On Secure Computation of Solitary Output Functionalities With and Without Broadcast

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Abstract

Solitary output secure computation models scenarios, where a single entity wishes to compute a function over an input that is distributed among several mutually distrusting parties. The computation should guarantee some security properties, such as correctness, privacy, and guaranteed output delivery. Full security captures all these properties together. This setting is becoming very important, as it is relevant to many real-world scenarios, such as service providers wishing to learn some statistics on the private data of their users.

In this paper, we study full security for solitary output three-party functionalities in the point-to-point model (without broadcast) assuming at most a single party is corrupted. We give a characterization of the set of three-party Boolean functionalities and functionalities with up to three possible outputs (over a polynomial-size domain) that are computable with full security in the point-to-point model against a single corrupted party. We also characterize the set of three-party functionalities (over a polynomial-size domain) where the output receiving party has no input. Using this characterization, we identify the set of parameters that allow certain functionalities related to private set intersection to be securely computable in this model.

Our main technical contribution is a reinterpretation of the hexagon argument due to Fischer et al. [Distributed Computing '86]. While the original argument relies on the agreement property (i.e., all parties output the same value) to construct an attack, we extend the argument to the solitary output setting, where there is no agreement. Furthermore, using our techniques, we were also able to advance our understanding of the set of solitary output three-party functionalities that can be computed with full security, assuming broadcast but where two parties may be corrupted. Specifically, we extend the set of such functionalities that were known to be computable, due to Halevi et al. [TCC '19].

Keywords: broadcast; point-to-point communication; secure multiparty computation; solitary output; impossibility result.

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1 Introduction

Solitary output secure computation [25] allows a single entity to compute a function over an input that is distributed among several parties, while guaranteeing security. The two most basic security properties are correctness and privacy. However, in many scenarios participating parties may also desire the output receiving party to always receive an output (also known as guaranteed output delivery or full security).[1] Examples include service providers that want to perform some analysis over their client's data, federal regulatory agencies wishing to detect fraudulent users/transactions among banks, researchers looking to collect statistics from users, or a government security agency wishing to detect widespread intrusions on different high-value state agencies. In cryptography, solitary output functionalities have been considered in privacy-preserving federated learning [10, 8, 11], and in designing minimal communication protocols via Private Simultaneous Messages Protocols [18] and its robust variant [7, 1].

Additionally, understanding solitary output functions have further theoretical motivation as this is a special case of secure multiparty computation (MPC). In particular, in order to understand the general MPC setting we first need to understand important special cases. As noted by [25], one reason to study solitary output as an important stepping stone, is due to the fact that fairness, where either all parties receive the output or none do, is not an issue. Indeed, in general MPC many impossibility results [28, 4] rely on the impossibility of fair two-party coin-tossing due to Cleve [13].

In the late 1980's, it was shown that every function (even non-solitary output) can be computed with full security in the presence of malicious adversaries corrupting a strict minority of the parties, assuming the existence of a *broadcast communication channel* (such a channel allows any party to reliably send its message to all other parties, guaranteeing that all parties receive the same message) and pairwise private channels (that can be established over broadcast using standard cryptographic techniques) [9, 30, 22].

Conversely, although fairness in not an issue for solitary output functions, without a broadcast channel or without an honest majority full security cannot be achieved. Indeed, Halevi et al. [25] presented a class of solitary output functionalities that cannot be computed with full security assuming the majority of the parties are corrupted (even assuming a broadcast channel). On the other hand, [2, 20] presented several examples of three-party solitary output functionalities that cannot be securely computed without a broadcast channel, even when only a single party may be corrupted. However, besides these handfuls of examples, no general class of functions was identified to be impossible to securely compute without broadcast. This raises the question of identifying the set of functions that can be computed with full security assuming either the availability of a broadcast channel but no honest majority, or vice versa.

In this paper, we investigate the above question for the important, yet already challenging, three-party case. Thus, we aim to study the following question:

Characterize the set of solitary output three-party functionalities that can be computed with full security, assuming either a broadcast channel and two corrupted parties, or assuming no broadcast channel and a single corrupted party.

[1] Formally, full security is defined via the real vs. ideal paradigm, where a (real-world) protocol is required to emulate an ideal setting, in which the adversary is limited to selecting inputs for the corrupted parties and receiving their outputs.

1.1 Our Results

Our main technical contribution is a reinterpretation of the hexagon argument due to Fischer et al. [19]. This argument (and its generalization known as the ring argument [15]) uses the agreement property, i.e., all parties obtain the same output, in order to derive an attack on a given three-party protocol, assuming there is *no* broadcast channel available. Since we consider solitary output functionalities, where only one party receives the output, we cannot rely on agreement. Thus, we cannot use this technique in a straightforward manner. Instead, we derive an attack by leveraging the correlation in the views between the parties.

Given this new interpretation, we are able to identify a large class of three-party solitary output functionalities that cannot be computed without a broadcast channel. Furthermore, we complement this negative result by showing a non-trivial class of solitary output functionalities that can be computed in this setting. Interestingly, for several important classes of functionalities, our results provide a complete characterization of which solitary output three-party functionality can be computed with full security. Examples include Boolean and even ternary-output functionalities over a domain of polynomial size.

Somewhat surprisingly, we are able to show that all functionalities captured by our positive results, can also be securely computed in the face of a dishonest majority (where two parties may be corrupted), assuming a broadcast channel *is available*. We do not know if this is a part of a more general phenomenon (i.e., if the ability to compute a functionality without broadcast channel against a single corruption implies the ability to compute it with a broadcast channel against two corruptions) and we leave it as an interesting open question. Still, our results do slightly improve the positive results of [25].

We next describe our positive and negative results, starting with the model, where a broadcast channel is not available and only a single party may be corrupted. We consider three-party solitary output functionalities $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$, where the first party A holds an input $x \in \mathcal{X}$, the second party B holds an input $y \in \mathcal{Y}$, and the third party C holds an input $z \in \mathcal{Z}$. We let the output receiving party be A.

To simplify the presentation, we will limit the following discussion to two families of functionalities, for which our results admit a characterization (a formal statement of the results for a more general class of functionalities is given in Section 3). Though the discussion is somewhat limited when compared to the rest of the paper, all of our techniques and ideas are still present.

The first family we characterize is that of *no-input output-receiving party* (NIORP) functionalities, where the output-receiving party A has no input. We further showcase the usefulness of the result by characterizing which parameters allow for secure computation of various functionalities related to private set intersection. The second family we characterize is the set of ternary-output functionalities, where the output of A is one of three values (with A possibly holding an input). In particular, this yields a characterization of Boolean functionalities. Below are the informal statements of the characterizations for *deterministic* functionalities. We handle randomized functionalities by a reduction to the deterministic case (see Section 1.2.3 below).

Functionalities with no input for the output-receiving party (NIORP). Before stating the theorem, we define a special partitioning of the inputs of B and C. The partition is derived from an equivalence relation, which we call ***c**ommon **o**utput **re**lation* (CORE), hence, we call the partition the *CORE partition*. To obtain some intuition for the definition, consider the matrix M

associated with a NIORP functionality f, defined as $M(y,z) = f(y,z)$ for all $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$.[2]

Before defining the equivalence relation, consider the following relation $\sim$. We say that two inputs $y, y' \in \mathcal{Y}$ satisfy $y \sim y'$ if the rows $M(y,\cdot)$ and $M(y',\cdot)$ contain a common output. Note that this relation is not transitive. The equivalence relation we define is the transitive closure of $\sim$, i.e., y and y' are equivalent if there exists a sequence of inputs starting at y and ending at y' such that every consecutive pair satisfy $\sim$. Formally, we define the relation as follows.

Definition 1.1 (CORE partition)**.** *Let $f : \{\lambda\} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party NIORP functionality. For inputs $y, y' \in \mathcal{Y}$, we say that $y \sim y'$ if and only if there exist (possibly equal) $z, z' \in \mathcal{Z}$ such that $f(y,z) = f(y',z')$. We define the equivalence relation $\equiv_{\mathsf{rel}}$ to be the* transitive closure *of $\sim$. That is, $y \equiv_{\mathsf{rel}} y'$ if and only if either $y \sim y'$ or there exist a sequence of inputs $y_1, \ldots, y_k \in \mathcal{Y}$ such that*

$$y \sim y_1 \sim \ldots \sim y_k \sim y'.$$

We partition the set of inputs $\mathcal{Y}$ according to the equivalence classes of $\equiv_{\mathsf{rel}}$, and we write the partition as $\mathcal{Y} = \{\mathcal{Y}_i : i \in [n]\}$. We partition $\mathcal{Z}$ into disjoint sets $\mathcal{Z} = \{\mathcal{Z}_j : j \in [m]\}$ similarly. We also abuse notations and use the relations $\sim$ and $\equiv_{\mathsf{rel}}$ over $\mathcal{Z}$ as well. We refer to these partitions as the CORE partitions *of $\mathcal{Y}$ and $\mathcal{Z}$, respectively, with respect to f. When $\mathcal{Y}$, $\mathcal{Z}$, and f are clear from context, we will simply refer to the partitions as CORE partitions.*

Observe that given a function f, finding its CORE partition can be done in time that is polynomial in the domain size. As an example, consider the following NIORP solitary output three-party functionality whose associated matrix is given by

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Here, the CORE partitions of both the rows and the columns result in the trivial partition, i.e., all rows are equivalent and all columns are equivalent. To see this, note that both the first and second rows contain the output 1. Therefore they satisfy the relation $\sim$. Similarly, the second and last row satisfy $\sim$ since 3 (and 4) are a common output. Thus, the first and last rows are equivalent (though they do not satisfy the relation $\sim$). Using a similar reasoning, one can verify that all columns are also equivalent.

We are now ready to state our characterization for NIORP functionalities.

Theorem 1.2 (Characterization of NIORP functionalities, informal)**.** *Let $f : \{\lambda\} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party NIORP functionality, and let $\mathcal{Y} = \{\mathcal{Y}_i : i \in [n]\}$ and $\mathcal{Z} = \{\mathcal{Z}_j : j \in [m]\}$ be the CORE partitions of $\mathcal{Y}$ and $\mathcal{Z}$, respectively. Then, f can be securely computed against a single corruption in the point-to-point model, if and only if there exist two families of distributions $\{Q_i\}_{i\in[n]}$ and $\{R_j\}_{j\in[m]}$, such that the following holds. For all $i \in [n]$, $j \in [m]$, $y \in \mathcal{Y}_i$, and $z \in \mathcal{Z}_j$, it holds that $f(y^*, z)$ where $y^* \leftarrow Q_i$, and that $f(y, z^*)$ where $z^* \leftarrow R_j$, are computationally indistinguishable.*

[2] We abuse notations and write $f(y,z)$ instead of $f(\lambda, y, z)$ where λ is the empty string (representing the input of A).

Stated differently, consider the partition of $\mathcal{Y} \times \mathcal{Z}$ into combinatorial rectangles[3] defined by $\mathcal{R} = \{\mathcal{Y}_i \times \mathcal{Z}_j : i \in [n], j \in [m]\}$, i.e., it is given by all Cartesian products of CORE partitions. Then f can be securely computed if and only if both B and C can each associate a distribution to each set in the partition of their respective set of inputs, such that the output distribution in each combinatorial rectangle in $\mathcal{R}$ looks fixed for any bounded algorithm. That is, if B samples an input $y \leftarrow Q_i$ for some $i \in [n]$, then the only way for C to affect the output of f is by choosing its own equivalence class $\mathcal{Z} \in \mathcal{Z}$, however, choosing a specific input within that class will not change the output distribution.

We briefly describe a few classes of functions that are captured by Theorem 1.2. Observe that any functionality, where there exists a value $w \in \mathcal{W}$ such that any single party (among B and C) can fix the output of A to be w, regardless of the other party's input, can be securely computed by the above theorem. This includes functionalities such as OR of y and z.[4] In fact, even if there exists a distribution D over $\mathcal{W}$, such that any single party among B and C can fix the output of A to be distributed according to D, can be securely computed. For example, this means that XOR and equality can be securely computed. Theorem 1.2 essentially refines the latter family of functionalities, by requiring the parties to be able to fix the distributions with respect to the combinatorial rectangles given by the CORE partition.

In Table 1, we illustrate the usefulness of Theorem 1.2 by considering various functionalities (which were also considered by [25]) related to private-set intersection (PSI), and mark whether each variant can be computed with full security. Define the NIORP functionality $\mathsf{PSI}^{\ell_1,\ell_2}_{k_1,k_2,m}$ to output to A the intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$, held by B and C, respectively. Here, $\mathcal{S}_i \subseteq \{1,\ldots,m\}$ and $k_i \leq |\mathcal{S}_i| \leq \ell_i$ for every $i \in \{1,2\}$. The variants we consider are those that apply some function g over the output of A, i.e., the functionality the parties compute is $g(\mathsf{PSI}^{\ell_1,\ell_2}_{k_1,k_2,m}(\mathcal{S}_1,\mathcal{S}_2))$. The proofs for which parameters allow each function to be computed are presented in Section 7. It is important to note that the domains of the functionalities are *constant* as otherwise some of the claims are provably false (e.g., [2] implicitly showed that $\mathsf{PSI}^{1,1}_{1,1,\kappa}$, where κ is the security parameter, can be securely computed).

Ternary-output functionalities. We next give our characterization for ternary-output functionalities. In this setting, party A also has an input, and its output is a value in $\{0,1,2\}$. We stress that this case is far more involved than the NIORP case, in both the analysis and in the description of the characterization. Nevertheless, we later demonstrate the usefulness of this characterization.

Similarly to the NIORP case, we consider partitions over the inputs of B and C. Here, however, each input $x \in \mathcal{X}$ is associated with a different CORE partition. For the characterization, we are interested in the *meet* of all such partitions. Intuitively, the meet of partitions of a is the partition given by using all partitions together. Formally, for partitions $\mathcal{P}_1,\ldots,\mathcal{P}_n$ over a set $\mathcal{S}$, their meet is defined as the collection of all non-empty intersections, i.e.,

$$\bigwedge_{i=1}^{n} \mathcal{P}_i := \left\{ \mathcal{T} \subseteq \mathcal{S} : \mathcal{T} \neq \emptyset, \exists \mathcal{T}_1 \in \mathcal{P}_1,\ldots,\mathcal{T}_n \in \mathcal{P}_n \text{ s.t. } \mathcal{T} = \bigcap_{i=1}^{n} \mathcal{T}_i \right\}.$$

[3] A combinatorial rectangle is subset $\mathcal{R} \subseteq \mathcal{Y} \times \mathcal{Z}$ that can be written as $\mathcal{R} = \mathcal{S} \times \mathcal{T}$ where $\mathcal{S} \subseteq \mathcal{Y}$ and $\mathcal{T} \subseteq \mathcal{Z}$.

[4] A similar condition was given by [15] for the symmetric case, where all parties output the same value. There, *every* party must be able to fix the output to be w.

Input restriction\Function g	$g(\mathcal{S}) = \mathcal{S}$	$g(\mathcal{S}) = \lvert\mathcal{S}\rvert$	$g(\mathcal{S}) = \begin{cases} 1 & \text{if } \mathcal{S} = \emptyset \\ 0 & \text{otherwise} \end{cases}$
$k_1 = k_2 = 0$, or $\ell_1 = 0$, or $\ell_2 = 0$, or $k_1 = m$, or $k_2 = m$	✓	✓	✓
$k_1 = \ell_1 \notin \{0, m\}$ and $k_2 = \ell_2 \notin \{0, m\}$	✗	✓	✓
$0 < k_1 < \ell_1$, $0 < k_2 < \ell_2$, and $\ell_1 + k_2,\ k_1 + \ell_2 > m$	✗	✗	✓
Any other choice	✗	✗	✗

Table 1: Summary of our results stated for various versions of the PSI functionality. Each row in the table above corresponds to a different choice of parameters. Each column corresponds to a different function g applied to the output of A. B holds set $\mathcal{S}_1$ and C hold set $\mathcal{S}_1$. We let $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$. The parameters k_1, k_2, ℓ_1, ℓ_2 correspond to bounds on the sizes of $\mathcal{S}_1$ and $\mathcal{S}_2$, and m is the size of the universe from which $\mathcal{S}_1$ and $\mathcal{S}_2$ are taken.

Before stating the theorem, we formalize the meet of the CORE partitions, which we call CORE$_\wedge$-partition, for a given solitary output functionality.

Definition 1.3 (CORE$_\wedge$-partition)**.** *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \{0, 1, 2\}$ be a deterministic solitary output three-party ternary-output functionality. For every $x \in \mathcal{X}$, we can view $f(x, \cdot, \cdot)$ as a NIORP functionality, and consider the same CORE partition as in Definition 1.1. We denote these partitions by $\mathcal{Y}_x = \{\mathcal{Y}_i^x : i \in [n(x)]\}$ and $\mathcal{Z}_x = \{\mathcal{Z}_j^x : j \in [m(x)]\}$. We define the* CORE$_\wedge$-partitions *of f as the meet of its CORE partitions, that is, we let $\mathcal{Y}_\wedge = \bigwedge_{x \in \mathcal{X}} \mathcal{Y}_x$ and $\mathcal{Z}_\wedge = \bigwedge_{x \in \mathcal{X}} \mathcal{Z}_x$. We denote their sizes as $n_\wedge = |\mathcal{Y}_\wedge|$ and $m_\wedge = |\mathcal{Z}_\wedge|$, and we write them as $\mathcal{Y}_\wedge = \{\mathcal{Y}_i^\wedge : i \in [n_\wedge]\}$ and $\mathcal{Z}_\wedge = \{\mathcal{Z}_j^\wedge : j \in [m_\wedge]\}$.*

As an example, consider the deterministic variant of the convergecast functionality [20], $\mathsf{CC} : (\{0,1\})^3 \to \{0,1\}$ defined as[5]

$$\mathsf{CC}(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{otherwise} \end{cases} \tag{1}$$

Equivalently, CC can be defined by the two matrices

$$M_0 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Here, A chooses a matrix, B chooses a row, and C chooses a column. The output of A is the value written in the chosen entry. Observe that in M_0, the rows are not equivalent while the columns

[5] Fitzi et al. [20] defined the convergecast functionality as the NIORP randomized solitary output functionality, where A outputs y with probability $1/2$, and outputs z with probability $1/2$.

are. In M_1, however, the converse holds, namely, the rows are equivalent while the columns are not. Thus, in the $\text{CORE}_\wedge$-partitions of CC any two inputs are in *different* sets.

We are now ready to state our characterization for ternary-output functions.

Theorem 1.4 (Characterization of ternary-output functionalities, informal)**.** *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \{0,1,2\}$ be a deterministic solitary output three-party ternary-output functionality, and let $\mathcal{Y}_\wedge = \{\mathcal{Y}_i^\wedge : i \in [n_\wedge]\}$ and $\mathcal{Z}_\wedge = \{\mathcal{Z}_j^\wedge : j \in [m_\wedge]\}$ be its $\text{CORE}_\wedge$-partitions. Then f can be securely computed against a single corruption in the point-to-point model, if and only if the following hold.*

1. *Either $\mathcal{Y}_x = \{\mathcal{Y}\}$ for all $x \in \mathcal{X}$, or $\mathcal{Z}_x = \{\mathcal{Z}\}$ for all $x \in \mathcal{X}$. In other words, either all $y \in \mathcal{Y}$ are equivalent for every $x \in \mathcal{X}$, or all $z \in \mathcal{Z}$ are equivalent for every $x \in \mathcal{X}$.*

2. *There exists an algorithm S, and there exist three families of distributions $\{P_x\}_{x\in\mathcal{X}}$, $\{Q_i\}_{i\in[n_\wedge]}$, and $\{R_j\}_{j\in[m_\wedge]}$, such that the following holds. For all $i \in [n_\wedge]$, $j \in [m_\wedge]$, $y \in \mathcal{Y}_i^\wedge$, $z \in \mathcal{Z}_j^\wedge$, and $x \in \mathcal{X}$, it holds that*

$$\mathsf{S}(x, x^*, f(x^*, y, z)), \text{ that } f(x, y^*, z), \text{ and that } f(x, y, z^*),$$

   are computationally indistinguishable from each other, where $x^* \leftarrow P_x$, where $y^* \leftarrow Q_i$, and where $z^* \leftarrow R_j$.

In fact, the positive direction holds even for functionalities that are not ternary-output.

At first sight, it might seem that the characterization is hard to use since it requires the existence of an algorithm S, which in spirit seems like a simulator for a corrupt A. However, note that we only require S to output what would become the output of (an honest) A, and not the entire view of an arbitrary adversary. Arguably, determining whether such an algorithm exists is much simpler than determining whether there exists a simulator for some adversary interacting in some protocol.

We next give two examples for using Theorem 1.4. As a first example, consider the deterministic convergecast functionality CC. Observe that it does *not* satisfy Item 1 since $\mathcal{Y}_0 \neq \{\mathcal{Y}\}$ and $\mathcal{Z}_1 \neq \{\mathcal{Z}\}$. Therefore it cannot be securely computed. To exemplify Item 2 of Theorem 1.4, consider the maximum function $\mathsf{Max} : \{0,1,2\}^3 \to \{0,1,2\}$. Similarly to CC, it can be defined by the three matrices

$$M_0 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix},$$

where A chooses a matrix, B chooses a row, and C chooses a column. The output of A is the value written in the chosen entry. Clearly, any two y's are equivalent, and any two z's are equivalent as well, for all $x \in \{0,1,2\}$. Therefore, Item 1 holds. As for Item 2, we let Q_1 and R_1 output 2 with probability 1 (recall that $n_\wedge = m_\wedge = 1$). Additionally, we let S ignore its inputs and output 2 with probability 1. It follows that Item 2 holds. Thus, Max can be securely computed. In fact, as the positive direction of Theorem 1.4 holds for functions that are not ternary-output, the same argument can be made when Max has a domain that is arbitrarily large, i.e., $\mathsf{Max} : \{1,\ldots,m\}^3 \to \{1,\ldots,m\}$ for some natural m.

To illustrate how the Theorem 1.4 can be used (in the positive direction), consider the following functionality $f : \{0,1\} \times (\{0,1\}^2 \cup \{2\}) \times \{0,1\} \to \{0,1,2\}$ defined as

$$f(x,y,z) = \begin{cases} y_1 \oplus z & \text{if } y = (y_0, y_1) \wedge y_0 = x \\ 2 & \text{otherwise} \end{cases}$$

Similarly to CC, f can be defined by the two matrices

$$M_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 0 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix}$$

where the first four rows corresponds to the set of inputs $\{0,1\}^2$ for party B, and last row corresponds to the input $y = 2$. We next show that f can be securely computed. First, observe that CORE partitions of the columns are trivial for both $x = 0$ and $x = 1$, i.e., $\mathcal{Z}_0 = \mathcal{Z}_1 = \{\mathcal{Z}\}$. As for the rows, in each matrix the two rows with Boolean values are in the same set, and the three rows with the fixed value of 2 belong to the same set, that is, $\mathcal{Y}_0 = \{\{(0,0),(0,1)\},\{(1,0),(1,1),2\}\}$, $\mathcal{Y}_1 = \{\{(0,0),(0,1),2\},\{(1,0),(1,1)\}\}$. As a result, Item 1 holds, and the $\mathrm{CORE}_\wedge$-partition of the rows is $\mathcal{Y}_\wedge = \{\{(0,0),(0,1)\},\{(1,0),(1,1)\},\{2\}\}$, while the $\mathrm{CORE}_\wedge$-partition of the columns is the trivial partitioning. To fix the output distributions, we take the distributions over the equivalence classes to be uniform, that is, for $i \in \{1,2\}$ let Q_i be uniform over $\{(i-1,0),(i-1,1)\}$ (Q_3 always outputs $y = 2$), and let R_1 be uniform over $\{0,1\}$. Then for $i \in \{1,2\}$ it holds that $f(x,y^*,z) \equiv f(x,(y_0,y_1),z^*)$, which is a uniform random bit if $y_0 = x$ and is equal to two otherwise. Furthermore, for $i = 3$ the two distributions always output 2. To show that f can be securely computed, using Theorem 1.4, it is left to define the algorithm S and the distributions P_0 and P_1. For all $x \in \{0,1\}$ we let P_x always output x, and define $\mathsf{S}(x,x^*,w)$ to output a uniform random bit if $w \in \{0,1\}$, and output 2 otherwise. Thus Item 2 holds. Therefore, f can be securely computed.

Randomized functionalities. So far, we have only dealt with deterministic functionalities. To handle the randomized case, we show how to reduce it to the deterministic case. That is, for any randomized solitary output three-party functionality f, we define a deterministic solitary output three-party functionality f', such that f can be securely computed if and only if f' can be securely computed.

Proposition 1.5 (Reducing randomized functionalities to deterministic functionalities, informal)**.** *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a (randomized) solitary output three-party functionality, and let $\mathcal{R}$ denote the domain of its randomness. Define the deterministic solitary output three-party functionality $f' : (\mathcal{X} \times \mathcal{R}) \times (\mathcal{Y} \times \mathcal{R}) \times (\mathcal{Z} \times \mathcal{R}) \to \mathcal{W}$ as*

$$f'((x,r_1),(y,r_2),(z,r_3)) = f(x,y,z;r_1+r_2+r_3),$$

where addition is done over $\mathcal{R}$ when viewed as an additive group. That is, the parties receive a share of the randomness in a 3-out-of-3 secret sharing scheme. Then f can be securely computed if and only if f' can be securely computed.

Assuming a broadcast channel. Surprisingly, we are also able to show that any (randomized) solitary output three-party functionality that can be securely computed, as captured by Theorems 1.2 and 1.4, can also be securely computed assuming the availability of a broadcast channel with security holding against two corrupted parties. In particular, any solitary output three-party

ternary-output functionality and any NIORP functionality, that can be securely computed without broadcast against a single corruption, can be securely computed with broadcast against two corruptions. Moreover, the set of functions captured by Theorems 1.2 and 1.4 extends the set of three-party functionalities previously known to be computable with broadcast, due to Halevi et al. [25] (see [25, Theorem 4.4]).

On the other hand, we claim that the converse is false, i.e., there exists a solitary output NIORP Boolean three-party functionality that can be securely computed with broadcast against two corruptions, yet it cannot be securely computed without broadcast against a single corruption. Indeed, consider the following solitary output three-party variant of the GHKL functionality,[6] denoted soGHKL, defined by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{2}$$

where B chooses a row, C chooses a column, and the output of A is the value written in the chosen entry.

Observe that soGHKL is a NIORP functionality that does not satisfy the necessary conditions given by Theorem 1.2. Thus, it cannot be securely computed in the point-to-point model. On the other hand, Halevi et al. [25] showed that soGHKL can be computed assuming a broadcast channel.[7] The theorem below summarizes our results.

Theorem 1.6 (Informal)**.** *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality. Assume that oblivious transfer exists, and that f is either a NIORP or a ternary-output functionality. Suppose that f can be securely computed assuming an honest majority in the point-to-point model. Then, assuming the availability of a broadcast channel, f can be securely computed tolerating two corruptions.*

Moreover, the converse is false. That is, there exists a NIORP Boolean three-party functionality that can be securely computed against two corruptions, assuming a broadcast channel in the OT-hybrid model, however, it cannot be securely computed in the point-to-point model against a single corruption.

In fact, Theorem 1.6 can be improved by slightly relaxing some of the conditions the function has to satisfy (see Section 1.2.4 below for more details). Furthermore, Theorem 1.6 captures NIORP functionalities whose status was previously unknown, e.g., the NIORP functionality $f_{\mathsf{special}} : \{\lambda\} \times (\{0,1,2,3\})^2 \to \{0,\ldots,7\}$ defined by the matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 4 & 5 & 6 & 7 \\ 5 & 4 & 7 & 6 \end{pmatrix} \tag{3}$$

can be securely computed assuming a broadcast channel, tolerating two corruptions.

In Table 2 below, we present several examples of three-party functionalities, and compare their status assuming no broadcast channel and one corruption, to the case where such a channel is available with two possible corruptions.

[6] Gordon et al. [24] showed that the symmetric two-party variant of this functionality can be computed with full security.

[7] In fact, [25] gave three different protocols for computing soGHKL securely.

Function\Model	Without broadcast (honest majority)	With broadcast (no honest majority)
Millionaires' Problem: $\mathsf{GT}(x,y,z) = \begin{cases} 0 & \text{if } x > y,z \\ 1 & \text{if } y > z, y \geq x \\ 2 & \text{otherwise} \end{cases}$	✗ Thm. 1.4	✓ [25]
NIORP Millionaires' Problem: $\mathsf{cGT}(y,z) = \begin{cases} 0 & \text{if } y > z \\ 1 & \text{otherwise} \end{cases}$	✓ Thm. 1.4	✓ [25]
$\mathsf{CC}(x,y,z)$ (see Eq. 1)	✗ Thm. 1.4	✓ [25]
$\mathsf{soGHKL}(y,z)$ (see Eq. 2)	✗ Thm. 1.4 (also Thm. 1.2)	✓ [25]
$\mathsf{Max}(x,y,z)$	✓ Thm. 1.4	✓ [25]
$\mathsf{EQ}(y,z) = \begin{cases} 1 & \text{if } y = z \\ 0 & \text{otherwise} \end{cases}$	✓ Thm. 1.4	✓ [25]
$\mathsf{EQ}(y,z) = \begin{cases} y & \text{if } y = z \\ 0 & \text{otherwise} \end{cases}$	✗ Thm. 1.4 (also Thm. 1.2)	✗ [25]
f_{special} (see Eq. 3)	✓ Thm. 1.4 (also Thm. 1.2)	✓ Thm. 1.4

Table 2: Comparing the landscape of functionalities that can be computed without broadcast but with an honest majority, to functionalities that can be computed with broadcast but no honest majority. All functions above have a constant domain. It is important that the domain of EQ does not include 0. Finally, the results for the computation of NIORP functionalities without broadcast also follow from Theorem 1.2.

1.2 Our Techniques

We now turn to describe our techniques. In Section 1.2.1 we handle NIORP functionalities. Then, in Section 1.2.2 we handle ternary-output functionalities. Then, in Section 1.2.3 we show how to reduce the randomized case to the deterministic case. Finally, in Section 1.2.4 we prove Theorem 1.6, showing that for the families of functions considered, the broadcast assumption is strictly stronger than the honest majority assumption. To simplify the proofs in this introduction, we only consider perfect security and functionalities with finite domain and range.

1.2.1 Characterizing NIORP Functionalities

We start with the negative direction of Theorem 1.2. Our argument is split into two parts. In the first part, we adapt the hexagon argument, due to Fischer et al. [19], to the MPC setting. Roughly, for every secure three-party protocol we attribute six distributions, all of which are identically distributed by the perfect security of the protocol. The second part of the proof is dedicated to the

analysis of these six distributions, resulting in necessary conditions for perfect security.

The hexagon argument for NIORP functionalities. In the following, let f be a solitary output three-party NIORP functionality (no input for the output receiving party), and let π be a three-party protocol computing f securely over point-to-point channels, tolerating a single corrupted party. At a high level, the hexagon argument is as follows.

1. First, we construct a new six-party protocol π'. This is the same hexagon protocol from [19] (see below for a formal definition).

2. Then, we consider six different *semi-honest* adversaries for π' corrupting four parties, and observe that each of them can be emulated by a *malicious* adversary in the original three-party protocol π. In more detail, for each of the semi-honest adversaries we consider for π', we show there exists a malicious adversary corrupting a single party in π satisfying the following: The transcript between the two honest parties and the transcript between each honest party and the adversary, are identically distributed in both protocols. We stress that π' is *not* secure, but rather any attacker for it can be emulated by an attacker for the three-party protocol π.

3. Observe that as the adversaries for π' are semi-honest, the view of each party (both corrupted and honest) is identically distributed across all six scenarios.

4. We then translate the above observation to π using the fact that each of the semi-honest adversaries for π' can be emulated in π. Thus, we obtain a certain correlation between the six malicious adversaries for π.

5. By the assumed security of π, each of the malicious adversaries can be simulated in the ideal world of f. Therefore, we can translate the correlation from the previous step to the ideal world, and obtain a necessary property f has to satisfy. This results in six distributions with differing definitions, all of which are identically distributed. Looking ahead, the second part of our argument is dedicated to analyzing these distributions.

We next provide a more formal argument. Consider the following six-party protocol π'. For each party $\mathsf{P} \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ in π we have two copies P and P' in π', both use the same code as an honest P does in π. Furthermore, the parties are connected via the following undirected cycle graph: (1) A is connected to B and C, (2) A' is connected to B' and C', (3) B is also connected to C', and (4) C is also connected to B'. See Figure 1 below for a pictorial definition. Finally, we let B, B', C, and C' hold inputs y, y', z, and z', respectively.

Now, consider the following 6 attack-scenarios for the six-party protocol, where in each scenario a *semi-honest* adversary corrupts four adjacent parties, as depicted in Figure 2. Observe that each attacker can be emulated in the original three-party protocol π, by a *malicious* adversary emulating the corresponding four parties in its head. For example, in Scenario 2a, an adversary in π can emulate the attack by corrupting C, and emulating in its head two virtual copies of C, a copy of A, and a copy B.

