\!]$

In the end, both parties obtain element-wise secret shares of the merge of their two sorted lists such that the resulting list is also in sorted order. We prove Merge securely computes $\mathcal{F}_{\text{merge}}$ in Lemma 5.

Our algorithm runs in time linear in the length of the two lists requires only linear communication between the two parties assuming the underlying encryption scheme produces ciphertexts with constant factor expansion. The concrete costs are outlined in Theorem 2.

6 Conclusion

In this paper, we presented the first linear-communication 2-party secure merge protocol. The protocol is asymptotically optimal, and efficient enough for prac-

Protocol 5 Merge: Securely Merge Sorted Lists

Input: Party P_i holds input list v_i of size n_i.

Output: Parties obtain a secret sharing of the merge of the lists in sorted order.

1: For $i \in \{0,1\}$, P_i locally generates random permutation, $\pi_i : [n_0 + n_1] \to [n_0 + n_1]$.
2: For $i \in \{0,1\}$, run $\mathsf{ShuffleLL}_i(v_i, \pi_{i+1})$ so that P_i obtains ciphertext list, c_i, linked list, t_i and secret shares, $[\![\mathfrak{p}_j]\!]_i$ and $[\![\mathfrak{e}_j]\!]_i$ for $j \in (0,1)$.
3: For $i \in \{0,1\}$, $[\![b_i]\!] \leftarrow [\![1]\!]$ ▷ b_i indicates real or dummy list
4: For $i \in \{0,1\}$, $[\![cur_i]\!] \leftarrow [\![\perp]\!]$ ▷ cur_i is the current value in the *real list*
5: $[\![end]\!] \leftarrow [\![\mathfrak{e}_0]\!]$ ▷ position of the *end-of-list* element
6: $[\![fin]\!] \leftarrow [\![0]\!]$ ▷ $fin = 1$ if either real list is exhausted
7: $k \leftarrow 0$
8: **while** $k < n_0 + n_1$ **do**
9: For $i \in \{0,1\}$, $[\![pos_i]\!] \leftarrow \mathsf{mux}([\![b_i]\!], [\![\mathfrak{p}_i[0]]\!], [\![\mathfrak{p}_i[1]]\!])$ ▷ Choose pos_i based on b_i
10: $[\![fin]\!] \leftarrow [\![fin]\!] \oplus [\![pos_0 = end]\!]$ ▷ If $fin = 1$ it will remain 1
11: For $i \in \{0,1\}$, $\mathsf{Reveal}_i([\![pos_i]\!])$
12: For $i \in \{0,1\}$, $[\![\mathfrak{p}_i]\!] \leftarrow \mathsf{UpdateHead}_i([\![b_i]\!], [\![\mathfrak{p}_i]\!], t_i)$ ▷ Move to new head
13: For $i \in \{0,1\}$, $[\![temp_i]\!] \leftarrow \mathsf{EncToSS}_i(c_i[pos_i])$ ▷ Access next position
14: For $i \in \{0,1\}$, $[\![val_i]\!] \leftarrow \mathsf{mux}([\![b_i]\!], [\![cur_i]\!], [\![temp_i]\!])$ ▷ Choose real values
15: $[\![b_0]\!] \leftarrow [\![val_0 < val_1]\!] \oplus [\![fin]\!]$ ▷ Compare real values
16: $[\![b_1]\!] \leftarrow [\![1 - b_0]\!]$
17: $[\![l[k]]\!] \leftarrow \mathsf{mux}([\![b_0]\!], [\![val_1]\!], [\![val_0]\!])$ ▷ $l[k]$ is the smaller value
18: For $i \in \{0,1\}$, $[\![cur_i]\!] \leftarrow [\![val_i]\!]$ ▷ Store most recent real value
19: $k \leftarrow k + 1$
20: **end while**
21: **return** $([\![l[0]]\!], \ldots, [\![l[n_0 + n_1 - 1]]\!])$ ▷ secret-sharing of sorted merged list

tical applications. To achieve this protocol, we introduced a 2-party method to obliviously traverse a permuted list using a novel linked list construction and an extremely efficient technique to convert ciphertexts to secret shares.

Our secure merge protocol makes only black-box use of an additively homomorphic cryptosystem, and a secure computation protocol supporting comparisons, equality tests, and multiplexing on secret shared values.

7 Analysis and Security

7.1 Correctness

Lemma 1. *In Protocol 5 (*Merge*), the variable fin is assigned the value* 1 *at some iteration and remains 1 for every subsequent iteration.*

Proof. It suffices to show $pos_0 = end$ exactly once. By construction, we know that if either party exhausts their real list, then pos_0 is set to end (see Section 5.3). Moreover, the other direction is also true i.e., if $pos_0 = end$, then some party exhausted their real list. This is easy to see as pos_0 stores the position of either the real or dummy pointer. Suppose the real pointer reaches end. Then we know that party 0 has exhausted their real list. Alternatively, suppose the

dummy pointer reaches *end*. Then, we will have consumed $n_1 - 1$ dummies and by construction we always consume one dummy element for each real element of the other party seen after the first one, it must be that party 1 has seen n_1 real elements i.e., exhausted their real list.

