FuLeeca: A Lee-based Signature Scheme

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Abstract. In this work we introduce a new code-based signature scheme, called FuLeeca, based on the NP-hard problem of finding codewords of given Lee-weight. The scheme follows the Hash-and-Sign approach applied to quasi-cyclic codes. Similar approaches in the Hamming metric have suffered statistical attacks, which revealed the small support of the secret basis. Using the Lee metric, we are able to thwart such attacks. We use existing hardness results on the underlying problem and study adapted statistical attacks. We propose parameters for FuLeeca and compare them to an extensive list of proposed post-quantum secure signature schemes including the ones already standardized by NIST. This comparison reveals that FuLeeca is competitive. For example, for NIST category I, i.e., 160 bit of classical security, we obtain an average signature size of 1100 bytes and public key sizes of 1318 bytes. Comparing the total communication cost, i.e., the sum of the signature and public key size, we see that FuLeeca is only outperformed by Falcon while the other standardized schemes Dilithium and SPHINCS+ show larger communication costs than FuLeeca.

Keywords: Post-Quantum cryptography · Signature scheme · Code-Based cryptography · Lee metric

1 Introduction

Due to the threat coming from capable quantum computers, NIST initialized in 2016 a standardization call for post-quantum alternatives.

Since the standardization call several of the submitted cryptosystems have been broken or removed from the competition as the proposed parameters seemed inferior to other schemes. Recently, several cryptosystems have been selected for standardization, both for key encapsulation and digital signatures. The most competitive and already selected signature schemes in terms of parameter sizes (keys and signature sizes) are based on structured lattices, namely CRYSTALS-Dilithium [28] and Falcon [36]. If signature sizes or signing times are not a major concern, hash-based signatures like SPHINCS+[5] provide even smaller key sizes. Since it is desirable to have a broader variety of schemes to choose from, NIST reopened the standardization call for digital signatures.

One possibility to build quantum-secure signature schemes is to rely on hard problems from coding theory which have been examined over decades [15,10,9,18]. While classical code-based cryptography considers vector spaces endowed with the Hamming metric, other metrics, such as the rank metric, have attracted attention in the context of cryptography and show a great potential for smaller key sizes. To the best of our knowledge this work marks the first Lee-based cryptographic primitive.

In general, there are two main methods to construct code-based signature schemes: the first one applies the Fiat-Shamir transform [35] to a code-based zero-knowledge protocol and the second one is called Hash-and-Sign approach [14]. The former approach usually suffers from large signature sizes, due to large cheating probabilities within the zero-knowledge protocol, while the latter features small signature sizes at the cost of larger public key sizes.

The signature scheme we present in this paper is based on the Hash-and-Sign approach, which has been introduced in 2001 by Courtois, Finiasz and Sendrier [24] (following the idea of [14]). This so called CFS scheme is a direct adaption of the McEliece public-key encryption scheme. In fact, the rationale is to start with an algebraically structured secret code that comes with an efficient decoding algorithm. The public key is a disguised version of the secret code. To generate a signature, the message and a salt is hashed until the digest results in a syndrome of a low-weight error vector. This approach has some potential drawbacks that have been exploited for attacks in the past: on the one hand, the public code might be distinguishable from a random code and thus leak information on the secret code[1]. On the other hand, the event that the hash of a message is a syndrome of a low weight error vector is highly unlikely and therefore this process has to be repeated many times. This causes the signing time of CFS to be impractically high. Additionally, as the public key is a disguised version of an algebraically structured code, the public key size of CFS tends to be rather large. The CFS scheme was the starting point for several Hash-and-Sign signature schemes, such as [6,40,23], which have not survived cryptanalysis [46,48]. The code-based scheme WAVE [26] also follows the same blueprint but translated into the theoretical framework of [39]. Additionally, it is based on the hardness of finding errors having large Hamming weights instead of small ones, thereby preventing all aforementioned attacks and so far no successful cryptanalysis has been mounted.

Code-based signature schemes based on quasi-cyclic structures with low density codes in the Hamming metric, e.g., [6,47], are vulnerable to statistical key recovery attacks [51,27]. These attacks have in common that they make use of the small support of the secret key. An attacker can recover the sparse secret key by observing the distribution of many signatures and comparing it to a random distribution. The use of the Lee metric thwarts such attacks, as even if the Lee

[1] See for example [31], where the CFS scheme using high rate Goppa codes has been attacked.

weight of the secret basis is low, the number of non-zero entries is relatively large.

FuLeeca is based on quasi-cyclic codes in the Lee metric. In a nutshell, the signature scheme works as follows: the secret key is a quasi-cyclic generator matrix, where the generators have Lee weight according to the Lee-metric Gilbert-Varshamov (GV) bound and the public key is its systematic form. Note that recovering the original generators is as hard as the problem of finding codewords of given Lee weight, which has been proven to be NP-hard [56]. The binary hash output of the message $\boldsymbol{m}$ is mapped onto $\{\pm 1\}$ and is considered as the target vector $\boldsymbol{c}$ for the main step of the scheme: the signer uses the secret generators to find a codeword, which is connected to the signature for $\boldsymbol{m}$, with two properties: firstly, the Lee weight should be low and secondly, the signum of the codeword should have many 1s, respectively -1s, in the same places as the target vector $\boldsymbol{c}$. This second property, called sign matching, is used to bind the message to the signature, while the first property is essential for the schemes security.

For our chosen parameters targeting NIST security level I, the public key and signature sizes are only 1318 bytes and 1100 bytes, respectively. The total size, which is the sum of the public key and signature sizes is often used for comparisons, since in certificates one would need to download both. The total size of FuLeeca is smaller than that of Dilithium [28] and SPHINCS$^+$ [5], and slightly larger than Falcon [36], which are the three signature schemes currently selected for standardization by NIST.

A multiple-use signature scheme should have an existential unforgeability under adaptive chosen message attacks (EUF-CMA) security proof. For code-based signatures constructed from a zero-knowledge protocol this property is assured by using a sufficiently large number of rounds with respect to the desired security level and the cheating probability of the zero-knowledge protocol. However, the EUF-CMA security proof is notoriously difficult for Hash-and-Sign approaches. To the best of our knowledge, WAVE [26] is the only known code-based Hash-and-Sign signature scheme that provides such a proof. Unfortunately, the achieved public-key size of more than 2 megabytes for 128 bit classical security is very large compared to Falcon's 897 bytes.

The capability of breaking FuLeeca (e.g. recovering the secret key from a polynomial number of collected signatures) is crucially based on the fact that signatures do not leak any useful information on the secret key. An EUF-CMA security proof would immediately follow by assuming that such problem is hard. However, we feel like such proof would be immature, given the current state of affairs. Indeed, such problem is somewhat non-standard, since this problem has not been used in cryptography before. EUF-CMA security proofs for novel problems can also lead to concrete breaks. In fact, Durandal [2], a promising code-base signature scheme with an EUF-CMA security proof, was recently attacked in [3]. Thus, an EUF-CMA security proof does not prevent from cryptanalysis. Consequently, we choose not to provide such security proof, which would be artificial, i.e., based on a new, ad-hoc problem. Instead, we consider attacks exploiting the leakage via published hash/signature pairs, and design our scheme integrating countermeasures for those attacks.

This paper is structured as follows: In Section 2, we introduce the notation that is used throughout this paper and recall the required coding-theoretic basics. In Section 3, we describe the proposed Lee metric signature scheme FuLeeca. In Section 4, we analyze the security of the proposed scheme. We first consider the best known solver to find d codewords of given Lee weight codewords, and secondly we provide heuristics for EUF-CMA security, which allow the signature scheme to be used multiple times. Finally, in Section 5, we analyze the performance of the scheme. We also compare the key sizes, the signature size and the computation time for signing and verification to other post-quantum signature schemes. Section 6 concludes the paper.