We now focus on party A in the six-party protocol. First, note that in Scenarios 2a and 2b, where A is honest, its output is identically distributed[8] since the adversaries are semi-honest. Second, in

[8]Note that even though f is assumed to be deterministic, it is not guaranteed that the output of an honest A is a fixed value even when interacting with a semi-honest adversary. This is due to the fact that the semi-honest adversaries are emulated in the three-party protocol using malicious adversaries.

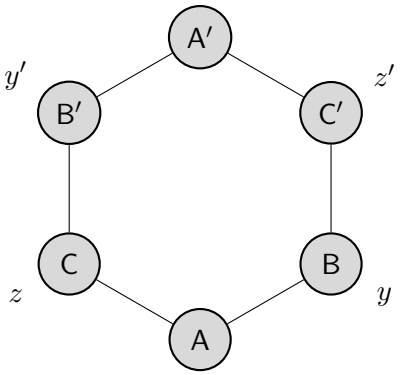
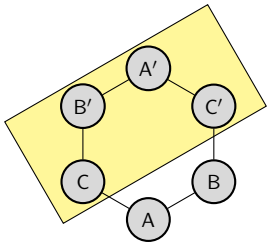
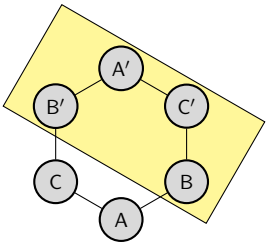


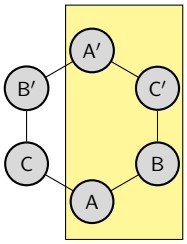
Figure 1: The six-party protocol



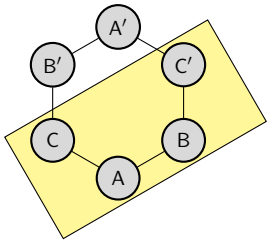
(a) Scenario 1



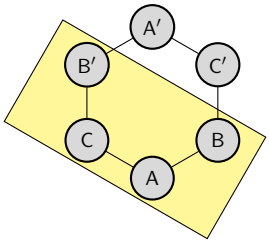
(b) Scenario 2



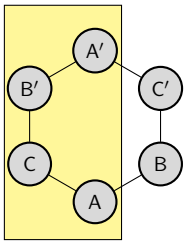
(c) Scenario 3



(d) Scenario 4



(e) Scenario 5



(f) Scenario 6

Figure 2: The six adversaries in the hexagon argument. The shaded yellow areas in each scenario correspond to the (virtual) parties the adversary controls.

the other four scenarios, where A is corrupted, the adversary's view contains the same view that an honest A has in an honest execution of π'. Therefore, it can compute an output with an identical distribution to the output distribution an honest A has in Scenarios 2a and 2b.

Next, we use the fact that the six semi-honest adversaries in π' can be emulated by malicious adversaries in π. We obtain that there exists some distribution D (that depends on all inputs y, y', z, and z' in the six-party protocol) over the set of possible outputs $\mathcal{W}$ of A, such that the following hold.

Scenarios 1 and 2: There exist two malicious adversaries for π, one corrupting C and holding (y', z, z'), and one corrupting B and holding (y, y', z'), such that the output of A in both scenarios is distributed according to D.

Scenarios 4 and 5: There exist two malicious adversaries for π, one corrupting C and holding (y, z, z'), and one corrupting B and holding (y, y', z), both of which can generate a sample

from D at the end of the execution.

Scenarios 3 and 6: There exist two malicious adversaries for π, both corrupting A, where one is holding (y, z') and the other is holding (y', z), such that both can generate a sample from D at the end of the execution.

By the assumed security of π, each of the adversaries can be simulated in the corresponding ideal world of the three-party functionality f. Thus, we obtain six different expressions for the distribution D, representing the output of A. The six expressions are described as follows.

Scenarios 1 and 2: There exist two malicious simulators in the ideal world of f, one corrupting C and holding (y', z, z'), and one corrupting B and holding (y, y', z'), such that the output of A in both ideal world executions is distributed according to D. Recall that the only way for the simulators to affect the output of A is by choosing the input they send to the trusted party. It follows the first simulator corrupting C, defines a distributed distribution $R_{y',z,z'}$ that depends only on y', z, and z', such that $f(y, z^*) \equiv D$, where $z^* \leftarrow R_{y',z,z'}$. Similarly, the second simulator corrupting B, defines a distributed distribution $Q_{y,y',z'}$ that depends only on y', z, and z', such that $f(y^*, z) \equiv D$, where $y^* \leftarrow Q_{y,y',z'}$.

Scenarios 4 and 5: There exist two malicious simulators, one corrupting C and holding (y, z, z'), and one corrupting B and holding (y, y', z), both of which can generate a view that is identical to their corresponding real world adversary. In particular, since both adversaries can generate a sample from D, it follows that both simulators must be able to do the same at the end of their respective ideal world execution. Since the simulators do not receive any output from the trusted party, it follows there exist two algorithms S_B and S_C, such that both $\mathsf{S}_\mathsf{C}(y, z, z')$ and $\mathsf{S}_\mathsf{B}(y, y', z)$ output a sample from D.

Scenarios 3 and 6: There exist two malicious simulators, both corrupting A, where one is holding (y, z') and the other is holding (y', z), such that both can generate a sample from D at the end of the execution. Unlike the previous case, this time the two simulators do receive an output from the trusted party. This implies there exist two algorithms S_3 and S_6, such that both $\mathsf{S}_3(y, z', f(y', z))$ and $\mathsf{S}_6(y', z, f(y, z'))$ output a sample from D.

We conclude that for all $y, y' \in \mathcal{Y}$ and $z, z' \in \mathcal{Z}$, there exist two efficiently samplable distributions $Q_{y,y',z'}$ and $R_{y',z,z'}$ over $\mathcal{Y}$ and $\mathcal{Z}$, respectively, and four algorithms S_B, S_C, S_3, and S_6, such that

$$f\left(y^*, z\right) \equiv f\left(y, z^*\right) \equiv \mathsf{S}_\mathsf{B}\left(y, y', z\right) \equiv \mathsf{S}_\mathsf{C}\left(y, z, z'\right) \equiv \mathsf{S}_3\left(y, z', f\left(y', z\right)\right) \equiv \mathsf{S}_6\left(y', z, f\left(y, z'\right)\right), \quad (4)$$

where $y^* \leftarrow Q_{y,y',z'}$ and where $z^* \leftarrow R_{y',z,z'}$.

Analyzing the six distributions over the output of A. We now turn to the analysis of Equation (4), which results in the necessary conditions stated in Theorem 1.2. Recall that our goal is to show that for all $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, it holds that

$$f(y, z^*) \equiv f(y^*, z),$$

where y^* and z^* are sampled according to specific distributions that depend on the equivalence classes containing y and z, respectively.

First, observe that as $\mathsf{S_B}$ is independent of z', it follows that all other distributions are also independent of it. For example, for any $z'' \neq z'$ it holds that

$$\mathsf{S}_3\left(y, z', f\left(y', z\right)\right) \equiv \mathsf{S_B}\left(y, y', z\right) \equiv \mathsf{S}_3\left(y, z'', f\left(y', z\right)\right).$$

Similarly, since $\mathsf{S_C}$ is independent of y' it follows that all other distributions are also independent of it as well. From this, we conclude the following: Let y_0 and z_0 be the lexicographically smallest elements of $\mathcal{Y}$ and $\mathcal{Z}$, respectively, and define the distributions $Q'_y := Q_{y,y_0,z_0}$ and $R'_z := R_{y_0,z,z_0}$.[9] Then, the above observation implies that

$$f\left(y^*, z\right) \equiv f\left(y, z^*\right) \equiv \mathsf{S}_3\left(y, z', f\left(y', z\right)\right) \equiv \mathsf{S}_6\left(y', z, f\left(y, z'\right)\right), \tag{5}$$

for all $y' \in \mathcal{Y}$ and $z' \in \mathcal{Z}$, where $y^* \leftarrow Q'_y$ and $z^* \leftarrow R'_z$.

Let us focus on S_3, and fix $\tilde{z} \in \mathcal{Z}$ such that $z \sim \tilde{z}$. Recall that the relation $\sim$ is defined as $z \sim \tilde{z}$ if and only if there exist $\tilde{y}, \tilde{y}' \in \mathcal{Y}$ such that $f(\tilde{y}, z) = f(\tilde{y}', \tilde{z})$. Since S_3 is independent of y', it follows that

$$\mathsf{S}_3\left(y, z', f\left(y', z\right)\right) \equiv \mathsf{S}_3\left(y, z', f\left(\tilde{y}, z\right)\right) \equiv \mathsf{S}_3\left(y, z', f\left(\tilde{y}', \tilde{z}\right)\right) \equiv \mathsf{S}_3\left(y, z', f\left(y', \tilde{z}\right)\right),$$

where the first and last transition follows from the previously made observation that the output distribution of S_3 is independent of the value of y', and the second transition follows from the fact that $f(\tilde{y}, z) = f(\tilde{y}', \tilde{z})$, hence S_3 receives the same inputs in both cases. Therefore, changing z to $\tilde{z}$ where $z \sim \tilde{z}$ does not change the output distribution of S_3. Note that the argument can be repeated to show that replacing $\tilde{z}$ with any other $\tilde{z}'$, where $\tilde{z} \sim \tilde{z}'$, does not change the distribution. It follows that changing z to any $\tilde{z}'$ satisfying $z \equiv_{\mathsf{rel}} \tilde{z}'$ does not change the output distribution of S_3. Thus, all distributions in Equation (5) are not affected by such change.

Plugging this back to Equation (5), results in the following. For every $j \in [m]$, every $y \in \mathcal{Y}$, and every equivalent $z, \tilde{z}' \in \mathcal{Z}_j$ (recall that $\mathcal{Z}_j$ is the j^{th} equivalence class with respect to the relation $\equiv_{\mathsf{rel}}$), it holds that

$$f(y, z^*) \equiv f(y^*, z) \equiv f(y^*, \tilde{z}') \equiv f(y, \tilde{z}^*),$$

where $y^* \leftarrow Q'_y$, where $z^* \leftarrow R'_z$, and where $\tilde{z}^* \leftarrow R'_{\tilde{z}}$. In particular, the distributions depend only on the index j, and not on the specific choice of input from the equivalence class $\mathcal{Z}_j$. Thus, if for any $j \in [m]$ we define the distribution $R''_j := R'_{z_j}$, where z_j is the lexicographically smallest element in $\mathcal{Z}_j$, it then follows that for every $j \in [m]$, every $y \in \mathcal{Y}$, and every $z \in \mathcal{Z}_j$, that

$$f(y, z^*) \equiv f(y^*, z),$$

where $y^* \leftarrow Q'_y$ and $z^* \leftarrow R'_j$.

Finally, an analogous argument starting by focusing on S_6, implies that the distributions depend only on the equivalence class containing y, rather than depending on y directly. Therefore, for any $i \in [n]$ we can define the distribution $Q''_i := Q'_{y_i}$, where y_i is the lexicographically smallest element in $\mathcal{Y}_i$. It then follows that for every $i \in [n]$, $j \in [m]$, $y \in \mathcal{Y}_i$, and $z \in \mathcal{Z}_j$ it holds that

$$f(y^*, z) \equiv f(y, z^*),$$

where $y^* \leftarrow Q''_i$ and $z^* \leftarrow R''_j$, as claimed.

[9] Note that the choice of taking the lexicographically smallest elements of $\mathcal{Y}$ and $\mathcal{Z}$ is arbitrary, and any other element would work.

The positive direction for NIORP functionalities. We now present a protocol for any solitary output three-party NIORP functionality f, satisfying the conditions stated in Theorem 1.2. Our starting point is the same as that of [15, 2], namely, computing f *fairly* (i.e., either all parties obtain the output or none do). This follows from the fact that, by the honest-majority assumption, the protocol of Rabin and Ben-Or [30] computes f assuming a *broadcast channel*; hence by [14] it follows that f can be computed with fairness over a point-to-point network.

We now describe the protocol. The parties start by computing f with fairness. If they receive outputs, then they can terminate, and output what they received.[10] If the protocol aborts, then B finds the unique $i \in [n]$ such that $y \in \mathcal{Y}_i$ and sends i to A. Similarly, C finds the unique $j \in [m]$ such that $z \in \mathcal{Z}_j$ and sends j to A. Observe that this can be done efficiently since the domain of f is of constant size. Party A then samples $y^* \leftarrow Q_i$ and outputs $f(y^*, z_j)$, where z_j is the lexicographically smallest element in $\mathcal{Z}_j$.

Observe that correctness holds since when all parties are honest, the fair protocol will never abort (note that without the fair computation of f the above protocol is not correct since A would always output $f(y^*, z_j)$ instead of $f(y, z)$). Now, consider a corrupt B (the case of a corrupt C is similar). First, note that the adversary does not obtain any information from the fair computation of f. Next, if the adversary sends some i' to A, then the simulator sends $y^* \leftarrow Q_{i'}$ to the trusted party. Then the output of A in the ideal world is $f(y^*, z)$. By our assumption on f this is identical to $f(y^*, z_j)$ – the output of A in the real world.

Next, consider a corrupt A. Since it does not obtain any information from the (failed) fair computation of f, it suffices to show how a simulator that is given $f(y, z)$ can compute the corresponding i and j. Observe that by our definition for the partition of the inputs, any two distinct combinatorial rectangles $\mathcal{Y}_i \times \mathcal{Z}_j$ and $\mathcal{Y}_{i'} \times \mathcal{Z}_{j'}$, where $(i, j) \neq (i', j')$, have no common output. Indeed, if $f(y, z) = f(y', z')$, where $(y, z) \in \mathcal{Y}_i \times \mathcal{Z}_j$ and $(y', z') \in \mathcal{Y}_{i'} \times \mathcal{Z}_{j'}$, then $y \sim y'$ and $z \sim z'$, hence they belong to the same sets. Therefore, the simulator for the corrupt A can compute the corresponding i and j given the output by simply looking them up (which can be done efficiently since the domain is of constant size).

1.2.2 Characterizing Ternary-Output Functionalities

We now explain our techniques for proving Theorem 1.4. We begin with the negative direction. Similarly to the proof of Theorem 1.2 presented earlier, the argument is comprised of the hexagon argument and the analysis of the six distributions that are obtained. However, since A now has an input, the argument is much more involved.

A generalized hexagon argument. Unlike in the previous proof, here the hexagon argument (as used there) does not suffice. To show where the argument falls short, let us first describe the six distributions obtained from the hexagon argument. In this setting, where A now has an input, the six-party protocol described earlier will now have A and A' hold inputs x and x', respectively. The two inputs are then given to the correct adversaries from the six scenarios. Furthermore, observe that the algorithms S_3 and S_6, which came from the two simulators for a corrupt A in the ideal world, can also send to the trusted party an input that is not necessarily the same input that the simulators hold. Let x_3^* and x_6^* denote the inputs used by S_3 and S_6, respectively, each sampled

[10] Although B and C are supposed to receive no output from f, in a fair computation they either receive the empty string indicating that A received its output, or a special symbol $\perp$ indicating abort.

according to a distribution that depends on the simulator's inputs. Thus, to adjust the hexagon argument to this case, Equation (4) should now be replaced with

$$\begin{aligned} f(x, y^*, z) &\equiv f(x, y, z^*) \\ &\equiv \mathsf{S}_\mathsf{B}(x, y, y', z) \\ &\equiv \mathsf{S}_\mathsf{C}(x, y, z, z') \\ &\equiv \mathsf{S}_3(x, x', y, z', x_3^*, f(x_3^*, y', z)) \\ &\equiv \mathsf{S}_6(x, x', y', z, x_6^*, f(x_6^*, y, z')), \end{aligned} \tag{6}$$

where $y^* \leftarrow Q_{x',y,y',z'}$, where $z^* \leftarrow R_{x',y',z,z'}$, where $x_3^* \leftarrow P^3_{x,x',y,z'}$, and where $x_6^* \leftarrow P^6_{x,x',y',z}$.

We now show where the argument falls short using an example: recall that we defined the deterministic variant of the convergecast functionality [20], $\mathsf{CC} : (\{0,1\})^3 \to \{0,1\}$ as

$$\mathsf{CC}(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{otherwise} \end{cases}$$

We claim that there exist distributions and algorithms satisfying Equation (6), hence the argument is insufficient to show the impossibility of securely computing CC. Indeed, take $Q_{x',y,y',z'}$ to always output $y^* = y$, take $R_{x',y',z,z'}$ to always output $z^* = z$, define $P^3_{x,x',y,z'}$ to always output $x_3^* = 1$ (causing S_3 to obtain z), define $P^6_{x,x',y',z}$ to always output $x_6^* = 0$ (causing S_6 to obtain y), and define S_B and S_C, both of which hold x, y, and z, to compute $\mathsf{CC}(x, y, z)$. Then all six distributions always output $\mathsf{CC}(x, y, z)$.

However, as we next explain, the functionality CC cannot be computed securely in our setting. Intuitively, this is because the adversary corrupting A as in Scenario 2c using inputs $x = 1$ and $x' = 0$, learns both y' and z. Indeed, in Scenario 2b (where B is corrupted) the output of an *honest* A is z, and in Scenario 2d (where C is corrupted) the output of an *honest* A' is y'. Since the adversaries are semi-honest, the adversary corrupting A as in Scenario 2c can compute both z and y' by computing the output of the honest A and A', respectively. However, in the ideal world, a simulator (for the malicious adversary emulating Scenario 2c) can only learn one of the inputs.

To generalize this intuition, we consider the joint distribution of the outputs of A and A' in the six-party protocol, rather than only the distribution of the output of A. Doing a similar analysis to the NIORP case results in the existence of six distributions $P^3_{x,x',y,z'}$, $P^6_{x,x',y',z}$, $Q_{x,y,y',z}$, $Q'_{x',y,y',z'}$, $R_{x,y,z,z'}$, and $R'_{x',y',z,z'}$, and the existence of six algorithms S_3, S_6, S_B, S'_B, S_C, and S'_C, where S_3 and S_6 output two values (corresponding to the outputs of A and A'), such that the following six distributions are identically distributed:

1. $\mathsf{S}_3(x, x', y, z', x_3^*, f(x_3^*, y', z))$, where $x_3^* \leftarrow P^3_{x,x',y,z'}$.
2. $\mathsf{S}_6(x, x', y', z, x_6^*, f(x_6^*, y, z'))$, where $x_6^* \leftarrow P^6_{x,x',y',z}$.
3. $(\mathsf{S}_\mathsf{B}(x, y, y', z, y_1^*), f(x', y_1^*, z'))$, where $y_1^* \leftarrow Q_{x,y,y',z}$.
4. $(f(x, y_2^*, z), \mathsf{S}'_\mathsf{B}(x', y, y', z', y_2^*))$, where $y_2^* \leftarrow Q'_{x',y,y',z'}$.
5. $(\mathsf{S}_\mathsf{C}(x, y, z, z', z_1^*), f(x', y', z_1^*))$, where $z_1^* \leftarrow R_{x,y,z,z'}$.
6. $(f(x, y, z_2^*), \mathsf{S}'_\mathsf{C}(x', y', z, z', z_2^*))$, where $z_2^* \leftarrow R'_{x',y',z,z'}$.

We stress that both S_3 and S_6 output two values from the set of outputs $\{0,1,2\}$, while S_B, S'_B, S_C, and S'_C, each output a single value from $\{0,1,2\}$.

Observe that for the function CC, the above distributions and algorithms do not exist for all possible choice of inputs. Indeed, for $x = 1$ and $x' = 0$, it holds that $\mathsf{CC}(x, y_2^*, z) = z$ (from the fourth distribution) and that $\mathsf{CC}(x', y', z_1^*) = y'$ (from the fifth distribution). Therefore, the marginal distribution of the first value must be z, and the marginal distribution of the second value must be y', both with probability 1. However, note that S_3 is given only one of y' or z, depending on the value of x_3^*, hence it cannot output both of them correctly.

Analyzing the six joint distributions over the outputs of A and A'. We now analyze the new six distributions described earlier. First, similarly to the case of NIORP functionalities, we make the observation that the marginal distribution of the first entry is independent of x', y', and z', and the marginal distribution of the second entry is independent of x, y, and z. Let us focus on S_3 and the distribution $P^3_{x,x',y,z'}$.

Our next goal is to analyze the support of $P^3_{x,x',y,z'}$, namely, analyze which inputs x_3^* can be used by S_3. This results in a necessary condition for f to be securely computable, since if the input x_3^* must satisfy some condition, in particular, this implies an input satisfying such condition must exist. We do this analysis by comparing the first (i.e., left) output of S_3 to the distribution in Item 4 above, where the first value is $f(x, y_2^*, z)$, and by comparing the second (i.e., right) output of S_3 to the distribution in Item 5 above, where the second value is $f(x', y', z_1^*)$. In fact, rather than directly comparing the outputs, we compare the *information on the equivalence class* of z and y' with respect to the CORE partitions that can be inferred from the outputs. We next focus on comparing to $f(x, y_2^*, z)$ (comparing to $f(x', y', z_1^*)$ is analogous).

Let us first recall the definition of the CORE partitions. Recall that for every x we can view $f(x, \cdot, \cdot)$ as a NIORP function. Thus, we can partition $\mathcal{Y}$ and $\mathcal{Z}$ according to the CORE partition for the given x. Since we focus on S_3 it suffices, for now, to only consider the partition of $\mathcal{Z}$. Let $M_x \in \{0,1,2\}^{|\mathcal{Y}|\times|\mathcal{Z}|}$ be the matrix associated with $f(x, \cdot, \cdot)$, defined as $M_x(y, z) = f(x, y, z)$ for all $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$. Recall that we denote the partition as $\mathcal{Z}_x = \{\mathcal{Z}_j^x : j \in [m(x)]\}$, and we let z and $\tilde{z}$ be in the same equivalence class if and only if there exist $z_1, \ldots, z_k \in \mathcal{Z}$ such that the columns $M_x(\cdot, z)$ and $M_x(\cdot, z_1)$ have a common output, for all $i \in [k-1]$ the columns $M_x(\cdot, z_i)$ and $M_x(\cdot, z_{i+1})$ have a common output, and the columns $M_x(\cdot, z_k)$ and $M_x(\cdot, \tilde{z})$ have a common output. Observe that for any $x \in \mathcal{X}$ and every $y \in \mathcal{Y}$ it holds that if $z \in \mathcal{Z}$ and $\tilde{z} \in \mathcal{Z}$ are in *different* classes, then $f(x, y, z) \neq f(x, y, \tilde{z})$.

Now, consider the distribution in Item 4 above, where the first value is $f(x, y_2^*, z)$. It follows that S_3 must be able to output $f(x, y_2^*, z)$.[11] Next, observe that from $f(x, y_2^*, z)$ it is possible to infer the (unique) $j \in [m(x)]$ satisfying $z \in \mathcal{Z}_j^x$. This is due to the fact that, as noted earlier, for z and $\tilde{z}$ in two different classes, the output of f on each of them (with the same x and y) is always different. Thus, for any fixed value for y_2^*, from the output $f(x, y_2^*, z)$ we can compute the equivalence class of z.

However, the only information that S_3 can obtain on the class $j \in [m(x)]$ can come from the output $f(x_3^*, y', z)$ (which corresponds to the output it receives from the trusted party). That is, the *only* information that S_3 can have is the equivalence class of z with respect to the partition of x_3^* rather than the partition with respect to x. Since the first entry in the output of S_3 must be identically distributed to $f(x, y_2^*, z)$, the value x_3^* it uses must be such that $f(x_3^*, y', z)$ reveals

[11] Formally, the marginal distribution of the first value in the output of S_3 is identically distributed to $f(x, y_2^*, z)$.

at least the same information on j as $f(x, y_2^*, z)$ does. This implies that x_3^* must be such that if $z \in \mathcal{Z}_{j_3^*}^{x_3^*}$ then $z \in \mathcal{Z}_j^x$, with probability 1. Furthermore, this must hold for all $z \in \mathcal{Z}$ and the distribution $P^3_{x,x',y,z'}$, from which x_3^* is drawn from, is independent of z, it follows that the partition $\mathcal{Z}_{x_3^*}$ must be a *refinement* of $\mathcal{Z}_x$. That is, any $\mathcal{Z} \in \mathcal{Z}_{x_3^*}$ must be a subset of some $\mathcal{Z}' \in \mathcal{Z}_x$. Similarly, since S_3 must also output $f(x', y', z_1^*)$ from the fifth distribution in Item 5, it follows that $\mathcal{Y}_{x_3^*}$ is a refinement of $\mathcal{Y}_{x'}$. As a result, we conclude that for any $x, x' \in \mathcal{X}$ there exists $x_3^* \in \mathcal{X}$ such that $\mathcal{Y}_{x_3^*}$ is a refinement of $\mathcal{Y}_{x'}$ and such that $\mathcal{Z}_{x_3^*}$ is a refinement of $\mathcal{Z}_x$. We stress that so far, we have not used the fact that f is ternary-output, thus the existence of such x_3^* holds for any function that can be securely computed. We formalize this in Lemma 4.9, where we also consider functions with polynomial-sized domain (note that it is important for the domain to be small in order to claim that the equivalence classes can be inferred efficiently from the output).

We now have all the necessary tools to prove Items 1 and 2 of Theorem 1.4. Let us start with the former. Recall that we need to show that either $\mathcal{Y}_x = \{\mathcal{Y}\}$ for all x, or $\mathcal{Z}_x = \{\mathcal{Z}\}$ for all x. First, since f is ternary-output, for every x it holds that either $\mathcal{Y}_x = \{\mathcal{Y}\}$ or $\mathcal{Z}_x = \{\mathcal{Z}\}$. Note that this is *weaker* than what we wish to show since for one x it might be the case that $\mathcal{Y}_x = \{\mathcal{Y}\}$, while for another x it might be the case that $\mathcal{Z}_x = \{\mathcal{Z}\}$. Let us assume that Item 1 of Theorem 1.4 does not hold. Then there exist x and x' such that $\mathcal{Y}_x \neq \{\mathcal{Y}\}$ and $\mathcal{Z}_{x'} \neq \{\mathcal{Z}\}$. Then, as argued above, there exists x^* such that $\mathcal{Y}_{x^*}$ refines $\mathcal{Y}_x$ and $\mathcal{Z}_{x^*}$ refines $\mathcal{Z}_{x'}$. However, this implies that $\mathcal{Y}_{x^*} \neq \{\mathcal{Y}\}$ and $\mathcal{Z}_{x^*} \neq \{\mathcal{Z}\}$, which is impossible for ternary-output functions.

We now prove Item 2 of Theorem 1.4. From here on, we will only focus on the first (i.e., left) entry in each of the above 6 distributions (there is no need to consider the second entry anymore). The proof follows similar ideas to that of the NIORP case. In more detail, we consider the $\text{CORE}_\wedge$-partition of the inputs $\mathcal{Y}_\wedge$ and $\mathcal{Z}_\wedge$, and we show that changing, say, z to any $\tilde{z}$ that belongs to the same equivalence class $\mathcal{Z}_j^\wedge \in \mathcal{Z}_\wedge$, does not change the distribution. Let us first recall the definition of $\text{CORE}_\wedge$-partition. We define $\mathcal{Z}_\wedge$ to be the meet of the partitions $\{\mathcal{Z}_x\}_{x\in\mathcal{X}}$, defined as

$$\mathcal{Z}_\wedge := \left\{ \mathcal{Z}^\wedge \subseteq \mathcal{Z} : \mathcal{Z}^\wedge \neq \emptyset, \text{ and } \forall x \in \mathcal{X}\ \exists \mathcal{Z}_x \in \mathcal{Z}_x \text{ s.t. } \mathcal{Z}^\wedge = \bigcap_{x\in\mathcal{X}} \mathcal{Z}_x \right\}.$$

For the sake of brevity, we will abuse notations and let S_3 only output the first entry rather than two values.

First observe that if $z, \tilde{z} \in \mathcal{Z}_j^\wedge$ for some $j \in [m_\wedge]$, then for any x there exists $j_x \in [m(x)]$ such that $z, \tilde{z} \in \mathcal{Z}_{j_x}^x$. Then, a similar analysis to the NIORP case shows that for any *fixed* $x_3^* \in \mathcal{X}$ satisfying $\mathcal{Z}_{x_3^*}$ refines $\mathcal{Z}_x$, it holds that

$$\mathsf{S}_3(x, x', y, z', x_3^*, f(x_3^*, y', z)) \equiv \mathsf{S}_3(x, x', y, z', x_3^*, f(x_3^*, y', \tilde{z})).$$

As the support of $P^3_{x,x',y,z'}$ is contains only those x_3^* where $\mathcal{Z}_{x_3^*}$ refines $\mathcal{Z}_x$, it follows that

$$\mathsf{S}_3(x, x', y, z', x_3^*, f(x_3^*, y', z)) \equiv \mathsf{S}_3(x, x', y, z', x_3^*, f(x_3^*, y', \tilde{z})),$$

where $x_3^* \leftarrow P^3_{x,x',y,z'}$. Therefore, the same must hold for all of the six distributions, i.e., they depend on the equivalence classes of y and z with respect to the $\text{CORE}_\wedge$-partition, rather than depending on the actual values themselves.

In the following we let x_0, y_0, and z_0 be the lexicographically smallest elements of $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, respectively. For $i \in [n_\wedge]$ let $Q_i'' := Q_{x_0,y_i,y_0,z_0}$, where y_i is the lexicographically smallest elements

of $\mathcal{Y}_i^\wedge$. Similarly, for $j \in [m_\wedge]$ we let $R''_j := R'_{x_0,y_0,z_j,z_0}$, where z_j is the lexicographically smallest element of $\mathcal{Z}_j^\wedge$. Then, similarly to the NIORP case, it follows that for all $i \in [n_\wedge]$, all $j \in [m_\wedge]$, all $x \in \mathcal{X}$, all $y \in \mathcal{Y}_i^\wedge$, and all $z \in \mathcal{Z}^\wedge$, it holds that

$$f(x, y^*, z) \equiv f(x, y, z^*), \tag{7}$$

where $y^* \leftarrow Q''_i$ and $z^* \leftarrow R''_j$. Note that the proof of Equation (7) did not use the fact that f is ternary-output (see Claim 4.7 for a formal treatment of the general case).

It is left to show the existence of an algorithm S that given x, x^* sampled from an appropriate distribution P_x, and $f(x, y, z)$ can generate the distribution in Equation (7). Here we use the fact that we showed that for ternary-output functions, either $\mathcal{Y}_x = \{\mathcal{Y}\}$ for all $x \in \mathcal{X}$, or $\mathcal{Z}_x = \{\mathcal{Z}\}$ for all $x \in \mathcal{X}$. Assume first the former. In this case we let $\mathsf{S}(x, x^*, w) = \mathsf{S}_3(x, x_0, y_0, z_0, x^*, w)$. Then, for $P_x := P^3_{x,x_0,y_0,z_0}$ it holds that

$$\mathsf{S}\left(x, x^*, f(x^*, y, z)\right) \equiv \mathsf{S}_3\left(x, x_0, y_0, z_0, x^*, f(x^*, y, z)\right) \equiv f(x, y^*, z) \equiv f(x, y, z^*),$$

where $x^* \leftarrow P_x$, $y^* \leftarrow Q''_1$ (recall we assume that $\mathcal{Y}_x = \{\mathcal{Y}\}$ for all x which implies that $n_\wedge = 1$), and $z^* \leftarrow R''_j$, as claimed. Now, if we assume that $\mathcal{Z}_x = \{\mathcal{Z}\}$ for all $x \in \mathcal{X}$, we will define $\mathsf{S}(x, x^*, w)$ using S_6 rather than S_3. In more details, we let $\mathsf{S}(x, x^*, w) = \mathsf{S}_6(x, x_0, y_0, z_0, x^*, w)$. Then, for $P_x := P^6_{x,x_0,y_0,z_0}$ it holds that

$$\mathsf{S}\left(x, x^*, f(x^*, y, z)\right) \equiv \mathsf{S}_6\left(x, x_0, y_0, z_0, x^*, f(x^*, y, z)\right) \equiv f(x, y^*, z) \equiv f(x, y, z^*),$$

where $x^* \leftarrow P_x$, $y^* \leftarrow Q''_i$, and $z^* \leftarrow R''_1$ (recall we assume that $\mathcal{Z}_x = \{\mathcal{Z}\}$ for all x which implies that $m_\wedge = 1$), as claimed.

