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An implicit numerical method of a new time distributed-order and two-sided space-fractional advection-dispersion equation

Xiuling Hu · F. Liu · I. Turner · V. Anh

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Abstract Distributed-order differential equations have recently been investigated for complex dynamical systems, which have been used to describe some important physical phenomena. In this paper, a new time distributed-order and two-sided space-fractional advection-dispersion equation is considered. Firstly, we transform the time distributed-order fractional equation into a multi-term time-space fractional partial differential equation by applying numerical integration. Then an implicit numerical method is constructed to solve the multi-term fractional equation. The uniqueness, stability and convergence of the implicit numerical method are proved. Some numerical results are presented to demonstrate the effectiveness of the method. The method and techniques can be extended to other time distributed-order and space-fractional partial differential equations.

Keywords Implicit numerical method · Distributed-order fractional derivative · Two-sided space-fractional derivative · Stability and convergence · Advection-dispersion equation

Xiuling Hu
School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, Jiangsu, China

F. Liu
School of Mathematical Sciences, Queensland University of Technology, Qld. 4001. Australia
E-mail: f.liu@qut.edu.au

I. Turner
School of Mathematical Sciences, Queensland University of Technology, Qld. 4001. Australia

V. Anh
School of Mathematical Sciences, Queensland University of Technology, Qld. 4001. Australia

1 Introduction

In recent years, anomalous diffusion is discussed at all scales, for example, in pore scale simulations of fluid flow and solute movement [14, **?**] and in field-scale simulations of dispersion in a heterogeneous aquifer [1]. However, many physical processes involving decelerating subdiffusion and decelerating superdiffusion lack power-law scaling over the whole time-domain. Such processes may be described by derivatives of distributed-order, which first introduced by Caputo (see [24] and the references therein). In some applications, a more complicated process cannot be described by a single power law and a mixture of power laws leads to a time distributed-order fractional derivative (see [20] and the references therein). Furthermore, some materials have rheological properties that depend on temperature, electrostatic field strength, or magnetic field strength. Lorenzo and Hartley [18] pointed out that if a sample of this material was subject to a temperature distribution, a corresponding order-distribution would exist throughout the material. Distributed-order differential models have recently been reported as a more powerful tool to describe the complex dynamical systems than the classical and fractional-order models.

Many details about a distributed-order dynamical system can be found in [12] and the references therein. Two generic applications in engineering practice are also discussed in their paper. One application is for distributed-order filters in signal processing and the other is for optimal distributed-order damping in control systems.

Recently, Morgado and her collaborators studied some numerical solutions of the time distributed-order partial differential equations [3–5]. In [3], they presented an implicit scheme for the distributed-order time-fractional reaction-diffusion equations with a nonlinear source term and analyzed the stability and convergence of the numerical scheme. However, to the best knowledge of the authors, numerical studies for the distributed-order equations are relatively limited, especially for the time distributed-order and space-fractional equations [6, 12, 13, 23].

Based on the above observations, in this paper, we focus on deriving a numerical method for the new time distributed-order and two-sided space-fractional advection-dispersion equation (TDO-TSSFADE). This model is obtained by replacing the fractional derivative with the time distributed-order derivative in a time-space fractional model which is used modeling the dispersion of aqueous tracers in heterogeneous soils, aquifers, and rivers [**?**] and is believed to be describe the multiple anomalous dispersion adequately than the original model. We first transform the distributed-order fractional equation into a multi-term fractional partial differential equation by adopting the numerical integration method of mid-point quadrature [6]. Liu et al. [17] proposed some numerical methods for solving the multi-term fractional partial differential equations. Jiang et al. [11] discussed the fundamental solutions for the multi-term modified power law wave equations. Ye et al. [26] derived series expansion solutions for the multi-term time and space fractional partial differential equations in two and three dimensions. Jiang et al. [9] also derived analytical solutions for the multi-term time-space Caputo-Riesz fractional advection-diffusion equations on a finite domain. In this paper, by using L_1 discretization to approximate the fractional-order derivative and using the shifted Grünwald-Letnikov formulae to approximate the two-sided space-fractional derivative in the multi-term fractional equation, we obtain an implicit difference method with convergence order $O(\tau^{1+\frac{\sigma}{2}}+h+\sigma^2)$. The uniqueness, stability and convergence of the method are analyzed and numerical experiments are provided.