Since the number of iterations is equal to the length of the lists ($n_0 + n_1$, $n_0, n_1 \geq 1$), and we advance a real list and secret-share exactly one real element every iteration, both parties will exhaust their real lists at distinct iterations. Furthermore, party 0 exhausts its real list between iteration n_0 and $n_0 + n_1$ and party 1 exhausts its real list between iteration n_1 and $n_0 + n_1$. Therefore, $pos_0 = end$ at least once. Without loss of generality, suppose party 0 exhausts its real list first. Then by construction, we set $fin = 1$ and continue to advance the dummy pointer of party 0 and the real pointer of party 1, essentially secret-sharing the remaining elements of party 1's real list. Since there are exactly $n_0 + n_1$ real elements, party 1 will only exhaust its list at iteration $n_0 + n_1$ but then, the protocol terminates and we never access pos_0 again. Therefore, $pos_0 = end$ exactly once.

Lemma 2. *In Protocol 5 (*Merge*), every element in the permuted lists of both parties is accessed exactly once.*

Proof. Note that by construction, we will have exhausted both real lists when the protocol terminates. Consider the iterations before either party reaches the *end-of-list* element i.e., neither party has exhausted their real list. As we always advance a pointer in each iteration and pointers never loop back before we reach the *end-of-list*, it is clear that each element accessed before the *end-of-list* is encountered, is accessed exactly once.

Now, consider the iterations after some party reaches its *end-of-list* element. Without loss of generality, suppose party 0 exhausted its real list first. Since we advance a party's dummy pointer for each real element accessed from the other party after the first one, we will have accessed all $n_0 - 1$ dummies in P_1's list and arrived at the *end-of-list* dummy element as well. Until this iteration, party 0 has accessed n_0 real elements and suppose party 1 has accessed n_1' of its real elements (which implies party 0 has accessed $n_1' - 1$ of its dummy elements).

Then, both parties access their *end-of-list* element. Once this happens however, fin is set to 1 and by construction, party 1 switches to the head of its real list while party 0 switches to the head of its dummy list. Therefore, as soon as both parties arrive to their *end-of-list* element, they switch over to consuming the remaining elements of the other list until the end of the protocol preventing either pointer to move from the *end-of-list* element into the other list which would have caused duplicate accesses. Party 0 now accesses its remaining $n_1 - n_1'$ dummy elements and party 1 accesses its remaining $n_1 - n_1'$ real elements (for the first time). Therefore, both parties access their *remaining* dummy and real elements. As soon as party 1 updates its real pointer (resp. party 0 updates is dummy pointer) to the position of the *end-of-list* element, the protocol will terminate as this will only happen when all real elements have been accessed (after n_0 or n_1 iterations) and the *end-of-list* element is not accessed a second time.

Theorem 2. *Protocol 5 (*Merge*) outputs a secret-shared, sorted list, using* $11n$ *multiplexes,* n *comparisons,* n *equality tests, and* $13n + 4$ *(local) encryptions, where* $n = n_0 + n_1$*, the sum of the length of the two lists. Furthermore,* Merge *runs in* $O(n)$ *time and requires* $O(n)$ *communication between both parties.*

Proof. Correctness follows directly from construction (Section 5.3) and the fact that each element is accessed exactly once (proved in Lemma 2). The exact number of operations can be calculated by simple counting as there is only one computation path and no alternate branches of computation.

The linear-time and communication complexity of the Merge algorithm follows from two points: (1) the $\mathsf{ShuffleLL}_i$ protocol can be run efficiently by easily generating a random permutation in linear-time using an algorithm such as Fisher-Yates [39], and only requires sending ciphertexts linear in the length of the inputs a constant number of times; and (2) $\mathsf{UpdateHead}_i$ and $\mathsf{EncToSS}_i$ perform only a constant number of $O(1)$-time operations with $O(1)$ communication overhead, and therefore each iteration of the Merge protocol (lines 8 to 20) requires only $O(1)$-time and communication.

For a more concrete analysis, we also calculate the communication cost, i.e., the number of messages sent as a function of n. Observe that the protocol secret shares exactly $21n + 10$ values and sends exactly $11n$ ciphertexts. As these are the only messages sent, the communication cost is $32n + 10$.

7.2 Security

Assuming the security of the underlying cryptographic primitives (homomorphic encryption, secure equality tests, comparisons and multiplexes) it is fairly straightforward to show that our protocols are indeed secure against semi-honest adversaries.