The positive direction for ternary-output functionalities. We now turn to the positive direction. Here we show that the protocol suggested by [2] securely computes f. Roughly speaking, in their protocol, in case an attack is detected (without the identity of the attacker being revealed) party A interacts either B or C while ignoring the other party, where the decision is based only on the function being computed (this is done even if the ignored party is honest).[12]

However, in [2], determining which party should interact with A (given the function f) is rather difficult. In contrast, as we show below, in our setting this is only determined by Item 1. Specifically, if $\mathcal{Y}_x = \{\mathcal{Y}\}$ for all $x \in \mathcal{X}$ then A interacts with C, and if $\mathcal{Z}_x = \{\mathcal{Z}\}$ for all $x \in \mathcal{X}$ then A interacts with B. In fact, if both $\mathcal{Y}_x = \{\mathcal{Y}\}$ and $\mathcal{Z}_x = \{\mathcal{Z}\}$ hold for all $x \in \mathcal{X}$, then A does not interact with any party in case of an attack. Additionally, in this case, the assumption of the existence of the algorithm S and the distributions $\{P_x\}_{x\in\mathcal{X}}$ is made redundant.

We next present the protocol. We assume without loss of generality that $\mathcal{Z}_x = \{\mathcal{Z}\}$ for all $x \in \mathcal{X}$. First, similarly to the NIORP case, by the honest-majority assumption, the parties can compute f fairly. If the parties receive an output, they can terminate; otherwise, similarly to [2] we let A and B compute the two-party functionality $f(x, y, z^*)$, where $z^* \leftarrow R_1$, ignoring C in the process (recall that since $\mathcal{Z}_x = \{\mathcal{Z}\}$ for all $x \in \mathcal{X}$ there is only one distribution given by Item 2 of Theorem 1.4).

[12] For the general case, where the domain of f is not constant, the protocol we use is a slight generalization of the one suggested by [2]. Specifically, the decision of whether A interacts with B or C in case of an attack depends on the security parameter κ. Assuming the domain of f is of polynomial size in κ, the decision can be computed efficiently and locally by every party.

Similarly to the NIORP case, correctness holds due to the correctness of the fair protocol. Furthermore, it is clear that a corrupt C cannot attack the protocol. Indeed, it does not gain any information in the fair computation of f; hence, if it aborts in this phase then the output of A is $g(x,y) = f(x,y,z^*)$, where $z^* \leftarrow R_1$. Similarly, a corrupt B cannot attack the protocol since its simulator can send $y^* \leftarrow Q_i$, where $i \in [n_\wedge]$ is such that $y \in \mathcal{Y}_i^\wedge$. By Item 2 of Theorem 1.4 the output of A in the ideal world is

$$f(x,y^*,z) \equiv f(x,y,z^*) \equiv g(x,y),$$

where $y^* \leftarrow Q_i$ and $z^* \leftarrow R_1$.

Next, consider a corrupt A. Similarly to the previous two cases, we only need to consider the case where A aborts during the fair computation of f. Observe that the only information it receives is $g(x,y) = f(x,y,z^*)$, where $z^* \leftarrow R_1$. The simulator will simply send $x^* \leftarrow P_x$ to the trusted party and receive back w as the output. Then, the corrupt A will output whatever $\mathsf{S}(x,x^*,w)$ outputs. By Item 2 of Theorem 1.4, the simulator output is identically distributed as $g(x,y)$.

1.2.3 Reducing the Randomized Case to the Deterministic Case

We next explain how to reduce the randomized case to the deterministic case. The reduction works in both the positive and the negative directions. Thus, we obtain characterizations for randomized NIORP functionalities, and randomized ternary-output functionalities as well.

Recall that for a randomized solitary output three-party $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$, we define the deterministic solitary output three-party functionality

$$f'((x,r_1),(y,r_2),(z,r_3)) = f(x,y,z;r_1+r_2+r_3).$$

Namely, the parties hold a share of the randomness of f in a 3-out-of-3 secret sharing scheme. We next show that f can be securely computed in the point-to-point model if and only if f' can.

Let us first assume that f' can be securely computed. Then in order to compute f, the parties will compute f' with their original inputs, and where r_1, r_2, and r_3 are sampled uniformly at random. Security follows from the fact that at least one party is honest, hence either r_1, r_2, or r_3 are sampled uniformly at random.

Let us now assume that f can be securely computed. First, similarly to the previous protocols, by the honest-majority assumption, the parties can compute f' with fairness. If the parties receive an output, they can terminate; otherwise, they compute f on their respective inputs. Correctness is given by the fact that the parties first compute f' fairly. Security is guaranteed since the adversary obtains no information from the fair computation, and since the simulator can send a uniform random r as part of the input, in case the fair computation is aborted.

1.2.4 When a Broadcast Channel is Available

In this section, we show that any NIORP or ternary-output functionality that can be securely computed in the point-to-point model against a single corrupted party, can also be securely computed assuming a broadcast channel against two corruptions. Similarly to the point-to-point model, we will only handle deterministic functionalities, as the randomized case can be handled using the same reduction from Section 1.2.3.

The NIORP case. Let us start with describing a protocol for the NIORP functionalities captured by Theorem 1.2. Recall that for these functionalities there exist two families of efficiently samplable distributions $\{Q_i\}_{i\in[n]}$ and $\{R_j\}_{j\in[m]}$ such that the following holds. For all $i \in [n]$, $j \in [m]$, $y \in \mathcal{Y}_i$, and $z \in \mathcal{Z}_j$, it holds that

$$f(y^*, z) \equiv f(y, z^*),$$

where $y^* \leftarrow Q_i$ and $z^* \leftarrow R_j$.

In the following, we show that a larger class of functionalities than those described above, can be securely computed against two corruptions. Specifically, it suffices to assume the existence of only a single efficiently samplable distribution, one for B *or* one for C. By symmetry, we only consider the latter case. That is, we assume there exists $j^* \in [m]$ and there exists a distribution R_{j^*} over $\mathcal{Z}_{j^*}$ such that the following holds. For every $i \in [n]$ and every $y, y' \in \mathcal{Y}_i$ it holds that

$$f(y, z^*) \equiv f(y', z^*), \tag{8}$$

where $z^* \leftarrow R_{j^*}$.

The protocol is an extension of one of the protocols suggested by [25], and it proceeds as follows. First, the parties compute a 3-out-of-3 secret sharing of the output $f(x, y, z)$ using a secure-with-identifiable-abort protocol (i.e., the adversary can force an abort after obtaining the output but at the expense of revealing the identity of a corrupted party).[13] In case a single party aborts, the remaining two parties compute the function on their inputs and with the input of the aborting party set to a default value. Observe that since f is solitary output, this can be done securely using the protocol of Kilian [26]. If both B and C abort, then A outputs $f(x, y_0, z_0)$, where $y_0 \in \mathcal{Y}$ and $z_0 \in \mathcal{Z}$ are default inputs.

If no abort occurs, then first B sends its share to A, and additionally, it sends the (unique) index $i \in [n]$ such that $y \in \mathcal{Y}_i$. If B aborts, then A and C compute $f(x, y_0, z)$. Otherwise, C sends its share to A. If C aborts, then A outputs $f(x, y_i, z^*)$, where y_i is the lexicographically smallest element in $\mathcal{Y}_i$, and where $z^* \leftarrow R_{j^*}$.

Similarly to the point-to-point case, corrupting A will not provide the adversary with any information since the index i can be inferred from the output given by the trusted party. Additionally, B and C obtain no information from the execution, since their views contain only secret shares of the output. Furthermore, if a corrupt B aborts (at any point during the computation), then it can be simulated by sending y_0 to the trusted party. Finally, if a corrupt C aborts after B sent its share, then this attack can be simulated by sending $z^* \leftarrow R_{j^*}$ to the trusted party. Then the output of A in the ideal world is $f(x, y, z^*)$, while its output in the real world is $f(x, y_i, z^*)$. By Equation (8), the two distributions are identical.

The ternary-output case. We now turn to ternary-output functionalities. In fact, we show that a much larger class of functionalities than those captured by Theorem 1.4, can be securely computed. Similarly to the point-to-point case, the protocol we present securely computes non-ternary-output functionalities. First, recall that by Item 1 of Theorem 1.4, it holds that either $\mathcal{Y}_x = \{\mathcal{Y}\}$ for all $x \in \mathcal{X}$, or $\mathcal{Z}_x = \{\mathcal{Z}\}$ for all $x \in \mathcal{X}$. We assume the latter without loss of generality. We next present a relaxation of Item 2 of Theorem 1.4 that suffices for f to be securely

[13] Additionally, the shares are also signed using a MAC to ensure that B and C won't change their values. For simplicity, we assume that a malicious adversary does not modify these values, but can abort the execution.

computable against two corruptions. Specifically, we assume that there exist two distributions Q and R over $\mathcal{Y}_i^\wedge$ and $\mathcal{Z}$, respectively, for some $i \in [n_\wedge]$, such that the following holds. There exists $y_i \in \mathcal{Y}_i^\wedge$, such that for all $z \in \mathcal{Z}$, and $x \in \mathcal{X}$, it holds that

$$f(x, y^*, z) \equiv f(x, y_i, z^*), \tag{9}$$

where $y^* \leftarrow Q$ and $z^* \leftarrow R$. Observe that this is indeed a relaxation since we do not require the assumption of the existence of S, and the distributions $\{P_x\}_{x\in\mathcal{X}}$ and $\{Q_{i'}\}_{i'\in[n_\wedge]\setminus\{i\}}$, and since the quantifier over y_i is replaced with an existential quantifier.

The protocol proceeds as follows. The parties first compute a secret sharing of the output of $f(x, y, z)$ using a secure-with-identifiable-abort protocol. The sharing scheme is a 2-out-of-2 scheme, with the shares given only to A and B. Assuming the computation followed through, B sends its share to A, which reconstructs the output. If B aborts at any point in the computation, then A outputs $f(x, y_i, z^*)$ where $z^* \leftarrow R$. If C aborts during the secure-with-identifiable-abort computation, then its input is replaced with a default value and the protocol restarts.

Clearly, corrupting C will not provide the adversary with any advantage. Additionally, corrupting A and (possibly) B will provide the adversary with only the output. The only case left is when B is corrupted and A is honest. In this case, the adversary gains no information from the secure-with-identifiable-abort computation, since it obtains only one share of the output. Now, if B aborts then we let its simulator send to the trusted party the input $y^* \leftarrow Q$. Then A outputs $f(x, y^*, z)$ in the ideal world. On the other hand, in the real world, the output of A is $f(x, y_i, z^*)$. By Equation (9), the two distributions are identical.

1.3 Related Work

For non-solitary output functionalities, Cleve [13] showed that without an honest majority, full security cannot be achieved even for the simple task of fair coin-tossing (even with a broadcast channel). On the other hand, even if two-thirds of the parties are honest, there is no fully secure protocol for computing the broadcast functionality in the plain model (i.e., without setup/proof-of-work assumptions) [29, 27, 19].[14]

For the two-party setting a characterization was given for the set of two-party, Boolean, symmetric (i.e., where all parties receive the same output) functions over a constant size domain [24, 3, 28, 5]. The cases of asymmetric functions and of multiparty functions assuming broadcast but no honest majority, were also investigated [23, 5, 17, 25, 16], but both characterizations are open.

The hexagon argument has been first used in the context of Byzantine agreement to rule out three-party protocols tolerating one corruption [19]. Cohen et al. [15] considered *symmetric* (possibly randomized) functionalities in the point-to-point model, and showed that a symmetric n-party functionality f can be computed against t corruptions, if and only if f is $(n - 2t)$-dominated, i.e., there exists y^* such that any $n-2t$ of the inputs can fix the output of f to be y^*. They generalized the hexagon argument to the ring argument to obtain their results.

Recently, Alon et al. [2] extended the discussion to consider asymmetric functionalities in the point-to-point model. They provided various necessary and sufficient conditions for a functionality

[14] Note that if strictly more than two-thirds of the parties are honest any functionality can be computed with full security [9].

to be securely computable. They considered some interesting examples for the special case of solitary-output functionalities, however, provided no characterization for any class of functions.

The investigation of the set of solitary output functionalities that can be securely computed assuming a broadcast channel but no honest majority was initiated in the work of Halevi et al. [25]. They provided various negative and positive results, and further investigated the round complexity required to securely compute solitary output functionalities. Badrinarayanan et al. [6] investigated the round complexity required to compute solitary output functionalities, assuming the availability of a broadcast channel and no PKI, and vice versa.

1.4 Organization

The preliminaries and definition of the model of computation appear in Section 2. In Section 3 we state our results in the point-to-point model. Then, in Sections 4 and 5 we prove the negative and positive results, respectively. Finally, in Section 6 we state and prove our results assuming a broadcast channel.

2 Preliminaries

2.1 Notations

We use calligraphic letters to denote sets, uppercase for random variables and distributions, lowercase for values, and we use bold characters to denote vectors. For $n \in \mathbb{N}$, let $[n] = \{1, 2 \dots n\}$. For a set $\mathcal{S}$ we write $s \leftarrow \mathcal{S}$ to indicate that s is selected uniformly at random from $\mathcal{S}$. Given a random variable (or a distribution) X, we write $x \leftarrow X$ to indicate that x is selected according to X. A PPT algorithm is probabilistic polynomial time, and a PPTM is a polynomial time (interactive) Turing machine.

A function $\mu \colon \mathbb{N} \to [0,1]$ is called negligible, if for every positive polynomial $p(\cdot)$ and all sufficiently large n, it holds that $\mu(n) < 1/p(n)$. We write neg for an unspecified negligible function and write poly for an unspecified positive polynomial. For a randomized function (or an algorithm) f we write $f(x)$ to denote the random variable induced by the function on input x, and write $f(x;r)$ to denote the value when the randomness of f is fixed to r.

A *distribution ensemble* $X = \{X_{a,n}\}_{a \in \mathcal{D}_n, n \in \mathbb{N}}$ is an infinite sequence of random variables indexed by $a \in \mathcal{D}_n$ and $n \in \mathbb{N}$, where $\mathcal{D}_n$ is a domain that might depend on n. When the domains are clear, we will sometimes write $\{X_{a,n}\}_{a,n}$ in order to alleviate notations.

The statistical distance between two finite distributions is defined as follows.

Definition 2.1. *The statistical distance between two finite random variables X and Y is*

$$\mathrm{SD}(X,Y) = \max_{\mathcal{S}} \{\Pr[X \in \mathcal{S}] - \Pr[Y \in \mathcal{S}]\}.$$

For a function $\varepsilon : \mathbb{N} \to [0,1]$, the two ensembles $X = \{X_{a,n}\}_{a \in \mathcal{D}_n, n \in \mathbb{N}}$ and $Y = \{Y_{a,n}\}_{a \in \mathcal{D}_n, n \in \mathbb{N}}$ are said to be ε-close, if for all sufficiently large n and $a \in \mathcal{D}_n$, it holds that

$$\mathrm{SD}(X_{a,n}, Y_{a,n}) \leq \varepsilon(n),$$

and are said to be ε-far otherwise. X and Y are said to be statistically close*, denoted $X \overset{\mathrm{S}}{\equiv} Y$, if they are ε-close for some negligible function ε. If X and Y are 0-close then they are said to be equivalent, denoted $X \equiv Y$.*

Computational indistinguishability is defined as follows.

Definition 2.2. *Let $X = \{X_{a,n}\}_{a\in\mathcal{D}_n,n\in\mathbb{N}}$ and $Y = \{Y_{a,n}\}_{a\in\mathcal{D}_n,n\in\mathbb{N}}$ be two ensembles. We say that X and Y are computationally indistinguishable, denoted $X \overset{\text{C}}{\equiv} Y$, if for every non-uniform* PPT *distinguisher D, there exists a negligible function $\mu(\cdot)$, such that for all n and $a \in \mathcal{D}_n$, it holds that*

$$|\Pr[\mathsf{D}(X_{a,n}) = 1] - \Pr[\mathsf{D}(Y_{a,n}) = 1]| \leq \mu(n).$$

The following simple fact states that whenever two ensembles with polynomial-size supports are computationally indistinguishable, they are also statistically close.

Fact 2.3. *Let $X = \{X_{a,n}\}_{a\in\mathcal{D}_n,n\in\mathbb{N}}$ and $Y = \{Y_{a,n}\}_{a\in\mathcal{D}_n,n\in\mathbb{N}}$ be two computationally indistinguishable ensembles over a set-family $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$, of size $|\mathcal{S}_n| \leq \text{poly}(n)$. Then $X \overset{\text{S}}{\equiv} Y$.*

Proof sketch. Assume for the sake of contradiction that the claim is false. It follows that there exists a set-family $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ where $\Pr[X_{a,n} \in \mathcal{T}_n] - \Pr[Y_{a,n} \in \mathcal{T}_n] \geq 1/p(n)$, for some polynomial p. Since $\mathcal{T}_n \subseteq \mathcal{S}_n$, it follows that $\mathcal{T}_n$ is of polynomial size, hence it can be given as auxiliary input to a bounded distinguisher D. Then, D can distinguish X from Y by outputting 1 if its input belongs to $\mathcal{T}_n$, and outputting 0 otherwise, thus contradicting the assumption that $X \overset{\text{C}}{\equiv} Y$. □

The following fact states an equivalent definition for statistical distance.

Fact 2.4. *Let $X = \{X_{a,n}\}_{a\in\mathcal{D}_n,n\in\mathbb{N}}$ and $Y = \{Y_{a,n}\}_{a\in\mathcal{D}_n,n\in\mathbb{N}}$ be two ensembles. Then $X \overset{\text{S}}{\equiv} Y$ if and only if for every unbounded distinguisher D, there exists a negligible function $\mu(\cdot)$, such that for all n and $a \in \mathcal{D}_n$, it holds that*

$$|\Pr[\mathsf{D}(X_{a,n}) = 1] - \Pr[\mathsf{D}(Y_{a,n}) = 1]| \leq \mu(n).$$

Definition 2.5 (Minimal and minimum elements)**.** *Let $\mathcal{S}$ be a set and let $\preceq$ be a partial order over $\mathcal{S}$. An element $s \in \mathcal{S}$ is called minimal, if no other element is smaller than s, that is, for any $s' \in \mathcal{S}$, if $s' \preceq s$ then $s' = s$.*

An element $s \in \mathcal{S}$ is called minimum if it is smaller than any other element, that is, for any $s' \in \mathcal{S}$ it holds that $s \preceq s'$.

We next define a refinement of a partition of some set.

Definition 2.6 (Refinement of partitions)**.** *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of some set $\mathcal{S}$. We say that* $\mathcal{P}_1$ refines *$\mathcal{P}_2$, if for every $\mathcal{S}_1 \in \mathcal{P}_1$ there exists $\mathcal{S}_2 \in \mathcal{P}_2$ such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$.*

The meet of two partitions is the partition formed by taking all non-empty intersections. Formally, it is defined as follows.

Definition 2.7 (Meet of partitions)**.** *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of some set $\mathcal{S}$. The* meet *of $\mathcal{P}_1$ and $\mathcal{P}_2$, denoted $\mathcal{P}_1 \wedge \mathcal{P}_2$, is defined as*

$$\mathcal{P}_1 \wedge \mathcal{P}_2 := \{\mathcal{S}_1 \cap \mathcal{S}_2 \mid \forall i \in \{1,2\} : \mathcal{S}_i \in \mathcal{P}_i \text{ and } \mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset\}.$$

Observe that $\wedge$ is associative, thus we can naturally extend the definition for several partitions.

Definition 2.8 (Equivalence class and quotient sets)**.** *For an equivalence relation $\equiv$ over some set $\mathcal{S}$, and an element $s \in \mathcal{S}$ we denote by $[s]_\equiv$ the equivalence class of s, i.e.,*

$$[s]_\equiv := \{s' \in \mathcal{S} : s \equiv s'\}.$$

We let $\mathcal{S}/\equiv$ denote the quotient set with respect to $\equiv$ defined as the set of all equivalence classes. Stated differently, it is the partition of $\mathcal{S}$ induced by the equivalence relation $\equiv$.

2.2 The Model of Computation

We provide the basic definitions for secure multiparty computation according to the real/ideal paradigm, for further details see [21]. Intuitively, a protocol is considered secure if whatever an adversary can do in the real execution of the protocol, can be done also in an ideal computation, in which an uncorrupted trusted party assists the computation.

In this paper we consider solitary output three-party functionalities. A functionality is a sequence of function $f = \{f_\kappa\}_{\kappa\in\mathbb{N}}$, where $f_\kappa \colon \mathcal{X}_\kappa \times \mathcal{Y}_\kappa \times \mathcal{Z}_\kappa \to \mathcal{W}_\kappa$ for every $\kappa \in \mathbb{N}$.[15] The functionality is called solitary output if only one party obtains an output. We denote the parties by A, B and C, holding inputs x, y, and z, respectively, and let A receive the output, denoted w. To alleviate notations, we will remove κ from f and its domain and range, and simply write it as $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$.

Although we focus on solitary output functionalities and deal with adversaries that corrupt a single party, we present the definition for the general case, as it will be useful later.

The Real Model

A three-party protocol π is defined by a set of three PPT interactive Turing machines A, B, and C. Each Turing machine (party) holds at the beginning of the execution the common security parameter 1^κ, a private input, and random coins. The *adversary* $\mathcal{A}$ is another PPT interactive Turing machine describing the behavior of the corrupted parties. It starts the execution with input that contains the identities of the corrupted parties, their inputs, and an additional auxiliary input aux.

The parties execute the protocol over a synchronous network. That is, the execution proceeds in rounds: each round consists of a *send phase* (where parties send their messages for this round) followed by a *receive phase* (where they receive messages from other parties).

We consider a fully connected point-to-point network, where every pair of parties is connected by a communication line. We will consider the *secure-channels* model, where the communication lines are assumed to be ideally private (and thus the adversary cannot read or modify messages sent between two honest parties). Depending on the context, we may assume the parties have access to a broadcast channel. We note that our upper bounds (protocols) can also be stated in the *authenticated-channels* model, where the communication lines are assumed to be ideally authenticated but not private (and thus the adversary cannot modify messages sent between two honest parties but can read them) via standard techniques, assuming public-key encryption. On the other hand, stating our lower bounds assuming secure channels will provide stronger results.

Throughout the execution of the protocol, all the honest parties follow the instructions of the prescribed protocol, whereas the corrupted parties receive their instructions from the adversary. The adversary is considered to be *malicious*, meaning that it can instruct the corrupted parties to deviate from the protocol in any arbitrary way. Additionally, the adversary has full access to the view of the corrupted parties, which consists of their inputs, their random coins, and the messages they see throughout this execution. At the conclusion of the execution, the honest parties output their prescribed output from the protocol, the corrupted parties output nothing, and the adversary outputs a function of its view. In some of our proofs, we consider *semi-honest* adversaries that always instruct the corrupted parties to honestly execute the protocol but may try to learn more information than they should.

[15] The typical convention in secure computation is to let $f : (\{0,1\}^*)^3 \to \{0,1\}^*$. However, we will mostly be dealing with functionalities whose domain is of polynomial size in κ, which is why we introduce this notation.

We next define the real-world global view for security parameter $\kappa \in \mathbb{N}$, inputs $x, y, z \in \{0,1\}^*$, and an auxiliary input $\mathsf{aux} \in \{0,1\}^*$ with respect to some adversary $\mathcal{A}$ controlling a subset $\mathcal{I} \subseteq \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ of the parties. Let $\text{OUT}^{\mathsf{real}}_{\pi,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z))$ denote the outputs of the honest parties in a random execution of π on inputs (x,y,z) and security parameter κ interacting with $\mathcal{A}$ with auxiliary input aux corrupting the parties in $\mathcal{I}$. Further let $\text{VIEW}^{\mathsf{real}}_{\pi,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z))$ be the adversary's output, being a function of its view (i.e., its auxiliary input, its random coins, the input of the corrupted party, and the messages it sees during the execution of the protocol) during an execution of π. We denote the global view in the real model by

$$\text{REAL}_{\pi,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z)) = \left(\text{VIEW}^{\mathsf{real}}_{\pi,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z)),\ \text{OUT}^{\mathsf{real}}_{\pi,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z))\right).$$

The Ideal Model

We consider an ideal computation with *guaranteed output delivery* (also referred to as *full security*), where a trusted party performs the computation on behalf of the parties, and the ideal-model adversary *cannot* abort the computation. An ideal computation of a three-party functionality $f = (f_1, f_2, f_3)$, with $f_1, f_2, f_3 : (\{0,1\}^*)^3 \to \{0,1\}^*$, on inputs $x, y, z \in \{0,1\}^*$ and security parameter κ, with an ideal-world adversary $\mathcal{A}$ running with an auxiliary input aux and corrupting a subset $\mathcal{I} \subseteq \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ of the parties, proceeds as follows:

Parties send inputs to the trusted party: Each honest party sends its input to the trusted party. The adversary $\mathcal{A}$ sends a value v from its domain as the input for the corrupted party. Let (x', y', z') denote the inputs received by the trusted party.

The trusted party performs computation: The trusted party selects a random string r, computes $(w_1, w_2, w_3) = f(x', y', z'; r)$, and sends w_1 to A, sends w_2 to B, and sends w_3 to C.

Outputs: Each honest party outputs whatever output it received from the trusted party and the corrupted party outputs nothing. The adversary $\mathcal{A}$ outputs some function of its view (i.e., the auxiliary input, its randomness, and the input and output of the corrupted party).

We next define the ideal-world global view for security parameter $\kappa \in \mathbb{N}$, inputs $x, y, z \in \{0,1\}^*$, and an auxiliary input $\mathsf{aux} \in \{0,1\}^*$ with respect to some adversary $\mathcal{A}$ controlling a subset $\mathcal{I}$ of the parties. Let $\text{OUT}^{\mathsf{ideal}}_{f,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z))$ denote the output of honest parties in a random execution of the above ideal-world process, interacting with $\mathcal{A}$. Further let $\text{VIEW}^{\mathsf{ideal}}_{f,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z))$ be the *output* (a simulated view) of $\mathcal{A}$ in such a process. We denote the global view in the ideal model by

$$\text{IDEAL}_{f,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z)) = \left(\text{VIEW}^{\mathsf{ideal}}_{f,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z)),\ \text{OUT}^{\mathsf{ideal}}_{f,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z))\right).$$

The Security Definition

Having defined the real and ideal models, we can now define security of protocols according to the real/ideal paradigm.

Definition 2.9 (Malicious security)**.** *Let f be a three-party functionality and let π be a three-party protocol. For $t \in \{1,2\}$, we say that π computes f with* computational t-security*, if for every non-uniform* PPT *adversary $\mathcal{A}$, controlling a subset $\mathcal{I} \subseteq \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ of size at most t in the real world,*

there exists a non-uniform PPT *adversary* Sim*, controlling the same subset* $\mathcal{I}$ *in the ideal-world such that*

$$\left\{\text{IDEAL}_{f,\mathsf{Sim}(\mathsf{aux})}\left(\kappa,(x,y,z)\right)\right\}_{\kappa\in\mathbb{N},x,y,z,\mathsf{aux}\in\{0,1\}^*} \stackrel{\text{C}}{\equiv} \left\{\text{REAL}_{\pi,\mathcal{A}(\mathsf{aux})}\left(\kappa,(x,y,z)\right)\right\}_{\kappa\in\mathbb{N},x,y,z,\mathsf{aux}\in\{0,1\}^*}.$$

We define statistical t*-security similarly, by replacing computational indistinguishability with statistical distance.*

When $t=2$ *we will sometimes say that* π *computes* f *will* full security*.*

Ideal computation with fairness. Although all our results are stated with respect to guaranteed output delivery, in our proofs we will consider a weaker security variant, where the adversary may cause the computation to prematurely abort, but only before it learns any new information from the protocol. Formally, security with fairness is defined by only modifying the ideal computation. Specifically, the difference is that during the Parties send inputs to the trusted party step, the adversary can send a special abort symbol. In this case, the trusted party sends $\perp$ to all parties instead of computing the function.

Ideal computation with security-with-identifiable-abort. We also use a security notion called security-with-identifiable-abort where, similarly to fairness,the adversary can cause the computation to prematurely abort. However, it can do so after learning the output, at the expense of revealing the identity of a corrupted party (see Appendix A for a formal definition).

The Hybrid Model

The *hybrid model* is a model that extends the real model with a trusted party that provides ideal computation for specific functionalities. The parties communicate with this trusted party in exactly the same way as in the ideal models described above.

Let f be a functionality. Then, an execution of a protocol π computing a functionality g in the f-hybrid model involves the parties sending normal messages to each other (as in the real model) and in addition, having access to a trusted party computing f. It is essential that the invocations of f are done sequentially, meaning that before an invocation of f begins, the preceding invocation of f must finish. In particular, there is at most a single call to f per round, and no other messages are sent during any round in which f is called.

Let $\mathsf{type}\in\{\mathsf{g.o.d.},\mathsf{fair},\mathsf{id\text{-}abort}\}$, and let $\mathcal{A}$ be a non-uniform PPT machine with auxiliary input aux controlling a subset of the parties. We denote by $\text{HYBRID}^{f,\mathsf{type}}_{\pi,\mathcal{A}(\mathsf{aux})}(\kappa,(x,y,z))$ the random variable consisting of the view of the adversary and the output of the honest parties, following an execution of a protocol π with ideal calls to a trusted party computing f according to the ideal model "type," on input vector (x,y,z), auxiliary input aux to $\mathcal{A}$, and security parameter κ. We call this the (f,type)-hybrid model.

The sequential composition theorem of Canetti [12] states the following. Let ρ be a protocol that securely computes f in the ideal model "type." Then, if a protocol π computes g in the (f,type)-hybrid model, then the protocol π^{ρ}, that is obtained from π by replacing all ideal calls to the trusted party computing f with the protocol ρ, securely computes g in the real model.

Theorem 2.10 ([12])**.** *Let* $t\in\{1,2\}$*, let* f *be a three-party functionality, let* $\mathsf{type}_1,\mathsf{type}_2\in\{\mathsf{g.o.d.},\mathsf{fair},\mathsf{id\text{-}abort}\}$*, let* ρ *be a protocol that* t*-securely computes* f *with* type_1*, and let* π *be a*

protocol that t-securely computes g with type_2 *in the* (f, type_1)*-hybrid model. Then, protocol* π^ρ *t-securely computes g with* type_2 *in the real model.*

We make use of a known fact stating that any functionality can be computed fairly assuming an honest majority.

Fact 2.11. *Let f be a three-party functionality. Then* (f, fair) *can be computed with statistical 1-security.*

This follows from the results of [30] and [14]. Specifically, Rabin and Ben-Or [30] showed how to compute any functionality with full security assuming an honest majority and a *broadcast* channel. On the other hand, Cohen and Lindell [14] showed that any protocol computing some functionality f with full security assuming a broadcast channel can be transformed into a protocol computing f fairly over a point-to-point network without the use of broadcast.

3 Our Main Results in the Point-to-Point Model

In this section, we present the statement of our main results in the point-to-point model. We present a necessary condition and two sufficient conditions for solitary output three-party functionalities with polynomial-sized domains, that can be computed with 1-security without broadcast. In Section 3.2.1, we present several corollaries of our results. In particular, we show that various interesting families of functionalities, such as deterministic NIORP and (possibly randomized) ternary-output functionalities, our necessary and sufficient conditions are equivalent, thus we obtain a characterization.

3.1 Useful Definitions

Before stating the result, we first present several important definitions. Throughout the entire subsection, we let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality.

The first definition introduces an equivalence relation over the domains $\mathcal{Y}$ and $\mathcal{Z}$ with respect to any fixed input $x \in \mathcal{X}$. We call this relation the ***c**ommon **o**utput **r**elation* (CORE). Note that the relation depends on the security parameter κ as well. We will not write κ as part of the notations in order to alleviate them.

Definition 3.1 (CORE and CORE partition)**.** *For an input* $x \in \mathcal{X}$ *we define the relation* $\sim_x$ *over* $\mathcal{Y}$ *as follows.*

$$y \sim_x y' \text{ if there exist } z, z' \in \mathcal{Z} \text{ such that } f(x, y, z) = f(x, y', z').$$

We define relation $\equiv_x$*, called* CORE*, to be the* transitive closure *of* $\sim_x$*, i.e.,* $y \equiv_x y'$ *if either* $y \sim_x y'$ *or if there exist* $y_1, \ldots, y_k \in \mathcal{Y}$ *such that*

$$y \sim_x y_1 \sim_x \ldots \sim_x y_k \sim_x y'.$$

Observe that $\equiv_x$ *is an equivalence relation. We let* $\mathcal{Y}_x$ *denote the set of equivalence classes of* $\mathcal{Y}$ *formed by* $\equiv_x$*. We also abuse notations, and define the relations* $z \sim_x z'$ *and* $z \equiv_x z'$ *over* $\mathcal{Z}$ *similarly, and let* $\mathcal{Z}_x$ *denote the set of equivalence classes over* $\mathcal{Z}$ *formed by* $\equiv_x$*.*

Additionally, we denote $n(x) = |\mathcal{Y}_x|$, $m(x) = |\mathcal{Z}_x|$, *and we write*

$$\mathcal{Y}_x = \{\mathcal{Y}_i^x : i \in [n(x)]\} \quad \text{and} \quad \mathcal{Z}_x = \{\mathcal{Z}_j^x : j \in [m(x)]\}.$$

Finally, we let

$$\mathcal{R}_x = \{\mathcal{Y}_i^x \times \mathcal{Z}_j^x : i \in [n(x)], j \in [m(x)]\}$$

be the partition of $\mathcal{Y} \times \mathcal{Z}$ *into the combinatorial rectangles formed by* $\mathcal{Y}_x$ *and* $\mathcal{Z}_x$. *We call* $\mathcal{Y}_x$, $\mathcal{Z}_x$, *and* $\mathcal{R}_x$ *the* CORE partitions of f with respect to x.