By comparing and contrasting our work with the work in [3], we found that the discrtization for the Caputo derivative used in [3] and used in this paper is actually the same. This can be proved by some computations on these two discretizations [7,8]. The difference between this paper and [3] is that this we studied the two-sided space-fractional advection-dispersion equation and Morgado and her coworkers studied the reaction-diffusion equations. The remainder of the paper is arranged as follows. An implicit numerical method is proposed for the TDO-TSSFADE in Section 2. The uniqueness, stability and convergence of the implicit numerical method are discussed in Section 3. Some numerical results are presented to demonstrate the effectiveness of the method in Section 4. Finally, some conclusions are given.

2 Implicit numerical method

Consider the following TDO-TSSFADE:

$$\mathcal{D}_t^{P(\beta)}u(x,t) = -V\frac{\partial u(x,t)}{\partial x} + D(\frac{1}{2}+\frac{q}{2})\frac{\partial u^\alpha(x,t)}{\partial x^\alpha} + D(\frac{1}{2}-\frac{q}{2})\frac{\partial u^\alpha(x,t)}{\partial(-x)^\alpha} + f(x,t), \quad (1)$$

with boundary conditions

$$u(a,t)=0,\ u(b,t)=0,\ t\in[0,T], \quad (2)$$

and initial condition

$$u(x,0)=\phi_0(x),\ x\in[a,b], \quad (3)$$

where $V>0$ is the drift of the process, $D>0$ is the coefficient of dispersion, $1<\alpha<2$ is the order of fractional differentiation, $-1\le q\le 1$ indicates the relative weight of forward versus backward transition probability and the function $f(x,t)$ is a source/sink term. The time distributed-order operator is defined as follows [24,12]

$$\mathcal{D}_t^{P(\beta)}u(x,t) = \int_0^1 P(\beta)\,{}_0^C\mathcal{D}_t^\beta u(x,t)d\beta, \quad (4)$$

with ${}_0^C\mathcal{D}_t^{\beta}$ is the fractional derivative defined by the Caputo operator

$$ {}_0^C\mathcal{D}_t^{\beta}u(x,t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial u(x,\tau)}{\partial \tau}\frac{d\tau}{(t-\tau)^{\beta}},\ 0<\beta<1, \tag{5} $$

and the non-negative weight function $P(\beta)$ used in (4) satisfies the conditions

$$ 0 \le P(\beta),\ P(\beta)\not\equiv 0,\ \beta\in[0,1],\ 0<\int_0^1 P(\beta)d\beta<\infty. \tag{6} $$

The two-sided space-fractional derivative operators are defined by [16,22]

$$ \frac{\partial u^{\alpha}(x,t)}{\partial x^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_a^x \frac{u(\xi,t)d\xi}{(x-\xi)^{\alpha+1-n}},\ n-1<\alpha\le n, \tag{7} $$

$$ \frac{\partial u^{\alpha}(x,t)}{\partial (-x)^{\alpha}} = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_x^b \frac{u(\xi,t)d\xi}{(\xi-x)^{\alpha+1-n}},\ n-1<\alpha\le n, \tag{8} $$

which are called the left- and right-handed Riemann-Liouville derivatives of order α respectively with n being a positive integer.

We first discretize the integral interval [0,1] by the grid $0=\xi_0<\xi_1<\cdots<\xi_S=1$ and take $\Delta\xi_m=\xi_m-\xi_{m-1}=\frac{1}{S}=\sigma$, $\beta_m=\frac{\xi_m+\xi_{m-1}}{2}=\frac{2m-1}{2S}$, $m=1,2,\cdots,S,\ S\in\mathbb{N}$. Then using the mid-point quadrature rule, we obtain

$$ \mathcal{D}_t^{P(\beta)}u(x,t)=\sum_{m=1}^{S}P(\beta_m){}_0^C\mathcal{D}_t^{\beta_m}u(x,t)\Delta\xi_m+O(\sigma^2) \tag{9} $$

where σ is the step size of the discretization of the numerical integration. Thus the distributed-order fractional equation (1) is now transformed into the following multi-term fractional equation:

$$ \sum_{m=1}^{S}\frac{P(\beta_m)}{S}\,{}_0^C\mathcal{D}_t^{\beta_m}u(x,t) = -V\frac{\partial u(x,t)}{\partial x}+D(\frac{1}{2}+\frac{q}{2})\frac{\partial u^{\alpha}(x,t)}{\partial x^{\alpha}} + D(\frac{1}{2}-\frac{q}{2})\frac{\partial u^{\alpha}(x,t)}{\partial (-x)^{\alpha}}+f(x,t). \tag{10} $$

Next, we discretize the domain $[a,b]\times[0,T]$ using $x_i=a+ih$, $i=0,1,\cdots,M$ and $t_k=k\tau$, $k=0,1,\cdots,N$, where $h=(b-a)/M$ and $\tau=T/N$ are the space and time steps respectively. We assume that $u(x,t)\in C^2([a,b]\times[0,T])$.