The biggest challenge is to show that the memory access pattern, as revealed in line 2 in Protocol 4 and line 11 in Protocol 5 are data-oblivious, and this follows from Lemma 2.

For completeness, we include the proofs of security below.

Lemma 3. $\mathsf{EncToSS}_i$ *(Protocol 3) securely computes* $\mathcal{F}^i_{decrypt}$ *in the presence of a semi-honest adversary.*

Proof. **Correctness:** It is straightforward to check that the secret shared value outputted by the protocol is $v = v + r_i - r_i$ which is the same as the input plaintext, v.

Security: To prove security, we show that there is a simulator, that can simulate the view of each party given only that party's inputs and the outputs of the ideal functionality. Since the protocol is asymmetric, we give two separate simulators for the two parties.

First, we show that there is an efficient simulator, Sim_i, that can simulate the view of party i, together with the output of the ideal functionality, given only party i's input ($\langle\!\langle v \rangle\!\rangle_{i+1}$) and party i's output, $[\![v'']\!]_i$. Since party i receives

only one message during the protocol (line 6), and this message is $[\![v']\!]_i$, which is a uniformly random element in $\mathcal{G}$, the simulator can *perfectly* simulate the view of party i, by sampling a random element from $\mathcal{G}$, and the simulator simulates party $(i+1)$'s output by sampling a random element from $\mathcal{G}$. To see this, note that party i's view of the protocol is $\langle\!\langle v \rangle\!\rangle_{i+1}, [\![v']\!]_i, [\![v'']\!]_i$, and the output of the protocol is $[\![v'']\!]_i$ and $[\![v'']\!]_{i+1}$ which are uniform subject to the constraint that $[\![v'']\!]_i + [\![v'']\!]_{i+1} = v$.

Next, we show that there is an efficient simulator, Sim_{i+1}, that can simulate the view of party $i+1$, together with the output of the ideal functionality, given only party $(i+1)$'s input ($\varnothing$) and party $(i+1)$'s output ($\langle\!\langle v \rangle\!\rangle_{i+1}$). During the execution of the protocol $\mathsf{EncToSS}_i$, party $i+1$ receives a single message $c' = [\![v+r_i]\!]_{i+1}$, thus the simulator must simulate three messages, c', party $(i+1)$'s output, $[\![v'']\!]_{i+1}$, and party i's output, $[\![v'']\!]_i$. The simulator proceeds as follows: Since r_i is chosen uniformly at random in the real protocol, the plaintext $v+r_i$ is also uniformly random in $\mathcal{G}$, and by the security of the Add function the distribution of the ciphertext $c' = \mathsf{Add}(pk_{i+1}, \langle\!\langle v \rangle\!\rangle_{i+1}, \langle\!\langle r_i \rangle\!\rangle_{i+1})$ is indistinguishable from the distribution of $c' \leftarrow \mathsf{Enc}(pk_{i+1}, v + r_i)$, the simulator can simulate c' by generating a random value r_s, and creating the single message $c_s = \mathsf{Enc}(pk_{i+1}, r_s)$. The simulator simulates the values $[\![v'']\!]_i$ and $[\![v'']\!]_{i+1}$ by generating uniformly random values $[\![v'']\!]_i$ and $[\![v'']\!]_{i+1}$. As above, the semantic security of the underlying cryptosystem shows that is computationally indistinguishable from the joint view of party $i+1$, together with the output of the protocol.

Lemma 4. $\mathsf{ShuffleLL}_i$ *(Protocol 2) securely computes* $\mathcal{F}^i_{shuffle}$ *in the presence of a semi-honest adversary.*

Proof. **Correctness:** By construction, $\mathsf{ShuffleLL}_i$ outputs a permutation of P_i's input list padded with m dummies, and the corresponding linked list both encrypted under pk_{i+1} to P_i, and a secret-sharing of the first dummy ($\pi(n+1)$), first real ($\pi(0)$), and *end-of-list* ($\pi(n)$) element's position to both parties. This is straightforwardly identical to the functionality, $\mathcal{F}^i_{\text{shuffle}}$.

Security: To prove security, we show that there is a simulator, that can simulate the view of each party together with the output of the ideal functionality, given only that party's inputs and the outputs. Since the protocol is asymmetric, we give two separate simulators for the two parties.