We next introduce equivalence relations over $\mathcal{X}$ that correspond to the CORE partitions formed by the inputs. In addition, we define partial orders over the quotient sets associated with these equivalence relations. Roughly, both the equivalence relations and the partial orders are defined by comparing the corresponding CORE partitions. Similarly to Definition 3.1, the following definition also depends κ, which is omitted from the notations to alleviate them.

Definition 3.2 (Equivalence relations and partial orders over $\mathcal{X}$)**.** *We define three equivalence relations* $\equiv_{\mathsf{B}}$, $\equiv_{\mathsf{C}}$, *and* $\equiv$, *over* $\mathcal{X}$ *as follows.*

- *We say that* $x \equiv_{\mathsf{B}} x'$ *if* $\mathcal{Y}_x = \mathcal{Y}_{x'}$.
- *We say that* $x \equiv_{\mathsf{C}} x'$ *if* $\mathcal{Z}_x = \mathcal{Z}_{x'}$
- *We say that* $x \equiv x'$ *if* $\mathcal{R}_x = \mathcal{R}_{x'}$. *Equivalently,* $x \equiv x'$ *if* $x \equiv_{\mathsf{B}} x'$ *and* $x \equiv_{\mathsf{C}} x'$.

We define partial orders $\preceq_{\mathsf{B}}$, $\preceq_{\mathsf{C}}$, *and* $\preceq$ *over the quotient sets* $\mathcal{X}/\equiv_{\mathsf{B}}$, $\mathcal{X}/\equiv_{\mathsf{C}}$, *and* $\mathcal{X}/\equiv$, *respectively, as follows.*

- *We say that* $[x]_{\equiv_{\mathsf{B}}} \preceq_{\mathsf{B}} [x']_{\equiv_{\mathsf{B}}}$ *if* $\mathcal{Y}_x$ *refines* $\mathcal{Y}_{x'}$.
- *We say that* $[x]_{\equiv_{\mathsf{C}}} \preceq_{\mathsf{C}} [x']_{\equiv_{\mathsf{C}}}$ *if* $\mathcal{Z}_x$ *refines* $\mathcal{Z}_{x'}$.
- *We say that* $[x]_{\equiv} \preceq [x']_{\equiv}$ *if* $\mathcal{R}_x$ *refines* $\mathcal{R}_{x'}$. *Equivalently,* $[x]_{\equiv} \preceq [x']_{\equiv}$ *if* $[x]_{\equiv_{\mathsf{B}}} \preceq_{\mathsf{B}} [x']_{\equiv_{\mathsf{B}}}$ *and* $[x]_{\equiv_{\mathsf{C}}} \preceq_{\mathsf{C}} [x']_{\equiv_{\mathsf{C}}}$.

For brevity, we write the partial orders as if they are over $\mathcal{X}$, *e.g., we write* $x \preceq_{\mathsf{B}} x'$ *instead of* $[x]_{\equiv_{\mathsf{B}}} \preceq_{\mathsf{B}} [x']_{\equiv_{\mathsf{B}}}$.[16] *Finally,* $\chi \in \mathcal{X}$ *is called* B*-minimal if* $[\chi]_{\equiv_{\mathsf{B}}}$ *is minimal with respect to* $\preceq_{\mathsf{B}}$, χ *is called* C*-minimal if* $[\chi]_{\equiv_{\mathsf{C}}}$ *is minimal with respect to* $\preceq_{\mathsf{C}}$, *and* χ *is called* R*-minimal if* $[\chi]_{\equiv}$ *is minimal with respect to* $\preceq$.

As mentioned in Section 1, we are interested in the meet of all CORE partitions. We call this new partition the CORE$_\wedge$-partition of f. Similarly to previous notations, CORE$_\wedge$-partition also depends on κ, and we will omit it for brevity.

Definition 3.3 (CORE$_\wedge$-partition)**.** *We denote*

$$\mathcal{Y}_\wedge := \bigwedge_{x \in \mathcal{X}} \mathcal{Y}_x = \bigwedge_{\substack{\chi \in \mathcal{X}: \\ \chi \text{ is } \mathsf{R}\text{-minimal}}} \mathcal{Y}_\chi \quad \text{and} \quad \mathcal{Z}_\wedge := \bigwedge_{x \in \mathcal{X}} \mathcal{Z}_x = \bigwedge_{\substack{\chi \in \mathcal{X}: \\ \chi \text{ is } \mathsf{R}\text{-minimal}}} \mathcal{Z}_\chi,$$

[16] Note that if we had defined $\preceq_{\mathsf{B}}$, $\preceq_{\mathsf{C}}$, and $\preceq$ directly over $\mathcal{X}$, then they would not correspond to partial orders. Indeed, for the relations to be partial orders, it required that they are antisymmetric, i.e., if $x \preceq x'$ and $x' \preceq x$ then $x = x'$. Observe that this is not generally the case, as the only guarantee we have is that $x \equiv x'$.

and call these two partitions the $\mathrm{CORE}_\wedge$-partitions *of* f. *We let* $n_\wedge = |\mathcal{Y}_\wedge|$ *and* $m_\wedge = |\mathcal{Z}_\wedge|$, *and we write the partitions as*

$$\mathcal{Y}_\wedge := \{\mathcal{Y}_i^\wedge : i \in [n_\wedge]\} \quad \textit{and} \quad \mathcal{Z}_\wedge := \{\mathcal{Z}_j^\wedge : j \in [m_\wedge]\}.$$

Finally, we let

$$\mathcal{R}_\wedge = \{\mathcal{Y}_i^\wedge \times \mathcal{Z}_j^\wedge : i \in [n_\wedge], j \in [m_\wedge]\},$$

be the partition of $\mathcal{Y} \times \mathcal{Z}$ *into the combinatorial rectangles formed by* $\mathcal{Y}_\wedge$ *and* $\mathcal{Z}_\wedge$.

The partitions $\mathcal{Y}_\wedge$ *and* $\mathcal{Z}_\wedge$ *are naturally associated with an equivalence relation* $\equiv_\wedge$ *over* $\mathcal{Y}$ *and over* $\mathcal{Z}$, *respectively: We say that* $y \equiv_\wedge y'$ *if there exists* $\mathcal{Y}^\wedge \in \mathcal{Y}_\wedge$ *such that* $y, y' \in \mathcal{Y}^\wedge$. *Equivalently,* $y \equiv_\wedge y'$ *if* $y \equiv_\chi y'$ *for all* R*-minimal* $\chi \in \mathcal{X}$. *Similarly,* $z \equiv_\wedge z'$ *if there exists* $\mathcal{Z}^\wedge \in \mathcal{Z}_\wedge$ *such that* $z, z' \in \mathcal{Z}^\wedge$.

We next define an important special property of a functionality f, which we call $\mathrm{CORE}_\wedge$*-forced*. This property plays a central role in both our positive and negative results, and generalizes the forced property defined in [25], which states that any party can fix the distribution of the output, using an appropriate distribution over its input.

Roughly, f is called $\mathrm{CORE}_\wedge$-forced if both B and C can each associate a distribution to each set in the $\mathrm{CORE}_\wedge$-partition of their respective set of inputs, such that the output distribution of A in each combinatorial rectangle in $\mathcal{R}_\wedge$ is fixed for every input $x \in \mathcal{X}$.

Definition 3.4 ($\mathrm{CORE}_\wedge$-forced)**.** *The function* f *is said to be* $\mathrm{CORE}_\wedge$-forced *if there exist two ensembles of efficiently samplable distributions* $\mathcal{Q} = \{Q_{\kappa,i}\}_{\kappa\in\mathbb{N},i\in[n_\wedge]}$ *and* $\mathcal{R} = \{R_{\kappa,j}\}_{\kappa\in\mathbb{N},j\in[m_\wedge]}$ *over* $\mathcal{Y}$ *and* $\mathcal{Z}$, *respectively, such that the following holds.*

$$\begin{aligned}\{f(x, y^*, z_j)\}_{\kappa\in\mathbb{N},x\in\mathcal{X},i\in[n_\wedge],j\in[m_\wedge],y\in\mathcal{Y}_i^\wedge,z\in\mathcal{Z}_j^\wedge} &\overset{\mathrm{S}}{\equiv} \{f(x, y^*, z)\}_{\kappa\in\mathbb{N},x\in\mathcal{X},i\in[n_\wedge],j\in[m_\wedge],y\in\mathcal{Y}_i^\wedge,z\in\mathcal{Z}_j^\wedge} \\ &\overset{\mathrm{S}}{\equiv} \{f(x, y, z^*)\}_{\kappa\in\mathbb{N},x\in\mathcal{X},i\in[n_\wedge],j\in[m_\wedge],y\in\mathcal{Y}_i^\wedge,z\in\mathcal{Z}_j^\wedge} \\ &\overset{\mathrm{S}}{\equiv} \{f(x, y_i, z^*)\}_{\kappa\in\mathbb{N},x\in\mathcal{X},i\in[n_\wedge],j\in[m_\wedge],y\in\mathcal{Y}_i^\wedge,z\in\mathcal{Z}_j^\wedge},\end{aligned}$$

where $y^* \leftarrow Q_{\kappa,i}$, $z^* \leftarrow R_{\kappa,j}$, *and where* y_i *and* z_j *are the lexicographically smallest elements in* $\mathcal{Y}_i^\wedge$ *and* $\mathcal{Z}_j^\wedge$, *respectively.*

Remark 3.5. *Though our lowerbound shows that any securely computable solitary output functionality must be* $\mathrm{CORE}_\wedge$*-partition, this can be strengthened as follows. Instead of requiring that every rectangle in* $\mathcal{R}_\wedge$ *is fixed for every* x, *it suffices to consider the meet of partitions formed by the CORE partitions with respect to all* R*-minimal elements that are smaller than* x, *i.e.,* $\bigwedge_{\chi\preceq x:\chi \text{ is } \mathsf{R}\text{-minimal}} \mathcal{R}_\chi$. *Then our lowerbound shows that for any* x, *the output distributions in the above collections of rectangles are fixed.*

3.2 Our Main Results

We are now ready to state our results, providing both sufficient and necessary conditions for a deterministic solitary output three-party functionalities with polynomial-sized domain, to be computable with 1-security over point-to-point channels. The result for randomized functionalities, where the domain of the randomness is polynomial as well, is handled below in Proposition 3.10 by reducing it to the deterministic case. We start by stating our negative results.

Theorem 3.6. *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality. Assume that $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \mathrm{poly}(\kappa)$. If f can be computed with computational 1-security, then the following hold.*

1. *For all sufficiently large $\kappa \in \mathbb{N}$, all B-minimal χ_{B} and all C-minimal χ_{C}, there exists an R-minimal $\chi \in \mathcal{X}$ such that $\chi_{\mathsf{B}} \equiv_{\mathsf{B}} \chi \equiv_{\mathsf{C}} \chi_{\mathsf{C}}$.*

2. *f is $\mathrm{CORE}_{\wedge}$-forced.*

Moreover, suppose that f has the property that for all sufficiently large κ, it holds that either $y \equiv_x y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, or $z \equiv_x z'$ for all $x \in \mathcal{X}$ and $z, z' \in \mathcal{Z}$. Then there exists an ensemble of efficiently samplable distributions $\mathcal{P} = \{P_{\kappa,x}\}_{\kappa \in \mathbb{N}, x \in \mathcal{X}}$ and there exists a PPT algorithm S such that

$$\{\mathsf{S}\,(1^\kappa, x, x^*, f(x^*, y, z))\}_{\kappa \in \mathbb{N}, x \in \mathcal{X}, i \in [n_\wedge], j \in [m_\wedge], y \in \mathcal{Y}_i^\wedge, z \in \mathcal{Z}_j^\wedge} \stackrel{\mathrm{S}}{\equiv} \{f(x, y^*, z)\}_{\kappa \in \mathbb{N}, x \in \mathcal{X}, i \in [n_\wedge], j \in [m_\wedge], y \in \mathcal{Y}_i^\wedge, z \in \mathcal{Z}_j^\wedge},$$

where $x^* \leftarrow P_{\kappa,x}$ and $y^* \leftarrow Q_{\kappa,i}$, where $Q_{\kappa,i}$ is the distribution given the $\mathrm{CORE}_{\wedge}$-forced property.

The proof is given in Section 4.

We now state our two positive results. The first positive result considers functionalities that satisfy the property given in the "moreover" part of Theorem 3.6. Specifically, we get a characterization (see Corollary 3.8 below) for when such functionalities can be computed securely. Interestingly, the protocol used in the proof of the theorem below is a slight generalization of the protocol suggested by [2].

Theorem 3.7. *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality. Assume that oblivious transfer exists, that $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \mathrm{poly}(\kappa)$, and that the following hold.*

1. *For all sufficiently large κ, either $y \equiv_x y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, or $z \equiv_x z'$ for all $x \in \mathcal{X}$ and $z, z' \in \mathcal{Z}$.*

2. *f is $\mathrm{CORE}_{\wedge}$-forced.*

3. *There exists an ensemble of efficiently samplable distributions $\mathcal{P} = \{P_{\kappa,x}\}_{\kappa \in \mathbb{N}, x \in \mathcal{X}}$ and a PPT algorithm S such that*

   $$\{\mathsf{S}\,(1^\kappa, x, x^*, f(x^*, y, z))\}_{\kappa \in \mathbb{N}, x \in \mathcal{X}, i \in [n_\wedge], j \in [m_\wedge], y \in \mathcal{Y}_i^\wedge, z \in \mathcal{Z}_j^\wedge} \stackrel{\mathrm{S}}{\equiv} \{f(x, y^*, z)\}_{\kappa \in \mathbb{N}, x \in \mathcal{X}, i \in [n_\wedge], j \in [m_\wedge], y \in \mathcal{Y}_i^\wedge, z \in \mathcal{Z}_j^\wedge},$$

   where $x^* \leftarrow P_{\kappa,x}$ and $y^* \leftarrow Q_{\kappa,i}$, where $Q_{\kappa,i}$ is the distribution given the $\mathrm{CORE}_{\wedge}$-forced property.

Then f can be computed with computational 1-security.

We thus have the following corollary, stating a characterization for a special class of functionalities.

Corollary 3.8. *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality. Assume that oblivious transfer exists and that $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \mathrm{poly}(\kappa)$. Further assume that f has the property that for all sufficiently large κ, either $y \equiv_x y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, or $z \equiv_x z'$ for all $x \in \mathcal{X}$ and $z, z' \in \mathcal{Z}$. Then f can be computed with computational 1-security if and only if the following hold.*

1. *f is $\mathrm{CORE}_{\wedge}$-forced.*

2. *There exists an ensemble of efficiently samplable distributions* $\mathcal{P} = \{P_{\kappa,x}\}_{\kappa\in\mathbb{N},x\in\mathcal{X}}$ *and a* PPT *algorithm* S *such that*

$$\{\mathsf{S}(1^\kappa, x, x^*, f(x^*, y, z))\}_{\kappa\in\mathbb{N},x_\kappa\in\mathcal{X},i\in[n_\wedge],j\in[m_\wedge],y\in\mathcal{Y}_i^\wedge,z\in\mathcal{Z}_j^\wedge} \stackrel{\mathrm{S}}{\equiv} \{f(x, y^*, z)\}_{\kappa\in\mathbb{N},x_\kappa\in\mathcal{X},i\in[n_\wedge],j\in[m_\wedge],y\in\mathcal{Y}_i^\wedge,z\in\mathcal{Z}_j^\wedge},$$

where $x^* \leftarrow P_{\kappa,x}$ *and* $y^* \leftarrow Q_{\kappa,i}$*, where* $Q_{\kappa,i}$ *is the distribution given the* $\mathrm{CORE}_\wedge$*-forced property.*

The proof of Theorem 3.7 is given in Section 5.1. The next result gives another sufficient condition. In fact, it characterizes a special class of functionalities, which includes (deterministic) NIORP functionalities, where the output-receiving party A has no input (see Corollary 3.15 below). Here, instead of assuming the functionality satisfies the property stated in the "moreover" part of Theorem 3.6, we assume that A has a *minimum input*, i.e., smaller than all other inputs with respect to $\preceq$.

Theorem 3.9. *Let* $f : \mathcal{X}\times\mathcal{Y}\times\mathcal{Z}\to\mathcal{W}$ *be a deterministic solitary output three-party functionality. Assume that* $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \mathrm{poly}(\kappa)$*, and that for all sufficiently large* κ*, there exists* $\chi\in\mathcal{X}$ *such that for all* $x\in\mathcal{X}$ *it holds that* $\chi\preceq x$*.*[17] *Then* f *can be computed with computational 1-security if and only if it is* $\mathrm{CORE}_\wedge$*-forced. Moreover, the protocol in the positive direction admits statistical 1-security.*

The negative direction directly follows from Theorem 3.6. We prove both Theorem 3.7 and the positive direction of Theorem 3.9 in Section 5.

The next proposition reduces the randomized case to the deterministic case. We stress that the reduction holds for general domain sizes, and functionalities where every party obtains an output (in fact, the reduction can be easily generalized to the multiparty setting assuming an honest majority).

Proposition 3.10 (Reducing randomized functionalities to deterministic functionalities)**.** *Let* $f : (\{0,1\}^*)^3\to\{0,1\}^*$ *be a (randomized) three-party functionality. Define the deterministic functionality* $f' : (\{0,1\}^*)^2\times(\{0,1\}^*)^2\times(\{0,1\}^*)^2\to\{0,1\}^*$ *as follows.*

$$f'((x,r_1),(y,r_2),(z,r_3)) = f(x,y,z;r_1\oplus r_2\oplus r_3).$$

Then f *can be computed with computational (statistical) 1-security if and only if* f' *can be computed with computational (statistical) 1-security.*

Proof. Let us first assume that f' can be computed with 1-security. To compute f (in the $(f', \mathsf{g.o.d.})$-hybrid model), the parties will invoke $(f', \mathsf{g.o.d.})$ with their original inputs x, y, and z, and where $r_1, r_2, r_3 \leftarrow \{0,1\}^*$ are sampled uniformly at random. Security follows directly from the fact that either r_1, r_2, or r_3 are guaranteed to be a uniform random string. Indeed, a simulator for some corrupted party will send to the trusted party T the same input the corrupted party used in the protocol.

We next show that if f can be computed with 1-security, then so is f'. Using Fact 2.11 and the composition theorem, it suffices to present a protocol for f' in the $\{(f, \mathsf{g.o.d.}), (f', \mathsf{fair})\}$-hybrid model.

[17] Note that there may be several minimum inputs, however, the assumption implies that they are all equivalent.

Protocol 3.11.

Private inputs: party A *holds* $(x, r_1) \in (\{0,1\}^*)^2$*, party* B *holds* $(y, r_2) \in (\{0,1\}^*)^2$*, and party* C *holds* $(z, r_3) \in (\{0,1\}^*)^2$*.*

Common input: the parties hold the security parameter 1^κ*.*

1. *The parties invoke* (f', fair) *with their inputs. Let* w_1*,* w_2*, and* w_2 *be the outputs of* A*,* B*, and* C*, respectively.*
2. *If* $w_1, w_2, w_3 \neq \perp$ *then* A *outputs* w_1*,* B *outputs* w_2*, and* C *outputs* w_3*.*
3. *Otherwise, the parties invoke* $(f, \mathsf{g.o.d.})$ *on their inputs* x*,* y*, and* z*, and output the result.*

We next show that the protocol is secure. Consider an adversary $\mathcal{A}$ corrupting A. The other cases are analogous. The simulator $\mathsf{Sim}_{\mathcal{A}}$ will first query $\mathcal{A}$ to receive its input (x', r_1') to (f', fair).

- If $(x', r_1') \neq \mathsf{abort}$, then $\mathsf{Sim}_{\mathcal{A}}$ sends (x', r_1') to the trusted party.
- Otherwise, the adversary $\mathcal{A}$ chooses an input $x'' \in \{0,1\}^*$ to send to $(f, \mathsf{g.o.d.})$.[18] The simulator samples $r_1^* \leftarrow \{0,1\}^*$ and sends (x'', r_1^*) to the trusted party.

In both cases, $\mathsf{Sim}_{\mathcal{A}}$ forwards the output w_1 received from the trusted party to $\mathcal{A}$, outputs whatever $\mathcal{A}$ outputs, and halts.

Clearly, if $\mathcal{A}$ does not abort during the invocation of (f', fair), then its joint view and the output of the honest parties is $f(x', y, z; r_1' \oplus r_2 \oplus r_3)$ in both worlds. Observe that if $\mathcal{A}$ does abort, however, then the output in both worlds is distributed as $f(x'', y, z)$. □

3.2.1 Interesting Corollaries

Although our necessary and sufficient conditions do not coincide in general, for various interesting families of functionalities the results do form a characterization. In the following section, we consider several such interesting families and present a characterization for them, as can be derived from Theorems 3.6, 3.7 and 3.9.

We first state the characterization for functionalities with at most three possible outputs. For this class of functionalities, we make the observation that for every $x \in \mathcal{X}$, either $y \equiv_x y'$ for all $y, y' \in \mathcal{Y}$, or $z \equiv_x z'$ for all $z, z' \in \mathcal{Z}$.

Corollary 3.12 (Characterization of ternary-output functionalities)**.** *Let* $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \{0,1,2\}$ *be a deterministic solitary output three-party functionality. Assume that oblivious transfer exists and that* $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \mathrm{poly}(\kappa)$*. Then* f *can be computed with computational 1-security if and only if the following hold.*

1. *For all sufficiently large* $\kappa \in \mathbb{N}$*, all* B*-minimal* χ_{B} *and all* C*-minimal* χ_{C}*, there exists an* R*-minimal* $\chi \in \mathcal{X}$ *such that* $\chi_{\mathsf{B}} \equiv_{\mathsf{B}} \chi \equiv_{\mathsf{C}} \chi_{\mathsf{C}}$*.*
2. f *is* $\mathrm{CORE}_\wedge$*-forced.*

[18] If $\mathcal{A}$ sends an invalid value or does not send any value, the simulator sets x'' to be the default value used by the ideal functionality of f.

3. *There exists an ensemble of efficiently samplable distributions* $\mathcal{P} = \{P_{\kappa,x}\}_{\kappa\in\mathbb{N},x\in\mathcal{X}}$ *and a* PPT *algorithm* S *such that*

$$\{\mathsf{S}\,(1^\kappa, x, x^*, f(x^*, y, z))\}_{\kappa\in\mathbb{N},x_\kappa\in\mathcal{X},i\in[n_\wedge],j\in[m_\wedge],y\in\mathcal{Y}_i^\wedge,z\in\mathcal{Z}_j^\wedge} \overset{\text{S}}{\equiv} \{f(x, y^*, z)\}_{\kappa\in\mathbb{N},x_\kappa\in\mathcal{X},i\in[n_\wedge],j\in[m_\wedge],y\in\mathcal{Y}_i^\wedge,z\in\mathcal{Z}_j^\wedge},$$

where $x^* \leftarrow P_{\kappa,x}$ *and* $y^* \leftarrow Q_{\kappa,i}$*, where* $Q_{\kappa,i}$ *is the distribution given the* $\mathrm{CORE}_\wedge$*-forced property.*

Proof. It suffices to show that Item 1 from the above statement implies Item 1 from Theorem 3.7. That is, we show that for all sufficiently large κ, either $y \equiv_x y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, or $z \equiv_x z'$ for all $x \in \mathcal{X}$ and $z, z' \in \mathcal{Z}$. Assume towards contradiction that for infinitely many κ's, there exist $x, x' \in \mathcal{X}$, $y, y' \in \mathcal{Y}$, and $z, z' \in \mathcal{Z}$ such that $y \not\equiv_x y'$ and $z \not\equiv_{x'} z'$. Now, observe that as f is a ternary-output functionality, it holds that x and x' are B-minimal and C-minimal, respectively. Moreover, it holds that $z \equiv_x z'$ and that $y \equiv_{x'} y'$. By (the assumed) Item 1 there exists an R-minimal $\chi \in \mathcal{X}$ satisfying $x \equiv_\mathsf{B} \chi \equiv_\mathsf{C} x'$. However, such χ cannot exists since it satisfies $y \equiv_\chi y'$ and $z \equiv_\chi z'$. □

We now state a characterization for functionalities that are symmetric with respect to the inputs of B and C, i.e., where $f(x, y, z) = f(x, z, y)$ for all x, y, and z. Here, the characterization follows from the observation all y's are equivalent and z's are equivalent with respect to all x's. In particular, the $\mathrm{CORE}_\wedge$-forced property implies the simpler forced property (i.e., both B and C can fix the distribution of the output).

Corollary 3.13 (Characterization of (B, C)-symmetric functionalities)**.** *Let* $f : \mathcal{X} \times \mathcal{D} \times \mathcal{D} \to \mathcal{W}$ *be a deterministic solitary output three-party functionality. Assume that oblivious transfer exists, that* $|\mathcal{X}|, |\mathcal{D}| = \mathrm{poly}(\kappa)$*, and that for all sufficiently large* $\kappa \in \mathbb{N}$*, for all* $x \in \mathcal{X}$ *and for all* $y, z \in \mathcal{D}$ *it holds that* $f(x, y, z) = f(x, z, y)$*. Then* f *can be computed with computational 1-security if and only if it is forced.*

We next state a characterization for the case where the input of party A is a single bit. The proof follows from the observation that for such functionalities there exists a minimum χ, hence we can apply Theorem 3.9.

Corollary 3.14. *Let* $f : \{0,1\} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ *be a deterministic solitary output three-party functionality. Assume that* $|\mathcal{Y}|, |\mathcal{Z}| = \mathrm{poly}(\kappa)$*. Then* f *can be computed with computational 1-security if and only if the following hold.*

1. *For all sufficiently large* $\kappa \in \mathbb{N}$*, either* $0 \preceq 1$ *or* $1 \preceq 0$*.*
2. f *is* $\mathrm{CORE}_\wedge$*-forced.*

Moreover, the protocol in the positive direction admits statistical 1-security.

Proof. First observe that if $0 \preceq 1$ or $1 \preceq 0$ for all sufficiently large $\kappa \in \mathbb{N}$, then f can be computed due to Theorem 3.9. For the other direction, we consider two cases. First, if f is not $\mathrm{CORE}_\wedge$-forced then by Theorem 3.6 it cannot be computed with 1-security. Otherwise, if $0 \not\preceq 1$ and $1 \not\preceq 0$ infinitely often, then both are R-minimal inputs infinitely often. However, there is no R-minimal χ such that $0 \equiv_\mathsf{B} \chi \equiv_\mathsf{C} 1$. Therefore, f cannot be computed due to Theorem 3.6. □

If A has no input, then the first property of Corollary 3.14 holds vacuously. Thus we have the following.

Corollary 3.15 (Characterization of NIORP functionalities)**.** *Let $f : \{\lambda\} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality. Assume that $|\mathcal{Y}|, |\mathcal{Z}| = \text{poly}(\kappa)$. Then f can be computed with computational 1-security if and only if it is $\text{CORE}_\wedge$-forced. Moreover, the protocol in the positive direction admits statistical 1-security.*

4 Impossibility Results

In this section, we prove the necessary conditions stated in Theorem 3.6. Our proof is split into two parts. In the first part, presented in Section 4.1, we apply the hexagon argument over the secure protocol assumed to exist. This results in 6 ensembles of distributions, all of which are statistically close. The second part of the proof, presented in Section 4.2, is dedicated to the analysis of these 6 ensembles. Specifically, we show how the assumption that the ensembles are close implies the necessary conditions stated in Theorem 3.6.

4.1 The Hexagon Argument

In this section we present the hexagon argument, which is the first step in the proof of Theorem 3.6. For a fixed three-party protocol $\pi = (\mathsf{A}, \mathsf{B}, \mathsf{C})$ that is defined over secure point-to-point channels in the plain model (without a broadcast channel or trusted setup assumptions), we can associate a six-party protocol denoted $\mathsf{Hex}(\pi) = (\mathsf{B}, \mathsf{A}, \mathsf{C}, \mathsf{B}', \mathsf{A}', \mathsf{C}')$ as illustrated in Figure 1. Formally, $\mathsf{Hex}(\pi)$ is defined as follows.

Definition 4.1 (The hexagon protocol)**.** *Given a three-party protocol $\pi = (\mathsf{A}, \mathsf{B}, \mathsf{C})$ we denote by $\mathsf{Hex}(\pi) = (\mathsf{B}, \mathsf{A}, \mathsf{C}, \mathsf{B}', \mathsf{A}', \mathsf{C}')$ the following six-party protocol. Parties A and A' are set with the code of A from π, parties B and B' with the code of B from π, and parties C and C' with the code of C from π.*

The communication network of $\mathsf{Hex}(\pi)$ is a cycle. Party A is connected to C, which is connected to B', which is connected to A', which is connected to C', which is connected to B, which is connected to A.

The following lemma states that any attacker corrupting any four adjacent parties in the six-party protocol $\mathsf{Hex}(\pi)$, can be perfectly emulated by an adversary corrupting a single party in three-party π.

Lemma 4.2 (Mapping attackers for $\mathsf{Hex}(\pi)$ to attackers for π)**.** *Let $\pi = (\mathsf{A}, \mathsf{B}, \mathsf{C})$ be a three-party protocol and let $\mathsf{Hex}(\pi) = (\mathsf{B}, \mathsf{A}, \mathsf{C}, \mathsf{B}', \mathsf{A}', \mathsf{C}')$ be as in Definition 4.1. In the following, for possible inputs (x, x', y, y', z, z') for protocol $\mathsf{Hex}(\pi)$ we let $\mathbf{h} = (x, x', y, y', z, z')$. Then the following hold.*

1. *For every non-uniform PPT adversary $\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}$ corrupting $\{\mathsf{A}, \mathsf{B}, \mathsf{C}', \mathsf{A}'\}$ in $\mathsf{Hex}(\pi)$, there exists a non-uniform PPT adversary $\mathcal{A}$ corrupting A in π, receiving the inputs y, z', and x' for B, C', and A', respectively, as auxiliary information, that perfectly emulates $\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}$, namely*

$$\left\{\text{REAL}_{\pi,\mathcal{A}(y,z',x',\mathsf{aux})}\left(\kappa,(x,y',z)\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \equiv \left\{\text{REAL}_{\mathsf{Hex}(\pi),\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

2. *For every non-uniform* PPT *adversary* $\mathcal{A}_{\mathsf{H}}^{\mathsf{B}',\mathsf{C}}$ *corrupting* $\{\mathsf{A}',\mathsf{B}',\mathsf{C},\mathsf{A}\}$ *in* $\mathsf{Hex}(\pi)$*, there exists a non-uniform* PPT *adversary* $\mathcal{A}'$ *corrupting* A *in* π*, receiving the inputs* y'*,* z*, and* x *for* B'*,* C*, and* A*, respectively, as auxiliary information, that perfectly emulates* $\mathcal{A}_{\mathsf{H}}^{\mathsf{B}',\mathsf{C}}$*, namely*

$$\left\{\mathrm{REAL}_{\pi,\mathcal{A}'(y',z,x,\mathsf{aux})}\left(\kappa,(x',y,z')\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \equiv \left\{\mathrm{REAL}_{\mathsf{Hex}(\pi),\mathcal{A}_{\mathsf{H}}^{\mathsf{B}',\mathsf{C}}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

3. *For every non-uniform* PPT *adversary* $\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}$ *corrupting* $\{\mathsf{B},\mathsf{A},\mathsf{C},\mathsf{B}'\}$ *in* $\mathsf{Hex}(\pi)$*, there exists a non-uniform* PPT *adversary* $\mathcal{B}$ *corrupting* B *in* π*, receiving the inputs* x*,* z*, and* y' *for* A*,* C*, and* B'*, respectively, as auxiliary information, that perfectly emulates* $\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}$*, namely*

$$\left\{\mathrm{REAL}_{\pi,\mathcal{B}(x,z,y',\mathsf{aux})}\left(\kappa,(x',y,z')\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \equiv \left\{\mathrm{REAL}_{\mathsf{Hex}(\pi),\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

4. *For every non-uniform* PPT *adversary* $\mathcal{B}_{\mathsf{H}}^{\mathsf{A}',\mathsf{C}'}$ *corrupting* $\{\mathsf{B}',\mathsf{A}',\mathsf{C}',\mathsf{B}\}$ *in* $\mathsf{Hex}(\pi)$*, there exists a non-uniform* PPT *adversary* $\mathcal{B}'$ *corrupting* B *in* π*, receiving the inputs* x'*,* z'*, and* y *for* A'*,* C'*, and* B*, respectively, as auxiliary information, that perfectly emulates* $\mathcal{B}_{\mathsf{H}}^{\mathsf{A}',\mathsf{C}'}$*, namely*

$$\left\{\mathrm{REAL}_{\pi,\mathcal{B}'(x',z',y,\mathsf{aux})}\left(\kappa,(x,y',z)\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \equiv \left\{\mathrm{REAL}_{\mathsf{Hex}(\pi),\mathcal{B}_{\mathsf{H}}^{\mathsf{A}',\mathsf{C}'}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

5. *For every non-uniform* PPT *adversary* $\mathcal{C}_{\mathsf{H}}^{\mathsf{A},\mathsf{B}}$ *corrupting* $\{\mathsf{C}',\mathsf{B},\mathsf{A},\mathsf{C}\}$ *in* $\mathsf{Hex}(\pi)$*, there exists a non-uniform* PPT *adversary* $\mathcal{C}$ *corrupting* C *in* π*, receiving the inputs* y*,* x*, and* z *for* A*,* B*, and* C*, respectively, as auxiliary information, that perfectly emulates* $\mathcal{C}_{\mathsf{H}}^{\mathsf{A},\mathsf{B}}$*, namely*

$$\left\{\mathrm{REAL}_{\pi,\mathcal{C}(y,x,z,\mathsf{aux})}\left(\kappa,(x',y',z')\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \equiv \left\{\mathrm{REAL}_{\mathsf{Hex}(\pi),\mathcal{C}_{\mathsf{H}}^{\mathsf{A},\mathsf{B}}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

6. *For every non-uniform* PPT *adversary* $\mathcal{C}_{\mathsf{H}}^{\mathsf{A}',\mathsf{B}'}$ *corrupting* $\{\mathsf{C},\mathsf{B}',\mathsf{A}',\mathsf{C}'\}$ *in* $\mathsf{Hex}(\pi)$*, there exists a non-uniform* PPT *adversary* $\mathcal{C}'$ *corrupting* C *in* π*, receiving the inputs* y'*,* x'*, and* z' *for* A*,* B*, and* C*, respectively, as auxiliary information, that perfectly emulates* $\mathcal{C}_{\mathsf{H}}^{\mathsf{A}',\mathsf{B}'}$*, namely*

$$\left\{\mathrm{REAL}_{\pi,\mathcal{C}'(y',x',z',\mathsf{aux})}\left(\kappa,(x,y,z)\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \equiv \left\{\mathrm{REAL}_{\mathsf{Hex}(\pi),\mathcal{C}_{\mathsf{H}}^{\mathsf{A}',\mathsf{B}'}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

Proof. We will prove only Item 1 as the rest follows from a similar argument. Fix an adversary $\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}$ corrupting $\{\mathsf{A},\mathsf{B},\mathsf{C}',\mathsf{A}'\}$ in $\mathsf{Hex}(\pi)$. Define an adversary $\mathcal{A}$ corrupting A in π as follows. First, it initializes $\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}$ with input x for A, input y for B, input z' for C', input x' for A', and auxiliary information aux. Each round, it passes to $\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}$ the messages received from the honest parties B and C, and replies to them as $\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}$ does. Finally, $\mathcal{A}$ output whatever $\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}$ outputs.