2 Preliminaries

2.1 Notation

Throughout this work, we denote by $\mathbb{F}_p$ the finite field of order p, where p is a prime. We often choose to represent this prime field as $\{-\frac{p-1}{2}, \dots, 0, \dots, \frac{p-1}{2}\}$, which we call the symmetric representation. We denote vectors in bold lowercase and matrices in bold uppercase letters. We refer to the i-th element of the vector $\boldsymbol{v}$ by v_i and similarly, to the j-th row of a matrix $\boldsymbol{A}$ by $\boldsymbol{a}_j$ and we denote the element in the j-th row and k-th column by $a_{j,k}$. The identity matrix of size n is denoted by $\boldsymbol{I}_n$. We denote by uppercase letters sets and for a set $S \subset \{1, \dots, n\}$, we denote by $|S|$ the cardinality and by $S^C = \{1, \dots, n\} \setminus S$ the complement. For a set $S \subset \{1, \dots, n\}$ of size s and matrix $\boldsymbol{A} \in \mathbb{F}_p^{k \times n}$, we denote by $\boldsymbol{A}_S$ the $k \times s$ matrix formed by the columns of $\boldsymbol{A}$ indexed by S, similarly for a vector $\boldsymbol{x} \in \mathbb{F}_p^n$, we denote by $\boldsymbol{x}_S$ the vector of length s formed by the entries of $\boldsymbol{x}$ indexed by S.

The sampling of an element a from the uniform distribution over a set $\mathcal{K}$ is denoted by $a \xleftarrow{\$} \mathcal{K}$. While the sampling of an element a according to a distribution χ is given by $a \xleftarrow{\$} \chi$ and by a slight abuse of notation we denote sampling of a vector $\boldsymbol{v}$ independently and identically distributed (i.i.d.) from χ by $\boldsymbol{v} \xleftarrow{\$} \chi$.

The binary entropy function with parameter p is defined as $h_2(p) \coloneqq -p \log_2(p) - (1-p)\log_2(1-p)$.

2.2 Basic cryptographic tools

We denote the security parameter by λ. We use standard definitions of probabilistic polynomial time algorithms. We denote by "Hash" a Hash function in the perfect random oracle model.

In a digital signature scheme we have two parties, *the signer* and the *verifier*, and three efficiently computable algorithms: the key generation, the signature generation and the signature verification. In the key generation, the signer randomly samples a secret key sk and computes and publishes the connected public key pk. For the signature generation, given a message $\boldsymbol{m}$, the signer then uses the secret key sk to compute a signature $\boldsymbol{v}$. The signer then sends $(\boldsymbol{m}, \boldsymbol{v})$ to the verifier. The verifier checks the validity of the signature $\boldsymbol{v}$ for the message $\boldsymbol{m}$ under the constraints imposed by the scheme using the public key in the signature verification step. An adversary might try to construct a valid signature, either using just the knowledge of the public key, or after having observed several signatures corresponding to different messages. The adversary is only allowed to succeed with negligible probability, e.g., $< 2^{-\lambda}$.

2.3 Lee-metric codes

An $[n, k]$ *linear code* $\mathcal{C}$ is a k-dimensional linear subspace of $\mathbb{F}_p^n$ and can be compactly represented either through a *generator matrix* $\boldsymbol{G} \in \mathbb{F}_p^{k \times n}$, which has the code as its image or through a *parity-check matrix* $\boldsymbol{H} \in \mathbb{F}_p^{(n-k) \times n}$ having the code as its kernel. The elements of a code are called *codewords* and for any $\boldsymbol{x} \in \mathbb{F}_p^n$, we call $\boldsymbol{s} = \boldsymbol{x}\boldsymbol{H}^\top$ the *syndrome* of $\boldsymbol{x}$. The *rate* of an $[n, k]$ code is $R = \frac{k}{n}$.

For an $[n, k]$ linear code $\mathcal{C}$ and a set $I \subset \{1, \ldots, n\}$, we denote by $\mathcal{C}_I$ the set of restrictions on codewords restricted to the coordinates specified in I. We say that $I \subset \{1, \ldots, n\}$ of size k is an *information set*, if $|\mathcal{C}_I| = |\mathcal{C}|$. As a consequence, we have that for a generator matrix $\boldsymbol{G}$, respectively a parity-check matrix $\boldsymbol{H}$ of the code, $\boldsymbol{G}_I$ and $\boldsymbol{H}_{I^C}$ are invertible. We say that a generator matrix $\boldsymbol{G}$, respectively a parity-check matrix $\boldsymbol{H}$, is in systematic form (with respect to I), if $\boldsymbol{G}_I = \boldsymbol{I}_k$, respectively $\boldsymbol{H}_{I^C} = \boldsymbol{I}_{n-k}$.

Classically, we endow the vector space $\mathbb{F}_p^n$ with the Hamming metric, where the *Hamming weight* of a vector $\boldsymbol{v}$, denoted by $\mathrm{wt}_H(\boldsymbol{v})$, is given by the number of non-zero entries of $\boldsymbol{v}$. However, for this scheme, we are interested in a different metric, called the Lee metric.

The *Lee weight* of an element $a \in \mathbb{F}_p$ is defined as

$$\mathrm{wt}_L(a) := \min\{a, p - a\}, \tag{1}$$

where the representation of a is chosen to be in $\{0, \ldots, p-1\}$. In fact, one can think of the Lee weight as the L_1-norm modulo p. Clearly, the Lee weight of an element can be at most $(p-1)/2$, thus we will denote this value by M. For a vector $\boldsymbol{v} \in \mathbb{F}_p^n$ its Lee weight is defined as the sum of the Lee weights of its elements, i.e.,

$$\mathrm{wt}_L(\boldsymbol{v}) := \sum_{i=1}^{n} \mathrm{wt}_L(v_i). \tag{2}$$

Note that, $\mathrm{wt}_H(\boldsymbol{v}) \leq \mathrm{wt}_L(\boldsymbol{v}) \leq M\mathrm{wt}_H(\boldsymbol{v})$ and the average Lee weight of the vectors in $\mathbb{F}_p^n$ is given by $(M/2)n$.

The Lee weight induces the *Lee distance*, which we define by $d_L(\boldsymbol{x}, \boldsymbol{y}) := \mathrm{wt}_L(\boldsymbol{x} - \boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_p^n$. For a linear code $\mathcal{C}$ we define the *minimum Lee distance* as

$$d_L(\mathcal{C}) = \min\{\mathrm{wt}_L(\boldsymbol{c}) \mid \boldsymbol{c} \in \mathcal{C}, \boldsymbol{c} \neq 0\}.$$

We denote by δ the relative minimum Lee distance, that is $\delta = \frac{d_L(\mathcal{C})}{nM}$. Let us denote by $V_L(p, n, r)$ the *Lee sphere* of radius t

$$V_L(p, n, t) := \{\boldsymbol{x} \in \mathbb{F}_p^n \mid \mathrm{wt}_L(\boldsymbol{x}) = t\},$$

and by

$$F_L(p, T) = \lim_{n\to\infty} \frac{1}{n} \log_p(|V_L(p, n, TnM)|)$$

its asymptotic size. The exact formulas for the size of $V_L(p, n, t)$ and $F_L(p, T)$ can be found in [56,37].

Let us denote by $A(n, \delta)$ the maximal size of a code in $\mathbb{F}_p^n$ of minimum Lee distance $\delta M n$ and by

$$R(\delta) = \limsup_{n\to\infty} \frac{1}{n} \log_p(A(n, \delta)).$$

The Gilbert-Varshamov (GV) bound in the Lee-metric [4] then states:

$$R(\delta) \geq 1 - F_L(p, \delta).$$

In [21] it was shown that random Lee-metric codes attain with high probability the Lee-metric GV bound, i.e., a random code has with high probability a relative minimum Lee distance δ such that $R(\delta) = 1 - F_L(p, \delta)$. For the considered quasi-cyclic code of rate $1/2$, the corresponding minimum Lee distance δ of codes on the GV bound will only depend on p and is thus denoted by δ_p^{GV}.

If $\mathcal{C} \in \mathbb{F}_p^n$ is a random code of dimension k, we can also compute the expected number of codewords of a given Lee weight w as

$$|V_L(p, n, w)| p^{k-n}.$$

3 System Description

In this section, we describe how FuLeeca works.

For our scheme we represent the elements of $\mathbb{F}_p$ as

$$\left\{-\frac{p-1}{2}, \ldots, 0, \ldots, \frac{p-1}{2}\right\}$$

for $p > 3$ prime and $n \in \mathbb{N}$ even. As usual, we write M for the maximal Lee weight in $\mathbb{F}_p$, that is $M = \frac{p-1}{2}$. We define a function $\mathrm{sgn}(x)$, that gives us the sign of an element in $\mathbb{F}_p$.

Definition 1 (Signum). *For* $x \in \mathbb{F}_p = \left\{-\frac{p-1}{2}, \dots, 0, \dots, \frac{p-1}{2}\right\}$ *let*

$$\operatorname{sgn}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

For the symmetric representation of $\mathbb{F}_p$ this corresponds to the common signum function.