First, we show that there is an efficient simulator, Sim_i, that can simulate the view of party i, together with the output of the ideal functionality, given only party i's input and party i's output. Essentially since P_{i+1} uses a permutation unknown to P_i, Sim_i can generate any random permutation, π_r and follow the steps of the protocol exactly. It immediately follows that Sim_i simulates the messages received by P_i and the output of P_{i+1}, $([\![\pi_r(n+1)]\!]_{i+1}, [\![\pi_r(0)]\!]_{i+1})$, and $[\![\pi_r(n)]\!]_{i+1}$. The output of P_i includes (1) a permuted list and linked list encrypted under pk_{i+1}, (2) a secret-sharing of the head pointer, and (3) a secret-sharing of the end-of-list element. Since it follows the protocol exactly, it's immediately true that it can simulate (2) and (3). To simulate (1), encrypt $n+m$

random values (from $\mathcal{G}$) using pk_{i+1} to simulate the permuted list with dummies, and encrypt another $n+m$ random values (from $\mathcal{G}$) using pk_{i+1} to simulate the linked list. Output both these lists for (1) along with $(\llbracket \pi_r(n+1) \rrbracket_i, \llbracket \pi_r(0) \rrbracket_i)$ for (2), and $\llbracket \pi_r(n) \rrbracket_i$ for (3). This output is computationally indistinguishable from the output of an execution of the protocol on the same inputs assuming the underlying encryption scheme is semantically secure so we cannot distinguish between the ciphertexts. Furthermore, a distinguisher cannot distinguish between the secret-shared output of the functionality and that of the simulator since the distinguisher does not know the permutation and the secret shares are generated with uniform and fresh randomness.

Next, we show that there is an efficient simulator, Sim_{i+1}, that can simulate the view of party $i+1$, together with the output of the ideal functionality, given only party $(i+1)$'s input (permutation, π) and party $(i+1)$'s output $((\llbracket \pi(n+1) \rrbracket_{i+1}, \llbracket \pi(0) \rrbracket)_{i+1}$ and $\llbracket \pi(n) \rrbracket_{i+1})$.

To simulate the first message received by P_{i+1} (line 1), Sim_{i+1} generates n uniformly random elements, $w[0], \ldots, w[n-1]$ from $\mathcal{G}$, and encrypts them under pk_i. Then the simulator executes the protocol $\mathsf{ShuffleLL}_i$ using these values in place of $c[k]$.

To simulate party i's output (line 11), the simulator generates m dummy elements and encrypts the $n+m$ values $w[0], \ldots, w[n-1]$ together with the m dummies, permuted under π under the key pk_{i+1}. It also generates the corresponding linked list using π. Finally, it can easily generate the secret shares in party i's output, $(\pi(n+1) - \llbracket \pi(n+1) \rrbracket_i, \pi(0) - \llbracket \pi(0) \rrbracket_i)$, and $\pi(n) - \llbracket \pi(n) \rrbracket_i$ as it has access to π.

The fact that transcript produced by the simulator is indistinguishable from a real execution of the protocol follows immediately from the semantic security of the underlying cryptosystem.

Lemma 5. Merge *(Protocol 5) securely computes ideal functionality $\mathcal{F}_{merge}$ in the presence of a semi-honest adversary.*

Proof. Although the Protocol 5 is somewhat complex, the security follows almost immediately from the security of the underlying primitives. The key observation is that the memory access pattern is data-oblivious.

The data-access pattern is defined by the Reveal (line 11 in Merge and line 2 in $\mathsf{UpdateHead}_i$), we note that these reveal statements define the positions being accessed, and in the real execution of the protocol every element is accessed exactly once (Lemma 2). The shuffle ensures that these positions are uniform, and independent of the underlying data, thus this sequence of reveals can easily be simulated by simply revealing a random permutation.

Lines 3, 4, 6, 10, require basic operations on secret shares, either sharing new values, or updating existing shares which can either be done locally, or can result in a uniformly random value being sent to each party.

We assume that the underlying MPC protocol supports multiplexing (Line 9) and equality tests (Line 10), thus these messages can be simulated by calls to the underlying MPC simulator.

To see that the messages sent in each iteration (from line 8 to 20) can be efficiently simulated, note that it only involves basic operations on secret shares (lines 16, 18) multiplexing (lines 14, 17), comparisons (line 15), equality tests (Line 10), a call to $\mathsf{EncToSS}_i$ (line 13), and a call to $\mathsf{UpdateHead}_i$ (line 12). We assume that the underlying secret sharing scheme is secure, and supports multiplexing and comparisons, and Lemma 3 shows that $\mathsf{EncToSS}_i$ is secure. Finally, note that $\mathsf{UpdateHead}_i$ only makes calls to multiplexing and $\mathsf{EncToSS}_i$. Thus every iteration can be efficiently simulated using the simulators for the multiplexing, equality tests, comparisons which are assumed to exist, together with the simulator for $\mathsf{EncToSS}_i$ which was shown to exist in Lemma 3.

Finally, Lemma 4 shows that the views produced by $\mathsf{ShuffleLL}_i$ can be efficiently simulated.

Thus by stringing together these existing simulators, we can simulate the view of either party in the Merge protocol.

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