The Connected Facility Location Polytope

Markus Leitner[a], Ivana Ljubić[b], Juan-José Salazar-González[c], Markus Sinnl[a]

[a]*Department of Statistics and Operations Research, University of Vienna, Austria*
[b]*ESSEC Business School of Paris, France*
[c]*DMEIO, Universidad de La Laguna, Tenerife, Spain*

Abstract

We analyze the polytope associated with a combinatorial problem that combines the Steiner tree problem and the uncapacitated facility location problem. The problem, called *connected facility location problem*, is motivated by a real-world application in the design of a telecommunication network, and concerns with deciding the facilities to open, the assignment of customers to open facilities, and the connection of the open facilities through a Steiner tree. Several solution approaches are proposed in the literature, and the contribution of our work is a polyhedral analysis for the problem. We compute the dimension of the polytope, present valid inequalities, and analyze conditions for these inequalities to be facet defining. Some inequalities are taken from the Steiner tree polytope and the uncapacitated facility location polytope. Other inequalities are new.

Keywords: Valid inequalities, facets, facility location, Steiner trees

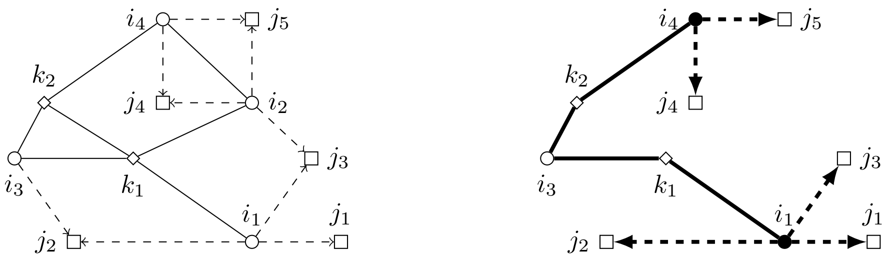
1. Introduction

This article concerns with the *connected facility location problem* (ConFL) arising in the design of a telecommunication network. It is defined as follows. Let I be the set of locations where a facility can be opened. Let J be the set of customers. Each customer must be assigned to an open facility. Let K be the set of *intermediate* nodes, i.e., locations that can be used for connecting open

Email addresses: `markus.leitner@univie.ac.at` (Markus Leitner), `ivana.ljubic@essec.edu` (Ivana Ljubić), `jjsalaza@ull.es` (Juan-José Salazar-González), `markus.sinnl@univie.ac.at` (Markus Sinnl)

facilities. All facility nodes can also be used for connection, regardless whether they are opened or not. In telecom terminology, J represents terminals and $S = I \cup K$ represents *Steiner* nodes. The ConFL problem consists of selecting a subset of I where facilities are opened, connecting these facilities through a tree structure that may use other Steiner nodes, and assigning the customers to open facilities. There are costs associated with opening facilities, connecting Steiner nodes, and assigning customers to facilities. The aim of ConFL is to find a minimum-cost solution.

Figure 1(a) shows an instance of the problem and Figure 1(b) gives a feasible solution. In the figure, $I = \{i_1, \ldots, i_4\}$, $J = \{j_1, \ldots, j_5\}$ and $K = \{k_1, k_2\}$. Open facilities in the feasible solution are indicated in black. Note that in the solution, facility i_3 is not opened, but it is used for the connection.



(a) Instance.

(b) Solution.

Figure 1: An instance of the ConFL problem in (a), and a feasible solution in (b).

The ConFL problem has been extensively addressed in the literature (see e.g., [3, 4, 6, 7, 8, 9]), but all works concern solution approaches and computer implementations. To our knowledge, this paper is the first investigation of the ConFL polytope and it contributes to the literature with new inequalities.

Section 2 describes the notation that is used in this work. Section 3 computes the dimension of the ConFL polytope. Section 4 adapts valid inequalities from the literature, and Section 5 introduces new inequalities. In all cases, conditions for the inequalities to define facets are investigated.

Some results in this work were adapted to the asymmetric ConFL problem and presented in the "International Symposium on Combinatorial Optimization" (Lisbon, 6-7 March 2014) [10].

2. Notation

Let $G = (V, E_S, A_J)$ be a mixed graph where $V = S \cup J$, the edge set E_S represents possible connections between Steiner nodes, and the arc set A_J represents possible assignments of customers to facilities. In the context of telecommunication, the edges represent the optical fiber cables in the core network, and the arcs represent the copper cables connecting the customers to the core network through servers. The graph (S, E_S) is called *core graph*, and it is assumed in this work to be complete, i.e., $E_S = \{\{s_1, s_2\} : s_1 \in S, s_2 \in S\}$. The graph $(I \cup J, A_J)$ is called *assignment graph*, and it is assumed to be complete bipartite, i.e., $A_J = \{(i, j) : i \in I, j \in J\}$. We also assume $|I| \geq 3$ and $|J| \geq 3$. Finally, let $c : E_S \cup A_J \to \mathbb{R}_0^+$ and $f : I \to \mathbb{R}_0^+$ be given cost functions.

The ConFL problem can be modeled by using the following binary variables:

$$x_e = \begin{cases} 1 & \text{if edge } e \text{ is part of the solution} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } e \in E_S;$$

$$y_s = \begin{cases} 1 & \text{if node } s \text{ is part of the solution} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } s \in S;$$

$$z_i = \begin{cases} 1 & \text{if facility } i \text{ is opened} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i \in I;$$

$$a_{ij} = \begin{cases} 1 & \text{if facility } i \text{ serves customer } j \text{ in the solution} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i \in I, j \in J.$$

For convenience of notation, we write $(H : L) := \{\{s_1, s_2\} \in E_S : s_1 \in H, s_2 \in L\}$ for $H, L \subset S$. For brevity, we write $E(H)$ instead of $(H : H)$ and $\delta(H)$ instead of $(H : S \setminus H)$. We also write $x(F) := \sum_{e \in F} x_e$ and $y(H) := \sum_{s \in H} y_s$ for $F \subseteq E_S$ and $H \subseteq S$.

Using this notation, a formulation for the ConFL problem is:

$$\min \sum_{e \in E_S} c_e x_e + \sum_{i \in I} f_i z_i + \sum_{(i,j) \in A_J} c_{ij} a_{ij} \tag{1}$$

$$\sum_{i \in I} a_{ij} = 1 \qquad \forall j \in J \tag{2}$$

$$a_{ij} \leq z_i \qquad \forall i \in I, \forall j \in J \tag{3}$$

$$z_i \leq y_i \qquad \forall i \in I \tag{4}$$

$$x(E(S)) = y(S) - 1 \tag{5}$$

$$x(E(H)) \leq y(H) - y_s \qquad \forall H \subset S, \forall s \in H : |H| \geq 2 \tag{6}$$

$$(x, y, z, a) \in \{0,1\}^{|E_S|+|S|+|I|+|A_J|} \tag{7}$$

Constraints (2) force that every customer is assigned to a facility. Constraints (3) ensure that a customer may be assigned to a facility when this facility is open. Constraints (4) guarantee that a Steiner node with an open facility must be in the solution. Constraints (6) are *generalized subtour elimination constraints* and, together with (5), ensure that the solution is a tree in the core network (see e.g. [13]). We will denote inequalities (6) as yGSECs and use ($yGSEC$) as abbreviation for formulation (1)–(7).

We now analyze the polyhedral structure of the convex hull of the solutions in (2)-(7). Let $\mathcal{P}$ be this polytope.

3. Dimension

The dimension of $\mathcal{P}$ can be derived by using a lifting theorem based on the dimensions of other known polytopes. Let $\mathcal{S}$ be the polytope of the *spanning tree* problem, $\mathcal{U}$ be the polytope of the *uncapacitated facility location* problem, and $\mathcal{P}_y^{x,z,a}(S') = conv\{(x,y,z,a) \in \mathcal{P} : y_s = 1, \forall s \in S'\}$ be an intermediate polytope for $S' \subseteq S$. The projection of $\mathcal{P}_y^{x,z,a}(S)$ on the x-space is $\mathcal{S}$, and on the (z,a)-space is $\mathcal{U}$. Since $\mathcal{P}_y^{x,z,a}(S) = \mathcal{S} \times \{y_s = 1, \forall s \in S\} \times \mathcal{U}$, all facets of $\mathcal{S}$ and $\mathcal{U}$ are also facets of $\mathcal{P}_y^{x,z,a}(S)$.

Starting from the dimension of $\mathcal{P}_y^{x,z,a}(S)$, we compute the dimension of the intermediate polytope, which leads to the dimension of $\mathcal{P} = \mathcal{P}_y^{x,z,a}(\emptyset)$.

Theorem 1. $dim(\mathcal{P}_y^{x,z,a}(S)) = |E_S| - 1 + |A_J| + |I| - |J|$.

Proof. The dimension of $\mathcal{S}$ is $|E_S| - 1$ (see [2]) and the dimension of $\mathcal{U}$ is $|A_J| + |I| - |J|$ (see [1]). $\square$

Theorem 2. *For each* $S' \subseteq S$, $dim(\mathcal{P}_y^{x,z,a}(S')) = |E_S| + |S| - 1 + |A_J| + |I| - |J| - |S'|$.

Proof. Clearly $dim(\mathcal{P}_y^{x,z,a}(S')) \leq |E_S| + |S| - 1 + |A_J| + |I| - |J| - |S'|$ since $\mathcal{P}_y^{x,z,a}(S') \subseteq \mathbb{R}^{|E_S|+|S|+|A_J|+|I|}$ and the $|J|$ equalities (2), equality (5) and $|S'|$ equalities $y_s = 1$ are linearly independent. For the other direction, i.e., to prove that $dim(\mathcal{P}_y^{x,z,a}(S')) \geq |E_S| + |S| - 1 + |A_J| + |I| - |J| - |S'|$, we claim that there are $|E_S| + |S| + |A_J| + |I| - |J| - |S'|$ affinely independent solutions for a given S'. This is proven next by induction on the cardinality of $S \setminus S'$.

When $|S'| = |S|$ the claim follows from Theorem 1. Suppose now that the claim holds for a set S' with $|S'| = \rho$ and consider the set $S'' = S' \setminus \{s\}$ for some $s \in S'$. By the induction hypothesis, there exist $|E_S| + |S| + |A_J| + |I| - |J| - \rho$ affinely independent solutions, all with $y_s = 1$. To prove the claim, we need a solution with $y_s = 0$. This solution exits by the assumption that the instance has at least two facilities and each facility is connected to all customers. $\square$

Corollary 1. $dim(\mathcal{P}) = |E_S| + |S| - 1 + |A_J| + |I| - |J|$.

Theorem 2 for $S' = \{s\}$ proves the following result.

Corollary 2. *Inequalities* $y_s \leq 1$ *are facet-inducing for* $\mathcal{P}$ *for all* $s \in S$.

4. Inequalities from the Uncapacitated Facility Location and Spanning Tree polytopes

The proof of Theorem 2 shows that every removal of a node from S increases the dimension of $\mathcal{P}_y^{x,z,a}(S)$ by one. Therefore the facets of $\mathcal{P}_y^{x,z,a}(S)$ can be lifted to $\mathcal{P}$ with the following result.

Lemma 1 ([12]). *Let* $1, 2, \ldots, u \in S$. *Let*

$$\sum_{e \in E_S} \alpha_e x_e + \sum_{i \in I} \delta_i z_i + \sum_{(i,j) \in A_J} \zeta_{ij} a_{ij} \geq \eta$$

be any facet-inducing inequality for $\mathcal{P}_y^{x,z,a}(S)$. *Then the lifted inequality*

$$\sum_{e \in E_S} \alpha_e x_e + \sum_{s=1}^{u} \beta_s (1 - y_s) + \sum_{i \in I} \delta_i z_i + \sum_{(i,j) \in A_J} \zeta_{ij} a_{ij} \geq \eta$$

is valid and facet-defining for $\mathcal{P}_y^{x,z,a}(S \setminus \{1, 2, \ldots, u\})$, *where*

$$\begin{aligned} \beta_s := &\eta - \min \left\{ \sum_{e \in E_S} \alpha_e x_e + \sum_{k=1}^{s-1} \beta_k (1 - y_k) + \sum_{i \in I} \delta_i z_i + \sum_{(i,j) \in A_J} \zeta_{ij} a_{ij} : \right. \\ &\left. (x, y, z, a) \in \mathcal{P}_y^{x,z,a}(S \setminus \{1, 2, \ldots, s-1\}) \text{ and } y_s = 0 \right\} \end{aligned}$$

for $1 \leq s \leq u$.

The previous lemma allows us to obtain facets of $\mathcal{P}$ from facets of the uncapacitated facility location polytope $\mathcal{U}$.

Theorem 3. *The following inequalities are facet-inducing for* $\mathcal{P}$*:*

(a) $a_{ij} \leq z_i$, *for all* $i \in I$ *and* $j \in J$*;*

(b) $a_{ij} \geq 0$, *for all* $i \in I$ *and* $j \in J$*;*

(c) $z_i \leq y_i$, *for all* $i \in I$.