Furthermore, we define a matching function $\operatorname{mt}(\boldsymbol{x}, \boldsymbol{y})$ that compares $\boldsymbol{x}$ and $\boldsymbol{y}$ and counts the number of symbols that hold the same sign.

Definition 2 (Sign Matches). *Let* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_p^n$ *and consider the number of matches in their sign such that*

$$\operatorname{mt}(\boldsymbol{x}, \boldsymbol{y}) = \left|\{i \in \{1, \dots, n\} \mid \operatorname{sgn}(x_i) = \operatorname{sgn}(y_i), x_i \neq 0, y_i \neq 0\}\right|.$$

We are interested in upper bounding the probability of an attacker being able to reuse any of the previously published signatures. For that, we introduce a function calculating the probability that a vector and a uniformly random hash digest (in $\{\pm 1\}^n$) have μ sign matches. When talking about the security of the signature scheme, we will usually consider the negative $\log_2$ of this probability.

Definition 3 (Logarithmic Matching Probability (LMP)). *For a fixed* $\boldsymbol{v} \in \mathbb{F}_p^n$ *and* $\boldsymbol{y} \xleftarrow{\$} \{\pm 1\}^n$, *the probability of* $\boldsymbol{y}$ *to have* $\mu := \operatorname{mt}(\boldsymbol{y}, \boldsymbol{v})$ *sign matches with* $\boldsymbol{v}$ *is*

$$B(\mu, \operatorname{wt}_H(\boldsymbol{v}), 1/2),$$

where $B(k, n, q)$ *is the binomial distribution defined as*

$$B(k, n, q) = \binom{n}{k} q^k (1-q)^{n-k} \quad .$$

To ease notation, we write $\operatorname{LMP}(\boldsymbol{v}, \boldsymbol{y}) = -\log_2(B(\mu, \operatorname{wt}_H(\boldsymbol{v}), 1/2))$.

Note that this function can be efficiently approximated via additions and subtractions of precomputed values of $\log_2(x!)$, i.e., using a look-up table.

In [11], the authors computed the marginal distribution of entries where vectors are uniformly distributed in $V_L(p, n, w)$. Let E denote a random variable corresponding to the realization of an entry of $\boldsymbol{x} \in \mathbb{F}_p^n$. As n tends to infinity we have the following result on the distribution of the elements in $\boldsymbol{x} \in \mathbb{F}_p^n$.

Lemma 4 ([11, Lemma 1]). *For any* $x \in \mathbb{F}_p$, *the probability that one entry of* $\boldsymbol{x}$ *is equal to* x *is given by*

$$p_w(x) = \frac{1}{Z(\beta)} \exp(-\beta \operatorname{wt}_L(x)),$$

where $Z(\beta) = \sum_{i=0}^{p-1} \exp(-\beta \operatorname{wt}_L(x))$ *denotes the normalization constant and* β *is the unique solution to* $w = \sum_{i=0}^{p-1} \operatorname{wt}_L(i) p_w(x)$.

Definition 5 (Typical Lee Set). *For a fixed weight* w*, let* $p_w(x)$ *be the probability from Lemma 4 of the element* $x \in \mathbb{F}_p$*. Then, we define the typical Lee set as*

$$T(p, n, w) = \left\{ \boldsymbol{x} \in \mathbb{F}_p^n \mid \boldsymbol{x}_i = x \text{ for } f(p_w(x)n) \text{ coordinates } i \in \{1, \ldots, n\} \right\},$$

for a rounding function f*. That is the set of vectors, for which the element* x *occurs* $f(p_w(x)n)$ *times.*

In principle, f could be simply chosen as the rounding function. This would, however, mean that the elements of $T(p, n, w)$ do in general not have Lee weight w. This effect is particularly evident when moderate values w are picked, for which number occurrences would be rounded to zero for many field elements.

Therefore, to obtain a closer approximation of the target weight, we design f as follows: if the expected number of occurrences for a symbol $x \in \mathbb{F}_p$ according to $p_w(x)n$ is at least 1, we always round down. If, however, the element x is expected to occur at most once, we round up or down according to a threshold τ. This τ allows us fine control over the Lee weight of the vector $\boldsymbol{x} \in T(p, n, w) \subset \mathbb{F}_p^n$. We choose this value such that the vector used to generate the secret key has Lee weight as close to the GV bound as possible.

3.1 Key Generation

The key generation of our signature scheme is presented in Algorithm 1. The basic idea to generate the secret key $\boldsymbol{G}_{sec}$ is to sample two cyclic matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{F}_p^{n/2 \times n/2}$ of Lee weight $w_{key} = \delta_p^{\mathrm{GV}} n$, where $\boldsymbol{A}$ has to fulfill the extra property of being an invertible matrix. Note that this property is satisfied for random matrices with large probability. The public key is obtained by computing the row reduced Echelon form of $\boldsymbol{G}_{sec}$, referred to as $\boldsymbol{G}_{sys}$. The public key is then formed by the non-trivial part of $\boldsymbol{G}_{sys}$, which we denote by $\boldsymbol{T}$.

Note that $|T(p, n/2, w_{key})|^2$ corresponds to the cardinality of our key space. In order to prevent brute force attacks this cardinality needs to be larger than 2^λ.

3.2 Signature Generation

Note that most of the Hash-and-Sign schemes require the Hash of a message to be a syndrome for a public parity-check matrix. In this Hash-and-Sign algorithm we proceed differently. We use the generator matrix to generate signatures which are codewords of Lee weight within a fixed range. The connection to the Hash of the message vector is established through the number of sign matches.

Algorithm 1: Key Generation

Input: Prime p, code length n, security level λ, Lee weight w_{key}

1 Choose $\boldsymbol{a}, \boldsymbol{b} \xleftarrow{\$} T(p, n/2, w_{key}/2)$.

2 Construct cyclic matrix $\boldsymbol{A} \in \mathbb{F}_p^{n/2 \times n/2}$ from all shifts of $\boldsymbol{a}$. $\boldsymbol{A}$ needs to be invertible. If this is not the case, resample $\boldsymbol{a}$ according to Line 1.

3 Construct cyclic matrix $\boldsymbol{B} \in \mathbb{F}_p^{n/2 \times n/2}$ from all shifts of $\boldsymbol{b}$.

4 Generate the secret key $\boldsymbol{G}_{\text{sec}} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \end{pmatrix} \in \mathbb{F}_p^{n/2 \times n}$.

5 Calculate the systematic form $\boldsymbol{G}_{\text{sys}} = \begin{pmatrix} \boldsymbol{I}_{n/2} & \boldsymbol{T} \end{pmatrix}$ of $\boldsymbol{G}_{\text{sec}}$ with $\boldsymbol{T} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

Output: public key $\boldsymbol{T}$, private key $\boldsymbol{G}_{\text{sec}}$

The signature generation takes as its input the message $\boldsymbol{m}$ to be signed and makes use of the private key $\boldsymbol{G}_{sec}$ and outputs the signature $\boldsymbol{y}$. To do so the algorithm utilizes the secret generators matrix of the code, namely the rows of $\boldsymbol{G}_{sec}$, to find a codeword $\boldsymbol{v} = [\boldsymbol{y}, \boldsymbol{yT}]$ of Lee weight in $[w_{sig} - \varepsilon_s, w_{sig}]$ with sign matches achieving a desired LMP between the hash of the message and the signature codeword. Without having access to a secret basis (the private key), it is already computationally hard to find codewords in the desired Lee weight range (even ignoring the LMP). Therefore, this property suffices to ensure that it is hard to generate fresh codewords that can function as signatures even for arbitrary hashes.

Loosely speaking, a high LMP value ensures that enough signs of the codeword $\boldsymbol{v}$ and challenge $\boldsymbol{c}$ match. This establishes the connection between the signature and the message and prevents reusing codewords contained in previously published signatures to sign freshly generated hashes. Sampling a fresh salt if a signing attempt does not work, guarantees that any message can be signed successfully.