By the definition of $\mathcal{A}$, in each round, the messages it receives from and sends to B and C in π, are identically distributed to the messages $\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}$ received from and sent to B' and C in $\mathsf{Hex}(\pi)$. Therefore the transcript in both executions are identically distributed. In particular, the joint distribution of the view of the adversary and the output of the honest parties are identical in both executions. □

An important use-case of the above lemma is for 1-secure protocols π computing some 3-party functionality f. Here, any attacker in π that emulates some attacker for $\mathsf{Hex}(\pi)$ as given by Lemma 4.2, can be simulated in the ideal world of f. Thus, we get the following corollary.

Corollary 4.3 (Mapping attackers for $\mathsf{Hex}(\pi)$ to simulators for f)**.** *Let $\pi = (\mathsf{A}, \mathsf{B}, \mathsf{C})$ be a three-party protocol computing some solitary output three-party functionality $f : (\{0,1\}^*)^3 \to \{0,1\}^*$ with computational 1-security. Then the following hold.*

1. *For every non-uniform* PPT *adversary $\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}$ corrupting $\{\mathsf{A}, \mathsf{B}, \mathsf{C}', \mathsf{A}'\}$ in $\mathsf{Hex}(\pi)$, there exists a non-uniform* PPT *simulator $\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B},\mathsf{C}'}$ in the ideal world of f corrupting A, such that*

$$\left\{\text{IDEAL}_{f,\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B},\mathsf{C}'}(y,z',x',\mathsf{aux})}\left(\kappa,(x,y',z)\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \stackrel{\text{C}}{\equiv} \left\{\text{REAL}_{\mathsf{Hex}(\pi),\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

2. *For every non-uniform* PPT *adversary $\mathcal{A}_{\mathsf{H}}^{\mathsf{B}',\mathsf{C}}$ corrupting $\{\mathsf{A}', \mathsf{B}', \mathsf{C}, \mathsf{A}\}$ in $\mathsf{Hex}(\pi)$, there exists a non-uniform* PPT *simulator $\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B}',\mathsf{C}}$ in the ideal world of f corrupting A, such that*

$$\left\{\text{IDEAL}_{\pi,\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B}',\mathsf{C}}(y',z,x,\mathsf{aux})}\left(\kappa,(x',y,z')\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \stackrel{\text{C}}{\equiv} \left\{\text{REAL}_{\mathsf{Hex}(\pi),\mathcal{A}_{\mathsf{H}}^{\mathsf{B}',\mathsf{C}}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

3. *For every non-uniform* PPT *adversary $\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}$ corrupting $\{\mathsf{B}, \mathsf{A}, \mathsf{C}, \mathsf{B}'\}$ in $\mathsf{Hex}(\pi)$, there exists a non-uniform* PPT *simulator $\mathsf{Sim}_{\mathcal{B}}^{\mathsf{A},\mathsf{C}}$ in the ideal world of f corrupting B, such that*

$$\left\{\text{IDEAL}_{\pi,\mathsf{Sim}_{\mathcal{B}}^{\mathsf{A},\mathsf{C}}(x,z,y',\mathsf{aux})}\left(\kappa,(x',y,z')\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \stackrel{\text{C}}{\equiv} \left\{\text{REAL}_{\mathsf{Hex}(\pi),\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

4. *For every non-uniform* PPT *adversary $\mathcal{B}_{\mathsf{H}}^{\mathsf{A}',\mathsf{C}'}$ corrupting $\{\mathsf{B}', \mathsf{A}', \mathsf{C}', \mathsf{B}\}$ in $\mathsf{Hex}(\pi)$, there exists a non-uniform* PPT *simulator $\mathsf{Sim}_{\mathcal{B}}^{\mathsf{A}',\mathsf{C}'}$ in the ideal world of f corrupting B, such that*

$$\left\{\text{IDEAL}_{\pi,\mathsf{Sim}_{\mathcal{B}}^{\mathsf{A}',\mathsf{C}'}(x',z',y,\mathsf{aux})}\left(\kappa,(x,y',z)\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \stackrel{\text{C}}{\equiv} \left\{\text{REAL}_{\mathsf{Hex}(\pi),\mathcal{B}_{\mathsf{H}}^{\mathsf{A}',\mathsf{C}'}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

5. *For every non-uniform* PPT *adversary $\mathcal{C}_{\mathsf{H}}^{\mathsf{A},\mathsf{B}}$ corrupting $\{\mathsf{C}', \mathsf{B}, \mathsf{A}, \mathsf{C}\}$ in $\mathsf{Hex}(\pi)$, there exists a non-uniform* PPT *simulator $\mathsf{Sim}_{\mathcal{C}}^{\mathsf{A},\mathsf{B}}$ in the ideal world of f corrupting C, such that*

$$\left\{\text{IDEAL}_{\pi,\mathsf{Sim}_{\mathcal{C}}^{\mathsf{A},\mathsf{B}}(y,x,z,\mathsf{aux})}\left(\kappa,(x',y',z')\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \stackrel{\text{C}}{\equiv} \left\{\text{REAL}_{\mathsf{Hex}(\pi),\mathcal{C}_{\mathsf{H}}^{\mathsf{A},\mathsf{B}}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

6. *For every non-uniform* PPT *adversary $\mathcal{C}_{\mathsf{H}}^{\mathsf{A}',\mathsf{B}'}$ corrupting $\{\mathsf{C}, \mathsf{B}', \mathsf{A}', \mathsf{C}'\}$ in $\mathsf{Hex}(\pi)$, there exists a non-uniform* PPT *simulator $\mathsf{Sim}_{\mathcal{C}}^{\mathsf{A}',\mathsf{B}'}$ in the ideal world of f corrupting C, such that*

$$\left\{\text{IDEAL}_{\pi,\mathsf{Sim}_{\mathcal{C}}^{\mathsf{A}',\mathsf{B}'}(y',x',z',\mathsf{aux})}\left(\kappa,(x,y,z)\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}} \stackrel{\text{C}}{\equiv} \left\{\text{REAL}_{\mathsf{Hex}(\pi),\mathcal{C}_{\mathsf{H}}^{\mathsf{A}',\mathsf{B}'}(\mathsf{aux})}\left(\kappa,\mathbf{h}\right)\right\}_{\kappa,\mathbf{h},\mathsf{aux}}.$$

One important use-case of Corollary 4.3 is when the six adversaries for $\mathsf{Hex}(\pi)$ are semi-honest. This is due to the fact that the views of the honest parties are identically distributed in all six cases, hence the same holds with respect to their outputs. Next, consider the joint distribution of the outputs of A and A' in $\mathsf{Hex}(\pi)$. Observe that for any adversary corrupting either of them, say A, its simulator given by Corollary 4.3 must be able to generate the output of A, as it is part of the view. Furthermore, if either A or A' is honest, then the simulator can force the output of A in the ideal world of f to be indistinguishable from the real world.

Now, recall that these simulators are for the *malicious* setting, hence they can send arbitrary inputs to the trusted party. Thus, the distributions over the outputs depend on the distribution over the input sent by each simulator to the trusted party. Notice that when considering semi-honest adversaries for $\mathsf{Hex}(\pi)$ that have no auxiliary input, these distributions depend only on the security parameter and the inputs given to the semi-honest adversary.

For example, in the case where $\{\mathsf{B}, \mathsf{A}, \mathsf{C}, \mathsf{B}'\}$ are corrupted, the simulator samples a random input y^* according to some distribution Q that depends only on the security parameter κ, and the inputs y, x, z, and y' given to the adversary. The input y^* must be such that the joint output of the simulator and the output of A in the ideal world of f, must be indistinguishable from the joint output of A and A' in $\mathsf{Hex}(\pi)$.

Lemma 4.4. *Let $f : (\{0,1\}^*)^3 \to \{0,1\}^*$ be a solitary output three-party functionality that can be computed with computational 1-security. Then the there exist*

- *two ensembles of efficiently samplable distributions*

$$\mathcal{P}^{\mathsf{B},\mathsf{C}'} = \{P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}\}_{\kappa\in\mathbb{N},x,x',y,z'\in\{0,1\}^*} \quad \textit{and} \quad \mathcal{P}^{\mathsf{B}',\mathsf{C}} = \{P^{\mathsf{B}',\mathsf{C}}_{\kappa,x',y',z,x}\}_{\kappa\in\mathbb{N},x,x',y,z'\in\{0,1\}^*}$$

  over $\mathcal{X}$,

- *two ensembles of efficiently samplable distributions*

$$\mathcal{Q} = \{Q_{\kappa,y',z,x,y}\}_{\kappa\in\mathbb{N},x,y,y',z\in\{0,1\}^*} \quad \textit{and} \quad \mathcal{Q}' = \left\{Q'_{\kappa,y,z',x',y'}\right\}_{\kappa\in\mathbb{N},x,y,y',z\in\{0,1\}^*}$$

  over $\mathcal{Y}$,

- *two ensembles of efficiently samplable distributions*

$$\mathcal{R} = \{R_{\kappa,z,x,y,z'}\}_{\kappa\in\mathbb{N},x,y,z,z'\in\{0,1\}^*} \quad \textit{and} \quad \mathcal{R}' = \left\{R'_{\kappa,z',x',y',z}\right\}_{\kappa\in\mathbb{N},x,y,z,z'\in\{0,1\}^*}$$

  over $\mathcal{Z}$,

- *and six* PPT *algorithms $\mathsf{S}^{\mathsf{B},\mathsf{C}'}$, $\mathsf{S}^{\mathsf{B}',\mathsf{C}}$, S_{B}, S'_{B}, S_{C}, and S'_{C},*

such that the following six distribution ensembles are computationally indistinguishable

1. $\{\mathsf{S}^{\mathsf{B},\mathsf{C}'}(x, y, z', x', x_1^*, f(x_1^*, y', z))\}_{\kappa,x,x',y,y',z,z'}$, *where* $x_1^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}$.
2. $\{\mathsf{S}^{\mathsf{B}',\mathsf{C}}(x', y', z, x, x_2^*, f(x_2^*, y, z'))\}_{\kappa,x,x',y,y',z,z'}$, *where* $x_2^* \leftarrow P^{\mathsf{B}',\mathsf{C}}_{\kappa,x',y',z,x}$.
3. $\{(\mathsf{S}_{\mathsf{B}}(y', z, x, y, y_1^*), f(x', y_1^*, z'))\}_{\kappa,x,x',y,y',z,z'}$, *where* $y_1^* \leftarrow Q_{\kappa,y',z,x,y}$.
4. $\{(f(x, y_2^*, z), \mathsf{S}'_{\mathsf{B}}(y, z', x', y', y_2^*))\}_{\kappa,x,x',y,y',z,z'}$, *where* $y_2^* \leftarrow Q'_{\kappa,y,z',x',y'}$.

5. $\{(\mathsf{S}_{\mathsf{C}}(z,x,y,z',z_1^*), f(x',y',z_1^*))\}_{\kappa,x,x',y,y',z,z'}$, *where* $z_1^* \leftarrow R_{\kappa,z,x,y,z'}$.

6. $\{(f(x,y,z_2^*), \mathsf{S}'_{\mathsf{C}}(z',x',y',z,z_2^*))\}_{\kappa,x,x',y,y',z,z'}$, *where* $z_2^* \leftarrow R'_{\kappa,z',x',y',z}$.

Moreover, if the domain of f is of polynomial size in κ, then the above ensembles are statistically close.

Proof. let π be a three-party protocol computing f with 1-security, and consider an honest execution of $\mathsf{Hex}(\pi)$. Let $(\text{OUT}(\kappa,\mathbf{h}), \text{OUT}'(\kappa,\mathbf{h}))$ denote the joint distribution of the outputs of A and A', respectively, in such execution of $\mathsf{Hex}(\pi)$, where $\mathbf{h} = (x,x',y,y',z,z')$ are the inputs of the parties. We will show how to obtain each of the Ensembles 1–6, such that each of them is computationally indistinguishable from (OUT,OUT').

We first show how to obtain Ensemble 1. Ensemble 2 can be obtained using a similar argument. Consider the semi-honest adversary $\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}$ corrupting $\{\mathsf{A},\mathsf{B},\mathsf{C}',\mathsf{A}'\}$ with no additional auxiliary information, that outputs the output of A and A' (note that this is well-defined since the adversary is semi-honest). By Item 1 from Corollary 4.3, there exists a non-uniform PPT simulator $\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B},\mathsf{C}'}$ in the ideal world of f corrupting A, such that

$$\left\{\text{IDEAL}_{f,\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B},\mathsf{C}'}(y,z',x')}(\kappa,(x,y',z))\right\}_{\kappa,\mathbf{h}} \overset{\text{C}}{\equiv} \left\{\text{REAL}_{\mathsf{Hex}(\pi),\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}}(\kappa,\mathbf{h})\right\}_{\kappa,\mathbf{h}}.$$

Since only A receives an output in the ideal world of f, it follows that

$$\begin{aligned}\left\{\text{VIEW}^{\mathsf{ideal}}_{f,\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B},\mathsf{C}'}(y,z',x')}(\kappa,(x,y',z))\right\}_{\kappa,\mathbf{h}} &\equiv \left\{\text{IDEAL}_{f,\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B},\mathsf{C}'}(y,z',x')}(\kappa,(x,y',z))\right\}_{\kappa,\mathbf{h}}\\ &\overset{\text{C}}{\equiv} \left\{\text{VIEW}^{\mathsf{real}}_{\mathsf{Hex}(\pi),\mathcal{A}_{\mathsf{H}}^{\mathsf{B},\mathsf{C}'}}(\kappa,\mathbf{h})\right\}_{\kappa,\mathbf{h}}\\ &\equiv \left\{(\text{OUT},\text{OUT}')\right\}_{\kappa,\mathbf{h}},\end{aligned}$$

where $\mathbf{h} = (x,x',y,y',z,z')$. We let $P_{\kappa,x,y,z',x'}$ denote the distribution over the input x^* that $\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B},\mathsf{C}'}$ sends to the trusted party T, and let $\mathsf{S}^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x_1^*,w)$ output whatever $\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B},\mathsf{C}'}$ outputs given that it sent x_1^* to T and received the output w. Therefore,

$$\begin{aligned}\left\{\mathsf{S}^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x_1^*,f(x_1^*,y',z))\right\}_{\kappa,\mathbf{h}} &\equiv \left\{\text{VIEW}^{\mathsf{ideal}}_{f,\mathsf{Sim}_{\mathcal{A}}^{\mathsf{B},\mathsf{C}'}(y,z',x')}(\kappa,(x,y',z))\right\}_{\kappa,\mathbf{h}}\\ &\overset{\text{C}}{\equiv} \left\{(\text{OUT}(\kappa,\mathbf{h}),\text{OUT}'(\kappa,\mathbf{h}))\right\}_{\kappa,\mathbf{h}},\end{aligned}$$

where $x_1^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}$.

We now show how to obtain Ensemble 3. The rest of the ensembles can be obtained using a similar argument. Similarly to the previous case, we consider the semi-honest adversary $\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}$ corrupting $\{\mathsf{B},\mathsf{A},\mathsf{C},\mathsf{B}'\}$ with no additional auxiliary information, that outputs the output of A. By Item 3 from Corollary 4.3, there exists a non-uniform PPT simulator $\mathsf{Sim}_{\mathcal{B}}^{\mathsf{A},\mathsf{C}}$ in the ideal world of f corrupting B, such that

$$\left\{\text{IDEAL}_{\pi,\mathsf{Sim}_{\mathcal{B}}^{\mathsf{A},\mathsf{C}}(x,z,y')}(\kappa,(x',y,z'))\right\}_{\kappa,\mathbf{h}} \overset{\text{C}}{\equiv} \left\{\text{REAL}_{\mathsf{Hex}(\pi),\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}}(\kappa,\mathbf{h})\right\}_{\kappa,\mathbf{h}}.$$

Since only A and A' receives an output in the execution of $\mathsf{Hex}(\pi)$, it follows that

$$\begin{aligned}\left\{\mathrm{REAL}_{\mathsf{Hex}(\pi),\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}}(\kappa,\mathbf{h})\right\}_{\kappa,\mathbf{h}} &\equiv \left\{\left(\mathrm{VIEW}^{\mathsf{real}}_{\mathsf{Hex}(\pi),\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}}(\kappa,(x',y,z')),\mathrm{OUT}^{\mathsf{ideal}}_{\mathsf{Hex}(\pi),\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}}(\kappa,(x',y,z'))\right)\right\}_{\kappa,\mathbf{h}}\\ &\equiv \left\{\left(\mathrm{OUT}(\kappa,\mathbf{h}),\mathrm{OUT}'(\kappa,\mathbf{h})\right)\right\}_{\kappa,\mathbf{h}}.\end{aligned}$$

Let $Q_{\kappa,y',z,x,y}$ denote the distribution over the input y^* that $\mathsf{Sim}_{\mathcal{B}}^{\mathsf{A},\mathsf{C}}$ sends to the trusted party T, and let $\mathsf{S}_{\mathsf{B}}(y',z,x,y,y_1^*)$ output whatever $\mathsf{Sim}_{\mathcal{B}}^{\mathsf{A},\mathsf{C}}$ outputs given that it sent y_1^* to T. Then

$$\begin{aligned}\left\{\left(\mathsf{S}_{\mathsf{B}}(y',z,x,y,y_1^*),f(x',y_1^*,z')\right)\right\}_{\kappa,\mathbf{h}} &\equiv \left\{\mathrm{IDEAL}_{\pi,\mathsf{Sim}_{\mathcal{B}}^{\mathsf{A},\mathsf{C}}(x,z,y')}(\kappa,(x',y,z'))\right\}_{\kappa,\mathbf{h}}\\ &\overset{\mathrm{C}}{\equiv} \left\{\mathrm{REAL}_{\mathsf{Hex}(\pi),\mathcal{B}_{\mathsf{H}}^{\mathsf{A},\mathsf{C}}}(\kappa,\mathbf{h})\right\}_{\kappa,\mathbf{h}}\\ &\equiv \left\{\left(\mathrm{OUT}(\kappa,\mathbf{h}),\mathrm{OUT}'(\kappa,\mathbf{h})\right)\right\}_{\kappa,\mathbf{h}},\end{aligned}$$

where $y_1^* \leftarrow Q_{\kappa,y',z,x,y}$.

As for the "moreover" part, observe that if the domain of f is of polynomial size, then the support of all ensembles is of polynomial size. Thus, by Fact 2.3 the ensembles are statistically close. □

4.2 Analyzing The Ensembles

In this section, we analyze the six distribution ensembles given by Lemma 4.4. For the sake of brevity, throughout the entire section, we fix a deterministic solitary output three-party functionality that can be computed with 1-security $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$, where $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \mathrm{poly}(\kappa)$. Additionally, we fix all distribution ensembles and PPT algorithms from Lemma 4.4, using the same notations.

It will be convenient in the proof to use the following notion of statistical independence. Roughly, a distribution ensemble is statistically independent of one of its variables, if changing the variable results in a statistically close distribution ensemble.

Definition 4.5 (Statistical independence)**.** *Let $X = \{X_{a,b,n}\}_{a\in\mathcal{D}_n,b\in\mathcal{D}'_n,n\in\mathbb{N}}$ be a distribution ensemble. We say that X is* statistically independent *of $\{\mathcal{D}'_n\}_{n\in\mathbb{N}}$ if*

$$\{X_{a,b,n}\}_{a\in\mathcal{D}_n,b,b'\in\mathcal{D}'_n,n\in\mathbb{N}} \overset{\mathrm{S}}{\equiv} \{X_{a,b',n}\}_{a\in\mathcal{D}_n,b,b'\in\mathcal{D}'_n,n\in\mathbb{N}}.$$

For the sake of simplifying the presentation, we will usually say that X is statistically independent of b, rather than referring to its domain.

Theorem 3.6 follows from the following two claims, stating the conditions specified in it.

Claim 4.6. *For all sufficiently large $\kappa \in \mathbb{N}$, if χ_{C} and χ_{B} are C-minimal and B-minimal, respectively, then there exists an R-minimal $\chi \in \mathcal{X}$ such that $\chi_{\mathsf{C}} \equiv_{\mathsf{C}} \chi \equiv_{\mathsf{B}} \chi_{\mathsf{B}}$.*

Claim 4.7. *For every $\kappa \in \mathbb{N}$ and every $i \in [n_\wedge]$ we let y_i denote the lexicographically smallest element of $\mathcal{Y}_i^\wedge$. Similarly, for $j \in [m_\wedge]$ we let z_j denote the lexicographically smallest element of*

$\mathcal{Z}_j^{\wedge}$. *Then there exist two ensembles of efficiently samplable distributions* $\mathcal{Q} = \{Q_{\kappa,i}\}_{\kappa\in\mathbb{N},i\in[n_\wedge]}$ *and* $\mathcal{R} = \{R_{\kappa,j}\}_{\kappa\in\mathbb{N},j\in[m_\wedge]}$ *over* $\mathcal{Y}$ *and* $\mathcal{Z}$*, respectively, such that the following holds.*

$$\{f(x,y^*,z_j)\}_{\kappa,x,i,j,y,z} \overset{\mathrm{S}}{\equiv} \{f(x,y^*,z)\}_{\kappa,x,i,j,y,z} \overset{\mathrm{S}}{\equiv} \{f(x,y,z^*)\}_{\kappa,x,i,j,y,z} \overset{\mathrm{S}}{\equiv} \{f(x,y_i,z^*)\}_{\kappa,x,i,j,y,z} \quad (10)$$

where $y^* \leftarrow Q_{\kappa,i}$, $z^* \leftarrow R_{\kappa,j}$.

Moreover, suppose that f *has the property that for all sufficiently large* κ*, it holds that either* $y \equiv_x y'$ *for all* $x \in \mathcal{X}$ *and* $y, y' \in \mathcal{Y}$*, or* $z \equiv_x z'$ *for all* $x \in \mathcal{X}$ *and* $z, z' \in \mathcal{Z}$*. Then there exists an ensemble of efficiently samplable distributions* $\mathcal{P} = \{P_{\kappa,x}\}_{\kappa\in\mathbb{N},x\in\mathcal{X}}$ *and a* PPT *algorithm* S *such that*

$$\{\mathsf{S}\,(1^\kappa, x, x^*, f(x^*,y,z))\}_{\kappa,x,i,j,y,z} \overset{\mathrm{S}}{\equiv} \{f(x,y^*,z)\}_{\kappa,x,i,j,y,z},$$

where $x^* \leftarrow P_{\kappa,x}$ *and* $y^* \leftarrow Q_{\kappa,i}$*, where* $Q_{\kappa,i}$ *is the distribution given the* $\mathrm{CORE}_\wedge$*-forced property.*

We prove Claims 4.6 and 4.7 below. We first make the following simple yet useful observation, which states that each of the marginal distributions of the ensembles are statistically independent of several of the inputs.

Claim 4.8. *Consider the* PPT *algorithms* $\mathsf{S}^{\mathsf{B},\mathsf{C}'}$ *and* $\mathsf{S}^{\mathsf{B}',\mathsf{C}}$ *from Lemma 4.4, and write them as* $\mathsf{S}^{\mathsf{B},\mathsf{C}'} = (\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}, \mathsf{S}_2^{\mathsf{B},\mathsf{C}'})$ *and* $\mathsf{S}^{\mathsf{B}',\mathsf{C}} = (\mathsf{S}_1^{\mathsf{B}',\mathsf{C}}, \mathsf{S}_2^{\mathsf{B}',\mathsf{C}})$*. Then both* $\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}$ *and* $\mathsf{S}_1^{\mathsf{B}',\mathsf{C}}$ *are statistically independent of* x'*,* y'*, and* z'*. Similarly, both* $\mathsf{S}_2^{\mathsf{B},\mathsf{C}'}$ *and* $\mathsf{S}_2^{\mathsf{B}',\mathsf{C}}$ *are statistically independent of* x*,* y*, and* z*.*

Proof. We prove that $\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}$ is statistically independent of x', y', and z'. The second statement can be proven using an analogous argument. Observe that by Lemma 4.4, it follows that

$$\begin{aligned}\{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\,(x,y,z',x',x^*,f(x^*,y',z))\}_{\kappa,x,x',y,y',z,z'} &\overset{\mathrm{S}}{\equiv} \{\mathsf{S}_{\mathsf{B}}\,(y',z,x,y,y^*)\}_{\kappa,x,x',y,y',z,z'} \\ &\overset{\mathrm{S}}{\equiv} \{\mathsf{S}_{\mathsf{C}}\,(z,x,y,z',z^*)\}_{\kappa,x,x',y,y',z,z'},\end{aligned}$$

where $y^* \leftarrow Q_{\kappa,y',z,x,y}$ and $z^* \leftarrow R_{\kappa,z,x,y,z'}$. As S_{B} and S_{C} are statistically independent of x', z' and x', y', respectively, it follows that $\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}$ is statistically independent of them as well. □

The following two lemmata are the main ingredients in our proof. The first lemma roughly identifies the support of the inputs x_1^* and x_2^* used by PPT algorithms $\mathsf{S}^{\mathsf{B},\mathsf{C}'}$ and $\mathsf{S}^{\mathsf{B}',\mathsf{C}}$ (up to negligible probability). The second lemma identifies when it is possible to change some of the inputs, such that at least one of the marginal distributions of the outcome of the PPT algorithms $\mathsf{S}^{\mathsf{B},\mathsf{C}'}$ and $\mathsf{S}^{\mathsf{B}',\mathsf{C}}$ remains similar.

Lemma 4.9. *Consider the distribution ensembles* $\mathcal{P}^{\mathsf{B},\mathsf{C}'}$ *and* $\mathcal{P}^{\mathsf{B}',\mathsf{C}}$ *from Lemma 4.4. Then for every* $x, x' \in \mathcal{X}$*, every* $y \in \mathcal{Y}$*, and every* $z' \in \mathcal{Z}$*, it holds that*

$$\Pr_{x_1^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}}\left[x_1^* \not\preceq_{\mathsf{C}} x \vee x_1^* \not\preceq_{\mathsf{B}} x'\right] = \mathrm{neg}(\kappa).$$

Similarly, for every $x, x' \in \mathcal{X}$*, every* $y' \in \mathcal{Y}$*, and every* $z \in \mathcal{Z}$*, it holds that*

$$\Pr_{x_2^* \leftarrow P^{\mathsf{B}',\mathsf{C}}_{\kappa,x',y',z,x}}\left[x_2^* \not\preceq_{\mathsf{B}} x \vee x_2^* \not\preceq_{\mathsf{C}} x'\right] = \mathrm{neg}(\kappa).$$

In particular, for all sufficiently large κ *and every* $x, x' \in \mathcal{X}$*, there exists* $x^* \in \mathcal{X}$ *such that*

$$x^* \preceq_{\mathsf{C}} x \wedge x^* \preceq_{\mathsf{B}} x'.$$

Lemma 4.10. *The following hold.*

1. $$\left\{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x_1^*,f(x_1^*,y',z_1))\right\}_{\kappa,x,x',y,y',z_1,z_2,z'} \overset{\text{S}}{\equiv} \left\{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x_1^*,f(x_1^*,y',z_2))\right\}_{\kappa,x,x',y,y',z_1,z_2,z'},$$

   where $z_1 \equiv_{\tilde{x}} z_2$ *for all* $\tilde{x} \preceq_{\mathsf{C}} x$*, and where* $x_1^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}$.

2. $$\left\{\mathsf{S}_2^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x_1^*,f(x_1^*,y_1',z))\right\}_{\kappa,x,x',y,y_1',y_2',z,z'} \overset{\text{S}}{\equiv} \left\{\mathsf{S}_2^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x_1^*,f(x_1^*,y_2',z))\right\}_{\kappa,x,x',y,y_1',y_2',z,z'},$$

   where $y_1' \equiv_{\tilde{x}} y_2'$ *for all* $\tilde{x} \preceq_{\mathsf{C}} x$*, and where* $x_1^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}$.

3. $$\left\{\mathsf{S}_1^{\mathsf{B}',\mathsf{C}}(x',y',z,x,x_2^*,f(x_2^*,y,z_1'))\right\}_{\kappa,x,x',y,y',z,z_1',z_2'} \overset{\text{S}}{\equiv} \left\{\mathsf{S}_1^{\mathsf{B}',\mathsf{C}}(x',y',z,x,x_2^*,f(x_2^*,y,z_2'))\right\}_{\kappa,x,x',y,y',z,z_1',z_2'},$$

   where $z_1' \equiv_{\tilde{x}} z_2'$ *for all* $\tilde{x} \preceq_{\mathsf{C}} x$*, and where* $x_2^* \leftarrow P^{\mathsf{B}',\mathsf{C}}_{\kappa,x',y',z,x}$.

4. $$\left\{\mathsf{S}_2^{\mathsf{B}',\mathsf{C}}(x',y',z,x,x_2^*,f(x_2^*,y_1,z'))\right\}_{\kappa,x,x',y_1,y_2,y',z,z'} \overset{\text{S}}{\equiv} \left\{\mathsf{S}_2^{\mathsf{B}',\mathsf{C}}(x',y',z,x,x_2^*,f(x_2^*,y_2,z'))\right\}_{\kappa,x,x',y_1,y_2,y',z,z'},$$

   where $y_1 \equiv_{\tilde{x}} y_2$ *for all* $\tilde{x} \preceq_{\mathsf{C}} x$*, and where* $x_2^* \leftarrow P^{\mathsf{B}',\mathsf{C}}_{\kappa,x',y',z,x}$.

Lemmas 4.9 and 4.10 are proved in Sections 4.2.1 and 4.2.2, respectively. Before providing the proofs, we first show that they imply Claims 4.6 and 4.7, and thus they imply Theorem 3.6.