Proof.

(a) The inequality induces a facet of $\mathcal{U}$, see [1]. Consider an arbitrary sequence of S to lift the β coefficients. For lifting node i, in any feasible solution, $z_i = 0$ when $y_i = 0$ due to (4), and also $a_{ij} = 0$ due to (3). A feasible solution exists since the core network is a complete graph and the customer network is bipartite. Thus, we get $\beta_i = 0$ as lifting coefficient. When lifting node s with $s \neq i$, one can either choose both $z_i = a_{ij} = 1$ or $z_i = a_{ij} = 0$ in a feasible solution; in both cases, we get $\beta_s = 0$.

(b) Every $a_{ij} \geq 0$ induces a facet of $\mathcal{U}$, see [1]. In an arbitrary lifting sequence, for each $s \in S$ there is a solution with $a_{ij} = 0$, thus $\beta_s = 0$.

(c) Every $z_i \leq 1$ induces a facet of $\mathcal{U}$, see [1]. Consider an arbitrary lifting sequence. When lifting node i, $z_i = 0$ due to (4) and thus we get $\beta_i = -1$. When lifting node s with $s \neq i$ there is a feasible solution with $z_i = 1$, thus $\beta_s = 0$. We obtain $-z_i - (1 - y_i) \geq -1$, which can be rewritten as $z_i \leq y_i$.

□

In a similar way, we can derive facet-inducing inequalities of $\mathcal{P}$ from the spanning tree polytope $\mathcal{S}$.

Theorem 4. *The following inequalities are facet-inducing for $\mathcal{P}$:*

(a) $x_e \geq 0$, *for all* $e \in E_S$*;*

(b)

$$x(E(H)) \leq y(H) - y_u, \tag{8}$$

for all $H \subset S : |H| \geq 2$, $u \in H, |H \cap I| \leq |I| - 1$*;*

(c)

$$x(E(H)) \leq y(H) - 1, \tag{9}$$

for all $H \subset S : |H| \geq 2$, $|H \cap I| = |I|$.

Proof.

(a) Every $x_e \geq 0$ induces a facet of $\mathcal{S}$, see [2]. Using the assumption that the graph G is complete and $|I| \geq 3$, there is a feasible solution of ConFL with $x_e = 0$, no matter the lifting sequence of S. Then $\beta_s = 0$ for all $s \in S$.

(b) $-x(E(H)) \geq -|H| + 1$ is a facet of $\mathcal{S}$ for $H \subset V, |H| \geq 2$, see [2]. Consider a lifting sequence for $s \in S$, where we first lift the coefficients of nodes in $S \setminus H$ and then in H, with node u lifted last. The lifting coefficient $\beta_s = 0$ for each $s \in S \setminus H$ because there exists a feasible solution with

$x(E(H)) = |H|-1$. For each $s \in H\setminus\{u\}$, we get $\beta_s = -1$ because the best value for $x(E(H))$ is $|H|-2$. When lifting node u, we get $\beta_u = 0$ because there exists a feasible solution not using any node in H by connecting the customers to a facility outside H. Therefore in the lifting minimization problem $x(E(H)) = 0$ and $\sum_{s\in H\setminus\{u\}} \beta_s(1-y_s) = -|H|+1$. Thus the resulting facet-defining inequality is $-x(E(H)) - \sum_{s\in H\setminus\{u\}}(1-y_s) \geq -|H|+1$, which can be rewritten as $x(E(H)) \leq y(H) - y_u$.

(c) Similar to the (b), except in the last step, when lifting node u, there exists no feasible solution not using any node in H. This is because all facilities are in H. Thus, we get $\beta_u = -1$ and therefore the resulting facet-defining inequality is $-x(E(H))-\sum_{s\in H}(1-y_s) \geq -|H|+1$, which can be rewritten as $x(E(H)) \leq y(H) - 1$.

□

Notice that we just proved that all the inequalities from the formulation (2)-(7) are facet-inducing, except inequalities (6) for $|H \cap I| = |I|$ which are dominated by (9).

Finally, the uncapacitated facility location polytope suggests a new family of facet-defining inequalities. Let us call *injective mapping* a function $h : I \to J$ such that $h(i_1) \neq h(i_2)$ when $i_1 \neq i_2$. Note that injective mappings exist when $|J| \geq |I|$.

Theorem 5. *Let h be an injective mapping. Then the inequality*

$$\sum_{i\in I}(z_i + a_{ih(i)}) \geq 2 \qquad (10)$$

is facet-inducing for $\mathcal{P}$.

Proof. The inequality induces a facet of $\mathcal{U}$, see [5]. Consider an arbitrary lifting sequence. Regardless of the lifted node s, the optimal objective value of the lifting problem is 2 because there is a solution where all customers are connected to the same facility. When s is a facility node, there are other facilities to connect the customers. Thus for any node $s \in S$, $\beta_s = 0$. □

5. New valid inequalities

We now present new valid inequalities for $\mathcal{P}$ and prove that some of them are facets. All the proofs make use of a methodology called *indirect approach* which is based on the following result.

Lemma 2 ([12]). *Let $(A^=, b^=)$ be the equality set of $\mathcal{P}$ containing m equations, and let $\mathcal{F} = \{(x, y, z, a) \in \mathcal{P} : \pi_x x + \pi_y y + \pi_z z + \pi_a a = \pi_0\}$ be a proper face of $\mathcal{P}$. Then the following two statements are equivalent*

1. *$\mathcal{F}$ is a facet of $\mathcal{P}$*
2. *if $\mathcal{F} \subseteq \mathcal{G} = \{(x, y, z, a) \in \mathcal{P} : \alpha x + \beta y + \gamma z + \delta a = \lambda_0\}$ then there exist some $s \in \mathbb{R}$ and some $t \in \mathbb{R}^m$, such that $(\alpha, \beta, \gamma, \delta) = s(\pi_x, \pi_y, \pi_z, \pi_a) + tA^=$ and $\lambda_0 = s\pi_0 + tb^=$.*

The equality set of the ConFL polytope $\mathcal{P}$ consists of (2) and (5), thus $m = |J| + 1$. In the proofs, we construct feasible solutions σ of the face $\mathcal{F}$ under consideration, and evaluate them with the equality defining $\mathcal{G}$ in order to determine the coefficients of this inequality. We denote by $\mathcal{L}(\sigma)$ the evaluation of $\alpha x + \beta y + \gamma z + \delta a$ on σ. These evaluations will make clear the existence of some s and t as in the lemma, thus proving that $\mathcal{F}$ is a facet of $\mathcal{P}$.

5.1. aGSEC Inequalities

The first family of inequalities is motivated by the yGSECs (8), where the node variable on the right hand side is replaced by a sum of assignment variables.

Theorem 6. *Inequalities*

$$x(E(H)) \leq y(H) - \sum_{i \in I \cap H} a_{ij} \tag{11}$$

with $H \subset S : 2 \leq |H| \leq |I| - 1$, and $j \in J$ are valid for $\mathcal{P}$.

Proof. Note that $\sum_{i \in I \cap H} a_{ij}$ is at most 1 for any H due to constraints (2). Thus, the smallest right hand side (rhs) we can get is $y(H) - 1$. This rhs is always non-negative, since $a_{ij} \leq y_i$. Thus $\sum_{i \in I \cap H} a_{ij} \leq \sum_{i \in I \cap H} y_i \leq \sum_{i \in H} y_i$.

Moreover, the rhs is always larger than the associated left hand side (lhs) since any feasible solution is always a tree due to constraints (5) and (6), and in any (sub-)tree with n nodes there are at most $n-1$ edges. Therefore, the inequality is valid for $\mathcal{P}$. $\square$

Note that inequality (11) with $|H \cap I| = |I|$ is equivalent to the corresponding inequality (9) because of equations (2). Furthermore, inequality (11) with $H \cap I = \{i\}$ is dominated by the inequality (8) defined for H and i. For the special case when $H = S \setminus \{i\}$ for some $i \in I$, the associated inequality (11) can be written as $a_{ij} + x(\delta(i)) \geq y_i$. Finally, observe that the constraints (11) suggest an alternative formulation for the ConFL problem, as they can replace (6) in (1)–(7), leading to a model that we will refer to as ($aGSEC$) formulation. Inequalities (11) will also be denoted as aGSECs in the following.

Theorem 7. *Inequalities* (11) *are facet-inducing for* $\mathcal{P}$ *if and only if* $H \subset S :$ $2 \leq |H \cap I| \leq |I| - 1$.

Proof. Let $\mathcal{F} = \{(x, y, z, a) \in \mathcal{G} : \ x(E(H)) - y(H) - \sum_{i \in I \setminus H} a_{ij} = -1\}$ be the proper face induced by (11) for some $j \in J$ and $H \subset K$: $2 \leq |H \cap I| \leq |I| - 1$. Note that we have rewritten the inequality using equation (2) for j.

The feasible solutions $\sigma \in \mathcal{F}$ used in the proof are described by tuples $L_q = (S_q \cap H, S_q \setminus H, I_q, E_q, A_q)$ where

- $S_q \subseteq S$: core nodes involved in the solution (y-variables with value one);
- $I_q \subseteq I$: open facilities in the solution (z-variables with value one);
- $E_q \subset E_S$: core edges in the solution (x-variables with value one);
- $A_q \subset A_J$: assignment arcs in the solution (a-variables with value one).

For each $i_1, i_2 \in I$; $s_1, s_2 \in S$; and $J' := J \setminus \{j\}$, the proof is based on a set of solutions depicted in Figure 2. In these figures, nodes in I, S and J are represented by circles, diamonds and squares, respectively. Open facilities are indicated in bold. Intermediate nodes and sometimes also closed facilities (i.e., nodes from S) are drawn as diamonds. In addition, the proof also uses other solutions constructed by small modifications of the solutions listed below.

- $L_1 = \big(\{i_1\}, \{i_2\}, \{i_1\}, \{\{i_1, i_2\}\}, (i_1 : J)\big)$
- $L_2 = \big(\{i_1\}, \{i_2\}, \{i_1, i_2\}, \{\{i_1, i_2\}\}, (i_1 : J)\big)$
- $L_3 = \big(\{i_1\}, \emptyset, \{i_1\}, \emptyset, (i_1 : J)\big)$
- $L_4 = \big(\{i_1, s_1\}, \emptyset, \{i_1\}, \{\{i_1, s_1\}\}, (i_1 : J)\big)$
- $L_{4'} = \big(\{i_1, s_1\}, \{s_2\}, \{i_1\}, \{\{i_1, s_1\}, \{s_1, s_2\}\}, (i_1 : J)\big)$
- $L_5 = \big(\{i_1\}, \{s_1, s_2\}, \{i_1\}, \{\{i_1, s_1\}, \{s_1, s_2\}\}, (i_1 : J)\big)$
- $L_6 = \big(\{i_1\}, \{s_1, s_2\}, \{i_1\}, \{\{i_1, s_2\}, \{s_1, s_2\}\}, (i_1 : J)\big)$
- $L_7 = \big(\{i_1, i_2\}, \emptyset, \{i_1\}, \{\{i_1, i_2\}\}, (i_1 : J)\big)$
- $L_8 = \big(\{i_1, i_2, s_1\}, \emptyset, \{i_1\}, \{\{i_1, s_1\}, \{s_1, i_2\}\}, (i_1 : J)\big)$
- $L_9 = \big(\{i_1, i_2\}, \emptyset, \{i_1, i_2\}, \{\{i_1, i_2\}\}, (i_1 : J') \cup \{(i_2, j)\}\big)$
- $L_{10} = \big(\emptyset, \{i_1\}, \{i_1\}, \emptyset, (i_1 : J)\big)$
- $L_{11} = \big(\{i_2\}, \{i_1\}, \{i_1, i_2\}, \{\{i_1, i_2\}\}, (i_1 : J') \cup \{(i_2, j)\}\big)$

Consider $\mathcal{F} \subseteq \mathcal{G}$ and recall that α relates to x, β relates to y, γ relates to z, and δ relates to a. Then:

T7a $\gamma_i = 0$, $\forall i \in I$:

To show that $\gamma_i = 0$ for $i \in I \setminus H$, we compare the solutions L_1 and L_2 (see Fig. 2(a) and 2(b)). Take any $i_2 \in I \setminus H$. The only difference between the two solutions is that in L_1 facility i_2 is closed, and in L_2 it is open. Since $\mathcal{L}(L_1) = \mathcal{L}(L_2)$ then $\gamma_{i_2} = 0$.

To show that $\gamma_i = 0$ for $i \in I \cap H$, we consider solutions $L_{1'}$ and $L_{2'}$, which are the same as L_1 and L_2, except that $i_2 \in I \cap H$. Note that these solutions can only be constructed under the assumption that $|I \cap H| \geq 2$. Also note that if in the following steps, an open facility $i \in I$ occurs, we will not mention γ_i explicitly again since the coefficient is zero.