In line 1, one takes the secret key $\boldsymbol{G}_{sec}$ from the Key Generation 1, and stacks it with its negative $-\boldsymbol{G}_{sec}$. In line 2, we hash the input message and get $\boldsymbol{m}'$, which will be fed together with a salt to CSPRNG in line 5 to get the target vector $\boldsymbol{c}$ for the number of sign matches, i.e., the $\mathrm{LMP}(\boldsymbol{v}, \boldsymbol{c})$, where $\boldsymbol{v}$ denotes the information vector of the signature $\boldsymbol{y}$. Line 6 assures that $\boldsymbol{c}$ is in $\{\pm 1\}^n$ making its signs comparable with the signs of vectors in $\mathbb{F}_p^n$. In line 9, we are checking how many matches the row $\boldsymbol{g}_i$ has with the target vector $\boldsymbol{c}$. We take into account how many of the signs of $\boldsymbol{c}$ and $\boldsymbol{g}_i$ are matching in line 10, where $\lfloor \cdot \rceil$ denotes truncation. We do this by setting the magnitude in the corresponding position of the information vector according to the number of matches and the scaling factor s. Thus, if the row has many matches with the target $\boldsymbol{c}$, we add this row multiple times. This results in the information vector $\boldsymbol{x}$ and in line 12 produces the preliminary codeword $\boldsymbol{v}$.

Lines 11-33, which we refer to as the *Concentrating* procedure, are necessary to ensure that the signatures vary as little as possible in Lee weight and sign matches.

$\mathcal{A}$ keeps track of which rows have already been added or subtracted from the codeword $\boldsymbol{v}$ and is updated respectively in line 26, 28. In line 14, we initiate the condition lf with 1, which keeps track whether the conditions of the signature (that is LMP and Lee weight) are satisfied, in which case lf will be set to 0. To ensure a constant time signature generation, the lines 16-28 will only run up to n_{con} times.

To have signatures with much lower Lee weight than other signatures is undesirable, as this might leak information on the secret key. Thus, the iterative approach in lines 16-20 is used to add or subtract the generator row minimizing the absolute difference to the desired LMP. For this we first add the row $\boldsymbol{g}_i'$ to $\boldsymbol{v}$ in line 17 and then check in line 18 if the difference of the LMP to the target is minimized by adding this row. Line 19 checks whether the row $\boldsymbol{g}_i'$ is within the set of allowed rows, i.e., in $\mathcal{A}$ or if the signature conditions are satisfied, i.e., $lf = 0$. This results in a codeword $\boldsymbol{v}'$ which is close enough to the target LMP.

Lines 21-24 aim at creating signatures of almost constant Lee weight. For this we compute in line 21 the Lee weight w' of $\boldsymbol{v}'$ and check in line 22 if it is close enough to the target Lee weight w_{sig}, i.e., at most has a w_{key} difference. In this case, we update the signature condition lf with 0. If the Lee weight w' is larger than the target, we reset $\boldsymbol{v}'$ with the initial $\boldsymbol{v}$ in lines 23, 24. The lines 25-28 update the set of rows which are allowed to be added. In fact, if $i' \leq n/2$, we added a row of $\boldsymbol{G}_{sec}$ and exclude the same row to be extracted again by excluding $i' + n/2$ from the allowed set $\mathcal{A}$. If $i' > n/2$, the added row was from $-\boldsymbol{G}_{sec}$ and we exclude $i' - n/2$ from $\mathcal{A}$ to avoid subtracting the same row again.

After all iterations have been completed, lines 29-33 check whether the resulting codeword is within the desired LMP/Lee weight range. If this is the case, we extract the information vector $\boldsymbol{y}$ from $\boldsymbol{v}$ in line 30 and publish the signature $(\text{salt}, \text{ENCODE}(\boldsymbol{y}))$. The encoding procedure $\text{ENCODE}(\cdot)$ is described in Section 3.4. Otherwise another salt is sampled and the signing procedure restarts.

The scaling parameter s used in line 10 is experimentally determined with the goal of minimizing the running time of the Signing algorithm. Its value is a trade-off between the probability of creating a valid signature for a specific hash value and the amount of iterations within the *Concentrating* procedure.

3.3 Signature Verification

The verification process is quite simple. In a first step, the received signature $\boldsymbol{y}'$ is decoded as explained in Section 3.4 to obtain the uncompressed vector $\boldsymbol{y}$. The verifier computes in line 3 and 4 $\boldsymbol{c}$ as CSPRNG from the hash of the message

Algorithm 2: Signing

Input: Secret key $\boldsymbol{a}, \boldsymbol{b}$, message $\boldsymbol{m}$, threshold ε, signature weight w_{sig}, key weight w_{key}, scaling factor $s \in \mathbb{R}$, security level λ, number of concentrating iterations n_{con}.

Output: salt, signature $\boldsymbol{y}$.

1 $\boldsymbol{G}_{sec} \leftarrow (\boldsymbol{A}, \boldsymbol{B})$, $\boldsymbol{G} = \begin{pmatrix} \boldsymbol{G}_{sec} \\ -\boldsymbol{G}_{sec} \end{pmatrix}$ with rows $\boldsymbol{g}'_i$
2 $\boldsymbol{m}' \leftarrow \mathrm{Hash}(\boldsymbol{m})$
3 **repeat**
4   salt $\xleftarrow{\$} \{0,1\}^{256}$ `// Simple signing starts`
5     $\boldsymbol{c} \leftarrow \mathrm{CSPRNG}(\boldsymbol{m}' \mid\mid \mathrm{salt})$
6     $c_i \leftarrow (-1)^{c_i} \quad \forall i$
7     $\boldsymbol{x} \leftarrow (0, \ldots, 0)$
8   **for** $i \leftarrow 1$ **to** $n/2$ **do**
9         $x_{mt} = \mathrm{mt}(\boldsymbol{g}_i, \boldsymbol{c}) - \frac{\mathrm{wt}_H(\boldsymbol{g}_i)}{2}$
10         $\boldsymbol{x}_i = \lfloor x_{mt} s \rceil$ `// Simple signing ends`
    end
11     $\mathcal{A} \leftarrow \{1, \ldots, n\}$ `// Allowed row index set`
12     $\boldsymbol{\nu} \leftarrow \boldsymbol{x}\boldsymbol{G}_{sec}$ `// Concentrating starts`
13     $\boldsymbol{\nu}' \leftarrow (0, ..., 0)$, $i' = 0$
14     $lf \leftarrow 1$
15   **for** $j \leftarrow 1$ **to** n_{con} **do**
16   **for** $i \in \{1, \ldots, n\}$ **do**
17             $\boldsymbol{\nu}'' \leftarrow \boldsymbol{\nu} + \boldsymbol{g}'_i$
18   **if** $|\mathrm{LMP}(\boldsymbol{\nu}'', \boldsymbol{c}) - (\lambda + 64 + \varepsilon)| \leq |\mathrm{LMP}(\boldsymbol{\nu}', \boldsymbol{c}) - (\lambda + 64 + \varepsilon)|$ **then**
19   **if** $i \in \mathcal{A} \mid\mid lf = 0$ **then**
20                     $\boldsymbol{\nu}' \leftarrow \boldsymbol{\nu}''$, $i' \leftarrow i$
        end
21         $w' \leftarrow \mathrm{wt}_L(\boldsymbol{\nu}')$
22   **if** $w' > w_{sig} - w_{key}$ **then**
            $lf \leftarrow 0$
23   **if** $w' \leq w_{sig}$ **then**
24             $\boldsymbol{\nu} \leftarrow \boldsymbol{\nu}'$
25   **if** $i' \leq \frac{n}{2}$ **then**
26             $\mathcal{A} \leftarrow \mathcal{A} \setminus \{i' + n/2\}$
27   **else**
28             $\mathcal{A} \leftarrow \mathcal{A} \setminus \{i' - n/2\}$
    end
29   **if** $\mathrm{wt}_L(\boldsymbol{\nu}) \leq w_{sig}$ && $\mathrm{wt}_L(\boldsymbol{\nu}) > w_{sig} - 2w_{key}$ && $\mathrm{LMP}(\boldsymbol{\nu}, \boldsymbol{c}) \geq \lambda + 64$ **then**
30         $[\boldsymbol{y}, \boldsymbol{y}\boldsymbol{T}] \leftarrow \boldsymbol{\nu}$
31   **return** salt, ENCODE($\boldsymbol{y}$)
32   **else**
33   go to Line 3 `// Concentrating ends`
end

and salt. Then, the verifier checks that $\boldsymbol{v}$ is indeed a codeword of the public code; this is ensured by computing $\boldsymbol{v}$ as $[\boldsymbol{y}\ \boldsymbol{yT}]$ in line 5.

Then, the verifier checks in line 6 that the codeword $\boldsymbol{v}$ has Lee weight of at most w_{sig}. Finally, one checks whether a sufficient amount of the signs of the signature $\boldsymbol{v}$ match the output $\boldsymbol{c}$ of the CSPRNG(Hash($\boldsymbol{m}$)||salt), i.e., $\mathrm{LMP}(\boldsymbol{v}, \boldsymbol{c}) \geq \lambda + 64$. This verification process is given in Algorithm 3.