Proof of Claim 4.6. Let χ_{B} and χ_{C} be B-minimal and C-minimal, respectively. We assume without loss of generality that $\chi_{\mathsf{B}} \not\equiv \chi_{\mathsf{C}}$, as otherwise the claim is trivial. By Lemma 4.9 there exists $\chi \in \mathcal{X}$ satisfying $\chi \preceq_{\mathsf{B}} \chi_{\mathsf{B}}$ and $\chi \preceq_{\mathsf{C}} \chi_{\mathsf{C}}$. By the minimality of χ_{B} and χ_{C} it follows that $\chi \equiv_{\mathsf{B}} \chi_{\mathsf{B}}$ and $\chi \equiv_{\mathsf{C}} \chi_{\mathsf{C}}$. It is left to show that χ is R-minimal. Let $\tilde{\chi} \preceq \chi$. By Lemma 4.9 there exists $\tilde{x}$ satisfying $\tilde{x} \preceq_{\mathsf{B}} \tilde{\chi} \preceq_{\mathsf{B}} \chi \equiv_{\mathsf{B}} \chi_{\mathsf{B}}$ and $\tilde{x} \preceq_{\mathsf{C}} \tilde{\chi} \preceq_{\mathsf{C}} \chi \equiv_{\mathsf{C}} \chi_{\mathsf{C}}$. By the minimality of χ_{B} and χ_{C} it follows that $\chi_{\mathsf{B}} \equiv_{\mathsf{B}} \tilde{x} \equiv_{\mathsf{C}} \chi_{\mathsf{C}}$. Therefore $\tilde{x} \equiv_{\mathsf{B}} \chi$ and $\tilde{x} \equiv_{\mathsf{C}} \chi$, hence $\tilde{x} \equiv \chi$. □

Proof of Claim 4.7. We first define the distributions $Q_{\kappa,i}$ and $R_{\kappa,j}$, for $i \in [n_\wedge]$ and $j \in [m_\wedge]$. Let $\mathcal{Q}'$ and $\mathcal{R}'$ be the distribution ensembles from Lemma 4.4. In the following, we fix x_0, y_0, and z_0 to be the lexicographically smallest elements of $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, respectively. We let $Q_{\kappa,i}$ be the distribution $Q'_{\kappa,y_i,z_0,x_0,y_0}$ and let $R_{\kappa,j}$ be the distribution $R'_{\kappa,z_0,x_0,y_0,z_j}$.

We now prove Equation (10). The second transition follows from Lemma 4.4. We prove the first transition. The last transition can be proved using an analogous argument. Let $\mathsf{S}^{\mathsf{B},\mathsf{C}'}$ be as in Lemma 4.4, and let $\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}$ be the first entry in its output. First, observe that if $z \equiv_\wedge z_j$, then $z \equiv_\chi z_j$ for all R-minimal χ. The minimality of all such χ implies that $z \equiv_{\tilde{x}} z_j$ for all $\tilde{x} \preceq_{\mathsf{C}} x$. Second, by Lemma 4.9 $x^* \preceq_{\mathsf{C}} x$ with probability at least $1 - \mathrm{neg}(\kappa)$. Thus, combining with Lemma 4.10 it follows that

$$\left\{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y',z))\right\}_{\kappa,j,x,x',y,y',z,z'} \overset{\text{S}}{\equiv} \left\{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y',z_j))\right\}_{\kappa,j,x,x',y,y',z,z'}, \tag{11}$$

where $x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}$. Furthermore, by Claim 4.8 the above ensembles are statistically independent of x', y', and z', thus the ensembles are statistically close for fixed $x' = x_0$, $y' = y_0$, and $z' = z_0$, i.e., it holds that

$$\left\{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z_0,x_0,x^*,f(x^*,y_0,z))\right\}_{\kappa,j,x,y,z} \stackrel{\mathrm{S}}{\equiv} \left\{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z_0,x_0,x^*,f(x^*,y_0,z_j))\right\}_{\kappa,j,x,y,z}.$$

Combined with Lemma 4.4 this implies that

$$\begin{aligned}\{f(x,y^*,z)\}_{\kappa,j,x,y,z} &\stackrel{\mathrm{S}}{\equiv} \left\{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z_0,x_0,x^*,f(x^*,y_0,z))\right\}_{\kappa,j,x,y,z}\\ &\stackrel{\mathrm{S}}{\equiv} \left\{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z_0,x_0,x^*,f(x^*,y_0,z_j))\right\}_{\kappa,j,x,y,z}\\ &\stackrel{\mathrm{S}}{\equiv} \{f(x,y^*,z_j)\}_{\kappa,j,x,y,z},\end{aligned}$$

where $x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z_0,x_0}$ and $y^* \leftarrow Q'_{\kappa,y,z_0,x_0,y_0}$. Finally, observe that this implies that

$$\{f(x,y^*,z_j)\}_{\kappa,x,i,j,y,z} \stackrel{\mathrm{S}}{\equiv} \{f(x,y^*,z)\}_{\kappa,x,i,j,y,z},$$

where $y^* \leftarrow Q'_{\kappa,y_i,z_0,x_0,y_0} \equiv Q_{\kappa,i}$.

We now prove the "moreover" part of the claim. Let $\mathcal{K}_{\mathsf{B}} \subseteq \mathbb{N}$ be the set of all $\kappa \in \mathbb{N}$ such that $y \equiv_x y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, and let $\mathcal{K}_{\mathsf{C}} \subseteq \mathbb{N}$ be the set of all $\kappa \in \mathbb{N}$ such that $z \equiv_x z'$ for all $x \in \mathcal{X}$ and $z, z' \in \mathcal{Z}$. For every $\kappa \in \mathcal{K}_{\mathsf{B}}$, we define the distribution $P_{\kappa,x}$ as $P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y_0,z_0,x_0}$ and let $\mathsf{S}(1^\kappa,x,x^*,w)$ output $\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y_0,z_0,x_0,x^*,w)$. Similarly, for any $\kappa \in \mathcal{K}_{\mathsf{C}}$ we define the distribution $P_{\kappa,x}$ as $P^{\mathsf{B}',\mathsf{C}}_{\kappa,x_0,y_0,z_0,x}$ and let $\mathsf{S}(1^\kappa,x,x^*,w)$ output $\mathsf{S}_1^{\mathsf{B}',\mathsf{C}}(x_0,y_0,z_0,x,x^*,w)$. Observe that since $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \mathrm{poly}(\kappa)$ identifying whether $\kappa \in \mathcal{K}_{\mathsf{B}}$ or $\kappa \in \mathcal{K}_{\mathsf{C}}$ can be done efficiently.

To conclude the proof, we now show that

$$\{\mathsf{S}(1^\kappa,x,x^*,f(x^*,y,z))\}_{\kappa,x,i,j,y,z} \stackrel{\mathrm{S}}{\equiv} \{f(x,y^*,z)\}_{\kappa,x,i,j,y,z}, \tag{12}$$

where $x^* \leftarrow P_{\kappa,x}$ and $y^* \leftarrow Q_{\kappa,i}$. Assume for the sake of contradiction that Equation (12) is false. Then the ensembles have a statistical distance of at least $1/\mathrm{poly}(\kappa)$, for infinitely many $\kappa \in \mathbb{N}$. By assumption, $\kappa \in \mathcal{K}_{\mathsf{B}} \cup \mathcal{K}_{\mathsf{C}}$ for all sufficiently large κ, hence the distance of $1/\mathrm{poly}(\kappa)$ holds for infinitely many $\kappa \in \mathcal{K}_{\mathsf{B}} \cup \mathcal{K}_{\mathsf{C}}$. We assume without loss of generality that all such infinitely many κ belong to $\mathcal{K}_{\mathsf{B}}$. However, by the definition of $P_{\kappa,x}$ and the PPT algorithm S, it holds that

$$\{\mathsf{S}(1^\kappa,x,x^*,f(x^*,y,z))\}_{\kappa\in\mathcal{K}_{\mathsf{B}},x,i,j,y,z} \equiv \{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y_0,z_0,x_0,x^*,f(x^*,y,z))\}_{\kappa\in\mathcal{K}_{\mathsf{B}},x,i,j,y,z},$$

where $x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y_0,z_0,x_0}$. Thus,

$$\{\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y_0,z_0,x_0,x^*,f(x^*,y,z))\}_{\kappa\in\mathcal{K}_{\mathsf{B}},x,i,j,y,z} \stackrel{\mathrm{S}}{\not\equiv} \{f(x,y^*,z)\}_{\kappa\in\mathcal{K}_{\mathsf{B}},x,i,j,y,z},$$

which contradicts Lemma 4.4. □

4.2.1 Proof of Lemma 4.9

Proof of Lemma 4.9. We prove the first part of the claim. The second part follows from an analogous argument. For brevity, we write x^* instead of x_1^*. Assume for the sake of contradiction that

there exist $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$, $z' \in \mathcal{Z}$, and a polynomial p, such that for infinitely many κ's it holds that

$$\Pr_{x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}} \left[x^* \not\preceq_{\mathsf{C}} x \vee x^* \not\preceq_{\mathsf{B}} x'\right] \geq 1/p(\kappa).$$

Then by the union bound it follows that for infinitely many κ's either

$$\Pr_{x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}} \left[x^* \not\preceq_{\mathsf{C}} x\right] \geq 1/2p(\kappa)$$

or

$$\Pr_{x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}} \left[x^* \not\preceq_{\mathsf{B}} x'\right] \geq 1/2p(\kappa).$$

Assume the former without loss of generality. We next show that Ensembles 1 and 4 are statistically far, that is, we show that

$$\{\mathsf{S}^{\mathsf{B},\mathsf{C}'}(x, y, z', x', x^*, f(x^*, y', z))\}_{\kappa,x,x',y,y',z,z'} \stackrel{\mathrm{S}}{\not\approx} \{(f(x, y^*, z), \mathsf{S}'_{\mathsf{B}}(y, z', x', y', y^*))\}_{\kappa,x,x',y,y',z,z'}, \tag{13}$$

where $x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}$ and $y^* \leftarrow Q'_{\kappa,y,z',x',y'}$, thus contradicting Lemma 4.4.

By Fact 2.4 it suffices to show a distinguisher. The distinguisher D will simply consider the first entry and infer the equivalence class of z. Formally, let $\mathsf{CClass}_x(w)$ output the unique $j \in [m(x)]$ such that $w \in f(x, \mathcal{Y}, \mathcal{Z}^x_j)$. Observe that CClass_x can be computed in polynomial time since $|\mathcal{Y}|$ and $|\mathcal{Z}|$ are polynomials. Then, given an output $w \in \mathcal{W}$ in the first entry, our distinguisher D outputs 1 if $\mathsf{CClass}_x(w) = j$, where $z \in \mathcal{Z}^x_j$, and outputs 0 otherwise. Clearly, given the output from the ensemble on the right-hand side of Equation (13), D outputs 1 with probability 1. We next analyze the probability of $\mathsf{S}^{\mathsf{B}',\mathsf{C}}$ outputting a value w' satisfying $\mathsf{CClass}_x(w') = j$, and show that it is significantly far from 1, thus proving that D has a noticeable distinguishing advantage.

Intuitively, if $x^* \not\preceq_{\mathsf{C}} x$ then $\mathsf{S}^{\mathsf{B},\mathsf{C}'}$ lacks information about the equivalence class of z with respect to the input x. Therefore it will have to guess it. In the following, we abuse notations and for $z \in \mathcal{Z}$ we let $\mathsf{CClass}_x(z)$ output the value $j \in [m(x)]$ satisfying $z \in \mathcal{Z}^x_j$. Let $\mathsf{S}^{\mathsf{B},\mathsf{C}'}_1$ be the first entry in the output of $\mathsf{S}^{\mathsf{B},\mathsf{C}'}$. We next formalize the above intuition. First observe that by the union bound, for each of the infinitely many κ's considered there exists $\tilde{x} \in \mathcal{X}$ satisfying $\tilde{x} \not\preceq_{\mathsf{C}} x$, such that

$$\Pr_{x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}} [x^* = \tilde{x}] \geq \frac{1}{2p(\kappa) \cdot |\mathcal{X}|}.$$

We let $z_1, z_2 \in \mathcal{Z}$ satisfy $z_1 \equiv_{\tilde{x}} z_2$ and $z_1 \not\equiv_x z_2$. The next claim roughly states that for $\mathsf{S}^{\mathsf{B},\mathsf{C}'}_1$, changing from $z = z_1$ to $z = z_2$ will not change its output with noticeable probability.

Claim 4.11. *For all $x' \in \mathcal{X}$, $y, y' \in \mathcal{Y}$, and $z' \in \mathcal{Z}$, it holds that*

$$\Pr\left[\mathsf{S}^{\mathsf{B},\mathsf{C}'}_1(x, y, z', x', x^*, f(x^*, y', z_1)) = \mathsf{S}^{\mathsf{B},\mathsf{C}'}_1(x, y, z', x', x^*, f(x^*, y', z_2))\right] \geq \frac{1}{2p(\kappa) \cdot |\mathcal{X}|} - \mathrm{neg}(\kappa),$$

where the probability is taken over the random coins of $\mathsf{S}^{\mathsf{B},\mathsf{C}'}_1$, and where $x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}$.

The claim is proven below. We first use it to show how D can distinguish with non-negligible probability. Consider the case where $z \leftarrow \{z_1, z_2\}$ is sampled uniformly at random and $x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}$, where both are sampled independently. We denote

$$q := \Pr\left[\mathsf{S}^{\mathsf{B},\mathsf{C}'}_1(x, y, z', x', x^*_1, f(x^*_1, y', z_1)) = \mathsf{S}^{\mathsf{B},\mathsf{C}'}_1(x, y, z', x', x^*_2, f(x^*_2, y', z_2))\right].$$

Then

$$\begin{aligned}
&\Pr\left[\mathsf{CClass}_x\left(\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y',z)\right)\right)=\mathsf{CClass}_x\left(z\right)\right]\\
&=\frac{1}{2}\cdot\Pr\left[\mathsf{CClass}_x\left(\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y',z_1)\right)\right)=\mathsf{CClass}_x\left(z_1\right)\right]\\
&+\frac{1}{2}\cdot\Pr\left[\mathsf{CClass}_x\left(\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y',z_2)\right)\right)=\mathsf{CClass}_x\left(z_2\right)\right]\\
&\leq\frac{1}{2}\cdot\left(q\cdot\Pr\left[\mathsf{CClass}_x\left(\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y',z_2)\right)\right)=\mathsf{CClass}_x\left(z_1\right)\right]+1-q\right)\\
&+\frac{1}{2}\cdot\Pr\left[\mathsf{CClass}_x\left(\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y',z_2)\right)\right)=\mathsf{CClass}_x\left(z_2\right)\right].
\end{aligned}$$

Now, let

$$a:=\Pr\left[\mathsf{CClass}_x\left(\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y',z_2)\right)\right)=\mathsf{CClass}_x\left(z_1\right)\right],$$

and let

$$b:=\Pr\left[\mathsf{CClass}_x\left(\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y',z_2)\right)\right)=\mathsf{CClass}_x\left(z_2\right)\right].$$

Then $a+b\leq 1$. Therefore

$$\begin{aligned}
\Pr\left[\mathsf{CClass}_x\left(\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y',z)\right)\right)=\mathsf{CClass}_x\left(z\right)\right]&\leq\frac{1}{2}\cdot(aq+1-q)+\frac{1}{2}\cdot b\\
&\leq\frac{1}{2}\cdot(aq+1-q)+\frac{1}{2}-\frac{1}{2}\cdot a\\
&=\frac{1}{2}\cdot(1-q)(1-a)+\frac{1}{2}\\
&\leq 1-\frac{1}{2}\cdot q\\
&\leq 1-\frac{1}{2}\cdot\left(\frac{1}{2p(\kappa)\cdot|\mathcal{X}|}-\mathrm{neg}(\kappa)\right),
\end{aligned}$$

where the last inequality follows from Claim 4.11. Since we assume $|\mathcal{X}|$ to be polynomial in κ, it follows that D has a noticeable distinguishing advantage. □

Proof of Claim 4.11. Since $z_1\equiv_{\tilde{x}} z_2$ there exist $z_{i_1},\ldots,z_{i_k}\in\mathcal{Z}$ such that

$$z_1\sim_{\tilde{x}} z_{i_1}\sim_{\tilde{x}}\ldots\sim_{\tilde{x}} z_{i_k}\sim_{\tilde{x}} z_2,$$

where $k=k(\kappa)$. For convenience, we let $z_{i_0}:=z_1$ and $z_{i_{k+1}}:=z_2$. This implies the existence of $y_{i_j},y'_{i_j}\in\mathcal{Y}$ for every $j\in\{0,\ldots,k+1\}$, such that the following hold.

1. $f(\tilde{x},y',z_{i_0})=f(\tilde{x},y_{i_0},z_{i_0})$.
2. For all $j\in\{0,\ldots,k\}$ it holds that $f(\tilde{x},y'_{i_j},z_{i_j})=f(\tilde{x},y_{i_{j+1}},z_{i_{j+1}})$.
3. $f(\tilde{x},y'_{i_{k+1}},z_{i_{k+1}})=f(\tilde{x},y',z_{i_{k+1}})$.

Now, observe that the event

$$\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y',z_{i_0})\right)=\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y',z_{i_{k+1}})\right)$$

is implied by the conjunction of the following four events:

1. $\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y',z_{i_0})) = \mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y_{i_0},z_{i_0}))$.
2. For all $j \in \{0,\ldots,k\}$ it holds that
$$\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y'_{i_j},z_{i_j})\right) = \mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y_{i_{j+1}},z_{i_{j+1}})\right).$$
3. For all $j \in [k]$ it holds that
$$\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y_{i_j},z_{i_j})\right) = \mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y'_{i_j},z_{i_j})\right).$$
4. $\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}\left(x,y,z',x',x^*,f(x^*,y'_{i_{k+1}},z_{i_{k+1}})\right) = \mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y',z_{i_{k+1}}))$.

Furthermore, observe that Event 2 is implied by the event $x^* = \tilde{x}$. Let E be the event that Events 1, 3, and 4 occur. Then

$$\Pr\left[\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y',z_{i_0})) = \mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y',z_{i_{k+1}}))\right] \geq \Pr[x^* = \tilde{x} \mid E] \cdot \Pr[E]$$

Additionally, by Claim 4.8 it follows that $\Pr[E] \geq 1 - \mathrm{neg}(\kappa)$, hence

$$\frac{1}{2p(\kappa)\cdot|\mathcal{X}|} \leq \Pr[x^* = \tilde{x}] \leq \Pr[x^* = \tilde{x} \mid E] \cdot \Pr[E] + \mathrm{neg}(\kappa).$$

Therefore

$$\Pr\left[\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y',z_{i_0})) = \mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y',z_{i_{k+1}}))\right] \geq \frac{1}{2p(\kappa)\cdot|\mathcal{X}|} - \mathrm{neg}(\kappa),$$

as claimed. □

4.2.2 Proof of Lemma 4.10

Proof of Lemma 4.10. We prove only the first item. The rest can be proved using a similar argument. We also write x^* instead of x_1^* for the sake of brevity. Assume towards contradiction that the claim is false. Then by Fact 2.4 there exists a distinguisher D such that for infinitely many κ, there exist $x, x' \in \mathcal{X}$, $y, y' \in \mathcal{Y}$, and $z_1, z_2, z' \in \mathcal{Z}$, satisfying

$$\Pr\left[\mathsf{D}\left(\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y',z_1))\right) = 1\right] - \Pr\left[\mathsf{D}\left(\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,f(x^*,y',z_2))\right) = 1\right] \geq \frac{1}{\mathrm{poly}(\kappa)}.$$

To alleviate notations, we will write $\mathsf{S}(x,y,x^*,w)$ instead of $\mathsf{S}_1^{\mathsf{B},\mathsf{C}'}(x,y,z',x',x^*,w)$. First, we claim that there exists $\tilde{x} \in \mathcal{X}$, such that $x^* = \tilde{x}$ occurs with noticeable probability, and D distinguishes the two ensembles for fixed $x^* = \tilde{x}$. That is, it holds that

$$\Pr[\mathsf{D}(\mathsf{S}(x,y,\tilde{x},f(\tilde{x},y',z_1))) = 1] - \Pr[\mathsf{D}(\mathsf{S}(x,y,\tilde{x},f(\tilde{x},y',z_2))) = 1] \geq \frac{1}{\mathrm{poly}(\kappa)}.$$

Indeed,

$$\begin{aligned}
\frac{1}{\mathrm{poly}(\kappa)} &\leq \Pr[\mathsf{D}(\mathsf{S}(x,y,x^*,f(x^*,y',z_1))) = 1] - \Pr[\mathsf{D}(\mathsf{S}(x,y,x^*,f(x^*,y',z_2))) = 1] \\
&= \sum_{\tilde{x}\in\mathcal{X}} \Pr[x^* = \tilde{x}] \cdot (\Pr[\mathsf{D}(\mathsf{S}(x,y,\tilde{x},f(\tilde{x},y',z_1))) = 1] - \Pr[\mathsf{D}(\mathsf{S}(x,y,\tilde{x},f(\tilde{x},y',z_2))) = 1]) \\
&\leq |\mathcal{X}| \cdot \max_{\tilde{x}\in\mathcal{X}} \{\Pr[x^* = \tilde{x}] \cdot (\Pr[\mathsf{D}(\mathsf{S}(x,y,\tilde{x},f(\tilde{x},y',z_1))) = 1] - \Pr[\mathsf{D}(\mathsf{S}(x,y,\tilde{x},f(\tilde{x},y',z_2))) = 1])\}.
\end{aligned}$$

Since $|\mathcal{X}|$ is polynomial in κ, it follows that such $\tilde{x}$ exists.

Now, let $z_{i_1}, \ldots, z_{i_k} \in \mathcal{Z}$ satisfy

$$z_1 \sim_{\tilde{x}} z_{i_1} \sim_{\tilde{x}} \ldots \sim_{\tilde{x}} z_{i_k} \sim_{\tilde{x}} z_2,$$

and denote $z_{i_0} := z_1$ and $z_{i_{k+1}} := z_2$. Since $|\mathcal{Z}| = \text{poly}(\kappa)$, by a hybrid argument there exists $\ell \in \{0, \ldots, k\}$ such that

$$\Pr\left[\mathsf{D}\left(\mathsf{S}\left(x, y, \tilde{x}, f(\tilde{x}, y', z_{i_\ell})\right)\right) = 1\right] - \Pr\left[\mathsf{D}\left(\mathsf{S}\left(x, y, \tilde{x}, f(\tilde{x}, y', z_{i_{\ell+1}})\right)\right) = 1\right] \geq \frac{1}{\text{poly}(\kappa)}.$$

Let $y'' \in \mathcal{Y}$ satisfy $f(\tilde{x}, y', z_{i_\ell}) = f(\tilde{x}, y'', z_{i_{\ell+1}})$. We now show that D can distinguish $\mathsf{S}(x, y, x^*, f(x^*, y', z_{i_{\ell+1}}))$ from $\mathsf{S}(x, y, x^*, f(x^*, y'', z_{i_{\ell+1}}))$, where $x^* \leftarrow P^{\mathsf{B},\mathsf{C}'}_{\kappa,x,y,z',x'}$, thus contradicting Claim 4.8. Indeed,

$$\begin{aligned}
&\Pr\left[\mathsf{D}\left(\mathsf{S}\left(x, y, x^*, f(x^*, y'', z_{i_{\ell+1}})\right)\right) = 1\right] - \Pr\left[\mathsf{D}\left(\mathsf{S}\left(x, y, x^*, f(x^*, y', z_{i_{\ell+1}})\right)\right) = 1\right] \\
&\quad \geq \Pr\left[x^* = \tilde{x}\right] \cdot \left(\Pr\left[\mathsf{D}\left(\mathsf{S}\left(x, y, \tilde{x}, f(\tilde{x}, y'', z_{i_{\ell+1}})\right)\right) = 1\right] - \Pr\left[\mathsf{D}\left(\mathsf{S}\left(x, y, \tilde{x}, f(\tilde{x}, y', z_{i_{\ell+1}})\right)\right) = 1\right]\right) \\
&\quad = \Pr\left[x^* = \tilde{x}\right] \cdot \left(\Pr\left[\mathsf{D}\left(\mathsf{S}\left(x, y, \tilde{x}, f(\tilde{x}, y', z_{i_\ell})\right)\right) = 1\right] - \Pr\left[\mathsf{D}\left(\mathsf{S}\left(x, y, \tilde{x}, f(\tilde{x}, y', z_{i_{\ell+1}})\right)\right) = 1\right]\right) \\
&\quad \geq \frac{1}{\text{poly}(\kappa)}.
\end{aligned}$$

□

5 Positive Results For the Point-to-Point Model

In this section, we prove Theorem 3.7 and the positive direction of Theorem 3.9, which give sufficient conditions for a functionality f to be computable with 1-security. We prove Theorem 3.7 in Section 5.1, and prove positive direction of Theorem 3.9 in Section 5.2.

5.1 Proving Theorem 3.7

In this section, we prove Theorem 3.7 by constructing a protocol for any functionality f satisfying the properties given in the theorem. Interestingly, our protocol is a slight variant of the one given by [2], where in case an attack is detected, A and one of the other parties interact in a two-party computation while ignoring the third party (even if it was honest). Our generalization chooses the other party that will interact with A in case of attack, depending on the security parameter. In particular, we get a characterization for all functionalities that can be securely computed with a such protocol.[19]

In the following section, we let $\mathcal{K}_\mathsf{B} \subseteq \mathbb{N}$ be the set of all $\kappa \in \mathbb{N}$ such that $y \equiv_x y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, and let $\mathcal{K}_\mathsf{C} \subseteq \mathbb{N}$ be the set of all $\kappa \in \mathbb{N}$ such that $z \equiv_x z'$ for all $x \in \mathcal{X}$ and $z, z' \in \mathcal{Z}$. Recall that we assume that f has the property where $\mathbb{N} \setminus (\mathcal{K}_\mathsf{B} \cup \mathcal{K}_\mathsf{C})$ is finite. Define the families of

[19] The slight variant we use, is that in our protocol, the identity of the party that will interact with A depends on the security parameter κ. To get a characterization as to which functionalities can be securely computed with the protocol of [2] directly, we need to reposition the quantifier over x in the first property from Theorem 3.7: For all sufficiently large κ and for all $x \in \mathcal{X}$, it holds that either $y \equiv_x y'$ for all $y, y' \in \mathcal{Y}$, or $z \equiv_x z'$ for all $z, z' \in \mathcal{Z}$.

sets $\mathcal{D} = \{\mathcal{D}_\kappa\}_{\kappa\in\mathbb{N}}$ as follows: let $\mathcal{D}_\kappa = \mathcal{Z}$ for all $\kappa \in \mathcal{K}_\mathsf{B}$ or $\kappa \in \mathbb{N} \setminus (\mathcal{K}_\mathsf{B} \cup \mathcal{K}_\mathsf{C})$, and let $\mathcal{D}_\kappa = \mathcal{Y}$ for all $\kappa \in \mathcal{K}_\mathsf{C}$. The two-party functionality $g : \mathcal{X} \times \mathcal{D}_\kappa \to \mathcal{W}$ is defined as

$$g(x,d) = \begin{cases} f(x,d,z^*) & \text{if } \kappa \in \mathcal{K}_\mathsf{C} \\ f(x,y^*,d) & \text{otherwise} \end{cases}$$

where $y^* \leftarrow Q_{\kappa,1}$ and $z^* \leftarrow R_{\kappa,1}$ (recall that in each case, there is only one equivalent class for the inputs of B or C). Observe that since we assume the domain of f to be of polynomial size in κ, it is possible to efficiently verify whether or not $\kappa \in \mathcal{K}_\mathsf{C}$, and efficiently compute g.

We now present a protocol for computing f in the $\{(f,\mathsf{fair}),(g,\mathsf{g.o.d.})\}$-hybrid model. By Fact 2.11 (f,fair) can be computed in the plain model, and since g is a solitary output two-party functionality, the result of Kilian [26] states it can be securely computed assuming OT. Thus, Theorem 3.7 follows from the composition theorem.

Protocol 5.1 (π_{ACOS})**.**

Private inputs: A *holds* $x \in \mathcal{X}$, B *holds* $y \in \mathcal{Y}$, *and* C *holds* $z \in \mathcal{Z}$.

Common input: the parties hold the security parameter 1^κ.

1. *The parties invoke* (f,fair) *with their inputs. Let* w_1, w_2, *and* w_2 *be the outputs of* A, B, *and* C, *respectively.*
2. *If* $w_1, w_2, w_3 \neq \perp$ *then* A *outputs* w_1.
3. *Otherwise, if* $\kappa \in \mathcal{K}_\mathsf{C}$ *then parties* A *and* B *invoke* $(g,\mathsf{g.o.d.})$ *with their inputs. If* $\kappa \notin \mathcal{K}_\mathsf{C}$ *then parties* A *and* C *invoke* $(g,\mathsf{g.o.d.})$ *with their inputs.*
4. *Party* A *outputs whatever it received from* g.

Theorem 3.7 follows from the following lemma, stating the security of π_{ACOS}.

Lemma 5.2. *Let* $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ *be a deterministic solitary output three-party functionality. Assume that* $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \text{poly}(\kappa)$, *and that the following hold.*

1. *For all sufficiently large* κ, *either* $y \equiv_x y'$ *for all* $x \in \mathcal{X}$ *and* $y, y' \in \mathcal{Y}$, *or* $z \equiv_x z'$ *for all* $x \in \mathcal{X}$ *and* $z, z' \in \mathcal{Z}$.
2. f *is* $\text{CORE}_\wedge$*-forced.*
3. *There exists an ensemble of efficiently samplable distributions* $\mathcal{P} = \{P_{\kappa,x}\}_{\kappa\in\mathbb{N},x\in\mathcal{X}}$ *and a* PPT *algorithm* S *such that*

$$\{\mathsf{S}(1^\kappa, x, x^*, f(x^*,y,z))\}_{\kappa,x,i,j,y,z} \overset{\text{S}}{\equiv} \{f(x,y^*,z)\}_{\kappa,x,i,j,y,z},$$

where $x^* \leftarrow P_{\kappa,x}$ *and* $y^* \leftarrow Q_{\kappa,i}$, *where* $Q_{\kappa,i}$ *is the distribution given the* $\text{CORE}_\wedge$*-forced property.*

Then π_{ACOS} *computes* f *with statistical 1-security in the* $\{(f,\mathsf{fair}),(g,\mathsf{g.o.d.})\}$*-hybrid model.*

Proof. π_{ACOS} is clearly correct since if all parties are honest, the fair computation of f will never abort. We next show that the protocol is secure against any adversary $\mathcal{B}$ corrupting B. The case of a corrupt C follows from a similar argument. We define the simulator $\mathsf{Sim}_{\mathcal{B}}$ as follows.

1. Query $\mathcal{B}$ for its input y' to (f, fair).
2. If $y' \neq \bot$, then send y' to the trusted party T, output whatever $\mathcal{B}$ outputs, and halt.
3. Otherwise, if $\kappa \in \mathcal{K}_{\mathsf{C}}$, query $\mathcal{B}$ for its input y'' to $(g, \mathsf{g.o.d.})$. If $\kappa \notin \mathcal{K}_{\mathsf{C}}$, then set y'' to be a default value.
4. Find the unique value $i \in [n_\wedge]$ such that $y'' \in \mathcal{Y}_i^\wedge$, sample $y^* \leftarrow Q_{\kappa,i}$, and send y^* to the trusted party.
5. Output whatever $\mathcal{B}$ outputs and halt.

Since B receives no messages in the protocol, the inputs y' and y'' chosen by the adversary in the real world, are identically distributed to their ideal world counterparts. Furthermore, it suffices to show that the output of A in both worlds are statistically close. Clearly, given that $y' \neq \bot$ or $\kappa \notin \mathcal{K}_{\mathsf{C}}$ the output of A in both worlds is identical. Otherwise, the output of A in the real world is $f(x, y'', z^*)$, where $z^* \leftarrow R_{\kappa,1}$ (recall that all z are equivalent with respect to $\equiv_x$). On the other hand, in the ideal world, the output of A is $f(x, y^*, z)$, where $y^* \leftarrow Q_{\kappa,i}$ and $i \in [n_\wedge]$ is such that $y'' \in \mathcal{Y}_i^\wedge$. By the $\mathrm{CORE}_\wedge$-forced property of f, the two distributions are statistically close.

We next fix an adversary $\mathcal{A}$ corrupting A. We define the simulator $\mathsf{Sim}_{\mathcal{A}}$ as follows.

1. Query $\mathcal{A}$ for its input x' to (f, fair).
2. If $x' \neq \bot$, then send x' to the trusted party T, pass the received output to $\mathcal{A}$, output whatever it outputs, and halt.
3. Otherwise, query $\mathcal{A}$ for its input x'' to $(g, \mathsf{g.o.d.})$.
4. Sample $x^* \leftarrow P_{\kappa,x''}$ and send it to the trusted party T.
5. Given an output w from T, send to $\mathcal{A}$ the result of $\mathsf{S}(1^\kappa, x'', x^*, w)$, output whatever $\mathcal{A}$ outputs, and halt.