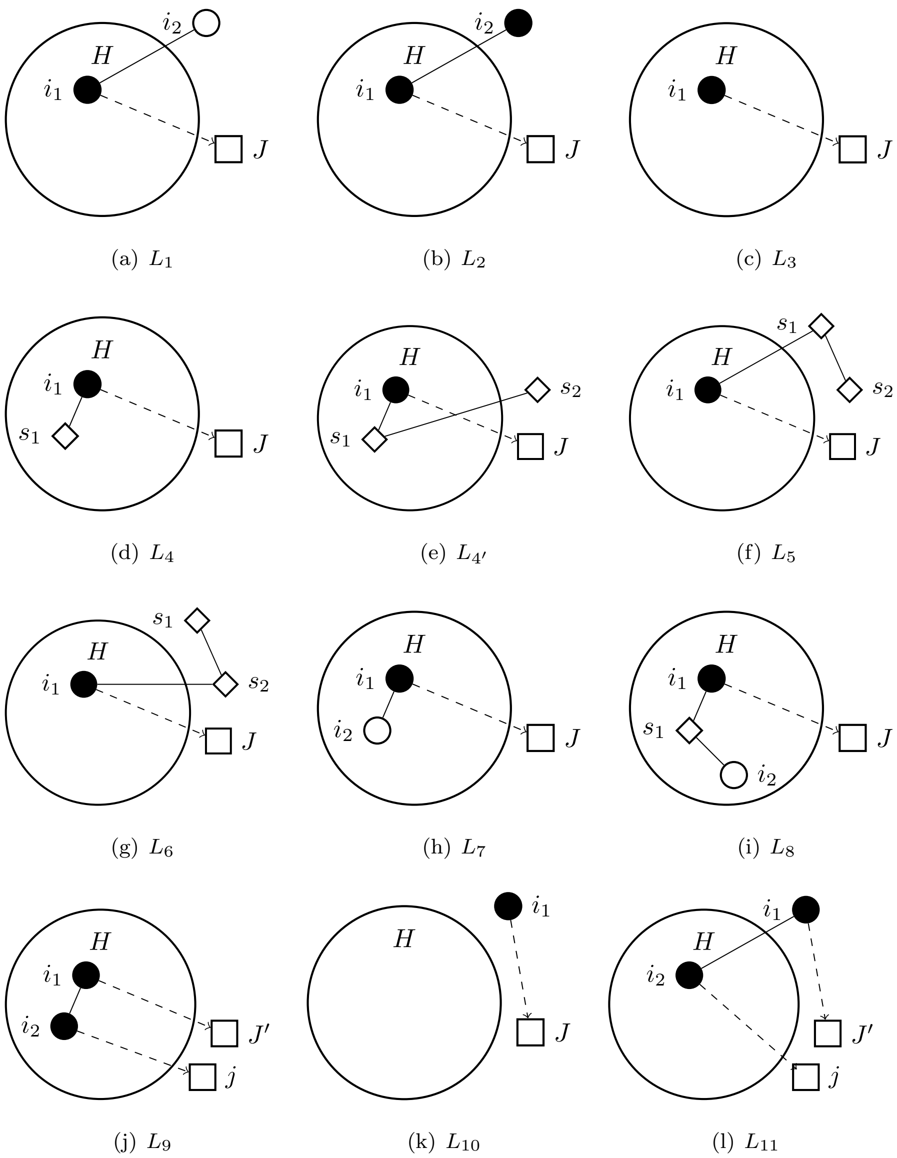


Figure 2: Feasible solutions for the proof of Theorem 7

T7b $\alpha_{ss'} = -\beta_{s'}$, $\forall s \in H$, $\forall s' \in S \setminus H$:

To show that $\alpha_{ss'} = -\beta_{s'}$, $\forall s \in H, s' \in S \setminus H$, we compare the two solutions L_4 and $L_{4'}$ (see Fig. 2(d) and 2(e)). L_4 consists of an open facility $i_1 \in H$, to which all customers are assigned to. Moreover, a core node $s_1 \in H$ is connected to i_1. $L_{4'}$ is nearly the same as L_4, except that there is an additional core node $s_2 \in S \setminus H$, which is connected to s_1. The result is obtained by considering $\mathcal{L}(L_4) = \mathcal{L}(L_{4'})$, which gives $\alpha_{s_1 s_2} + \beta_{s_2} = 0$ for $s_1 \in H, s_2 \in S \setminus H$. Thus $\alpha_{ss'} = -\beta_{s'}$ for all $s \in H, s' \in S \setminus H$.

T7c $\alpha_{ss'} = -\beta_{s'}, \forall s, s' \in S \setminus H$:

Let $\mathcal{L}(L_{5'})$ be L_5 without the node $\{s_2\}$ and edge $\{s_1, s_2\}$. The result follows from $\mathcal{L}(L_{5'}) = \mathcal{L}(L_5)$, which gives $\alpha_{s_1 s_2} + \beta_{s_2} = 0$ for $s_1, s_2 \in S \setminus H$.

T7d $\beta_s = -\bar{\alpha}$, $\forall s \in S \setminus H$:

$\mathcal{L}(L_5) = \mathcal{L}(L_6)$, gives $\alpha_{i_1 s_1} = \alpha_{i_1 s_2}$ for $i_1 \in H, s_1, s_2 \in S \setminus H$. Using the results from step (T7b), we see that all β_s, $s \in S \setminus H$ must have the same coefficient, denote it by $-\bar{\alpha}$. Note that steps (T7c)-(T7d) are only needed for $|S \setminus H| \geq 2$; otherwise, the result already follows from step (T7b).

T7e $\alpha_{ss'} = \hat{\alpha}$, $\forall s, s' \in H$ and $\beta_s = -\hat{\alpha}$, $\forall s \in H$:

First, suppose $H \subset I$. $\mathcal{L}(L_3) = \mathcal{L}(L_7)$ gives $\alpha_{i_1 i_2} + \beta_{i_2} = 0$ for $i_1, i_2 \in H \cap I$. We can switch i_1 and i_2 in L_3, L_7 and get $\alpha_{i_1 i_2} + \beta_{i_1} = 0$. Thus $\alpha_{ss'} = -\beta_s = -\beta_{s'}$ for $s, s' \in H \cap I$. Denote this value by $\hat{\alpha}$.

Now, for $|H \setminus I| = 1$, $\mathcal{L}(L_3) = \mathcal{L}(L_4)$ implies $\alpha_{i_1 k_1} + \beta_{k_1} = 0$ for $k_1 \in H \cap K, i_1 \in H \cap I$. Note that these coefficients are not yet related to $\hat{\alpha}$. To determine the relation, consider $\mathcal{L}(L_4) = \mathcal{L}(L_8)$. We get $\alpha_{k_1 i_2} + \beta_{i_2} = 0$ for $k_1 \in H \cap K, i_2 \in H \cap I$. Since $\beta_{i_2} = -\hat{\alpha}$ we get $\alpha_{k_1 i_2} = \hat{\alpha}$ (and also $\beta_{k_1} = -\hat{\alpha}$). Thus $\alpha_{ss'} = -\beta_s = -\beta_{s'}$ for $s \in H \cap I, s' \in H$.

Finally, for $|H \setminus I| \geq 2$ there are also coefficients $\alpha_{ss'}$ for $s, s' \in H \cap K$. Let $L_{8'}$ be L_8 with $k_2 \in H \cap K$ instead of $i_2 \in H \cap I$. Then $\mathcal{L}(L_4) = \mathcal{L}(L_{8'})$ gives $\alpha_{k_1 k_2} + \beta_{k_2} = 0$, from which $\alpha_{ss'} = \hat{\alpha}$ for $s, s' \in H \cap K$ follows since $\beta_{s'} = -\hat{\alpha}$.

T7f $\delta_{ij} = \bar{\delta}_j$, $\forall i \in H \cap I$:

Let $L_{7'}$ be L_7 with i_2 opened. From $\mathcal{L}(L_{7'}) = \mathcal{L}(L_9)$, it follows that $\delta_{i_1 j} = \delta_{i_2 j}$ for all $i_1, i_2 \in H \cap I$. Denote this value by $\bar{\delta}_j$.

T7g $\delta_{ij'} = \bar{\delta}_{j'}$, $\forall i \in H \cap I$, $\forall j' \in J$, $j' \neq j$:

Let $L_{9'}$ be L_9 where customer j', instead of customer j, is connected to i_2. From $\mathcal{L}(L_{7'}) = \mathcal{L}(L_{9'})$ it follows that $\delta_{i_1 j'} = \delta_{i_2 j'}$ for all $i_1, i_2 \in H \cap I$, $j' \in J$, $j' \neq j$. Denote this value by $\bar{\delta}_{j'}$ for $j' \in J$, $j' \neq j$.

T7h $\delta_{ij} = \bar{\delta}_j - \hat{\alpha} + \bar{\alpha}$, $\forall i \in I \setminus H$:

From $\mathcal{L}(L_{10}) = \mathcal{L}(L_{11})$ we have $\delta_{i_2 j} = \beta_{i_1} + \alpha_{i_1 i_2} + \delta_{i_1 j}$ for $i_2 \in I \setminus H, i_1 \in H \cap I$. Using results from steps (T7b), (T7d), (T7e) and (T7f), we get $\delta_{ij} = \bar{\delta}_j - \hat{\alpha} + \bar{\alpha}$, for $i \in I \setminus H$.

T7i $\delta_{ij'} = \bar{\delta}_{j'}$, $\forall i \in I \setminus H$, $\forall j' \in J$, $j' \neq j$:

Let $L_{11'}$ be L_{11} where also customer j' is connected to i_2 instead of i_1. Then $\mathcal{L}(L_{11}) = \mathcal{L}(L_{11'})$ gives $\delta_{i_2 j'} = \delta_{i_1 j'}$ for $i_2 \in I \setminus H, i_1 \in H \cap I$. Using the result from step (T7g), we get $\delta_{ij'} = \bar{\delta}_{j'}$ for $i \in I \setminus H$.

T7j Define $\rho := \bar{\alpha} - \hat{\alpha}$

Note that we can now write all coefficients in terms of $\bar{\alpha}$, ρ and $\bar{\delta}_{j'}$ for $j' \in J$. The equation defining $\mathcal{G}$ looks as follows:

$$\overbrace{0z(I)}^{\text{(T7a)}} - \overbrace{(\bar{\alpha} - \rho)y(H)}^{\text{(T7e),(T7j)}} - \overbrace{\bar{\alpha}y(S \setminus H)}^{\text{(T7d)}} + \overbrace{(\bar{\alpha} - \rho)x(E(H))}^{\text{(T7e),(T7j)}} + \overbrace{\bar{\alpha}x(E(S \setminus H))}^{\text{(T7c),(T7d)}} + \overbrace{\bar{\alpha}x(\delta(H))}^{\text{(T7b),(T7d)}} +$$
$$\overbrace{\bar{\delta}_j \sum_{i \in H \cap I} a_{ij}}^{\text{(T7f)}} + \overbrace{(\bar{\delta}_j + \rho) \sum_{i \in I \setminus H} a_{ij}}^{\text{(T7h),(T7j)}} + \overbrace{\sum_{j' \in J \setminus \{j\}} \bar{\delta}_{j'} \sum_{i \in I} a_{ij'}}^{\text{(T7g),(T7i)}} = \lambda_0.$$

By evaluating any feasible solution (e.g., L_3) we get $\lambda_0 = \mathcal{L}(L_3) = \rho - \bar{\alpha} + \sum_{j' \in J} \bar{\delta}_{j'}$. Rewriting the equation defining $\mathcal{G}$, we get

$$-\rho\Big(x(E(H)) - y(H) - \sum_{i \in I \setminus H} a_{ij}\Big) - \bar{\alpha} \overbrace{\Big(- x(E(S)) + y(S)\Big)}^{(5)} + \sum_{j' \in J} \bar{\delta}_{j'} \overbrace{\sum_{i \in I} a_{ij'}}^{(2)}$$
$$= \rho - \bar{\alpha} + \sum_{j' \in J} \bar{\delta}_{j'}.$$

Thus the equation defining $\mathcal{G}$ is a linear combination of the equation defining $\mathcal{F}$ and the equality set of $\mathcal{P}$. Therefore, inequalities (11) are facet-inducing when $2 \leq |H \cap I| \leq |I| - 1$.

To see that $2 \leq |H \cap I| \leq |I| - 1$ is also a necessary condition, consider the following cases:

1. $|H \cap I| = 0$: Inequalities (11) reduce to $x(E(H)) \leq y(H)$ and are dominated by inequalities (8).

2. $|H \cap I| = 1$: Inequalities (11) reduce to $x(E(H)) \leq y(H) - a_{ij}$, with i being the unique facility in H. Thus they are also dominated by inequalities (8) since $a_{ij} \leq y_i$.

3. $|H \cap I| = |I|$: Inequalities (11) are inequalities (9) (which are also facet-inducing).