Algorithm 3: Verification

Input: signature (salt, $\boldsymbol{y}'$) message $\mathbf{m}$, public key $\boldsymbol{T}$, Lee weight w_{sig}.
1 $\boldsymbol{y} \leftarrow \textsc{decode}(\boldsymbol{y}')$
2 $\boldsymbol{m}' \leftarrow \mathrm{Hash}(\boldsymbol{m})$
3 $\boldsymbol{c} \leftarrow \mathrm{CSPRNG}(\boldsymbol{m}' \mid\mid \mathrm{salt})$
4 $c_i \leftarrow (-1)^{c_i} \quad \forall i$
5 $\boldsymbol{v} = [\boldsymbol{y}\ \boldsymbol{yT}]$.
6 Accept if the following two conditions are satisfied:
(a) $\mathrm{wt}_L(\boldsymbol{v}) \leq w_{\mathrm{sig}}$,
(b) $\mathrm{LMP}(\boldsymbol{v}, \boldsymbol{c}) \geq \lambda + 64$.
Otherwise, Reject.
Output: Accept or Reject

3.4 Encoding and Decoding

The coefficients that constitute a signature before encoding follow a Gaussian-like distribution centered at zero. This fact allows to reduce the signature size by compressing the signature and encode it in a bitstring. For that, we use the same approach as proposed in the Falcon signature scheme [36]. That is, each coefficient is converted into its signed representation and split into a *tail* and *head*. The coefficient's sign bit is concatenated with the uncoded tail, as this tail is approximately uniformly distributed and thus cannot be compressed efficiently. The remaining bits in the coefficients head are then encoded in a 0^k1 fashion, that is a sequence of k zeroes and a one, where k is the value of the head.

4 Security Analysis

In this section, we assess the security of FuLeeca. The analysis consists of three parts. We begin by considering the generic solvers for finding codewords of given Lee weight. The second part describes known attacks and our countermeasures. The third part discusses the applicability of lattice reduction algorithms to solve the hard computational problems underlying this system. Taking all mentioned attacks into account we determine the presented parameters to achieve the security levels required by NIST.

4.1 Hardness of Underlying Problem and Generic Solvers

The adversary can attempt to recover the secret key from the public key, which is known as a key recovery attack. For FuLeeca, this is equivalent to finding any of the the rows of the secret generator matrix, which are of weight w_{key}. Alternatively, the attacker can try to forge a signature directly, without knowledge of the secret key. Forging a signature of FuLeeca is, therefore, equivalent to finding a codeword of given Lee weight that satisfies both the number of required matches and the weight restriction.

Hence, both attacks require solving instances of the finding a codeword of given Lee weight problem, which is formally defined as follows.

Problem 6 (Finding Codeword of Given Lee Weight). Given $\boldsymbol{H} \in \mathbb{F}_p^{(n-k)\times n}$ and $w \in \mathbb{N}$ find a $\boldsymbol{c} \in \mathbb{F}_p^n$ such that $\boldsymbol{c}\boldsymbol{H}^\top = \boldsymbol{0}$ and $\mathrm{wt}_L(\boldsymbol{c}) = w$.

This problem has first been studied in [42]. Problem 6, i.e., finding codewords of given weight is equivalent to the decoding problem. The decisional version of this problem has been proven to be NP-complete in [56].

Several algorithms have been proposed to solve this problem, they all belong to the family of Information Set Decoding (ISD) algorithms.

Remark 7. Note that ISD algorithms can be formulated such that they solve the syndrome decoding problem, that is: given a parity-check matrix $\boldsymbol{H} \in \mathbb{F}_p^{(n-k)\times n}$, a syndrome $\boldsymbol{s} \in \mathbb{F}_p^{n-k}$ and a target weight t, they find an error vector $\boldsymbol{e} \in \mathbb{F}_p^n$, such that $\boldsymbol{H}\boldsymbol{e}^\top = \boldsymbol{s}^\top$ and $\mathrm{wt}(\boldsymbol{e}) = t$. Thus, by setting $\boldsymbol{s} = \boldsymbol{0}$, we can use such solvers to find codewords of weight t. However, note that Prange's algorithm [49] searches for a transformed syndrome $\boldsymbol{s}' = \boldsymbol{s}\boldsymbol{U}$, for some invertible $\boldsymbol{U}$ and wants the transformed syndrome to have weight t. As this is never satisfied for $\boldsymbol{s} = \boldsymbol{0}$, Prange cannot be used to find codewords of given weight. However, all improvements upon Prange, such as Stern/Dumer [55,29], MMT [45], BJMM [13] try to first enumerate the error vector in the information set and then check whether the remaining vector has the remaining weight. This can also be applied to $\boldsymbol{s} = \boldsymbol{0}$.

ISD algorithms make use of an information set of the code, where one assumes a small weight and thus constructs lists of these partial solutions.

Let us quickly recall the main steps of an ISD algorithm. Given $\boldsymbol{H} \in \mathbb{F}_p^{(n-k)\times n}$, choose an information set I and bring $\boldsymbol{H}$ into a partial systematic form. For this, let J be a set of size $k+\ell$, which contains the information set I and transform $\boldsymbol{H}$ as

$$\boldsymbol{U}\boldsymbol{H}\boldsymbol{P} = \tilde{\boldsymbol{H}} = \begin{pmatrix} \boldsymbol{I}_{n-k-\ell} & \boldsymbol{H_1} \\ 0 & \boldsymbol{H}_2 \end{pmatrix},$$

where $\boldsymbol{U} \in \mathbb{F}_p^{(n-k)\times(n-k)}$ is an invertible matrix and $\boldsymbol{P} \in \mathbb{F}_p^{n\times n}$ is a permutation matrix. Thus, we also split the unknown solution $\boldsymbol{c}$ into the indices J and J^C, i.e., $\boldsymbol{c}\boldsymbol{P}^\top = (\boldsymbol{c}_1, \boldsymbol{c}_2)$. Assuming that $\boldsymbol{c}_2$ has Lee weight v, we get the following two equations:

$$\begin{aligned} \boldsymbol{c}_1 + \boldsymbol{c}_2\boldsymbol{H}_1^\top &= 0 \\ \boldsymbol{c}_2\boldsymbol{H}_2^\top &= 0. \end{aligned}$$

Thus, we can first solve the second equation, $\boldsymbol{c}_2\boldsymbol{H}_2^\top = 0$ with $\mathrm{wt}_L(\boldsymbol{c}_2) = v$ as we then can easily check if the missing part $\boldsymbol{c}_1$ has the remaining Lee weight, by $\mathrm{wt}_L(\boldsymbol{c}_2\boldsymbol{H}_1^\top) = w - v$.

In [56], several algorithms have been presented to solve the smaller instance, namely using Wagner's approach of a set partitioning and using representation technique. In [22], the authors presented the amortized Wagner's approach.

Finally, in [12] the authors presented an adaption of these algorithms, taking into account that a random low Lee weight codeword has the exponential weight distribution observed in [11]. In these papers, it has been observed, that the amortized BJMM approach attains the lowest computational cost, and thus we consider this algorithm to compute the security level of the proposed parameters.

For the details of the algorithm, we refer to [12]. Mathematica programs to compute the computational costs of BJMM are publicly available[2] or for Wagner's cost here[3].

We adapted the program which computes the classical asymptotic cost c in the form $2^{c \cdot n}$, by considering the cost $c/2$ on a capable quantum computer (see [16,22]).

Since we sample the secret vectors for the generator matrix using the typical Lee sets, i.e., any $x \in \mathbb{F}_p$ occurs in the sought-after codeword $f(p_{w_{key}}(x) \cdot n)$ number of times, it makes sense to use this information in an ISD algorithm. However, as shown in [12], the amortized BJMM algorithm outperforms even the attempts to use restricted balls in case, where we are beyond the unique decoding radius. Thus, we build our security analysis on this fastest known algorithm, taking into account also polynomial speedups due to the quasi-cyclic structure [53].

4.2 Analysis of the Algorithm with Respect to Known Attacks

We assume that an attacker has access to up to 2^{64} signatures for chosen messages. Such multi-use scenarios require an existential unforgeability under chosen message attack (EUF-CMA) security proof. For Hash-and-Sign approaches, EUF-CMA security proofs are notoriously difficult. Unfortunately, we cannot provide one at the moment. We prevented possible leakages and vulnerabilities via the *Concentrating* procedure. These considerations are described in more

[2] https://git.math.uzh.ch/isd/lee-isd/lee-isd-algorithm-complexities/-/blob/master/Lee-ISD-restricted.nb

[3] https://github.com/setinski/Information-Set-Decoding-Analysis

detail below. Note that the *Concentrating* procedure at the moment does not involve a threshold on how close the valid signatures have to be to the target LMP. This flexibility might be of use in a future EUF-CMA security proof. Additionally, the scheme does not involve rejection sampling, which might be helpful to strengthen security, as soon as new attack vectors are known.