Since no honest party has an output, it suffices to show that the view of $\mathcal{A}$ in both worlds are statistically close. First, since $\mathcal{A}$ does not receive any message before the invocation of g, it follows that x' and x'' are identically distributed in both worlds. Now, in the real world, the only message that $\mathcal{A}$ receives is $f(x, y, z^*)$ if $\kappa \in \mathcal{K}_{\mathsf{C}}$ or $f(x, y^*, z)$ if $\kappa \notin \mathcal{K}_{\mathsf{C}}$, where $y^* \leftarrow Q_{\kappa,1}$ and $z^* \leftarrow R_{\kappa,1}$. In the ideal world, on the other hand, it receives $\mathsf{S}(1^\kappa, x'', x^*, f(x^*, y, z))$, where $x^* \leftarrow P_{\kappa,x''}$. By the assumption on f, this is statistically close to $f(x, y^*, z)$, thus security holds with respect to all $\kappa \notin \mathcal{K}_{\mathsf{C}}$. For all $\kappa \in \mathcal{K}_{\mathsf{C}}$, the $\mathrm{CORE}_\wedge$-forced property implies that $f(x, y^*, z)$ is statistically close to $f(x, y, z^*)$, concluding the proof. □

5.2 Proving The Positive Direction of Theorem 3.9

In this section, we present a 1-secure protocol for computing the functionalities captured by Theorem 3.9. We first present an intuitive description of the protocol.

Similarly to π_{ACOS}, the parties first compute f fairly and if the computation followed through, then A outputs the result. Otherwise, the parties do the following. Both B and C (locally) compute the equivalence classes of their respective inputs with respect to the lexicographically smallest minimum input χ (i.e., smaller than all $x \in \mathcal{X}$ with respect to $\preceq$). They then send these values to A, who samples their inputs y^* and z^* according to the appropriate distribution given by the $\text{CORE}_\wedge$-forced assumption, and outputs $f(x, y^*, z^*)$.[20]

Intuitively, the only information a corrupt A obtains from the above interaction is the equivalence classes of the inputs of B and C with respect to χ. This can be simulated by sending χ to the trusted party and searching (by brute-force) for the equivalence classes. This can be done since by the definition of CORE partition, the equivalence classes are fully determined by the output and χ.

We next formalize the above intuition. We present the protocol in the (f, fair)-hybrid model. By Fact 2.11 (f, fair) can be computed in the plain model. Thus, Theorem 3.9 follows from the composition theorem. In the following we let χ be the lexicographically smallest minimum input with respect to $\preceq$.

Protocol 5.3 (π)**.**

Private inputs: A *holds* $x \in \mathcal{X}$, B *holds* $y \in \mathcal{Y}$, *and* C *holds* $z \in \mathcal{Z}$.

Common input: the parties hold the security parameter 1^κ.

1. *The parties invoke* (f, fair) *with their inputs. Let* w_1, w_2, *and* w_2 *be the outputs of* A, B, *and* C, *respectively.*
2. *If* $w_1, w_2, w_3 \neq \perp$ *then* A *outputs* w_1.
3. *Otherwise, party* B *finds the (unique) index* $i \in [n(\chi)]$ *such that* $y \in \mathcal{Y}_i^\chi$ *and sends it to* A. *Similarly,* C *sends the index* $j \in [m(\chi)]$ *such that* $z \in \mathcal{Z}_j^\chi$.
4. A *samples and outputs* $w = f(x, y^*, z_j)$, *where* $y^* \leftarrow Q_{\kappa,i}$ *and* $Q_{\kappa,i}$ *is the distribution given by the* $\text{CORE}_\wedge$*-forced property, and where* z_j *is the lexicographically smallest element of* $\mathcal{Z}_j^\chi$.

The next lemma immediately proves Theorem 3.9.

Lemma 5.4. *Let* $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ *be a deterministic solitary output three-party functionality. Assume that* $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \text{poly}(\kappa)$, *that for all sufficiently large* κ, *there exists* $\chi \in \mathcal{X}$ *such that for all* $x \in \mathcal{X}$ *it holds that* $\chi \preceq x$, *and that* f *is* $\text{CORE}_\wedge$*-forced. Then* π *computes* f *with statistical 1-security in the* (f, fair)*-hybrid model.*

Proof. Clearly, π is correct since if all parties are honest, the fair computation of f will never abort. We next show that the protocol is secure against any adversary $\mathcal{B}$ corrupting B. The case of a corrupt C follows from a similar argument. We define the simulator $\mathsf{Sim}_\mathcal{B}$ as follows.

[20] In the formal description of the protocol below, we let A set one of the random inputs to be the lexicographically smallest element in its equivalence class. This is only for the sake of presentation and it does not affect the security of the protocol.

1. Query $\mathcal{B}$ for its input y' to (f, fair).
2. If $y' \neq \perp$, then send y' to the trusted party T, output whatever $\mathcal{B}$ outputs, and halt.
3. Otherwise, $\mathcal{B}$ sends to A a value $i' \in [n(\chi)]$.
4. Sample $y^* \leftarrow Q_{\kappa,i'}$, send it to T, output whatever $\mathcal{B}$ outputs, and halt.

Since $\mathcal{B}$ receives no messages, it suffices to show that the outputs of A in both worlds are statistically close. Now, observe that the messages $\mathcal{B}$ sends are identically distributed in both worlds. For the case where $\mathcal{B}$ sends and input $y' \neq \perp$ to (f, fair), the output of A in both worlds is $f(x, y', z)$. Next, assume that $\mathcal{B}$ sends $y' = \perp$ and then sends i'. Then the output of A in the real world is $f(x, y^*, z_j)$, where $y^* \leftarrow Q_{\kappa,i'}$ and where z_j in the lexicographically smallest element in $\mathcal{Z}_j^\chi$. In the ideal world, the output of A is $f(x, y^*, z)$, where y^* is distributed as before. By the CORE$_\wedge$-forced property of f, the two distributions are statistically close.

We next consider an adversary $\mathcal{A}$ corrupting A. We define the simulator $\mathsf{Sim}_\mathcal{A}$ as follows.

1. Query $\mathcal{A}$ for its input x' to (f, fair).
2. If $x' \neq \perp$, then send x' to the trusted party T, pass the received output to $\mathcal{A}$, output whatever it outputs, and halt.
3. Otherwise, send χ to T.
4. Let w be the output sent by T.
5. Find the (unique) $i \in [n(\chi)]$ and $j \in [m(\chi)]$ such that there exist $y' \in \mathcal{Y}_i^\chi$ and $z' \in \mathcal{Z}_j^\chi$ satisfying $w = f(x, y', z')$.
6. Send i and j to $\mathcal{A}$, output whatever it outputs, and halt.

Since B and C have no output, it suffices to show that the view of $\mathcal{A}$ in both worlds are close (in fact, they are identically distributed). Now, if $\mathcal{A}$ sent an input $x' \neq \perp$ to (f, fair), then in both worlds the only message that $\mathcal{A}$ sees is $f(x,', y, z)$. Otherwise, it obtains two values i and j representing equivalence classes over $\mathcal{Y}$ and $\mathcal{Z}$, respectively. Observe that the classes are the same in both worlds since they were computed with respect to the minimum input χ.

□

6 Computation With Broadcast and a Dishonest Majority

In this section we show that all three-party functionalities captured by our positive results from the previous section, i.e., Theorems 3.7 and 3.9, can be computed given a broadcast channel, tolerating two corruptions. In fact, we can even relax some of the requirements. Both of our results (stated below) improve the results of Halevi et al. [25], which identify several classes of functionalities that can be securely computed.

We next state our two results. We state the results only for deterministic functionalities, as the randomized case can be handled with using a standard reduction. The first result states that a generalized class of functionalities of those captured by Theorem 3.7, can be computed with full security.

Theorem 6.1. *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality. Assume that oblivious transfer exists, that $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \text{poly}(\kappa)$, and that one of the following holds.*

1. *For all sufficiently large κ, either $y \equiv_x y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, or $z \equiv_x z'$ for all $x \in \mathcal{X}$ and $z, z' \in \mathcal{Z}$.*
2. *f is $\text{CORE}_\wedge$-forced.*

Then f can be computed with computational full security.

The proof is given in Section 6.1.1. We next state our second result, which states that a generalized class of functionalities of those captured by Theorem 3.9, can be computed with full security. This result directly improves one of the results by Halevi et al. [25], who showed that any *all-but-one forced* solitary output functionality (i.e., either B or C but not necessarily both, can fix the output distribution), can be computed with full security.

For this result, we need a stronger notion than (all-but-P) $\text{CORE}_\wedge$-forced. Intuitively, all-but-P strong $\text{CORE}_\wedge$-forced requires that the output distributions in some of the combinatorial rectangles in $\mathcal{R}_\wedge$ to be close. Roughly speaking, for every $x \in \mathcal{X}$ there exists a minimal input χ smaller than x with respect to an appropriate partial order, such that the output distribution in the rectangles in $\mathcal{R}_\chi$ can be fixed by the parties. Note that for $\text{CORE}_\wedge$-forced, the distributions for different rectangles could be far. Similarly to [25], for our construction it suffices to consider an all-but-P strong $\text{CORE}_\wedge$-forced functionality, for some $\mathsf{P} \in \{\mathsf{B}, \mathsf{C}\}$, where the remaining party in $\{\mathsf{B}, \mathsf{C}\} \setminus \{\mathsf{P}\}$ can fix the output distributions in the rectangles.

Definition 6.2 (All-but-P strong $\text{CORE}_\wedge$-forced)**.** *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality. We say that f is* all-but-B strong $\text{CORE}_\wedge$-forced*, if there exists an ensemble of efficiently samplable distributions $\{R_{\kappa,j}\}_{\kappa \in \mathbb{N}, j \in [m_\wedge]}$ such that the following holds. For every sequence of inputs $\mathbf{x} = \{x_\kappa \in \mathcal{X}\}_{\kappa \in \mathbb{N}}$, there exists a sequence of B-minimal inputs $\chi_\mathsf{B} = \{\chi_\kappa \in \mathcal{X}\}_{\kappa \in \mathbb{N}}$, such that $\chi_\kappa \preceq x_\kappa$ for all $\kappa \in \mathbb{N}$, and*

$$\{f(x_\kappa, y, z^*)\}_{\kappa \in \mathbb{N}, x_\kappa \in \mathcal{X}, i \in [n_\wedge], j \in [m_\wedge], y \in \mathcal{Y}_i^\wedge, z \in \mathcal{Z}_j^\wedge} \overset{\text{S}}{\equiv} \{f(x_\kappa, y_\chi, z^*)\}_{\kappa \in \mathbb{N}, x_\kappa \in \mathcal{X}, i \in [n_\wedge], j \in [m_\wedge], y \in \mathcal{Y}_i^\wedge, z \in \mathcal{Z}_j^\wedge},$$

where y_χ is the lexicographically smallest element such that $y_\chi \equiv_{\chi_\kappa} y$ and where $z^* \leftarrow R_{\kappa,j}$. We define C-strong $\text{CORE}_\wedge$-forced similarly.

Observe that for any functionality with a minimum element for A, strong-$\text{CORE}_\wedge$-forced is equivalent to standard $\text{CORE}_\wedge$-forced (since the minimum input satisfies the conditions stated above).

We are now ready to state the result.

Theorem 6.3. *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality. Assume that oblivious transfer exists, that $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \text{poly}(\kappa)$ and that f is all-but-P strong $\text{CORE}_\wedge$-forced, for some $\mathsf{P} \in \{\mathsf{B}, \mathsf{C}\}$. Then f can be computed with computational full security.*

The proof of Theorem 6.3 is given in Section 6.1.2. We first discuss an interesting consequence of Theorems 6.1 and 6.3. Observe that the conditions stated in Theorems 6.1 and 6.3 are relaxations of the conditions stated in Theorems 3.7 and 3.9, respectively. Therefore, for the families of functionalities discussed in Section 3.2.1 (e.g., ternary-output), for which we have a complete

characterization in the point-to-point model, it holds that if a functionality f can be computed assuming an honest majority but without a broadcast channel, then f can also be computed with a broadcast channel, but with no honest majority. Thus, we have the following corollary.

Corollary 6.4. *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality. Assume that oblivious transfer exists, that $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \text{poly}(\kappa)$ and that either $|\mathcal{W}| \leq 3$ or $|\mathcal{X}| \leq 2$. Then, if f can be computed with computational 1-security in the point-to-point model (without broadcast), then f can be computed with computational full security given a broadcast channel.*

Furthermore, since the conditions stated in Theorems 6.1 and 6.3 are *strict* relaxations of the conditions stated in Theorems 3.7 and 3.9, it follows that the converse is not true. Thus, the more common broadcast assumption is a *strictly stronger* than the honest majority assumption, for the above families of functionalities. A concrete example that showcases the separation is the following solitary output three-party variant of the GHKL function [24]. That is, the function $\mathsf{soGHKL} : \emptyset \times \{0,1,2\} \times \{0,1\} \to \{0,1\}$ given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

where B chooses a row, C chooses a column, and the output of A is the value written in the chosen entry. Indeed, all inputs of B and C are equivalent, yet soGHKL is not forced and thus cannot be computed in the point-to-point model. On the other hand, Theorem 6.3 requires that only one of the parties needs to be able to fix the distribution of the output.

6.1 Proofs of the Results

In this section, we prove Theorems 6.1 and 6.3. We start with proving Theorem 6.1 in Section 6.1.1, and then proving Theorem 6.3 in Section 6.1.2.

6.1.1 Proof of Theorem 6.1

The idea of the protocol is as follows. The parties first compute a 2-out-of-2 secret sharing of the output, where one share is given to party A, and the other share is given to either B or C depending on κ. This second party that holds a share, denoted P, then sends it to A who reconstructs the output. In case, P does not send the share, A replaces the aborting party's input with a default input, samples the input of the third party according to the distribution associated with it as given by the $\text{CORE}_\wedge$-forced property, and computes the function on these inputs.

Intuitively, the party that does not receive any share provides no advantage to the adversary. Additionally, corrupting A and P gives to the adversary only the output, hence it cannot attack the protocol. Finally, when A is honest, corrupting and aborting P can be simulated by sending an input sampled according to the appropriate distribution associated with the default input of P, as given by the $\text{CORE}_\wedge$-forced property of f.

We next formalize the above intuition. Similarly to Section 5.1, let $\mathcal{K}_\mathsf{B} \subseteq \mathbb{N}$ be the set of all $\kappa \in \mathbb{N}$ such that $y \equiv_x y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, and let $\mathcal{K}_\mathsf{C} \subseteq \mathbb{N}$ be the set of all $\kappa \in \mathbb{N}$ such that $z \equiv_x z'$ for all $x \in \mathcal{X}$ and $z, z' \in \mathcal{Z}$. Recall that we assume that f has the property where

$\mathbb{N} \setminus (\mathcal{K}_{\mathsf{B}} \cup \mathcal{K}_{\mathsf{C}})$ is finite. Let $\mathsf{P} = \mathsf{B}$ if $\kappa \in \mathcal{K}_{\mathsf{C}}$, and let $\mathsf{P} = \mathsf{C}$ otherwise. Further let P' be the remaining party in $\{\mathsf{B}, \mathsf{C}\} \setminus \{\mathsf{P}\}$.

We let $\mathsf{ShrGen}_f(x, y, z)$ be the three-party functionality that computes a 2-out-of-2 additive secret sharing of the output $f(x, y, z)$. The functionality gives the shares to only A either B or C depending on κ (see Algorithm 6.5 below for a formal description). Additionally, it signs each of the shares using a one-time MAC. To simplify the presentation, we assume that a corrupted party will not modify its share, but may abort and not send it at all.

Algorithm 6.5 (Functionality ShrGen_f)**.**

Private inputs: A *holds* $x \in \mathcal{X}$, B *holds* $y \in \mathcal{Y}$, *and* C *holds* $z \in \mathcal{Z}$.

Common input: the parties hold the security parameter 1^κ.

1. *Compute* $w = f(x, y, z)$.
2. *Share* w *in an 2-out-of-2 secret sharing scheme. For* $\mathsf{Q} \in \{\mathsf{A}, \mathsf{P}\}$, *let* $w[\mathsf{Q}]$ *denote the share associated with party* Q.
3. *Party* A *receives* $w[\mathsf{A}]$, *and party* P *receives* $w[\mathsf{P}]$.

We next present a protocol for computing f in the $\{(\mathsf{ShrGen}_f, \mathsf{id\text{-}abort})\}$-hybrid model.

Protocol 6.6 (π_{bc})**.**

Private inputs: A *holds* $x \in \mathcal{X}$, B *holds* $y \in \mathcal{Y}$, *and* C *holds* $z \in \mathcal{Z}$.

Common input: the parties hold the security parameter 1^κ.

1. *The parties invoke* $(\mathsf{ShrGen}_f, \mathsf{id\text{-}abort})$ *with their inputs.*
   - *If* A *aborts then the computation halts.*
   - *Otherwise, if* P' *aborts then the parties restart without it, and with their input being set to default values. If* P *aborts, go to Step 3.*
2. *If* P *is still active, it sends* $w[\mathsf{P}]$ *to* A.
3. *If* P *aborts during any step of the computation, then* A *does the following.*
   (a) *If* $\mathsf{P} = \mathsf{B}$, *set* $y_0 \in \mathcal{Y}$ *to be the lexicographically smallest element, sample* $z^* \leftarrow R_{\kappa,1}$, *and output* $f(x, y_0, z^*)$.
   (b) *If* $\mathsf{P} = \mathsf{C}$, *set* $z_0 \in \mathcal{Z}$ *to be the lexicographically smallest element, sample* $y^* \leftarrow Q_{\kappa,1}$, *and output* $f(x, y^*, z_0)$.
4. *Otherwise,* A *outputs* $w[\mathsf{A}] + w[\mathsf{P}]$.

The next lemma immediately proves Theorem 6.1

Lemma 6.7. *Let $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ be a deterministic solitary output three-party functionality. Assume that $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \mathrm{poly}(\kappa)$, and that the following hold.*

1. *For all sufficiently large κ, either $y \equiv_x y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, or $z \equiv_x z'$ for all $x \in \mathcal{X}$ and $z, z' \in \mathcal{Z}$.*
2. *f is $\mathrm{CORE}_\wedge$-forced.*

Then π_{bc} computes f with statistical full security in the $(\mathsf{ShrGen}_f, \mathsf{id\text{-}abort})$-hybrid model.

Proof. Clearly, the protocol is correct. Fix an adversary $\mathcal{A}$ corrupting a subset of the parties. Observe that if A is corrupted, then the adversary sees at most two shares whose sum is the output $f(x, y, z)$. Therefore, this case can be simulated.

We now assume that A is honest. In this case, $\mathcal{A}$ (possibly) sees only the random value $w[\mathsf{P}]$. If P' is corrupted and aborts during the call to $(\mathsf{ShrGen}_f, \mathsf{id\text{-}abort})$, then the simulator replaces its input with a default value. If P is corrupted and aborts during any step of the protocol, then the simulator replaced its input with a random input as follows: If $\mathsf{P} = \mathsf{B}$ then send $y^* \leftarrow Q_{\kappa,i}$ where $i \in [n_\wedge]$ is the unique index satisfying $y_0 \in \mathcal{Y}_i^\wedge$. Otherwise, if $\mathsf{P} = \mathsf{C}$ then send $z^* \leftarrow R_{\kappa,j}$ where $j \in [m_\wedge]$ is the unique index satisfying $z_0 \in \mathcal{Z}_j^\wedge$.

Then for all $\kappa \in \mathcal{K}_\mathsf{C}$ (i.e., $\mathsf{P} = \mathsf{B}$) it holds that the output of A in the ideal world is $f(x, y^*, z)$, which by the $\mathrm{CORE}_\wedge$-forced assumption, is statistically close to $f(x, y_0, z^*)$ that is the output of A in the real world. Similarly, for all $\kappa \notin \mathcal{K}_\mathsf{C}$ the outputs are statistically close. □

6.1.2 Proof of Theorem 6.3

We first present an intuitive description of the protocol. Towards constructing the protocol, we use an algorithm, denoted Str_P for $\mathsf{P} \in \{\mathsf{B}, \mathsf{C}\}$, which computes efficiently the sequence of minimal inputs that satisfy the conditions from Definition 6.2, for any all-but-P strong $\mathrm{CORE}_\wedge$-forced three-party solitary output functionality. We present the algorithm in Section 6.1.3 below. We first use it to construct the protocol.

Assume without loss of generality that the functionality f is all-but-B strong $\mathrm{CORE}_\wedge$-forced. The idea is for the parties to compute a 3-out-of-3 secret sharing of the output. Additionally, A and B will receive shares of the equivalence class of the input y held by B, with respect to the input $\chi \preceq x$ guaranteed to exist by the strong $\mathrm{CORE}_\wedge$-forced assumption.

The protocol proceeds as follows. First, B sends its two shares to A. In case of abort, A and C restart the protocol with the input of B set to a default value. Otherwise, C sends its input to A, which reconstructs the output. In case C aborts, A reconstructs B's equivalence class and chooses any input from it. It then samples an input for C according to a default distribution and computes f on these inputs. Intuitively, a corrupted B or C can be simulated by sending to the trusted party either a default input or sample an input according to the distribution guaranteed to exist by the $\mathrm{CORE}_\wedge$-forced assumption. Additionally, a corrupt A only learns the output and the equivalence class of y with respect to χ, which can be inferred from the output since $\chi \preceq x$.

We next present a formal description of the protocol. Denote $N = N_\kappa = \prod_{x \in \mathcal{X}} n(x)$. We next define the three-party share generator functionality $\mathsf{ShrGen}'_f(x, y, z)$. Roughly speaking, it computes a three-out-of-three additive secret sharing of the output $f(x, y, z)$, and it shares between A and B the equivalence class of y with respect to to χ, for the B-minimal χ that is computed by the algorithm from Claim 6.12 (see Algorithm 6.8 below for a formal description). Additionally,

it signs each of the shares using a one-time MAC. To simplify the presentation, we assume that a corrupted party will not modify its shares, but may abort and not send them at all. We now present a formal description of ShrGen'_f.

Algorithm 6.8 (Functionality ShrGen'_f)**.**

Private inputs: A *holds* $x \in \mathcal{X}$, B *holds* $y \in \mathcal{Y}$, *and* C *holds* $z \in \mathcal{Z}$.

Common input: the parties hold the security parameter 1^κ.

Computation:

1. *Compute* $w = f(x, y, z)$.
2. *Compute* $\chi = \mathsf{Str}_\mathsf{B}(1^\kappa, x)$.
3. *Find the unique index* $i \in [n(\chi)]$ *satisfying* $y \in \mathcal{Y}_i^\chi$.

Sharing phase:

1. *Share* w *in an 3-out-of-3 secret sharing scheme. For* $\mathsf{P} \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ *let* $w[\mathsf{P}]$ *denote the share associated with party* P.
2. *Sample* $i[\mathsf{A}] \leftarrow [N]$, *and let* $i[\mathsf{B}] = i - i[\mathsf{A}] \mod N$.[21]

Output: Party A *receives* $(\chi, w[\mathsf{A}], i[\mathsf{A}])$, *party* B *receives* $(w[\mathsf{B}], i[\mathsf{B}])$, *and party* C *receives* $w[\mathsf{C}]$ *(note that* χ *can also be computed locally by* A*).*

We next present a protocol for computing f in the $\{(\mathsf{ShrGen}'_f, \mathsf{id\text{-}abort})\}$-hybrid model.

Protocol 6.9 (π'_{bc})**.**

Private inputs: A *holds* $x \in \mathcal{X}$, B *holds* $y \in \mathcal{Y}$, *and* C *holds* $z \in \mathcal{Z}$.

Common input: the parties hold the security parameter 1^κ.

1. *The parties invoke* $(\mathsf{ShrGen}'_f, \mathsf{id\text{-}abort})$ *with their inputs.*
   - *If* A *aborts then the computation halts.*
   - *Otherwise, if any other party aborts the parties restart without it, and their input being set to a default value.*
2. *If* B *is still active, it sends* $(w[\mathsf{B}], i[\mathsf{B}])$ *to* A.
   - *In case* B *aborts, the parties set its input to a default value and restart the protocol without it.*
3. *If* C *is still active, it sends* $w[\mathsf{C}]$ *to* A.
   - *In case* C *aborts,* A *does the following.*

     (a) Set $i = i[\mathsf{A}] + i[\mathsf{B}] \mod n(\chi)$ *if* B *is active, and set* $i = 1$ *otherwise.*

[21] We let $\bmod n$ output in $[n]$ instead of $\{0, \ldots, n-1\}$ for convenience.

(b) *Compute and output* $w^* = f(x, y_\chi, z^*)$*, where* y_χ *is the lexicographically smallest element of* $\mathcal{Y}_i^\chi$*, and where* $z^* \leftarrow R_{\kappa,1}$.

4. *If no party aborts,* A *reconstructs the output.*

The next lemma immediately proves Theorem 6.3

Lemma 6.10. *Let* $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ *be a deterministic solitary output three-party functionality. Assume that* $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}| = \text{poly}(\kappa)$*, and that* f *is all-but-*B *strong. Then* π'_{bc} *computes* f *with statistical full security in the* $(\mathsf{ShrGen}'_f, \mathsf{id\text{-}abort})$*-hybrid model.*

Proof. Clearly, the protocol is correct. Fix an adversary $\mathcal{A}$ corrupting a subset of the parties. We separate the proof into two cases. For the first case, let us assume that A is honest. We assume that both B and C are corrupted. The case where exactly one of them is corrupted can be handled similarly. The simulator $\mathsf{Sim}_\mathcal{A}$ does the following.

1. Query $\mathcal{A}$ for its input y and z to $(\mathsf{ShrGen}'_f, \mathsf{id\text{-}abort})$.
2. Send to $\mathcal{A}$ the values $(w[\mathsf{B}], i[\mathsf{B}])$ and $w[\mathsf{C}]$, where $w[\mathsf{B}], w[\mathsf{C}] \leftarrow \mathcal{W}$ and where $i[\mathsf{B}] \leftarrow [N]$. If $\mathcal{A}$ replies with $(\mathsf{abort}, \mathsf{P})$, for some $\mathsf{P} \in \{\mathsf{B}, \mathsf{C}\}$, then go back to Step 1 with the input of P set to a default value.
3. Otherwise, if B aborts at Step 2, then go back to Step 1 with the input of B set to a default value.
4. If C aborts, sample $z^* \leftarrow R_{\kappa,1}$ and send to the trusted party (y', z^*), where $y' = y$ if B is active, and y' is a default value otherwise. Output whatever $\mathcal{A}$ outputs, and halt.
5. Otherwise, if C does not abort, send y and z to the trusted party T, output whatever $\mathcal{A}$ outputs, and halt.

It's clear that the views of $\mathcal{A}$ in both worlds are identically distributed, and in particular, its responses are identically distributed as well. First, consider the case where C does not abort at Step 3. Then the output of A in both worlds is $f(x, y', z)$, where $y' = y$ if B is active, and y' is a default value otherwise.

Now, consider the case where C does abort at Step 3. Let us first consider the real world. Then the value i set by A is $i = 1$ if B is inactive, and $i \in [n(\chi)]$ is the unique index satisfying $y \in \mathcal{Y}_i^\chi$ if B is active. Then the output of A is of the form $f(x, y'_{\mathsf{real}}, z^*)$, where $z' \leftarrow R_{\kappa,1}$, and where y'_{real} is the lexicographically smallest element in $\mathcal{Y}_i^\chi$. Let us now consider the ideal world. The of A in this case is $f(x, y'_{\mathsf{ideal}}, z^*)$, where $z^* \leftarrow R_{\kappa,1}$ as before, and where $y'_{\mathsf{ideal}} = y$ if B is active, and is a default value otherwise. Observe that, regardless of whether or not B is active, it holds that $y'_{\mathsf{real}} \equiv_\chi y'_{\mathsf{ideal}}$. By Claim 6.12, it follows that both outputs are statistically close.

We now assume that A is corrupted. We only deal with the case where C is also corrupted since the other cases are simpler. We define the simulator $\mathsf{Sim}_\mathcal{A}$ as follows.

1. Query $\mathcal{A}$ for its inputs to $(\mathsf{ShrGen}'_f, \mathsf{id\text{-}abort})$. If the adversary aborts then restart without the aborting party. Let x and z be the inputs used in the last call.

2. Send to $\mathcal{A}$ random shares of the output $w[\mathsf{A}], w[\mathsf{C}] \leftarrow \mathcal{W}$, the random share $i[\mathsf{A}] \leftarrow [N]$, and the R-minimal element χ as computed by ShrGen'_f (recall that χ depends only on x).
3. Send x and z to the trusted party T, and let w be the output received from T.
4. Set $w[\mathsf{B}] = w - w[\mathsf{A}] - w[\mathsf{C}]$.
5. Find $i \in [n(\chi)]$ for which there exist $y' \in \mathcal{Y}_i^{\chi}$ and $z' \in \mathcal{Z}$ such that $w = f(x, y', z')$.
6. Let $i[\mathsf{B}] = i - i[\mathsf{A}] \mod n(\chi)$.
7. Send to $\mathcal{A}$ the pair $(w[\mathsf{B}], i[\mathsf{B}])$, output whatever $\mathcal{A}$ outputs and halt.

Clearly, since $n(\chi)$ divides N, the view of $\mathcal{A}$ in both worlds are identically distributed. Since no honest party obtains an output from T, it follows that the real and ideal worlds are identically distributed. □

6.1.3 The $\mathsf{Str}_{\mathsf{P}}$ Algorithm For Finding χ_{B} and χ_{C}

We next present the idea behind the algorithm $\mathsf{Str}_{\mathsf{P}}$ for $\mathsf{P} = \mathsf{B}$ (the case where $\mathsf{P} = \mathsf{C}$ is analogous). For a given input x, the algorithm searches for a B-minimal input χ_{B} and two rectangles $\mathcal{Y}_i^{\chi_{\mathsf{B}}} \times \mathcal{Z}_j^{\chi_{\mathsf{B}}}$ and $\mathcal{Y}_{i'}^{\chi_{\mathsf{B}}} \times \mathcal{Z}_j^{\chi_{\mathsf{B}}}$ such that statistical distance between the output distributions that are associated with the rectangles (as given by sampling either y or z according to the corresponding distributions and computing f over these inputs) is maximized. The algorithm then outputs χ_{B}. Intuitively, if χ_{B} does not satisfy the properties from Definition 6.2, then this contradicts the maximality of the statistical distance. Note that the algorithm is efficient since we assume the domain of f to be of polynomial size. We now formalize the above intuition.

Algorithm 6.11 ($\mathsf{Str}_{\mathsf{P}}$)**.**

Setting: Suppose that $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$ is a three-party solitary output all-but-P strong $\mathrm{CORE}_{\wedge}$-forced functionality, and let $\mathcal{Q}$ and $\mathcal{R}$ be the associated distribution ensembles.

Input: The security parameter 1^{κ} and $x \in \mathcal{X}$.

Computation:

- *If $\mathsf{P} = \mathsf{B}$, find a B-minimal $\chi \preceq_{\mathsf{B}} x$, $\hat{y}, \hat{y}' \in \mathcal{Y}$, and $j \in [m(\chi_{\mathsf{B}})]$ such that $\hat{y} \not\equiv_{\chi} \hat{y}'$ and $\hat{y} \equiv_x \hat{y}'$, that maximizes*
$$\mathrm{SD}\left(f(x, \hat{y}, z_1^*),\ f(x, \hat{y}', z_2^*)\right),$$
where $z_1^*, z_2^* \leftarrow R_{\kappa,j}$ are independent.
- *If $\mathsf{P} = \mathsf{C}$ find a C-minimal $\chi \preceq_{\mathsf{C}} x$, $i \in [n(\chi_{\mathsf{C}})]$ and $\hat{z}, \hat{z}' \in \mathcal{Z}$ such that $\hat{z} \not\equiv_{\chi} \hat{z}'$ and $\hat{z} \equiv_x \hat{z}'$, that maximizes*
$$\mathrm{SD}\left(f(x, y_1^*, \hat{z}),\ f(x, y_2^*, \hat{z}')\right),$$
where $y_1^*, y_2^* \leftarrow Q_{\kappa,i}$ are independent.

Output: χ.

Claim 6.12. *Suppose that f is all-but-P strong $\mathrm{CORE}_\wedge$-forced, and fix a sequence of inputs $\mathbf{x} = \{x_\kappa \in \mathcal{X}\}_{\kappa\in\mathbb{N}}$. Define the sequence $\chi = \{\chi_\kappa\}_{\kappa\in\mathbb{N}}$, where χ_κ is the output of $\mathsf{Str}_\mathsf{P}(1^\kappa, x_\kappa)$. Then χ is the sequence guaranteed to exist by the all-but-P strong $\mathrm{CORE}_\wedge$-forced assumption. That is, if $\mathsf{P} = \mathsf{B}$ then*

$$\{f(x_\kappa, y, z^*)\}_{\kappa,x_\kappa,i,j,y,z} \overset{\mathrm{S}}{\equiv} \{f(x_\kappa, y_\chi, z^*)\}_{\kappa,x_\kappa,i,j,y,z}\,,$$

where y_χ is the lexicographically smallest element such that $y_\chi \equiv_{\chi_\kappa} y$ and where $z^* \leftarrow R_{\kappa,j}$. Similarly, if $\mathsf{P} = \mathsf{C}$ then an analogous statement holds.