$\square$

5.2. Partition Inequalities

The following two families of inequalities are based on a partition of the set of facilities I into two sets $\hat{I}$ and $I \setminus \hat{I}$. The second family also involves a partition of the set K of intermediate nodes. Moreover, both families also use an injective mapping h and assume that $|K| \geq 1$.

The first family will be referred to as *2+u partition inequalities*, since, aside from the partition of the facility set, a node $u \in K$ also plays an important role in the definition of the inequalities.

Theorem 8. *Let us consider $u \in K$, $\hat{I} \subset I$ and an injective mapping h. The inequality*

$$\sum_{i \in \hat{I}} z_i + \sum_{i \in I \setminus \hat{I}} (a_{ih(i)} + y_i) + x(\hat{I} : K) \geq 1 + y_u \tag{12}$$

is valid for $\mathcal{P}$.

Proof. If two (or more) facilities are opened, the inequality is clearly valid, since the right hand side is at most two and every facility is represented by either a

y or a z variable on the lhs. Thus, we only need to concentrate on feasible solutions with one open facility. Three cases are possible:

(a) $y_u = 0$: In a feasible solution, at least one facility must be opened and thus the lhs is at least one.

(b) $y_u = 1$: For this case, we make a further case distinction, depending on whether the open facility i is in $\hat{I}$ or $I \setminus \hat{I}$:

- $i \in \hat{I}$: Since both nodes i and u are in the solution, there must be a connection between i and u. In this connection, there must either be an edge from some node in $\hat{I}$ to a node in K and thus $x(\hat{I} : K) \geq 1$, or an edge from i to a node $i' \in I \setminus \hat{I}$ and thus $y_{i'} = 1$. Thus the lhs of the inequality is at least two.
- $i \in I \setminus \hat{I}$: As there is only one open facility, all customers must be connected to i, thus $a_{ih(i)}$ must be one, and the lhs is at least two.

$\square$

Theorem 9. *Inequalities* (12) *are facet-inducing for $\mathcal{P}$ if and only if $|I \setminus \hat{I}| \geq 2$ and $\hat{I} \neq \emptyset$.*

Proof. Let

$$\mathcal{F} = \{(x, y, z, a) \in \mathcal{G} : \sum_{i \in \hat{I}} z_i + \sum_{i \in I \setminus \hat{I}} (a_{ih(i)} + y_i) + x(\hat{I} : K) - y_u = 1\}$$

be the proper face induced by (12) for some $u \in K$ and $\hat{I} \subset I : |I \setminus \hat{I}| \geq 2, \hat{I} \neq \emptyset$. The feasible solutions $\sigma \in \mathcal{F}$ used in the proof are described by tuples $M_q = (S_q \cap \hat{I}, S_q \cap (I \setminus \hat{I}), S_q \cap K, I_q, E_q, A_q)$ where

- $S_q \subseteq S$: core nodes involved in the solution (y-variables with value one);
- $I_q \subseteq I$: open facilities in the solution (z-variables with value one);
- $E_q \subset E_S$: core edges in the solution (x-variables with value one);
- $A_q \subset A_J$: assignment arcs in the solution (a-variables with value one).

Let $K' := K \setminus \{u\}$, $J' := J \setminus \{j_1\}, J'' := J \setminus \{j_2\}$. The following solutions, where $i_1, i_4 \in \hat{I}$; $i_2, i_3 \in I \setminus \hat{I}$; $k_1 \in K'$, $h(i_2) := j_2$, $h(i_3) := j_3$, will be used. To help a reader, the solutions are depicted in Figure 3. Facilities from $\hat{I}$ and $I \setminus \hat{I}$ are shown as circles and triangles, respectively. Open facilities are indicated in bold. Intermediate nodes and sometimes also closed facilities (i.e., nodes from S) are drawn as diamonds. Customers are shown as squares. In addition, some more solutions, which can be constructed by small modifications of the solutions listed below, will also be used.

- $M_1 = (\{i_1\}, \{i_2\}, \{u\}, \{i_1, i_2\}, \{\{i_1, i_2\}, \{i_2, u\}\}, (i_1 : J))$
- $M_2 = (\{i_1\}, \{i_2\}, \{u\}, \{i_1, i_2\}, \{\{i_1, i_2\}, \{i_2, u\}\}, (i_1 : J') \cup \{(i_2, j_1)\})$
- $M_3 = (\emptyset, \{i_2, i_3\}, \{u\}, \{i_2, i_3\}, \{\{i_2, u\}, \{i_2, i_3\}\}, (i_2 : J'') \cup \{(i_3, j_2)\})$
- $M_4 = (\emptyset, \{i_2, i_3\}, \{u\}, \{i_2, i_3\}, \{\{i_3, u\}, \{i_2, i_3\}\}, (i_2 : J'') \cup \{(i_3, j_2)\})$
- $M_5 = (\emptyset, \{i_2, i_3\}, \{u\}, \{i_2, i_3\}, \{\{i_2, u\}, \{i_3, u\}\}, (i_2 : J'') \cup \{(i_3, j_2)\})$
- $M_6 = (\{i_1\}, \emptyset, \emptyset, \{i_1\}, \emptyset, (i_1 : J))$
- $M_7 = (\{i_1\}, \emptyset, \{u\}, \{i_1\}, \{\{i_1, u\}\}, (i_1 : J))$
- $M_8 = (\{i_1\}, \{i_2, i_3\}, \{u\}, \{i_2, i_3\}, \{\{i_2, i_1\}, \{i_2, i_3\}, \{i_2, u\}\}, (i_2 : J'') \cup \{(i_3, j_2)\})$
- $M_9 = (\{i_1\}, \{i_2, i_3\}, \{u\}, \{i_2, i_3\}, \{\{i_2, i_1\}, \{i_1, i_3\}, \{i_2, u\}\}, (i_2 : J'') \cup \{(i_3, j_2)\})$
- $M_{10} = (\{i_1, i_4\}, \emptyset, \emptyset, \{i_1\}, \{\{i_1, i_4\}\}, (i_1 : J))$
- $M_{11} = (\emptyset, \{i_2\}, \{u\}, \{i_2\}, \{\{i_2, u\}\}, (i_2 : J))$
- $M_{12} = (\{i_1\}, \{i_2\}, \{u\}, \{i_1, i_2\}, \{\{i_2, i_1\}, \{i_2, u\}\}, (i_2 : J'') \cup \{(i_1, j_2)\})$
- $M_{13} = (\{i_1\}, \emptyset, \{u, k_1\}, \{i_1\}, \{\{i_1, u\}, \{u, k_1\}\}, (i_1 : J))$
- $M_{14} = (\{i_1\}, \emptyset, \{u, k_1\}, \{i_1\}, \{\{i_1, k_1\}, \{u, k_1\}\}, (i_1 : J))$
- $M_{15} = (\emptyset, \{i_2\}, \{u, k_1\}, \{i_2\}, \{\{i_2, k_1\}, \{u, k_1\}\}, (i_2 : J))$

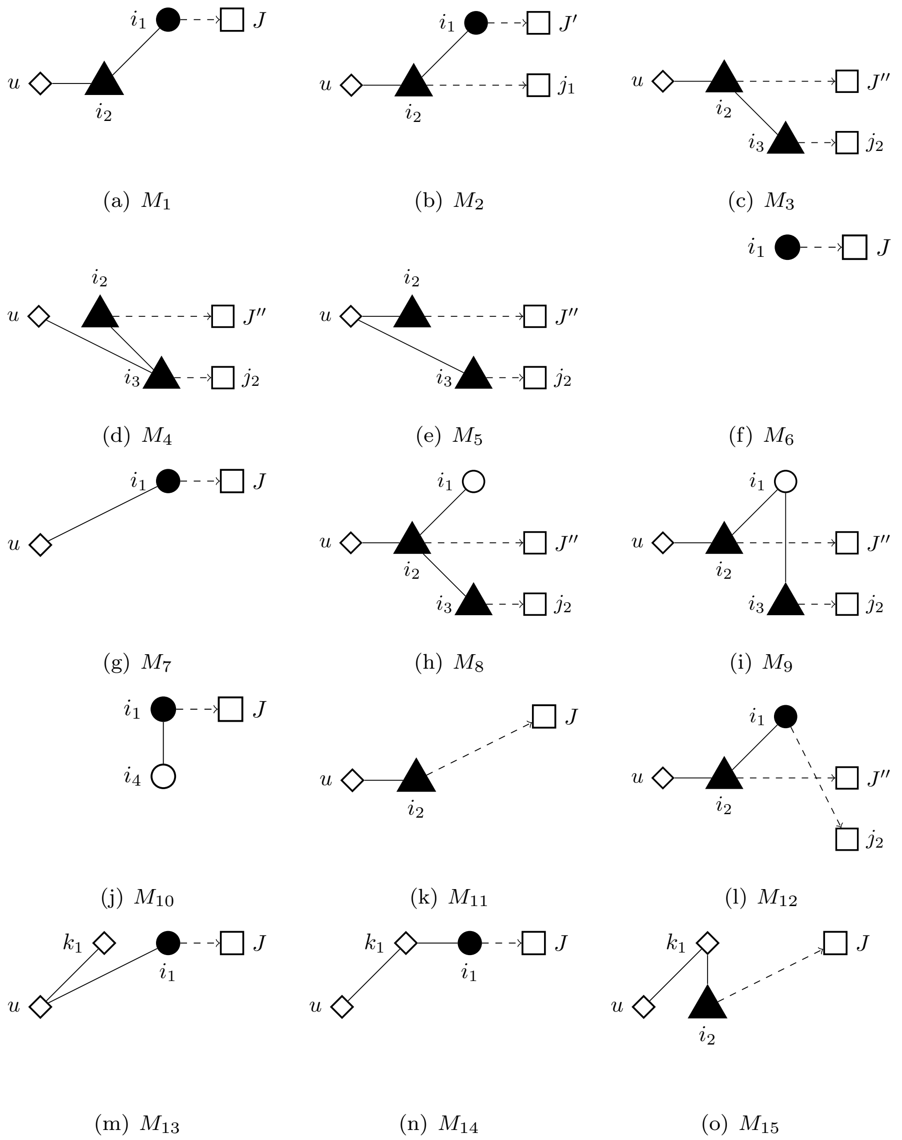


Figure 3: Feasible solutions for the proof of Theorem 9

Assume $\mathcal{F} \subseteq \mathcal{G}$ and recall that α relates to x, β relates to y, γ relates to z, and δ relates to a. Then:

T9a $\gamma_i = 0, \forall i \in I \setminus \hat{I}$:

Let $M_{1'}$ be M_1, where facility i_2 is not opened. The result follows from

$\mathcal{L}(M_1) = \mathcal{L}(M_{1'})$.

T9b $\delta_{ij} = \bar{\delta}_j$, $\forall j \in J$, $\forall i \in I \setminus \hat{I} : h(i) \neq j$ and $\forall i \in \hat{I}$:

$\mathcal{L}(M_1) = \mathcal{L}(M_2)$ gives $\delta_{i_1 j_1} = \delta_{i_2 j_1}$, for $i_1 \in \hat{I}$, $i_2 \in I \setminus \hat{I}$ and any customer $j_1 \neq h(i_2)$. Since this step can be repeated for any facility in I, it follows that all coefficients δ_{ij} associated with a customer j, except for the facility $i \in I \setminus \hat{I}$ with $h(i) = j$, have the same value. Denote this value by $\bar{\delta}_j$.

T9c $\alpha_{ii'} = \bar{\alpha}$, $\forall i, i' \in I \setminus \hat{I}$ and $\alpha_{iu} = \bar{\alpha}$, $\forall i \in I \setminus \hat{I}$:

Obtained from $\mathcal{L}(M_3) = \mathcal{L}(M_4) = \mathcal{L}(M_5)$, which gives $\alpha_{i_2 i_3} = \alpha_{i_2 u} = \alpha_{i_3 u}$. Denote the value of the coefficients by $\bar{\alpha}$.

T9d $\alpha_{iu} = -\beta_u$, $\forall i \in \hat{I}$:

Obtained from $\mathcal{L}(M_6) = \mathcal{L}(M_7)$.

T9e $\alpha_{ii'} = \bar{\alpha}$, $\forall i \in I \setminus \hat{I}$, $\forall i' \in \hat{I}$ and $\beta_{i'} = -\bar{\alpha}$, $\forall i' \in \hat{I}$:

$\mathcal{L}(M_8) = \mathcal{L}(M_9)$ gives $\alpha_{i_1 i_2} = \alpha_{i_2 i_3}$, for $i_2, i_3 \in I \setminus \hat{I}$, $i_1 \in \hat{I}$, thus $\alpha_{ii'} = \bar{\alpha}$, using the result from step (T9c). The result $\beta_{i'} = -\bar{\alpha}$ follows from $\mathcal{L}(M_8) = \mathcal{L}(M_3)$.

T9f $\alpha_{ii'} = \bar{\alpha}$, $\forall i, i' \in \hat{I}$:

Obtained from $\mathcal{L}(M_6) = \mathcal{L}(M_{10})$, which gives $\alpha_{i_1 i_2} = -\beta_{i_2}$, $\forall i_1, i_2 \in \hat{I}$, and the result follows from the result in step (T9e).