Exploiting additional knowledge given to the attacker in form of signatures is perhaps the most common way to attack Hash-and-Sign based signature schemes. In fact, information leaked by the signatures has repeatedly been used to retrieve the private key. To give an example, successful attacks on the schemes [6,47] have been presented in [51,27]. Specifically, these attacks exploit the fact that for the proposed schemes in the Hamming metric a basis vector as well as the signatures have low weight, i.e., a small support. The main problem in the design of these attacked schemes was that the supports of the published signatures correlate with the private key. We consider attacks exploiting leakage via published hash/signature pairs. Such support-based attacks cannot directly be applied to FuLeeca as in the Lee metric vectors of low Lee weight do not necessarily have a small Hamming support. In fact, by putting the weight of the secret generators on the GV bound, we may even treat the resulting code as a random code. This thwarts Hamming-metric attacks as the secret generators and the signatures have close to full Hamming weight.

Setting a sufficiently high threshold for the number of required sign matches prevents that a previously published signature can be directly used to sign another message. An obvious generalization of this reuse attack is creating linear combinations of existing signatures to forge new signatures. Note, however, that with overwhelming probability the Lee weight of the resulting vector will be too large to be accepted by the verifier. Hence, such an attack, which is similar to performing a sieving algorithm known from lattice-based cryptography, requires complexity which is exponential in the code parameters. Notably the works [38,44] show that finding a codeword of lower Lee weight in a quasi-cyclic code is significantly easier in case the code dimension $n/2$ is a composite number. In fact the security reduces to the codeword finding problem in a quasi-cyclic code with dimension equal to the smallest factor of $n/2$. Therefore, for all considered parameter sets in this work, we choose $n/2$ to be prime.

To avoid leakage via published hash/signature pairs we integrated a specific procedure into the signing algorithm, which we refer to as the *Concentrating* procedure. In the following, we first examine the signing algorithm without applying the specified *Concentrating* procedure. We randomly draw $k = 500$ salts and messages and observe the corresponding outputs of the hash-function $h_1, \ldots, h_k$, i.e., $h_\ell = \mathrm{Hash}(\mathrm{salt}||m_\ell)$. For two different private keys we compare the Lee weights and sign matches of the corresponding signatures after just applying "Simple Signing". Figure 1 shows the relation between the relative Lee weights and the LMP between the codeword and the target vector, which is the hash of the message. Since the signature algorithm effectively correlates the secret key and the hashes it appears to be possible to learn at least some information about the

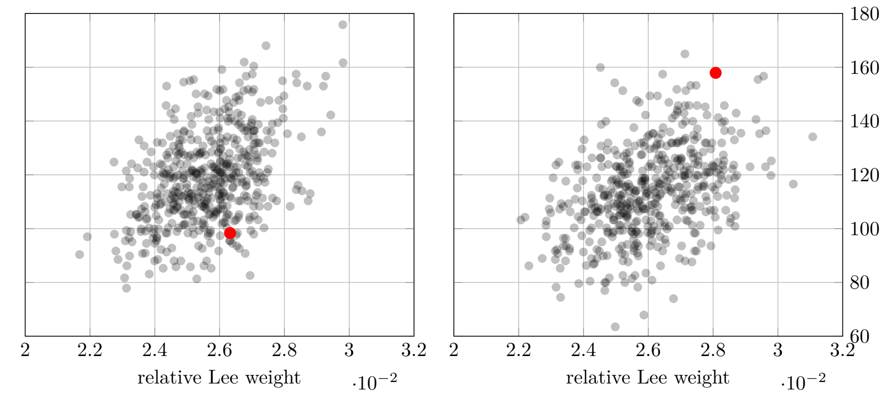

Fig. 1: Evaluation of 500 signatures for simulated hashes (i.i.d uniform) using *two different keys* (left and right) after application of "Simple Signing".

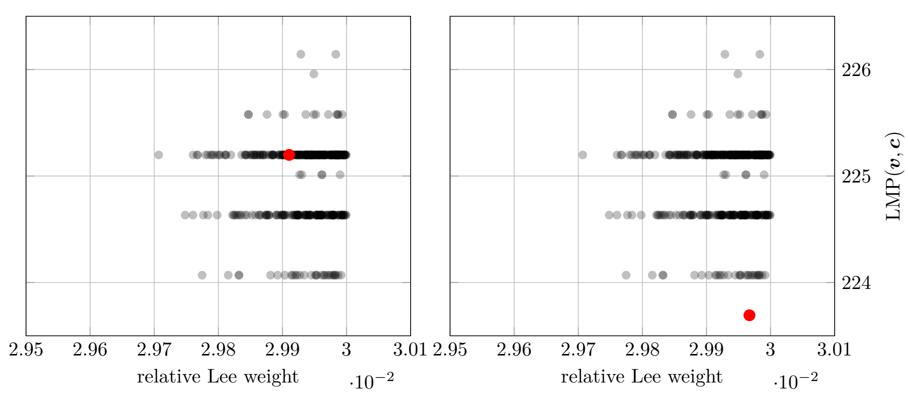

Fig. 2: Evaluation of 500 signatures for simulated hashes (i.i.d uniform) for *two different keys* after application of both "Simple Signing" and "Concentrating".

secret key based on the distribution of resulting codewords in this Lee weight / LMP space.

The distribution of signatures for both private keys of Figure 1 show that the LMP between hash and codeword as well as the resulting Lee weights vary significantly and depend on the secret key. Since we are using two different private keys, we obtain two different signatures for each of the hashes. To exemplify this, we marked the resulting signatures before the *Concentrating* procedure for the same hash (the red dots) but using different private keys in Figure 1. Even though we do not provide a specific attack exploiting this behavior, the results suggest that some information about the private key is leaked and can potentially be exploited to help in the process of recovering the secret key. Figure 2 shows the distribution of LMP values and relative Lee weights for the same hashes as

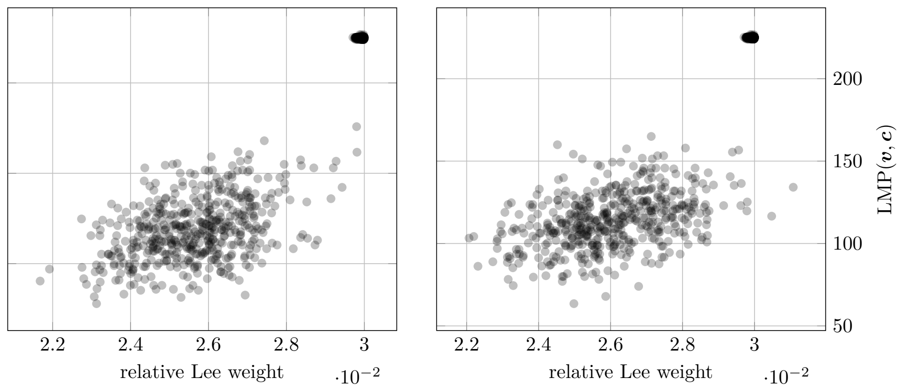

Fig. 3: Evaluation of 500 signatures for simulated hashes (i.i.d uniform) using *two different keys* both after application of the "Simple Signing" part of Algorithm 2 and as well as applying the "Concentrating" procedure (dense clusters in the upper right).

in Figure 1 after the *Concentrating* part of Algorithm 2 has been completed. The difference between the distributions for the different secret keys shall be as small as possible to minimize leakage about the secret key. As in Figure 1, we marked the signatures for the same hashes and different secret keys, this time after the *Concentrating* procedure in Figure 2. The results show that the *Concentrating* procedure significantly reduces the leakage observable via the relative Lee weight / LMP map. Figure 3 provides the information observable from Figure 1 and Figure 2 within a single plot to further illustrate the effect of the *Concentrating* procedure.