Proof. We prove the statement only for the case where $\mathsf{P} = \mathsf{B}$, as the other case is analogous. Assume that the claim is false. Then for infinitely many κ, there exist $i \in [n_\wedge]$, $j \in [m_\wedge]$, $y \in \mathcal{Y}_i^\wedge$, and $z \in \mathcal{Z}_j^\wedge$, such that

$$\mathrm{SD}\left(f(x_\kappa, y, z_1^*),\ f(x_\kappa, y_\chi, z_2^*)\right) > 1/\operatorname{poly}(\kappa),$$

where $z_1^*, z_2^* \leftarrow R_{\kappa,j}$ are independent. Since $y \equiv_{\chi_\kappa} y_\chi$, by the $\mathrm{CORE}_\wedge$-forced property of f, it follows that there exists a different sequence $\chi' = \{\chi'_\kappa\}_{\kappa\in\mathbb{N}}$ such that $y \not\equiv_{\chi'_\kappa} y_\chi$. However, this contradicts the maximality assumption over χ. □

7 Various Interesting Examples

In this section, we provide some interesting examples of functionalities and identify which can be securely computed with 1-security in the point-to-point model. Our examples include variants of private-set intersection. Throughout the section, for natural numbers $k, \ell, m \in \mathbb{N}$ satisfying $k \le \ell \le m$, we denote

$$\binom{[m]}{k} = \{\mathcal{S} \subseteq [m] : |\mathcal{S}| = k\}$$

and we denote

$$\binom{[m]}{k,\ell} = \{\mathcal{S} \subseteq [m] : k \le |\mathcal{S}| \le \ell\}\,.$$

Claim 7.1. *For two natural numbers $k \le m$, let $\mathsf{disj}_{k,m} : \binom{[m]}{k}^3 \to \{0,1\}$ be the solitary output three-party* disjointness *functionality defined as*

$$\mathsf{disj}_{k,m}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) = \begin{cases} 1 & \text{if } \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathsf{disj}_{k,m}$ can be computed with computational 1-security if and only if $k > 2m/3$ or $k = 0$.

Proof. Observe that if $k > 2m/3$ or $k = 0$, then $\mathsf{disj}_{k,m}$ is constant, and thus can be securely computed. We now assume that $0 < k \le 2m/3$ and show that $\mathsf{disj}_{k,m}$ is not $\mathrm{CORE}_\wedge$-forced, and thus cannot be computed securely. We separate the proof into two cases.

Case 1: $m/2 < k \leq 2m/3$**.** We first show that for any $\mathcal{S}_1$, it holds that $\mathcal{S}_2 \equiv_{\mathcal{S}_1} \mathcal{S}'_2$ for all $\mathcal{S}_2, \mathcal{S}'_2 \in \binom{[m]}{k}$. Indeed, since $k > m/2$ it follows that $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$ and that $\mathcal{S}_1 \cap \mathcal{S}'_2 \neq \emptyset$. Therefore, for any $\mathcal{S}_3 \supseteq \mathcal{S}_1 \cap \mathcal{S}_2$ it holds that $\mathsf{disj}_{k,m}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) = 0$. Similarly, for any $\mathcal{S}'_3 \supseteq \mathcal{S}_1 \cap \mathcal{S}'_2$ it holds that $\mathsf{disj}_{k,m}(\mathcal{S}_1, \mathcal{S}'_2, \mathcal{S}'_3) = 0$. By symmetry, $\mathcal{S}_3 \equiv_{\mathcal{S}_1} \mathcal{S}'_3$ for all $\mathcal{S}_3, \mathcal{S}'_3 \in \binom{[m]}{k}$ as well.

Now, assume towards contradiction that there exists a distribution $\mathcal{R} = \{R_\kappa\}_{\kappa \in \mathbb{N}}$ over $\binom{[m]}{k}$ such that

$$\left\{\mathsf{disj}_{k,m}\left(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}^*_3\right)\right\}_{\kappa, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}'_2} \overset{\text{S}}{\equiv} \left\{\mathsf{disj}_{k,m}\left(\mathcal{S}_1, \mathcal{S}'_2, \mathcal{S}^*_3\right)\right\}_{\kappa, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}'_2}, \tag{14}$$

where $\mathcal{S}^*_3 \leftarrow R_\kappa$. Since the domain of $\mathsf{disj}_{k,m}$ is finite, there exists $\mathcal{S}_3 \in \binom{[m]}{k}$ such that $\Pr_{\mathcal{S}^*_3 \leftarrow R_\kappa}[\mathcal{S}^*_3 = \mathcal{S}_3] \geq p$ infinitely often for some constant $p > 0$. Consider $\mathcal{S}_1$ that minimizes $|\mathcal{S}_1 \cap \mathcal{S}_3|$, i.e., it holds that $|\mathcal{S}_1 \cap \mathcal{S}_3| = 2k - m$. Since $2k - m \leq m/3$, there exists $\mathcal{S}_2$ such that $\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 = \emptyset$. Therefore

$$\Pr_{\mathcal{S}^*_3 \leftarrow R_\kappa}\left[\mathsf{disj}_{k,m}\left(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}^*_3\right) = 1\right] \geq \Pr_{\mathcal{S}^*_3 \leftarrow R_\kappa}\left[\mathcal{S}^*_3 = \mathcal{S}_3\right] \geq p,$$

holds infinitely often. On the other hand, for $\mathcal{S}'_2 = \mathcal{S}_1$ it holds that $\mathsf{disj}_{k,m}(\mathcal{S}_1, \mathcal{S}'_2, \cdot)$ is the constant 0 function, hence

$$\Pr_{\mathcal{S}^*_3 \leftarrow R_\kappa}\left[\mathsf{disj}_{k,m}\left(\mathcal{S}_1 \cap \mathcal{S}'_2 \cap \mathcal{S}^*_3\right) = 1\right] = 0$$

for all κ, contradicting Equation (14).

Case 2: $0 < k \leq m/2$**.** The proof follows similar arguments to the previous case. We first show that for any $\mathcal{S}_1$, it holds that $\mathcal{S}_2 \equiv_{\mathcal{S}_1} \mathcal{S}'_2$ for all $\mathcal{S}_2, \mathcal{S}'_2 \in \binom{[m]}{k}$. Indeed, since $k \leq m/2$ it follows that there exists $\mathcal{S}_3 \in \binom{[m]}{k}$ such that $\mathcal{S}_1 \cap \mathcal{S}_3 = \emptyset$, and in particular, $\mathsf{disj}_{k,m}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) = \mathsf{disj}_{k,m}(\mathcal{S}_1, \mathcal{S}'_2, \mathcal{S}_3) = 1$. By symmetry, $\mathcal{S}_3 \equiv_{\mathcal{S}_1} \mathcal{S}'_3$ for all $\mathcal{S}_3, \mathcal{S}'_3 \in \binom{[m]}{k}$ as well.

Assume towards contradiction that there exists a distribution ensemble $\mathcal{R} = \{R_\kappa\}_{\kappa \in \mathbb{N}}$ over $\binom{[m]}{k}$ such that

$$\left\{\mathsf{disj}_{k,m}\left(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}^*_3\right)\right\}_{\kappa, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}'_2} \overset{\text{S}}{\equiv} \left\{\mathsf{disj}_{k,m}\left(\mathcal{S}_1, \mathcal{S}'_2, \mathcal{S}^*_3\right)\right\}_{\kappa, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}'_2}, \tag{15}$$

where $\mathcal{S}^*_3 \leftarrow R_\kappa$. Since the domain of $\mathsf{disj}_{k,m}$ is finite, there exists $\mathcal{S}_3 \in \binom{[m]}{k}$ such that $\Pr_{\mathcal{S}^*_3 \leftarrow R_\kappa}[\mathcal{S}^*_3 = \mathcal{S}_3] \geq p$ infinitely often for some constant $p > 0$. Consider $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3$. Then, as $k \neq 0$ it follows that $\mathcal{S}_3 \neq \emptyset$. Thus

$$\Pr_{\mathcal{S}^*_3 \leftarrow R_\kappa}\left[\mathsf{disj}_{k,m}\left(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}^*_3\right) = 0\right] \geq \Pr_{\mathcal{S}^*_3 \leftarrow R_\kappa}[\mathcal{S}^*_3 = \mathcal{S}_3] \geq p,$$

holds infinitely often. On the other hand, since $k \leq m/2$ there exists $\mathcal{S}'_2 \in \binom{[m]}{k}$ such that $\mathcal{S}_1 \cap \mathcal{S}'_2 = \emptyset$, hence

$$\Pr_{\mathcal{S}^*_3 \leftarrow R_\kappa}\left[\mathsf{disj}_{k,m}\left(\mathcal{S}_1 \cap \mathcal{S}'_2 \cap \mathcal{S}^*_3\right) = 0\right] = 0$$

for all κ, contradicting Equation (15). □

Claim 7.2. *For $k_1, \ell_1, k_2, \ell_2, m \in \mathbb{N}$ where $0 \leq k_1 \leq \ell_1 \leq m$ and $0 \leq k_2 \leq \ell_2 \leq m$, let $\mathsf{PSI}^{\ell_1,\ell_2}_{k_1,k_2,m} : \{\lambda\} \times \binom{[m]}{k_1,\ell_1} \times \binom{[m]}{k_2,\ell_2} \to 2^{[m]}$ be the solitary output three-party* private set intersection *functionality defined as*

$$\mathsf{PSI}^{\ell_1,\ell_2}_{k_1,k_2,m}(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{S}_1 \cap \mathcal{S}_2.$$

Then $\mathsf{PSI}^{\ell_1,\ell_2}_{k_1,k_2,m}$ *can be computed with computational 1-security if and only if one of the following holds.*

1. $k_1 = k_2 = 0$, *or*
2. $\ell_1 = 0$ *or* $\ell_2 = 0$, *or*
3. $k_1 = m$ *or* $k_2 = m$.

Proof. We write PSI instead of $\mathsf{PSI}^{\ell_1,\ell_2}_{k_1,k_2,m}$ for brevity. We first show the positive direction. If $k_1 = k_2 = 0$, then PSI is forced since both B and C can fix the output to be $\emptyset$. If $\ell_1 = 0$ or $\ell_2 = 0$ then PSI is the constant $\emptyset$ function. If $k_1 = m$ or $k_2 = m$, then PSI is independent of one of its second argument and in particular, is $\mathrm{CORE}_\wedge$-forced.[22]

We now show the negative direction. We first show that $\mathcal{S}_1 \equiv \mathcal{S}'_1$ and $\mathcal{S}_2 \equiv \mathcal{S}'_2$, for all $\mathcal{S}_1, \mathcal{S}'_1 \in \binom{[m]}{k_1,\ell_1}$ and $\mathcal{S}_2, \mathcal{S}'_2 \in \binom{[m]}{k_2,\ell_2}$. We show only the former as the latter can be proved using an analogous argument. Observe that if $|\mathcal{S}_1 \cap \mathcal{S}'_1| \geq k_2$, then for any $\mathcal{S}_2 \subseteq \mathcal{S}_1 \cap \mathcal{S}'_1$ of size $k_2 \leq |\mathcal{S}_2| \leq \ell_2$ it holds that $\mathsf{PSI}(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{S}_2 = \mathsf{PSI}(\mathcal{S}'_1, \mathcal{S}_2)$. On the other hand, if $|\mathcal{S}_1 \cap \mathcal{S}'_1| < k_2$, then there exists $\mathcal{S}_2 \in \binom{[m]}{k_2}$ such that $\mathcal{S}_1 \cap \mathcal{S}'_1 \subseteq \mathcal{S}_2$ and $\mathcal{S}_2 \cap (\mathcal{S}_1 \setminus \mathcal{S}'_1) = \mathcal{S}_2 \cap (\mathcal{S}'_1 \setminus \mathcal{S}_1) = \emptyset$.

Now, assume towards contradiction that PSI is $\mathrm{CORE}_\wedge$-forced. since all inputs are equivalent, it follows that PSI is forced. Then there exists an ensemble of efficiently samplable distributions $\mathcal{R} = \{R_\kappa\}_{\kappa\in\mathbb{N}}$ such that

$$\{\mathcal{S}_1 \cap \mathcal{S}^*_2\}_{\kappa,\mathcal{S}_1,\mathcal{S}'_1} \overset{\text{S}}{\equiv} \{\mathcal{S}'_1 \cap \mathcal{S}^*_2\}_{\kappa,\mathcal{S}_1,\mathcal{S}'_1}, \tag{16}$$

where $\mathcal{S}^*_2 \leftarrow R_\kappa$. Since the domain of PSI is finite, there exists $\mathcal{S}_2 \in \binom{[m]}{k_2,\ell_2}$ such that $\Pr_{\mathcal{S}^*_2 \leftarrow R_\kappa}[\mathcal{S}^*_2 = \mathcal{S}_2] \geq p$ infinitely often for some constant $p > 0$. Now, recall that $k_1 \neq 0$ or $k_2 \neq 0$. We assume the latter without loss of generality. Thus, $\mathcal{S}_2 \neq \emptyset$. We next separate the proof into two cases.

Case 1: $|\mathcal{S}_2| \leq \ell_1$**.** In this case, there exists $\mathcal{S}_1 \in \binom{[m]}{k_1,\ell_1}$ such that $\mathcal{S}_2 \subseteq \mathcal{S}_1$. Therefore

$$\Pr_{\mathcal{S}^*_2 \leftarrow R_\kappa}[\mathcal{S}_1 \cap \mathcal{S}^*_2 = \mathcal{S}_2] \geq \Pr_{\mathcal{S}^*_2 \leftarrow R_\kappa}[\mathcal{S}^*_2 = \mathcal{S}_2] \geq p$$

infinitely often. However, since $\mathcal{S}_2 \neq \emptyset$ there exists $\mathcal{S}'_1 \in \binom{[m]}{k_1,\ell_1}$ such that $\mathcal{S}_2 \not\subseteq \mathcal{S}'_1$. Thus

$$\Pr_{\mathcal{S}^*_2 \leftarrow R_\kappa}[\mathcal{S}'_1 \cap \mathcal{S}^*_2 = \mathcal{S}_2] = 0$$

for all κ, contradicting Equation (16).

Case 2: $|\mathcal{S}_2| > \ell_1$**.** In this case, there exist $\mathcal{S}_1, \mathcal{S}'_1 \in \binom{[m]}{k_1,\ell_1}$ such that $\mathcal{S}_1, \mathcal{S}'_1 \subseteq \mathcal{S}_2$. Moreover, since $k_1 \neq m$ and $\ell_1 \neq 0$, it follows that we can take $\mathcal{S}_1 \neq \mathcal{S}'_1$. Thus

$$\Pr_{\mathcal{S}^*_2 \leftarrow R_\kappa}[\mathcal{S}_1 \cap \mathcal{S}^*_2 = \mathcal{S}_1] \geq \Pr_{\mathcal{S}^*_2 \leftarrow R_\kappa}[\mathcal{S}^*_2 = \mathcal{S}_2] \geq p$$

infinitely often. However,

$$\Pr_{\mathcal{S}^*_2 \leftarrow R_\kappa}[\mathcal{S}'_1 \cap \mathcal{S}^*_2 = \mathcal{S}_1] = 0$$

for all κ, contradicting Equation (16). □

[22] Note that in all cases we do not need to assume the existence of OT. For the first case, where $k_1 = k_2 = 0$, we can use the protocol where if the fair computation fails, we let A output $\emptyset$. For the other two cases the computation is trivial, since either the function is constant, or the protocol where B or C send their input to A is secure.

Claim 7.3. *For $k_1, \ell_1, k_2, \ell_2, m \in \mathbb{N}$ where $0 \leq k_1 \leq \ell_1 \leq m$ and $0 \leq k_2 \leq \ell_2 \leq m$, let $\mathsf{PSIZE}^{\ell_1,\ell_2}_{k_1,k_2,m} : \{\lambda\} \times \binom{[m]}{k_1,\ell_1} \times \binom{[m]}{k_2,\ell_2} \to \{0,\ldots,m\}$ be the solitary output three-party functionality defined as*

$$\mathsf{PSIZE}^{\ell_1,\ell_2}_{k_1,k_2,m}(\mathcal{S}_1, \mathcal{S}_2) = |\mathcal{S}_1 \cap \mathcal{S}_2| \,.$$

Then $\mathsf{PSIZE}^{\ell_1,\ell_2}_{k_1,k_2,m}$ can be computed with computational 1-security if and only if one of the following holds.

1. *$k_1 = k_2 = 0$, or*
2. *$\ell_1 = 0$ or $\ell_2 = 0$,*
3. *$k_1 = m$ or $k_2 = m$, or*
4. *$k_1 = \ell_1$ and $k_2 = \ell_2$.*

Proof. We write PSIZE instead of $\mathsf{PSIZE}^{\ell_1,\ell_2}_{k_1,k_2,m}$ for brevity. Similarly to the PSI functionality, if $k_1 = k_2 = 0$, then PSIZE is forced since both B and C can fix the output to be 0. If $\ell_1 = 0$ or $\ell_2 = 0$, then PSIZE is the constant 0. If $k_1 = m$ or $k_2 = m$, then PSIZE is independent of one of its arguments and in particular, is forced. Finally, if $k_1 = \ell_1$ and $k_2 = \ell_2$ then PSIZE is forced since the uniform distribution for both parties fixes the output distribution to be uniform.

We now show the negative direction. We first show that $\mathcal{S}_1 \equiv \mathcal{S}'_1$ and $\mathcal{S}_2 \equiv \mathcal{S}'_2$, for all $\mathcal{S}_1, \mathcal{S}'_1 \in \binom{[m]}{k_1,\ell_1}$ and $\mathcal{S}_2, \mathcal{S}'_2 \in \binom{[m]}{k_2,\ell_2}$. We show only the former as the latter can be proved using an analogous argument. Observe that the set of possible outputs for a fixed $\mathcal{S}_1 \in \binom{[m]}{k_1,\ell_1}$ is exactly

$$\{\max\{0, |\mathcal{S}_1| + k_2 - m\}, \ldots, \min\{|\mathcal{S}_1|, \ell_2\}\} \,.$$

Then, if $|\mathcal{S}_1| = |\mathcal{S}'_1|$ there are $\mathcal{S}_2, \mathcal{S}'_2 \in \binom{[m]}{k_2,\ell_2}$ such that $|\mathcal{S}_1 \cap \mathcal{S}_2| = |\mathcal{S}'_1 \cap \mathcal{S}_2|$. Next, consider the case where $\mathcal{S}'_1 = \mathcal{S} \cup \{a\}$, where $a \notin \mathcal{S}_1$. Then there are no such $\mathcal{S}_2$ and $\mathcal{S}'_2$ if and only if

$$\max\{0, |\mathcal{S}_1| + 1 + k_2 - m\} > \min\{|\mathcal{S}_1|, \ell_2\}.$$

However, since $|\mathcal{S}_1|, \ell_2 \geq 0$ and $|\mathcal{S}_1| + k_2 - m < |\mathcal{S}_1|$ as $k_2 \neq m$, it follows that $|\mathcal{S}_1| + k_2 - m \geq \ell_2$. This is clearly impossible since this implies that $|\mathcal{S}_1| \geq \ell_2 - k + m \geq m$. The case where $|\mathcal{S}'_1| > |\mathcal{S}_1| + 1$ can be done using an inductive argument (over $|\mathcal{S}'_1| - |\mathcal{S}_1|$).

We now show that PSIZE is not forced and hence cannot be computed with 1-security. Recall that for the negative direction, we assume that $k_1 \neq \ell_1$ or $k_2 \neq \ell_2$. Assume the former without loss of generality, and assume towards contradiction that PSIZE is forced. Then there exists an ensemble of efficiently samplable distributions $\mathcal{R} = \{R_\kappa\}_{\kappa \in \mathbb{N}}$ such that

$$\{|\mathcal{S}_1 \cap \mathcal{S}^*_2|\}_{\kappa, \mathcal{S}_1, \mathcal{S}'_1} \stackrel{\text{S}}{\equiv} \{|\mathcal{S}'_1 \cap \mathcal{S}^*_2|\}_{\kappa, \mathcal{S}_1, \mathcal{S}'_1} \,, \tag{17}$$

where $\mathcal{S}^*_2 \leftarrow R_\kappa$. Since the domain of PSIZE is finite, there exists $\mathcal{S}_2 \in \binom{[m]}{k_2,\ell_2}$ such that $\Pr_{\mathcal{S}^*_2 \leftarrow R_\kappa}[\mathcal{S}^*_2 = \mathcal{S}_2] \geq p$ infinitely often for some constant $p > 0$. We separate the proof into two cases.

Case 1: $k_1 < k_2$**.** Fix $\mathcal{S}_1 \in \binom{[m]}{k_1}$ such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$, and fix some $a \in \mathcal{S}_2 \setminus \mathcal{S}_1$. Let $n = |\mathcal{S}_1 \cap \mathcal{S}_2|$, and let $\mathcal{S}'_1 = \mathcal{S}_1 \cup \{a\}$ (note that $\mathcal{S}'_1 \in \binom{[m]}{k_1,\ell_1}$ since $k_1 \neq \ell_1$). Then

$$\Pr_{\mathcal{S}_2^* \leftarrow R_\kappa}\left[|(\mathcal{S}_1 \cup \{a\}) \cap \mathcal{S}_2^*| = n+1\right] \geq \Pr_{\mathcal{S}_2^* \leftarrow R_\kappa}\left[\mathcal{S}_2^* = \mathcal{S}_2\right] \geq p,$$

infinitely often. However, for those exact same κ it holds that

$$\Pr_{\mathcal{S}_2^* \leftarrow R_\kappa}\left[|\mathcal{S}_1 \cap \mathcal{S}_2^*| = n+1\right] = 0,$$

resulting in a contradiction.

Case 2: $k_1 \geq k_2$**.** Fix $\mathcal{S}_1 \in \binom{[m]}{k_1+1}$ such that $\mathcal{S}_2 \subseteq \mathcal{S}_1$, and fix some $a \in \mathcal{S}_1 \setminus \mathcal{S}_2$. Let $n = |\mathcal{S}_1 \cap \mathcal{S}_2|$. Then

$$\Pr_{\mathcal{S}_2^* \leftarrow R_\kappa}\left[|\mathcal{S}_1 \cap \mathcal{S}_2^*| = n\right] \geq \Pr_{\mathcal{S}_2^* \leftarrow R_\kappa}\left[\mathcal{S}_2^* = \mathcal{S}_2\right] \geq p,$$

infinitely often. However, since $\ell_2 \neq 0$ it follows that $\mathcal{S}_2 \neq \emptyset$, hence for those exact same κ it holds that

$$\Pr_{\mathcal{S}_2^* \leftarrow R_\kappa}\left[|(\mathcal{S}_1 \setminus \{a\}) \cap \mathcal{S}_2^*| = n\right] = 0,$$

resulting in a contradiction. □

Claim 7.4. *For $k_1, \ell_1, k_2, \ell_2, m \in \mathbb{N}$ where $0 \leq k_1 \leq \ell_1 \leq m$ and $0 \leq k_2 \leq \ell_2 \leq m$, let $\mathsf{disj}^{\ell_1,\ell_2}_{k_1,k_2,m} : \{\lambda\} \times \binom{[m]}{k_1,\ell_1} \times \binom{[m]}{k_2,\ell_2} \to \{0,\ldots,m\}$ be the solitary output three-party functionality defined as*

$$\mathsf{disj}^{\ell_1,\ell_2}_{k_1,k_2,m}(\mathcal{S}_1,\mathcal{S}_2) = \begin{cases} 1 & \text{if } \mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathsf{disj}^{\ell_1,\ell_2}_{k_1,k_2,m}$ can be computed with computational 1-security if and only if one of the following holds.

1. $k_1 = k_2 = 0$, *or*
2. $\ell_1 = 0$ *or* $\ell_2 = 0$, *or*
3. $k_1 = m$ *or* $k_2 = m$, *or*
4. $k_1 = \ell_1$ *and* $k_2 = \ell_2$.
5. $\ell_1 + k_2 > m$ *and* $k_1 + \ell_2 > m$.

Proof. We write disj instead of $\mathsf{disj}^{\ell_1,\ell_2}_{k_1,k_2,m}$ for brevity. Similarly to the PSI and PSIZE functionality, if $k_1 = k_2 = 0$, then disj is forced since both B and C can fix the output to be 1. If $\ell_1 = 0$ or $\ell_2 = 0$ then disj is the constant 1. If $k_1 = m$ or $k_2 = m$, then disj is independent of one of its arguments and in particular, is forced. If $k_1 = \ell_1$ and $k_2 = \ell_2$ then disj is forced since the uniform distribution for both parties fixes the output distribution to be uniform. Finally, if $\ell_1 + k_2 > m$ and $k_1 + \ell_2 > m$, then both parties can fix the output to be 0.

We now show the negative direction. We first show that $\mathcal{S}_1 \equiv \mathcal{S}'_1$ and $\mathcal{S}_2 \equiv \mathcal{S}'_2$, for all $\mathcal{S}_1, \mathcal{S}'_1 \in \binom{[m]}{k_1,\ell_1}$ and $\mathcal{S}_2, \mathcal{S}'_2 \in \binom{[m]}{k_2,\ell_2}$. We show only the former as the latter can be proved using an analogous argument. Since disj is Boolean, it suffices to show that there exists $\mathcal{S}_1 \in \binom{[m]}{k_1,\ell_1}$ for which $\mathsf{disj}(\mathcal{S}_1, \cdot)$

is not constant. Clearly, since we assume $\ell_1, \ell_2 \neq 0$, any $\mathcal{S}_1 \in \binom{[m]}{k_1,\ell_1}$ must intersect at least one $\mathcal{S}_2 \in \binom{[m]}{k_2,\ell_2}$. Now, recall that we assume that either $\ell_1 + k_2 \leq m$ or $k_1 + \ell_2 \leq m$. Either way, it follows that $k_1 + k_2 \leq m$. Therefore, for any $\mathcal{S}_1 \in \binom{[m]}{k_1}$ there exists $\mathcal{S}_2 \in \binom{[m]}{k_2}$ that does not intersect $\mathcal{S}_1$.

We now show that disj is not forced and hence cannot be computed with 1-security. Recall that for the negative direction, we assume that $k_1 \neq \ell_1$ or $k_2 \neq \ell_2$. Assume the former without loss of generality, and assume towards contradiction that disj is forced.

Then there exist two ensembles of efficiently samplable distributions $\mathcal{Q} = \{Q_\kappa\}_{\kappa\in\mathbb{N}}$ $\mathcal{R} = \{R_\kappa\}_{\kappa\in\mathbb{N}}$ such that, in particular

$$\{\mathsf{disj}\,(\mathcal{S}_1^* \cap \mathcal{S}_2)\}_{\kappa,\mathcal{S}_2,\mathcal{S}_2'} \overset{\text{S}}{\equiv} \{\mathsf{disj}\,(\mathcal{S}_1^* \cap \mathcal{S}_2')\}_{\kappa,\mathcal{S}_2,\mathcal{S}_2'}\,, \tag{18}$$

and

$$\{\mathsf{disj}\,(\mathcal{S}_1 \cap \mathcal{S}_2^*)\}_{\kappa,\mathcal{S}_1,\mathcal{S}_1'} \overset{\text{S}}{\equiv} \{\mathsf{disj}\,(\mathcal{S}_1' \cap \mathcal{S}_2^*)\}_{\kappa,\mathcal{S}_1,\mathcal{S}_1'}\,, \tag{19}$$

where $\mathcal{S}_1^* \leftarrow Q_\kappa$ and $\mathcal{S}_2^* \leftarrow R_\kappa$. We next separate the proof into two cases.

Case 1: $\ell_1 + k_2 \leq m$ **and** $k_1 + \ell_2 > m$**.** In this case, $\mathsf{disj}(\cdot, \mathcal{S}_2) \equiv 0$ for any $\mathcal{S}_2 \in \binom{[m]}{\ell_2}$, but $\mathsf{disj}(\mathcal{S}_1, \cdot) \not\equiv 0$ for any $\mathcal{S}_1 \in \binom{[m]}{k_1,\ell_1}$. Similarly to Claim 7.1, this immediately contradicts Equation (18).

Case 2: $k_1 + \ell_2 \leq m$**.** We show that for $\mathcal{S}_1 \leftarrow \binom{[m]}{k_1}$ and $\mathcal{S}_1' \leftarrow \binom{[m]}{k_1+1}$ sampled independently, the statistical distance between $\mathsf{disj}\,(\mathcal{S}_1 \cap \mathcal{S}_2^*)$ and $\mathsf{disj}\,(\mathcal{S}_1' \cap \mathcal{S}_2^*)$ is not negligible. This implies that there exist $\mathcal{S}_1$ and $\mathcal{S}_1'$ for which the statistical distance is not negligible, thus Equation (19) does not hold.

Observe that for any $\mathcal{S}_2 \in \binom{[m]}{n}$, for some $n \in \{k_2, \ldots, \ell_2\}$, it holds that

$$\Pr_{\mathcal{S}_1\leftarrow\binom{[m]}{k_1}}[\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset] = \Pr_{\mathcal{S}_1\leftarrow\binom{[m]}{k_1}}[\mathcal{S}_1 \subseteq [m] \setminus \mathcal{S}_2] = \frac{\binom{m-n}{k_1}}{\binom{m}{k_1}}.$$

Similarly,

$$\Pr_{\mathcal{S}_1'\leftarrow\binom{[m]}{k_1+1}}[\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset] = \frac{\binom{m-n}{k_1+1}}{\binom{m}{k_1+1}}.$$

Let $d(n) := \frac{\binom{m-n}{k_1}}{\binom{m}{k_1}} - \frac{\binom{m-n}{k_1+1}}{\binom{m}{k_1+1}}$. Then for any $n \leq m - k_1$ it holds that $d(n) > 0$. Since $k_1 + \ell_2 \leq m$, it follows that $d(n) > 0$ for any $n \in \{k_2, \ldots, \ell_2\}$. Now, for every $n \in \{k_2, \ldots, \ell_2\}$ and every $\kappa \in \mathbb{N}$, let $p_{n,\kappa} = \Pr_{\mathcal{S}_2^*\leftarrow R_\kappa}[|\mathcal{S}_2^*| = n]$. Then

$$\Pr_{\mathcal{S}_1\leftarrow\binom{[m]}{k_1},\mathcal{S}_2^*\leftarrow R_\kappa}[\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset] - \Pr_{\mathcal{S}_1'\leftarrow\binom{[m]}{k_1+1},\mathcal{S}_2^*\leftarrow R_\kappa}[\mathcal{S}_1' \cap \mathcal{S}_2 = \emptyset] = \sum_{n=k_2}^{\ell_2} p_{n,\kappa} \cdot d(n).$$

Since there exists n^* for which $p_{n^*,\kappa} \geq 1/n^*$ infinitely often, it follows that the above difference is at least $d(n^*)/n^*$, which is non-negligible. □

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A Definition of Security-With-Identifiable-Abort

We next define an ideal computation with *security-with-identifiable-abort*, where a trusted party performs the computation on behalf of the parties, and where the ideal-model adversary can abort the computation after learning the output, but at the expense of revealing the identity of a corrupted party. An ideal computation of a three-party functionality $f = (f_1, f_2, f_3)$, with $f_1, f_2, f_3 : (\{0,1\}^*)^3 \to \{0,1\}^*$, on inputs $x, y, z \in \{0,1\}^*$ and security parameter κ, with an ideal-world adversary $\mathcal{A}$ running with an auxiliary input aux and corrupting a (strict) subset $\mathcal{I} \subseteq \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ of the parties proceeds as follows:

Parties send inputs to the trusted party: Each honest party sends its input to the trusted party. For each corrupted party, the adversary $\mathcal{A}$ sends a value v from the corresponding domain as the input for the corrupted party. Let (x', y', z') denote the inputs received by the trusted party.

The trusted party performs computation: The trusted party selects a random string r, computes $(w_\mathsf{A}, w_\mathsf{B}, w_\mathsf{C}) = f(x', y', z'; r)$, and sends $\{w_\mathsf{P}\}_{\mathsf{P} \in \mathcal{I}}$ to $\mathcal{A}$.

Malicious adversary instructs trusted party to continue or halt: The adversary $\mathcal{A}$ sends either $\mathsf{continue}$ or $(\mathsf{abort}, \mathsf{P})$ for some $\mathsf{P} \in \mathcal{I}$ to T. If it sent $\mathsf{continue}$, then for every honest party Q the trusted party sends it w_Q. Otherwise, if $\mathcal{A}$ sent $(\mathsf{abort}, \mathsf{P})$, then T sends $(\mathsf{abort}, \mathsf{P})$ to the each honest party Q.

Outputs: Each honest party outputs whatever output it received from the trusted party and the corrupted parties output nothing. The adversary $\mathcal{A}$ outputs some function of its view (i.e., the auxiliary input, its randomness, and the input and output of the corrupted parties).