T9g $\delta_{ih(i)} = \bar{\alpha} + \bar{\beta} + \bar{\delta}_{h(i)}, \forall i \in I \setminus \hat{I}$ and $\beta_i = \bar{\beta}$, $\forall i \in I \setminus \hat{I}$:

From $\mathcal{L}(M_{11}) = \mathcal{L}(M_3)$, we get $\delta_{i_2 h(i_2)} = \alpha_{i_2 i_3} + \beta_{i_3} + \delta_{i_3 h(i_2)}$ for $i_2, i_3 \in I \setminus \hat{I}$. Using results from steps (T9b) and (T9c), we get $\delta_{i_2 h(i_2)} = \bar{\alpha} + \beta_{i_3} + \bar{\delta}_{h(i_2)}$. Since this must hold for any $i_3 \in I \setminus \hat{I}$, it follows that β_i has the same value for all $i \in I \setminus \hat{I}$. Denote this value by $\bar{\beta}$.

T9h Define $\rho := \bar{\alpha} + \bar{\beta}$.

T9i $\gamma_i = \rho$, $\forall i \in \hat{I}$:

$\mathcal{L}(M_{12}) = \mathcal{L}(M_8)$ gives $\gamma_{i_1} = \alpha_{i_2 i_3} + \beta_{i_3}$, for $i_1 \in \bar{I}, i_2, i_3 \in I \setminus \hat{I}$, using the result from step (T9b). The result $\gamma_i = \rho$, $\forall i \in \hat{I}$, is then obtained using results from steps (T9c) and (T9g) and the definition from step (T9h).

T9j $\alpha_{iu} = \bar{\alpha} + \rho$, $\forall i \in \hat{I}$ and $\beta_u = -(\bar{\alpha} + \rho)$:

$\mathcal{L}(M_7) = \mathcal{L}(M_1)$ gives $\alpha_{i_1 u} = \alpha_{i_2 u} + \beta_{i_2} + \alpha_{i_2 i_1}$ for $i_1 \in \hat{I}$, $i_2 \in I \setminus \hat{I}$. The result $\alpha_{iu} = \bar{\alpha} + \rho$, $\forall i \in \hat{I}$, follows from the results of steps (T9c), (T9e) and (T9g) and the definition from step (T9h). The result $\beta_u = -(\bar{\alpha} + \rho)$ is then obtained using the result from step (T9d).

T9k $\alpha_{ik} = \bar{\alpha} + \rho$, $\forall i \in \hat{I}$, $\forall k \in K'$:

Obtained from $\mathcal{L}(M_{13}) = \mathcal{L}(M_{14})$ using the result from step (T9j).

T9l $\alpha_{ik} = \bar{\alpha}$, $\forall k \in K', \forall i \in I \setminus \hat{I}$, $\alpha_{uk} = \bar{\alpha}$, $\forall k \in K'$ and $\beta_k = -\bar{\alpha}$, $\forall k \in K'$:

Let $M_{11'}$ be M_{11} with the additional edge $\{u, k_1\}$ and $M_{11''}$ be M_{11} with the additional edge $\{i_2, k_1\}$. The result $\alpha_{ik} = \alpha_{uk} = \bar{\alpha}$ is obtained by $\mathcal{L}(M_{11'}) = \mathcal{L}(M_{11''}) = \mathcal{L}(M_{15})$, using the result from step (T9c). The result $\beta_k = -\bar{\alpha}$ is then obtained by $\mathcal{L}(M_{11'}) = \mathcal{L}(M_{11})$.

T9m $\alpha_{kk'} = \bar{\alpha}$, $\forall k, k' \in K'$:

Let $M_{15'}$ be M_{15} with the additional edge $\{k_1, k_2\}$. We get $\alpha_{k_1 k_2} = -\beta_{k_2}$ for $k_1, k_2 \in K'$ and the result follows from the result in step (T9l).

Note that we can now write all coefficients in terms of $\bar{\alpha}$, ρ and $\bar{\delta}_j$ for $j \in J$. The equation defining $\mathcal{G}$ looks as follows:

$$\overbrace{\rho \sum_{i \in \hat{I}} z_i}^{\text{(T9i)}} + \overbrace{0 \sum_{i \in I \setminus \hat{I}} z_i}^{\text{(T9a)}} - \overbrace{\bar{\alpha} \sum_{i \in \hat{I}} y_i}^{\text{(T9e)}} + \overbrace{(\rho - \bar{\alpha}) \sum_{i \in I \setminus \hat{I}} y_i}^{\text{(T9g),(T9h)}} + \overbrace{\bar{\alpha} x(E(I))}^{\text{(T9c),(T9e),(T9f)}} + \overbrace{(\rho + \bar{\alpha}) x(\hat{I} : K)}^{\text{(T9j),(T9k)}} +$$

$$\overbrace{\bar{\alpha} x(I \setminus \hat{I} : K)}^{\text{(T9c),(T9l)}} + \overbrace{(-\rho - \bar{\alpha}) y_u}^{\text{(T9j)}} - \overbrace{\bar{\alpha} \sum_{k \in K'} y_k}^{\text{(T9l)}} + \overbrace{\bar{\alpha} x(E(K))}^{\text{(T9l),(T9m)}} +$$

$$\sum_{i \in I \setminus \hat{I}} \overbrace{(\rho + \bar{\delta}_{h(i)}) a_{ih(i)}}^{\text{(T9g),(T9h)}} + \sum_{j \in J} \overbrace{\bar{\delta}_j \sum_{i \in I : h(i) \neq j} a_{ij}}^{\text{(T9b)}} = \lambda_0.$$

By evaluating any feasible solution (e.g., M_6) we get

$$\lambda_0 = \rho - \bar{\alpha} + \sum_{j \in J} \bar{\delta}_j.$$

Rewriting the equation defining $\mathcal{G}$, we get:

$$\rho\Big(\sum_{i \in \hat{I}} z_i + \sum_{i \in I \setminus \hat{I}} (a_{ih(i)} + y_i) + x(\hat{I} : K) - y_u\Big)$$

$$-\bar{\alpha}\overbrace{\Big(-y(S) + x(E(S))\Big)}^{(5)} + \sum_{j \in J} \bar{\delta}_j \overbrace{\sum_{i \in I} a_{ij}}^{(2)} = \rho - \bar{\alpha} + \sum_{j \in J} \bar{\delta}_j.$$

Thus the equation defining $\mathcal{G}$ is a linear combination of the equation defining $\mathcal{F}$ and the equality set of $\mathcal{P}$. Therefore, inequalities (12) are facet-inducing, when $|I \setminus \hat{I}| \geq 2, \hat{I} \neq \emptyset$.

To see that $|I \setminus \hat{I}| \geq 2$ and $\hat{I} \neq \emptyset$ is also a necessary condition, consider the following cases:

1. $\hat{I} = \emptyset$: Inequalities (12) reduce to

$$\sum_{i \in I} (a_{ih(i)} + y_i) \geq 1 + y_u$$

and are dominated by inequalities (10).

2. $|I \setminus \hat{I}| = 0$: Inequalities (12) reduce to

$$\sum_{i \in I} z_i + x(I : K) \geq 1 + y_u,$$

which are dominated by

$$\sum_{i \in I} a_{ij} + x(I : K) \geq 1 + y_u$$

for some $j \in J$. The latter inequalities are a combination of an equation (2) and an inequality

$$x(I : K) \geq y_u.$$

Rewrite this remaining inequality as

$$x(I : K) \geq -y(S \setminus \{u\}) + x(E(S)) + 1$$

using equation (5). After further rewriting, we get

$$y(K') + y(I) - 1 \geq x(E(K)) + x(E(I)),$$

where $K' = K \setminus \{u\}$. This inequality is easily seen to be a combination of (6) and (9).

3. $|I \setminus \hat{I}| = 1$: Let $I \setminus \hat{I} = \{i\}$. Inequalities (12) reduce to

$$\sum_{i' \in I'} z_{i'} + y_i + a_{ih(i)} + x(I' : K) \geq 1 + y_u$$

for $I' = I \setminus \{i\}$. The inequalities are dominated by inequalities

$$\sum_{i' \in I} a_{i'h(i)} + y_i + x(I' : K) \geq 1 + y_u,$$

which are a combination of an equation (2) and an inequality

$$y_i + x(I' : K) \geq y_u.$$

Using equation (5), the latter inequality can be rewritten as

$$y(K') + y_i + y(I) - 1 \geq x(E(K \cup \{i\})) + x(E(I))$$

where $K' = K \setminus \{u\}$. Again, this inequality is easily seen to be a combination of (6) and (9).

□

For the other new family of facet-inducing inequalities, consider also the partition of the set of intermediate nodes K into three disjoint subsets $K = K_1 \cup K_2 \cup K_3$, with $|K_1| \geq 1$. Before we present the inequalities themselves, we give two lemmas, which we will need for the validity proof of the inequalities.

Lemma 3. *Let us consider $K' \subseteq K$. The inequality*

$$x(K \cup I : K') \geq y(K'). \tag{13}$$

is valid for $\mathcal{P}$.

Proof. The inequalities are special case of constraints (9) for $H = S \setminus K'$. This can be verified as follows: Inequality (9) for $H = S \setminus K'$ is $x(E(S \setminus K')) \leq y(S \setminus K') - 1$. The result is obtained by rewriting this inequality using equation (5). $\square$

Lemma 4. *Let us consider $\hat{I} \subseteq I$, $u \in I \setminus \hat{I}$ and an injective mapping h. The inequality*

$$\sum_{i \in \hat{I}} z_i + \sum_{i \in I \setminus \hat{I}} (a_{ih(i)} + y_i) \geq 1 + y_u \tag{14}$$

is valid for $\mathcal{P}$.

Proof. For the case when $z(\hat{I}) \geq 1$, the inequality is obviously valid, so assume that $z(\hat{I}) = 0$. For some $i \in I \setminus \hat{I}$ we have $y_i = 1$, thus the inequality is valid if $y_u = 0$. So, the only non-trivial case occurs when $y_u = 1$, and $y_i = 0$ for all other nodes $i \in I \setminus \hat{I}, i \neq u$. Since $u \in I \setminus \hat{I}$ is the only open facility, every customer has to be assigned to u. Thus $a_{uh(u)} = 1$ and therefore the left-hand side of the inequality is also equal to two, which concludes the proof. Note that the proof also works for $\hat{I} = \emptyset$. $\square$

Inequalities (14) are not facet inducing since they are dominated by inequalities $z(\hat{I}) + y((I \setminus \hat{I}) \setminus \{u\}) + a_{uh(u)} \geq 1, u \in I \setminus \hat{I}$. These inequalities are in turn dominated by inequalities $z(I \setminus \{u\}) + a_{uh(u)} \geq 1, u \in I \setminus \hat{I}$. Finally, in the latter inequalities, we can replace the z-variables with the a-variables going to customer $h(u)$ and end up with $\sum_{i \in I} a_{ih(u)} \geq 1$, which is implied by the equation (2) from the formulation of $\mathcal{P}$, associated with $h(u)$.

We are now ready to introduce the second family of facet-inducing inequalities, which we will refer to as *2+3 partition inequalities.* The name indicates that the sets I and K are partitioned into two and three subsets, respectively. The variables associated with the core network, which occur in inequalities (15) are illustrated in Figure 4.

Theorem 10. *Let us consider $\hat{I} \subset I$, the partition $K = K_1 \cup K_2 \cup K_3$, $K_1 \neq \emptyset$*

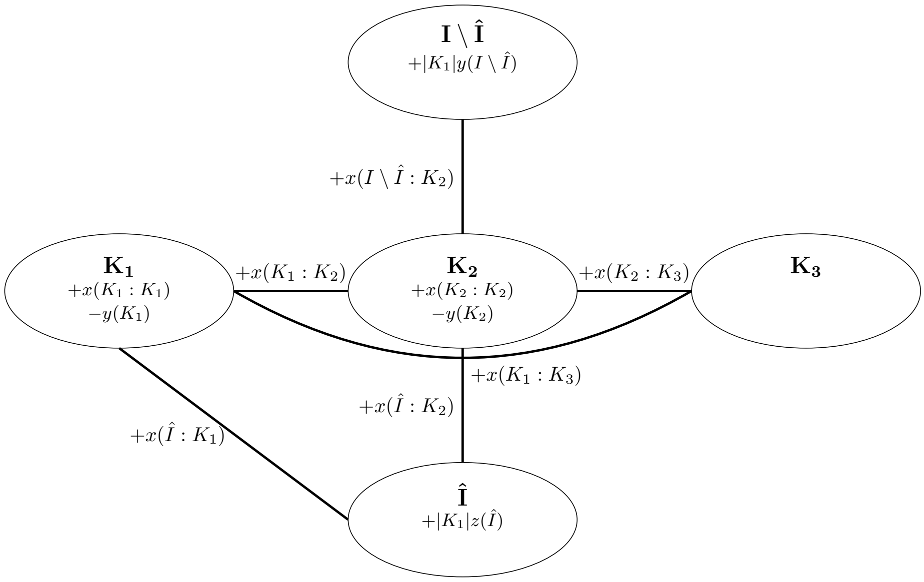


Figure 4: Illustration of the support graph of variables of the core network involved in inequalities (15), where $x(K : K_1 \cup K_2) = \sum_{i=1}^{2}(x(K_i : K_i) + x(K_i : K_3))$.

and an injective mapping h. The inequality

$$|K_1| \sum_{i \in \hat{I}} z_i + |K_1| \sum_{i \in I \setminus \hat{I}} (a_{ih(i)} + y_i) + x(\hat{I} : K_1 \cup K_2) + x(I \setminus \hat{I} : K_2) + \\ + x(K : K_1 \cup K_2) \geq |K_1| + \sum_{k \in K_1} y_k + \sum_{k \in K_2} y_k \quad (15)$$

is valid for $\mathcal{P}$.