Similarly, we also observe that the shape of the distribution of signatures in the Lee weight / LMP space does not appear to meaningfully depend on the distribution of the same signatures after "Simple Signing". This is demonstrated in Figure 4 and 5 where for a single key we apply "Simple Signing" to the same set of hashes as before but split the signatures into two groups of almost equal size. For group one (left hand side of Fig. 4) obtaining a codeword with the required LMP after application of the *Concentrating* procedure is expected to be easier than for group two (right hand side of Fig. 4) since in terms of the ratio between the log probability (LMP) and the Lee weight all of these are above average, while group two is below average. In fact, the percentage of hashes in group two that lead to a valid signature (right hand side of Fig. 5) in the end is slightly lower than for group one (left hand side of Fig. 5). However, this behaviour is to be expected for effectively every private key and, thus, this does not reveal any useful information about any chosen key in particular.

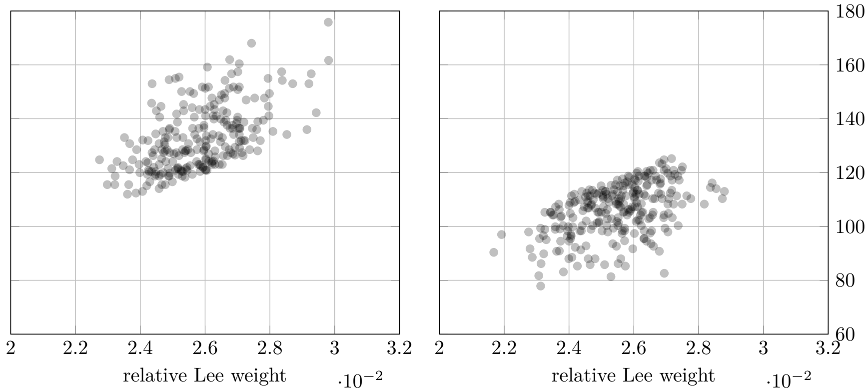

Fig. 4: Evaluation of 500 signatures for simulated hashes (i.i.d uniform) before applying the *Concentrating* procedure. Unlike the previous figures, all of the displayed signatures were created using a *single key*. The vectors are divided into two (nearly equally large) groups, where the ratio between the log probability (LMP) and the Lee weight is above average (left), respectively below average (right).

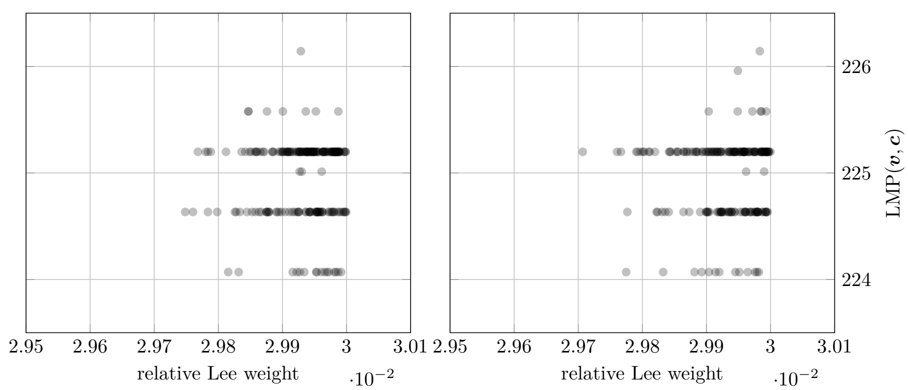

Fig. 5: The same two sets of hashes for *the same key* (as in figure 4) after applying the “Concentrating” algorithm.

4.3 Lattice-based Attacks

Since the Lee metric is close to the Euclidean metric used in lattice-based cryptography, one has to study the known combinatorial attacks therein. In fact, the Lee metric corresponds to the L_1-norm, whereas the Euclidean metric corresponds to the L_2-norm. It is well known [50] that problems with respect to the L_2-norm can be reduced to problems with respect to any other L_p-norm. This result translates to: any algorithm solving a problem in the L_p-norm can also be used to solve the problem in the L_2-norm. Or as stated in [50]: “our main result shows that for lattice problems, the L_2-norm is the easiest." Thus, one can use the Lee-metric ISD algorithms to solve lattice-based problems in the Euclidean metric. It is unknown, whether the reverse direction is also possible, i.e., whether there exists a reduction from problems with respect to the L_1-norm to problems with respect to the L_2-norm. This is, however, exactly the direction required in order to use lattice-based algorithms to solve problems in the Lee metric.

To the best of our knowledge the only sieving algorithm in the L_1-norm is provided in [20], where the authors provide an $(1+\varepsilon)$ approximation algorithm for the closest vector problem for all L_p-norms that runs in $(2+1/\varepsilon)^{\mathcal{O}(n)}$. The asymptotic cost of this algorithm does not outperform the considered Lee-metric ISD algorithms.

Another lattice-based approach is to search for the codeword of lowest Euclidean weight, e.g., using the BKZ algorithm [52]. Since we set the weight of the secret generators on the GV bound and thus assume that our code behaves like a random code, it is not known whether the codeword of lowest Euclidean weight is also the codeword of lowest Lee weight, i.e., the secret key. Under the conservative assumption that this is indeed the case, we estimate the cost of BKZ for the full rank lattice to be in $\mathcal{O}\left(2^{0.292n}\right)$. We observe that the parameter sets we choose attain the target security levels also according to this attack.

Assumption 1: Let us use BJMM to find a vector $\boldsymbol{v}$ of Lee weight w_{sig}. We assume that finding another vector $\boldsymbol{v'}$ of equal Euclidean length, i.e., $||\boldsymbol{v}||_2 = ||\boldsymbol{v'}||_2$, by using BKZ has a lower complexity than finding $\boldsymbol{v}$ using BJMM. If this assumption did not hold, then using BJMM we would be able to achieve a speedup in solving SVP compared to using BKZ, which would in turn affect all lattice-based cryptosystems.

Assumption 2: We assume that the complexity of using BKZ to find a vector having Lee weight less than or equal to w_{sig} is higher compared to using BJMM for this task.

For a Lee weight of w_{sig} the consequence of Assumption 2 not holding is that BKZ would outperform all known ISD algorithms for solving the given weight codeword finding problem at that weight. BKZ requires orthogonal projections within the LLL step. However, the L_1 norm is not induced by a scalar product and, therefore, we assume that the best way to use BKZ for finding short vectors in the L_1 norm is to use it for finding short vectors in L_2 norm and to hope that those are also short enough in the L_1 norm. We assume that using BJMM to find short vectors in the L_1 norm is more efficient than this.

5 Efficiency and performance

5.1 Parameters

Due to the quasi-cyclic structure of the private matrix $\boldsymbol{G}_{sec}$ it is sufficient to store only one of its rows. Therefore, the size of the private key is in the order $\mathcal{O}_p(n)$, where the constant depends on the parameter p.

We take a conservative choice for the NIST security levels [43], as shown in Table 1.

Table 1: Conservative NIST Categories

NIST Security Level	Classical Cost	Quantum Cost
I	160	80
III	224	112
V	288	144

Table 2: Parameters for the proposed signature scheme FuLeeca. All sizes are given in Bytes.

NIST cat.	n	s	n_{con}	secret key size	public key size	sign. size
I	1318	0.046 875	100	2636	1318	1100
III	1982	0.035 156 25	90	3964	1982	1620
V	2638	0.023 437 5	178	5276	2638	2130

The chosen parametersand associated data sizes for the NIST categories I, III and V are given in Table 2. Note that for all parameter sets, we fix $p = 65521, \varepsilon = 1$ and the relative Lee weights $\omega_{sig} = w_{sig}/(nM) = 0.001437$, and $\omega_{key} = w_{key}/(nM) = 0.03$ is on the GV bound, where we recall that $M = \lfloor \frac{p-1}{2} \rfloor$ is the maximal Lee weight in $\mathbb{F}_p$.

The signature sizes are averaged over 1k generated compressed signatures and include the size of the salt. For compression, we have adapted the mechanisms as used in the Falcon signature scheme. Although the signature size is not constant, it can be padded to obtain a fix size. As proposed in [30], it is possible to compress the signatures resulting from Algorithm 2 even further.

5.2 Reason for choice of parameters

Recall that the choice to set w_{key} on the Lee-metric GV bound is necessary, to treat the public code as a random code and thus estimate the BKZ algorithms cost at $2^{0.292n}$.

We choose $p = 65\,521$, in order to set the Lee weight w_{key} of the secret generators on the Lee-metric GV bound and still have a large enough distance to the Lee weight of the signatures w_{sig}. In fact, for smaller choices of p and setting w_{key} on the Lee-metric GV bound we cannot find enough sign matches to signatures of Lee weight w_{sig} with $w_{sig} < 0.2$. The bound $w_{sig} < 0.2$ is mandatory to avoid a polynomial time cost of ISD algorithms.

Table 3: Comparison of post-quantum signature schemes for NIST level I (except for Dilithium which achieves NIST level II). All sizes are given in kB.

scheme	public key size	signature size	total size	variant
Falcon [36]	0.9	0.6	1.5	-
FuLeeca [This work]	1.3	1.1	2.4	-
Dilitihium [28]	1.3	2.4	3.7	-
R-BG [7]	0.1 0.1	7.7 7.2	7.8 7.3	Fast Short
Rank SDP Fen[32]	0.9 0.9	7.4 5.9	8.3 6.8	Fast Short
Ideal Rank BG[19]	0.5 0.5	8.4 6.1	8.9 6.6	Fast Short
PKP BG [19]	0.1 0.1	9.8 8.8	9.9 8.9	Fast Short
SDItH [34]	0.1 0.1	11.5 8.3	11.6 8.4	Fast Short
Ret. of SDitH [1]	0.1 0.1	12.1 5.7	12.1 5.8	Fast, V3 Shortest, V3
SPHINCS$^+$ [5]	<0.1 <0.1	16.7 7.7	16.7 7.7	Fast Short
Beu [17]	0.1 0.1	18.4 12.1	18.5 12.2	Fast Short
Durandal [2]	15.2	4.1	19.3	-
FJR [33]	0.1 0.1	22.6 16.0	22.7 16.1	Fast Short
GPS [41]	0.1 0.1	24.0 19.8	24.1 19.9	Fast Short
MinRank Fen [32]	18.2 18.2	9.3 7.1	27.5 25.3	Fast Short
LESS-FM [8]	10.4 205.7	11.6 5.3	23.0 211.0	Balanced Short sign
WAVE [26]	3200	2.1	3202	-

The parameters are also chosen according to the best known attack to find a codeword of given Lee weight given our public key $\boldsymbol{G}_{pub}$, namely the quantum, amortized BJMM algorithm in the Lee metric.

For the choice of $p = 65\,521$, one cannot explicitly compute the cost of the BJMM algorithm using the program[4] due to numerical instabilities. A conservative extrapolation from results for smaller choices of p suggests that the cost for BJMM at $w_{sig} = 0.03$ lies at $2^{0.08n}$. We want to note here that Wagner's algorithm implies a cost of $2^{0.5n}$.

We choose the length n according to the BKZ algorithm on full-rank lattices, which runs with a cost of $2^{0.292n}$. We aim at the conservative classical security levels $\lambda_1 = 160, \lambda_3 = 224, \lambda_5 = 288$ and set n at least such that

$$2\lambda_i + 64 = 0.292n.$$

This choice is conservative in two ways. Not only the security levels λ_i have been chosen conservatively but also assuming a loss in the security level of λ_i for each of the provided 2^{64} signature vectors is a very conservative approach within the estimation of the resulting security level. In fact, the parameters are chosen in such a way that even for the aforementioned loss of $\lambda_i + 64$ bits a security level of of at least λ_i bits is maintained for the respective parameter sets. It is possible to speed up solving the SVP using BKZ by providing the algorithm with short Euclidean lattice vectors [25]. The obtainable speedup is upper bounded by the cost of finding the provided lattice vectors since otherwise we would have found an improved lattice reduction algorithm. The exact speedup obtained from integrating the short (codeword) vectors depends on their Euclidean length, but we assume that a vector of comparable Euclidean length can be obtained at a lower cost using BKZ compared to using BJMM. We conservatively add 64 to account for the maximum possible speedup once 2^{64} signatures have been published.

In fact, we choose n even slightly larger to ensure that we reach the necessary LMP with good probability. This leads to the following lengths: $n_1 = 1318$, which ensures that $n_1/2 = 659$ is prime, $n_3 = 1982$, which ensures that $n_3/2 = 991$ and finally $n_5 = 2638$, which ensures that $n_5/2 = 1319$ is prime.

Parameter Choice I The parameter choice $p = 65\,521$, $n = 1318$, $\omega_{sig} = w_{sig}/(nM) = 0.03$, $\omega_{key} = w_{key}/(nM) = 0.001437$ leads at least to the desired quantum cost of 2^{80}, since BJMM's algorithm indicates a quantum complexity of $2^{80} = 2^{0.08n}$ operations and the BKZ algorithm requires at least a classical complexity of $2^{384} = 2^{0.292n}$.

Parameter Choice III The parameter choice $p = 65\,521$, $n = 1982$, $\omega_{sig} = w_{sig}/(nM) = 0.03$, $\omega_{key} = w_{key}/(nM) = 0.001437$ leads to the desired quantum cost of 2^{112}, since BJMM's algorithm indicates a quantum complexity of $2^{112} = 2^{0.08n}$ operations and the BKZ algorithm requires at least a classical complexity of $2^{578} = 2^{0.292n}$.

[4] `https://git.math.uzh.ch/isd/lee-isd/lee-isd-algorithm-complexities/-/blob/master/Lee-ISD-restricted.nb`

Parameter Choice V The parameter choice $p = 65\,521$, $n = 2638$, $\omega_{sig} = w_{sig}/(nM) = 0.03$, $\omega_{key} = w_{key}/(nM) = 0.001437$ leads to the desired quantum cost of 2^{144}, since BJMM's algorithm indicates a quantum complexity of $2^{144} = 2^{0.08n}$ operations and the BKZ algorithm requires at least a classical complexity of $2^{770} = 2^{0.292n}$.

5.3 Detailed Performance Analysis

To evaluate FuLeeca, we provide a constant-time C reference implementation that is publicly available at https://gitlab.lrz.de/tueisec/fuleeca-signature. Both the hash functions as well as the CSPRNGs were instantiated with SHA-3 primitives. More precisely, we use the SHA-3 hash functions, as specified in FIPS 202 [54], with digest size of 2λ for the message hashing and expand this message digest together with a salt using the eXtendable-Output Function (XOF) SHAKE256 from the FIPS 202 specification as CSPRNG.

Table 4 shows the required clock cycles and run time in milliseconds for the reference implementation of the algorithm averaged over 10 000 runs. These values were obtained on an Ubuntu 22.04 machine with an Intel Comet Lake (Intel Core i7-10700) CPU at its base frequency of 2900 MHz and 64 GB of RAM using GCC version 11.3.0 and an O3 optimization. In order to generate reliable results, all dynamic performance enhancement and power management features like hyper threading, turbo boost, and dynamic undervolting of the CPU were disabled. Clock cycles are measured using the internal performance registers of the CPU using the library *libcpucycles*[5].

Table 4: Runtime of the constant-time reference implementation in kilocycles and milliseconds on an Intel Comet Lake with a base frequency of 2900 MHz averaged over 10000 runs.

NIST cat.	Unit	Keygen	Sign	Verify
I	kCycles	53 913	1 803 104	1452
	ms	18	621	0.49
III	kCycles	111 937	2 139 170	2534
	ms	38	737	0.86
V	kCycles	195 729	11 805 175	3845
	ms	67	4070	1.32

6 Conclusion

In this paper, we proposed a Hash-and-Sign signature scheme based on the hardness of finding a codeword of given Lee weight. Taking known statistical

[5]The implementation is publicly available at https://cpucycles.cr.yp.to/.

attacks into account, we refined the simple signing process to render the scheme multiple-use. We keep the EUF-CMA security proof as an open problem. The scheme can be efficiently implemented as it only uses simple arithmetics and is able to achieve short signatures of 1100 bytes and public keys of 1318 bytes for the NIST category I security level. This compares favorably to state-of-the-art of lattice-based and code-based schemes.

7 Acknowledgments

We would like to thank Sabine Pircher, Georg Sigl, Thomas Debris-Alazard and Wessel van Woerden for meaningful discussions.

Violetta Weger is supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement no. 899987. Sebastian Bitzer, Georg Maringer, Stefan Ritterhoff and Antonia Wachter-Zeh were supported by the German Research Foundation (Deutsche Forschungsgemeinschaft, DFG) under Grant No. WA3907/4-1, the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 801434), and acknowledge the financial support by the Federal Ministry of Education and Research of Germany in the programme of "Souverän. Digital. Vernetzt.". Joint project 6G-life, project identification number: 16KISK002. Patrick Karl acknowledges the financial support by the Federal Ministry of Education and Research of Germany in the programme of "Souverän. Digital. Vernetzt.". Joint project 6G-life, project identification number: 16KISK002.

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