Proof. The proof is based on a connectivity argument: When some of the nodes occurring on the rhs (i.e, nodes from K_1 or K_2) are in a solution, they must be connected to the rest of this solution. Thus also edges (which occur on the lhs of the inequality) must be selected.

We make a case distinction depending on the value of $y(I \setminus \hat{I})$.

- $y(I \setminus \hat{I}) = 0$ (i.e., no node from $I \setminus \hat{I}$ is in the solution):

  At least one facility in $\hat{I}$, say i_1, must be opened, thus the term $|K_1|z(\hat{I})$ on the lhs is at least $|K_1|$, which takes care of the $|K_1|$ on the rhs. It only remains to show that $x(\hat{I}:K_1\cup K_2)+x(K:K_1\cup K_2)\geq y(K_1)+y(K_2)$ (note that opening more than one facility only increases the lhs and thus it is enough to focus on the case, where $z(\hat{I})=1$). We can reformulate the lhs of the latter inequality as $x(\hat{I}\cup K:K_1\cup K_2)=x(I\cup K:K_1\cup K_2)$ (since $y(I\setminus\hat{I})=0$), and so by Lemma 3, for $K'=K_1\cup K_2$, the desired result follows.

- $y(I\setminus\hat{I})\geq 1$: By Lemma 4, we have that $z(\hat{I})+\sum_{i\in I\setminus\hat{I}}(a_{ih(i)}+y_i)\geq 2$, and therefore it only remains to show that $x(\hat{I}:K_1\cup K_2)+x(I\setminus\hat{I}:K_2)+x(K:K_1\cup K_2)\geq y(K_2)$. The latter inequality obviously holds by Lemma 3, for $K'=K_2$.

□

Observe that for $K_1=\emptyset$, inequalities (15) only make sense for $K_2\neq\emptyset$. In this case, (15) reduce to the facet-inducing inequalities (9) for $H=S\setminus K_2$ (shown in the form given in Lemma 3). The following theorem provides necessary and sufficient conditions for these inequalities to be facet-inducing.

Theorem 11. *Inequalities* (15) *are facet-inducing for* $\mathcal{P}$ *if and only if* $|I\setminus\hat{I}|\geq 2$ *and* $\hat{I}\neq\emptyset$.

Proof. Let

$$\mathcal{F}=\{(x,y,z,a)\in\mathcal{G}:\sum_{i\in\hat{I}}z_i+\sum_{i\in I\setminus\hat{I}}(a_{ih(i)}+y_i)+\frac{1}{|K_1|}x(\hat{I}:K_1\cup K_2)+\frac{1}{|K_1|}x(I\setminus\hat{I}:K_2)+$$
$$+\frac{1}{|K_1|}x(K_1\cup K_2\cup K_3:K_1\cup K_2)-\frac{1}{|K_1|}\sum_{k\in K_1}y_k-\frac{1}{|K_1|}\sum_{k\in K_2}y_k=1\}$$

be the proper face induced by (15) for some partition $K=K_1\cup K_2\cup K_3:|K_1|\geq 1$ and $\hat{I}\subset I:|I\setminus\hat{I}|\geq 2,\hat{I}\neq\emptyset$. Note that we divided the inequality by $|K_1|$.

The feasible solutions $\sigma\in\mathcal{F}$ used in the proof are described by tuples $N_q=(S_q\cap\hat{I},S_q\cap(I\setminus\hat{I}),S_q\cap K_1,S_q\cap K_2,S_q\cap K_3,I_q,E_q,A_q)$ where

- $S_q\subseteq S$: core nodes involved in the solution (y-variables with value one);

- $I_q \subseteq I$: open facilities in the solution (z-variables with value one);
- $E_q \subset E_S$: core edges in the solution (x-variables with value one);
- $A_q \subset A_J$: assignment arcs in the solution (a-variables with value one).

Let $K_1' = K_1 \backslash \{k_1\}$, $J' := J \backslash \{j_1\}$, $J'' := J \backslash \{j_2\}$. The following solutions, where $i_1, i_4 \in \hat{I}$; $i_2, i_3 \in I \setminus \hat{I}$; $k_1 \in K_1$, $k_2 \in K_2$, $k_3 \in K_3$, $h(i_2) := j_2$, $h(i_3) := j_3$, will be used. To help a reader, the solutions are depicted in Figure 5. Nodes from K_1, K_2 and K_3 are shown as diamonds, pentagons and stars, respectively. Circles represent facilities and squares represent customers. Open facilities are indicated in bold. In addition, other solutions, which can be constructed by small modifications of the solutions listed below, will be used.

- $N_1 = (\{i_1\}, \{i_2\}, K_1, \emptyset, \emptyset, \{i_1, i_2\}, \{\{i_1, i_2\}\} \cup (i_2 : K_1), (i_1 : J))$
- $N_2 = (\{i_1\}, \{i_2\}, K_1, \emptyset, \emptyset, \{i_1, i_2\}, \{\{i_1, i_2\}\} \cup (i_2 : K_1), (i_1 : J') \cup \{(i_2, j_1)\})$
- $N_3 = (\emptyset, \{i_2, i_3\}, K_1, \emptyset, \emptyset, \{i_2, i_3\}, \{\{i_2, i_3\}\} \cup (i_2 : K_1), (i_2 : J'') \cup \{(i_3, j_2)\})$
  Note that $K_1 = \{k_1\} \cup K_1'$ as depicted in Figure (5(c)).
- $N_4 = (\emptyset, \{i_2, i_3\}, K_1, \emptyset, \emptyset, \{i_2, i_3\}, \{\{i_3, k_1\}, \{i_2, i_3\}\} \cup (i_2 : K_1'), (i_2 : J'') \cup \{(i_3, j_2)\})$
- $N_5 = (\emptyset, \{i_2, i_3\}, K_1, \emptyset, \emptyset, \{i_2, i_3\}, \{\{i_2, k_1\}, \{i_3, k_1\}\} \cup (i_2 : K_1'), (i_2 : J'') \cup \{(i_3, j_2)\})$
- $N_6 = (\{i_1\}, \emptyset, \emptyset, \emptyset, \emptyset, \{i_1\}, \emptyset, (i_1 : J))$
- $N_7 = (\{i_1\}, \emptyset, \{k_1\}, \emptyset, \emptyset, \{i_1\}, \{(i_1, k_1)\}, (i_1 : J))$
- $N_8 = (\{i_1\}, \{i_2, i_3\}, K_1, \emptyset, \emptyset, \{i_2, i_3\}, \{\{i_2, i_1\}, \{i_2, i_3\}\} \cup (i_2 : K_1), (i_2 : J'') \cup \{(i_3, j_2)\})$
- $N_9 = (\{i_1\}, \{i_2, i_3\}, K_1, \emptyset, \emptyset, \{i_2, i_3\}, \{\{i_2, i_1\}, \{i_1, i_3\}\} \cup (i_2 : K_1), (i_2 : J'') \cup \{(i_3, j_2)\})$
- $N_{10} = (\{i_1, i_4\}, \emptyset, \emptyset, \emptyset, \{i_1\}, \{\{i_1, i_4\}\}, (i_1 : J))$

- $N_{11} = (\emptyset, \{i_2\}, K_1, \emptyset, \emptyset, \{i_2\}, (i_2 : K_1), (i_2 : J))$
- $N_{12} = (\{i_1\}, \{i_2\}, K_1, \emptyset, \emptyset, \{i_1, i_2\}, \{\{i_2, i_1\}\} \cup (i_2 : K_1), (i_2 : J'') \cup \{(i_1, j_2)\})$
- $N_{13} = (\{i_1\}, \emptyset, \{k_1\}, \{k_2\}, \emptyset, \{i_1\}, \{\{i_1, k_2\}, \{k_1, k_2\}\}, (i_1 : J))$
- $N_{14} = (\{i_1\}, \emptyset, \{k_1\}, \{k_2\}, \emptyset, \{i_1\}, \{\{i_1, k_1\}, \{i_1, k_2\}\}, (i_1 : J))$
- $N_{15} = (\{i_1\}, \emptyset, \{k_1\}, \{k_2\}, \emptyset, \{i_1\}, \{\{i_1, k_1\}, \{k_1, k_2\}\}, (i_1 : J))$
- $N_{16} = (\{i_1\}, \emptyset, \emptyset, \emptyset, \{k_3\}, \{i_1\}, \{\{i_1, k_3\}\}, (i_1 : J))$
- $N_{17} = (\emptyset, \{i_2, i_3\}, K_1, \emptyset, \{k_3\}, \{i_2, i_3\}, \{\{i_2, i_3\}, \{i_2, k_3\}\} \cup (i_2 : K_1), (i_2 : J'') \cup \{(i_3, j_2)\})$
- $N_{18} = (\emptyset, \{i_2, i_3\}, K_1, \emptyset, \{k_3\}, \{i_2, i_3\}, \{\{i_2, k_3\}, \{i_3, k_3\}\} \cup (i_2 : K_1), (i_2 : J'') \cup \{(i_3, j_2)\})$

We now suppose $\mathcal{F} \subseteq \mathcal{G}$ and determine the following properties of coefficients of $\mathcal{G}$. Recall that α relates to x, β relates to y, γ relates to z, and δ relates to a. Note that some of the steps are almost similar to the previous proof.

T11a $\gamma_i = 0, \forall i \in I \setminus \hat{I}$:

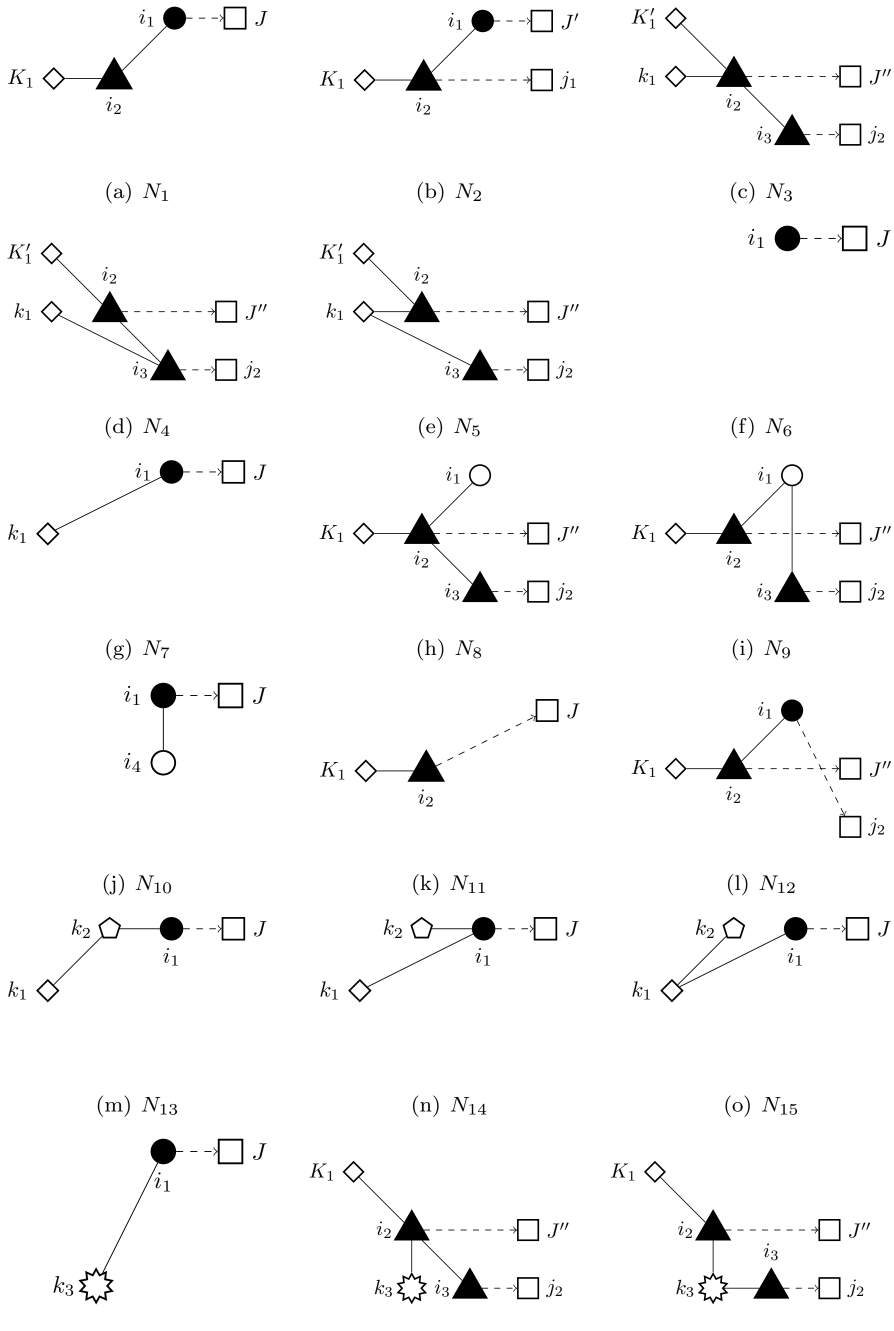
Let $N_{1'}$ be N_1, where facility i_2 is not opened. The result follows from $\mathcal{L}(N_1) = \mathcal{L}(N_{1'})$. Note that if in the following steps, an open facility $i \in I \setminus \hat{I}$ occurs, we will not mention γ_i explicitly again, since the coefficient is zero.

T11b $\delta_{ij} = \bar{\delta}_j$, $\forall j \in J$, $\forall i \in I \setminus \hat{I} : h(i) \neq j$ and $\forall i \in \hat{I}$:

$\mathcal{L}(N_1) = \mathcal{L}(N_2)$ gives $\delta_{i_1 j_1} = \delta_{i_2 j_1}$, for $i_1 \in \hat{I}$, $i_2 \in I \setminus \hat{I}$ and any customer $i_1 \neq h(i_2)$. Since this step can be repeated for any facility in I, it follows, that all coefficients δ_{ij} associated with a customer j, except for the facility $i \in I \setminus \hat{I}$ with $h(i) = j$, have the same value, denote it by $\bar{\delta}_j$.

T11c $\alpha_{ii'} = \bar{\alpha}$, $\forall i, i' \in I \setminus \hat{I}$ and $\alpha_{ik} = \bar{\alpha}$, $\forall i \in I \setminus \hat{I}, \forall k \in K_1$:

Obtained from $\mathcal{L}(N_3) = \mathcal{L}(N_4) = \mathcal{L}(N_5)$, which gives $\alpha_{i_2 i_3} = \alpha_{i_2 k_1} = \alpha_{i_3 k_1}$ for $i_2, i_3 \in I \setminus \hat{I}, k_1 \in K_1$. Denote the value of the coefficients by $\bar{\alpha}$.



(a) N_1 (b) N_2 (c) N_3 (d) N_4 (e) N_5 (f) N_6 (g) N_7 (h) N_8 (i) N_9 (j) N_{10} (k) N_{11} (l) N_{12} (m) N_{13} (n) N_{14} (o) N_{15} (p) N_{16} (q) N_{17} (r) N_{18}

Figure 5: Feasible solutions for the proofs of Theorem 11

T11d $\alpha_{ik} = \hat{\alpha}, \forall i \in \hat{I}, \forall k \in K_1$, $\alpha_{kk'} = \hat{\alpha}$, $\forall k, k' \in K_1$, and $\beta_k = -\hat{\alpha}$, $\forall k \in K_1$:

From $\mathcal{L}(N_6) = \mathcal{L}(N_7)$, we get $\alpha_{i_1 k_1} = -\beta_{k_1}$, for $i_1 \in \hat{I}, k_1 \in K_1$, which means all coefficients for a particular $k_1 \in K_1$ are the same. Thus, if $|K_1| = 1$, we are already done. For $|K_1| \geq 2$, we also have edges $\alpha_{kk'}$: Let $N_{7'}$ be N_7 with the additional edge $\{k_1, k_1'\}$ for $k_1' \in K_1'$. From $\mathcal{L}(N_{7'}) = \mathcal{L}(N_7)$, we get $\alpha_{k_1 k_1'} = -\beta_{k_1'}$, for $k_1, k_1' \in K_1$. It follows, that the coefficients α_{ik}, $\forall i \in \hat{I}, \forall k \in K_1$ and $\alpha_{kk'}$, $\forall k, k' \in K_1$ are all the same, denote their value by $\hat{\alpha}$. The result $\beta_k = -\hat{\alpha}$, $\forall k \in K_1$ follows immediately.

T11e $\alpha_{ii'} = \bar{\alpha}$, $\forall i \in I \setminus \hat{I}$, $\forall i' \in \hat{I}$ and $\beta_{i'} = -\bar{\alpha}$, $\forall i' \in \hat{I}$:

$\mathcal{L}(N_8) = \mathcal{L}(N_9)$ gives $\alpha_{i_1 i_2} = \alpha_{i_2 i_3}$, for $\forall i_2, i_3 \in I \setminus \hat{I}$, $i_1 \in \hat{I}$, thus $\alpha_{ii'} = \bar{\alpha}$, using the result from step (T11c). The result $\beta_{i'} = -\bar{\alpha}$ follows from $\mathcal{L}(N_8) = \mathcal{L}(N_3)$.

T11f $\alpha_{ii'} = \bar{\alpha}$, $\forall i, i' \in \hat{I}$:

Obtained from $\mathcal{L}(N_6) = \mathcal{L}(N_{10})$, which gives $\alpha_{i_1 i_2} = -\beta_{i_2}$, $\forall i_1, i_2 \in \hat{I}$ and the result follows from the result in step (T11e).

T11g $\delta_{ih(i)} = \bar{\alpha} + \bar{\beta} + \bar{\delta}_{h(i)}, \forall i \in I \setminus \hat{I}$ and $\beta_i = \bar{\beta}$, $\forall i \in I \setminus \hat{I}$:

From $\mathcal{L}(N_{11}) = \mathcal{L}(N_3)$, we get $\delta_{i_2 h(i_2)} = \alpha_{i_2 i_3} + \beta_{i_3} + \delta_{i_3 h(i_2)}$ for $i_2, i_3 \in I \setminus \hat{I}$. Using results from steps (T11b) and (T11c), we get $\delta_{i_2 h(i_2)} = \bar{\alpha} + \beta_{i_3} + \bar{\delta}_{h(i_2)}$. Since this must hold for any $i_3 \in I \setminus \hat{I}$, it follows that β_i has the same value for all $i \in I \setminus \hat{I}$, denote it by $\bar{\beta}$.

T11h Define $\rho := \bar{\alpha} + \bar{\beta}$.

T11i $\gamma_i = \rho$, $\forall i \in \hat{I}$:

$\mathcal{L}(N_{12}) = \mathcal{L}(N_8)$ gives $\gamma_{i_1} = \alpha_{i_2 i_3} + \beta_{i_3}$, for $i_1 \in \bar{I}, i_2, i_3 \in I \setminus \hat{I}$, using the result from step (T11b). The result $\gamma_i = \rho$, $\forall i \in \hat{I}$ is then obtained using results from steps (T11c) and (T11g) and the definition from step (T11h)

T11j $\hat{\alpha} = \bar{\alpha} + \frac{\rho}{|K_1|}$:

$\mathcal{L}(N_7) = \mathcal{L}(N_1)$ gives $\alpha_{i_1k_1} + \beta_{k_1} = \sum_{k\in K_1} \alpha_{i_2k} + \sum_{k\in K_1} \beta_u + \beta_{i_2} + \alpha_{i_2i_1}$ for $i_1 \in \hat{I}$, $i_2 \in I\backslash\hat{I}$. Using the results from steps (T11c), (T11d), (T11e), (T11g), (T11h), we get $0 = |K_1|\bar{\alpha} - |K_1|\hat{\alpha} + \rho$ and the result follows from rewriting this equation.

T11k $\alpha_{kk'} = \bar{\alpha} + \frac{\rho}{|K_1|}$, $\forall k \in K_2, \forall k' \in K_1$,$\alpha_{ik} = \bar{\alpha} + \frac{\rho}{|K_1|}$, $\forall k \in K_2, \forall i \in \hat{I}$, $\beta_k = -(\bar{\alpha} + \frac{\rho}{|K_1|})$, $\forall k \in K_2$:

The first two results are obtained from $\mathcal{L}(N_{13}) = \mathcal{L}(N_{14}) = \mathcal{L}(N_{15})$, which gives $\alpha_{k_1k_2} = \alpha_{i_1k_1} = \alpha_{i_1k_2}$ for $i_1 \in \hat{I}, k_1 \in K_1, k_2 \in K_2$ and using the results from steps (T11d), (T11j). To get the result $\beta_k = -\left(\bar{\alpha} + \frac{\rho}{|K_1|}\right)$, let $N_{14'}$ be N_{14} without the edge $\{i_1, k_2\}$ and node k_2. $\mathcal{L}(N_{14}) = \mathcal{L}(N_{14'})$ which gives $\alpha_{i_1k_2} = -\beta_{k_2}$ for $i_1 \in \hat{I}, k_2 \in K_2$ and the result follows.

T11l $\alpha_{ik} = \bar{\alpha} + \frac{\rho}{|K_1|}$, $\forall k \in K_2, \forall i \in I \setminus \hat{I}$:

Let $N_{11'}$ be N_{11} with the additional node k_2 and edge $\{i_2, k_2\}$ for $k_2 \in K_2$. $\mathcal{L}(N_{11}) = \mathcal{L}(N_{11'})$ gives $\alpha_{i_2k_2} = -\beta_{k_2}$ for $i_2 \in I \setminus \hat{I}, k_2 \in K_2$ and the results follows by using the results from step (T11k).

T11m $\alpha_{kk'} = \bar{\alpha} + \frac{\rho}{|K_1|}$, $\forall k, k' \in K_2$:

Let $N_{14'}$ be N_{14} with the additional node k_2' and edge $\{k_2, k_2'\}$ for $k_2' \in K_2$. $\mathcal{L}(N_{14}) = \mathcal{L}(N_{14'})$ gives $\alpha_{k_2k_2'} = -\beta_{k_2'}$ for $k_2, k_2' \in K_2$ and the results follows by using the results from step (T11k).

T11n $\alpha_{kk'} = \bar{\alpha} + \frac{\rho}{|K_1|}$, $\forall k \in K_1, \forall k' \in K_3$:

Let $N_{16'}$ be N_{16} with the additional node k_1 and edge $\{k_3, k_1\}$ for $k_1 \in K_1$. $\mathcal{L}(N_{16'}) = \mathcal{L}(N_{16})$ gives $\alpha_{k_1k_3} = -\beta_{k_1}$ for $k_3 \in K_3, k_1 \in K_1$ and the result follows from using the results from steps (T11d), (T11j).

T11o $\alpha_{ik} = \bar{\alpha}$, $\forall k \in K_3, \forall i \in I$ and $\beta_k = -\bar{\alpha}$, $\forall k \in K_3$:

$\mathcal{L}(N_{17}) = \mathcal{L}(N_{18})$ gives $\alpha_{i_2i_3} = \alpha_{i_2k_3}$ for $i_2, i_3 \in I \setminus \hat{I}, k_3 \in K_3$. The first result, for the case $i \in I \setminus \hat{I}$, follows by using the results from step (T11c). Let $\mathcal{L}(N_{17'})$ be $\mathcal{L}(N_{17})$ without the edge $\{i_2, k_3\}$ and node k_3. $\mathcal{L}(N_{17}) = \mathcal{L}(N_{17'})$ gives $\alpha_{i_2k_3} = -\beta_{k_3}$ for $i_2 \in I\backslash\hat{I}, k_3 \in K_3$ and the result $\beta_k = -\bar{\alpha}$,

$\forall k \in K_3$ follows. The first result, for the case $i \in \hat{I}$, then follows from $\mathcal{L}(N_6) = \mathcal{L}(N_{16})$, which gives $\alpha_{i_1 k_3} = -\beta_{k_3}$ for $i_1 \in \hat{I}, k_3 \in K_3$.

T11p $\alpha_{kk'} = \bar{\alpha} + \frac{\rho}{|K_1|}, \forall k \in K_2, \forall k' \in K_3$:

Let $N_{16''}$ be N_{16} with the additional node k_2 and edge $\{k_3, k_2\}$ for $k_2 \in K_2$. $\mathcal{L}(N_{16''}) = \mathcal{L}(N_{16})$ gives $\alpha_{k_3 k_2} = -\beta_{k_2}$ for $k_2 \in K_2, k_3 \in K_3$ and the result follows by using the result from step (T11k).

T11q $\alpha_{kk'} = \bar{\alpha}, \forall k, k' \in K_3$: being a mammal (N) is necessary but not sufficient to being human (S), Let $N_{16'''}$ be N_{16} with the additional node k_3' and edge $\{k_3, k_3'\}$ for $k_3' \in K_3$. $\mathcal{L}(N_{16'''}) = \mathcal{L}(N_{16})$ gives $\alpha_{k_3 k_3'} = -\beta_{k_3}$ for $k_3, k_3' \in K_3$ and the result follows by using the result from step (T11o).

Note that we can now write all coefficients in terms of $\bar{\alpha}$, ρ and $\bar{\delta}_j$, for $j \in J$. The equation defining $\mathcal{G}$ looks as follows:

$$\overbrace{\rho \sum_{i \in \hat{I}} z_i}^{\text{(T11i)}} + \overbrace{0 \sum_{i \in I \setminus \hat{I}} z_i}^{\text{(T11a)}} - \overbrace{\bar{\alpha} \sum_{i \in \hat{I}} y_i}^{\text{(T11e)}} + \overbrace{(\rho - \bar{\alpha}) \sum_{i \in I \setminus \hat{I}} y_i}^{\text{(T11g),(T11h)}} + \overbrace{\bar{\alpha} x(E(I))}^{\text{(T11c).(T11e),(T11f)}} + \overbrace{\Big(\bar{\alpha} + \frac{\rho}{|K_1|}\Big) x(\hat{I} : K_1)}^{\text{(T11d),(T11j)}} +$$

$$\overbrace{\Big(\bar{\alpha} + \frac{\rho}{|K_1|}\Big) x(\hat{I} : K_2)}^{\text{(T11k)}} + \overbrace{\bar{\alpha} x(\hat{I} : K_3)}^{\text{(T11o)}} + \overbrace{\bar{\alpha} x(I \setminus \hat{I} : K_1)}^{\text{(T11c)}} + \overbrace{\Big(\bar{\alpha} + \frac{\rho}{|K_1|}\Big) x(I \setminus \hat{I} : K_2)}^{\text{(T11l)}}$$

$$\overbrace{\bar{\alpha} x(I \setminus \hat{I} : K_3)}^{\text{(T11o)}} + \overbrace{\Big(-\bar{\alpha} - \frac{\rho}{|K_1|}\Big) \sum_{k \in K_1} y_k}^{\text{(T11d),(T11j)}} + \overbrace{\Big(-\bar{\alpha} - \frac{\rho}{|K_1|}\Big) \sum_{k \in K_2} y_k}^{\text{(T11k)}} - \overbrace{\bar{\alpha} \sum_{k \in K_3} y_k}^{\text{(T11o)}} +$$

$$\overbrace{\Big(\bar{\alpha} + \frac{\rho}{|K_1|}\Big) x(E(K_1))}^{\text{(T11d),(T11j)}} + \overbrace{\Big(\bar{\alpha} + \frac{\rho}{|K_1|}\Big) x(K_1 : K_2)}^{\text{(T11k)}} + \overbrace{\Big(\bar{\alpha} + \frac{\rho}{|K_1|}\Big) x(K_1 : K_3)}^{\text{(T11n)}} +$$

$$\overbrace{\Big(\bar{\alpha} + \frac{\rho}{|K_1|}\Big) x(E(K_2))}^{\text{(T11m)}} + \overbrace{\hat{\alpha} x(K_2 : K_3)}^{\text{(T11p)}} + \overbrace{\bar{\alpha} x(E(K_3))}^{\text{(T11q)}} +$$

$$\sum_{i \in I \setminus \hat{I}} \overbrace{(\rho + \bar{\delta}_{h(i)}) a_{ih(i)}}^{\text{(T11g),(T11h)}} + \sum_{j \in J} \overbrace{\bar{\delta}_j \sum_{i \in I : h(i) \neq j} a_{ij}}^{\text{(T11b)}} = \lambda_0.$$

By evaluating any feasible solution (e.g., N_6) we get

$$\lambda_0 = \rho - \bar{\alpha} + \sum_{j\in J} \bar{\delta}_j.$$

Rewriting the equation defining $\mathcal{G}$, we get:

$$\rho\Big(\sum_{i\in \hat{I}} z_i + \sum_{i\in I\setminus \hat{I}} (a_{ih(i)} + y_i) + \frac{1}{|K_1|} x(\hat{I} : K_1 \cup K_2) + \frac{1}{|K_1|} x(I \setminus \hat{I} : K_2) +$$

$$+\frac{1}{|K_1|} x(K_1 \cup K_2 \cup K_3 : K_1 \cup K_2) - \frac{1}{|K_1|} \sum_{k\in K_1} y_k - \frac{1}{|K_1|} \sum_{k\in K_2} y_k\Big)$$

$$-\bar{\alpha} \overbrace{(-y(S) + x(E(S)))}^{(5)} + \sum_{j\in J} \bar{\delta}_j \overbrace{\sum_{i\in I} a_{ij}}^{(2)} = \rho - \bar{\alpha} + \sum_{j\in J} \bar{\delta}_j.$$

Thus the equation defining $\mathcal{G}$ is a linear combination of the equation defining $\mathcal{F}$ and the equality set of $\mathcal{P}$. Therefore, inequalities (15) are facet-inducing, when $|I \setminus \hat{I}| \geq 2, \hat{I} \neq \emptyset, K_1 \neq \emptyset$.

To see that $|I \setminus \hat{I}| \geq 2, \hat{I} \neq \emptyset$ is also a necessary condition, consider the following cases:

1. $\hat{I} = \emptyset$: Inequalities (15) reduce to

   $$|K_1| \sum_{i\in I} (a_{ih(i)} + y_i) + x(I : K_2) + x(K : K_1 \cup K_2) \geq |K_1| + \sum_{k\in K_1} y_k + \sum_{k\in K_2} y_k.$$

   By Lemma 3 for $K' = K_2$, we get

   $$|K_1| \sum_{i\in I} (a_{ih(i)} + y_i) + x(K \setminus K_2 : K_1) \geq |K_1| + \sum_{k\in K_1} y_k.$$

   Replace the y-variables by the z-variables to obtain the stronger inequalities

   $$|K_1| \sum_{i\in I} (a_{ih(i)} + z_i) + x(K \setminus K_2 : K_1) \geq |K_1| + \sum_{k\in K_1} y_k.$$

   For the given injective mapping h, we now subtract inequalities (10) $|K_1|$ times to obtain

   $$x(K \setminus K_2 : K_1) \geq \sum_{k\in K_1} y_k - |K_1|.$$

   This is obviously an aggregation of the upper bound inequalities $y_k \leq 1$ of the relaxation of binary variables y_k, $k \in K_1$, plus the term $x(K \setminus K_2 : K_1)$.

2. $|I \setminus \hat{I}| = 0$: Inequalities (15) reduce to

$$|K_1| \sum_{i \in I} z_i + x(I : K_1 \cup K_2) + x(K : K_1 \cup K_2) \geq |K_1| + \sum_{k \in K_1} y_k + \sum_{k \in K_2} y_k.$$

By Lemma 3 for $K' = K_1 \cup K_2$, we get

$$|K_1| \sum_{i \in I} z_i \geq |K_1|.$$

Replace the z-variables by a-variables for some fixed $j \in J$. We get

$$|K_1| \sum_{i \in I} a_{ij} \geq |K_1|.$$

This inequality is easily seen to be implied by $|K_1|$ times the equation (2) for customer j.

3. $|I \setminus \hat{I}| = 1$: Let $I \setminus \hat{I} = \{i\}$. Inequalities (15) reduce to

$$|K_1| \sum_{i' \in I'} z_{i'} + |K_1| a_{ih(i)} + |K_1| y_i + x(I' : K_1 \cup K_2) + x(i : K_2) +$$
$$x(K : K_1 \cup K_2) \geq |K_1| + \sum_{k \in K_1} y_k + \sum_{k \in K_2} y_k$$

for $I' = I \setminus \{i\}$. Replace $|K_1| y_i$ by $x(i : K_1)$ to get the stronger inequalities (since $x_{ss'} \leq y_s$ due to inequalities (6) for $H = \{s, s'\}$)

$$|K_1| \sum_{i' \in I'} z_{i'} + |K_1| a_{ih(i)} + x(I : K_1 \cup K_2) + x(K : K_1 \cup K_2) \geq |K_1| + \sum_{k \in K_1} y_k + \sum_{k \in K_2} y_k.$$

By Lemma 3 for $K' = K_1 \cup K_2$, we get

$$|K_1| \sum_{i' \in I'} z_{i'} + |K_1| a_{ih(i)} \geq |K_1|.$$

Replace the z-variables by a-variables for $h(i) \in J$. We get

$$|K_1| \sum_{i' \in I} a_{i'h(i)} \geq |K_1|.$$

This inequality is easily seen to be implied by $|K_1|$ times the equation (2) for customer $h(i)$.

$\square$

Two special cases of inequalities (15) are of particular interest. One case is given by the *2+2 partition inequalities* that are obtained for $K_2 = \emptyset$ and $K_3 = \emptyset$. They are given as:

$$|K_1| \sum_{i \in \hat{I}} z_i + |K_1| \sum_{i \in I \setminus \hat{I}} (a_{ih(i)} + y_i) + x(\hat{I} : K_1) + x(K : K_1) \geq |K_1| + \sum_{k \in K_1} y_k \quad (16)$$

The other case is given by the *2+1 partition inequalities* that are obtained for $K_2 = \emptyset$ and $K_3 = \emptyset$ (i.e., $K_1 = K$). They are given as:

$$|K| \sum_{i \in \hat{I}} z_i + |K| \sum_{i \in I \setminus \hat{I}} (a_{ih(i)} + y_i) + x(\hat{I} : K) + x(E(K)) \geq |K| + \sum_{k \in K} y_k \quad (17)$$

6. Conclusions

This article analyzes the polytope defined by the feasible solutions of the connected facility location problem. This problem combines the uncapacitated facility location problem and the Steiner tree problem, and has been motivated by a telecommunication application. The article computes the dimension of the polytope and shows several families of valid inequalities. Some of these inequalities are lifted variants from the uncapacitated facility location polytope. Other inequalities are taken from the polytope of the Spanning Tree problem. In addition, the article also presents new inequalities exploiting the interaction of the two combinatorial structures, like what we call partition inequalities. The article also study conditions under which these inequalities are facet defining. The proofs are based on the so-called indirect method. Some of the inequalities analyzed in this article are used in [11] to describe a branch-and-cut approach to design telecommunication networks with a tree-star topology.

Acknowledgements

This work is supported by the Vienna Science and Technology Fund (WWTF) through project ICT15-014. M. Leitner, I. Ljubić, and M. Sinnl are supported by the Austrian Research Fund (FWF) under grants I892-N23 and P26755-N19. J.J. Salazar-González is supported by the Spanish Government through the project MTM2015-63680-R (MINECO/FEDER, UE).

References

[1] G. Cornuejols and J.-M. Thizy. Some facets of the simple plant location polytope. *Math. Programming*, 23(1):50–74, 1982.

[2] J. Edmonds. Submodular functions, matroids, and certain polyhedra. *Combinatorial structures and their applications*, pages 69–87, 1970.

[3] S. Gollowitzer and I. Ljubić. MIP models for connected facility location: A theoretical and computational study. *Comput. & Oper. Res.*, 38(2):435–449, 2011.

[4] F. Grandoni and T. Rothvoß. Approximation algorithms for single and multi-commodity connected facility location. In O. Günlük and G. Woeginger, editors, *IPCO 2011, New York, NY, USA, Proceedings*, volume 6655 of *LNCS*, pages 248–260. Springer, 2011.

[5] M. Guignard. Fractional vertices, cuts and facets of the simple plant location problem. In M. Padberg, editor, *Combinatorial Optimization*, volume 12 of *Mathematical Programming Studies*, pages 150–162. Springer Berlin Heidelberg, 1980.

[6] F. Havet and M. Wennink. The push tree problem. *Networks*, 44(4):281–291, 2004.

[7] D. Karger and M. Minkoff. Building Steiner trees with incomplete global knowledge. In *FOCS*, pages 613–623. IEEE Computer Society, 2000.

[8] C. Krick, H. Räcke, and M. Westermann. Approximation algorithms for data management in networks. *Theory Comput. Syst.*, 36(5):497–519, 2003.

[9] M. Leitner and G. R. Raidl. Branch-and-cut-and-price for capacitated connected facility location. *J. of Math. Modelling and Algorithms*, 10(3): 245–267, 2011.

[10] M. Leitner, I. Ljubić, J. J. Salazar-González, and M. Sinnl. On the asymmetric connected facility location polytope. *Lecture Notes in Computer Science*, 8596:371–383, 2014.

[11] M. Leitner, I. Ljubić, J. J. Salazar-González, and M. Sinnl. An algorithmic framework for the exact solution of tree-star problems. *Working paper*, 2016.

[12] G. L. Nemhauser and L. A. Wolsey. *Integer and Combinatorial Optimization*, volume 18. Wiley New York, 1988.

[13] J. J. Salazar. A note on the generalized Steiner tree polytope. *Discrete Appl. Math.*, 100(1):137–144, 2000.