Lemma 1 *[25] Suppose* $0<\beta<1$, $y(t)\in\mathcal{C}^2[0,t_n]$, *it holds that*

$$ \Bigg|\frac{1}{\Gamma(1-\beta)}\int_0^{t_n}\frac{y'(s)ds}{(t_n-s)^{\beta}} - \frac{\tau^{-\beta}}{\Gamma(2-\beta)}\left[b_0y(t_n)-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})y(t_k)-b_{n-1}y(0)\right]\Bigg| \le \frac{1}{\Gamma(2-\beta)}\left[\frac{1-\beta}{12}+\frac{2^{2-\beta}}{2-\beta}-(1+2^{-\beta})\right]\max_{0\le t\le t_n}|y''(t)|\tau^{2-\beta}, $$

where $b_k=(k+1)^{1-\beta}-k^{1-\beta}$.

Lemma 2 *[21, 19] For $1 < \alpha < 2$, suppose that $u \in L_1(\mathbb{R})$, $u \in C^{\alpha+p}(\mathbb{R})$, the following two shifted Grünwald-Letnikov formulae hold:*

$$\frac{\partial^\alpha}{\partial x^\alpha}u(x_{i,t_{k+1}}) = \frac{1}{h^\alpha}\sum_{j=0}^{i+1}\omega_j^\alpha u(x_{i+1-j}, t_{k+1}) + O(h^p), \tag{11}$$

$$\frac{\partial^\alpha}{\partial(-x)^\alpha}u(x_{i,t_{k+1}}) = \frac{1}{h^\alpha}\sum_{j=0}^{M-i+1}\omega_j^\alpha u(x_{i-1+j}, t_{k+1}) + O(h^p). \tag{12}$$

For different α, the above two formulae can lead to approximations of different order p. In this paper, we take

$$\omega_j^\alpha = (-1)^j\frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!},\ j = 0, 1, \cdots, \tag{13}$$

hence $p = 1$ in the two formulae $(11) - (12)$.

Define the grid function $U_i^k = u(x_i, t_k)$ as exact solution of the equations (1)-(3), $f_i^k = f(x_i, t_k)$. By applying (5), Lemma 1, Lemma 2 and the backward difference formula, we obtain the following discrete form of equation (10) at the point (x_i, t_{k+1}):

$$\begin{aligned}
&\sum_{m=1}^{S}\frac{P(\beta_m)}{\mu_m S}\left[U_i^{k+1} - \sum_{j=1}^{k}(b_{k-j}^{\beta_m} - b_{k-j+1}^{\beta_m})U_i^j - b_k^{\beta_m}U_i^0\right]\\
&= -\frac{V}{h}(U_i^{k+1} - U_{i-1}^{k+1}) + (\frac{1}{2} + \frac{q}{2})\frac{D}{h^\alpha}\sum_{j=0}^{i+1}\omega_j^\alpha U_{i+1-j}^{k+1}\\
&+ (\frac{1}{2} - \frac{q}{2})\frac{D}{h^\alpha}\sum_{j=0}^{M-i+1}\omega_j^\alpha U_{i-1+j}^{k+1} + f_i^{k+1} + r_i^{k+1},
\end{aligned} \tag{14}$$

where $b_k^{\beta_m} = (k+1)^{1-\beta_m} - k^{1-\beta_m}$, $\mu_m = \tau^{\beta_m}\Gamma(2-\beta_m)$, $\omega_j^\alpha = (-1)^j\begin{pmatrix}\alpha\\ j\end{pmatrix}$ and r_i^{k+1} is the local truncation error. Noting that $\beta_m = (2m-1)/2S,\ 1 \le m \le S$ and $1/S = \sigma$, we have

$$1 + \frac{\sigma}{2} = 2 - S\sigma + \frac{1}{2}\sigma \le 2 - \beta_m = 2 - m\sigma + \frac{1}{2}\sigma \le 2 - \sigma + \frac{1}{2}\sigma = 2 - \frac{1}{2}\sigma.$$

Then, we derive the inequality (see [25, 19] for further details)

$$|r_i^{k+1}| = |O(\tau^{2-\beta_m} + h + \sigma^2)| \le C(\tau^{1+\frac{\sigma}{2}} + h + \sigma^2). \tag{15}$$

The coefficients ω_j^α defined in equation (13) satisfy the properties given in Lemma 3 (see [15] for further details).

Lemma 3 *([15]) The coefficients ω_j^α, $j = 0, 1, \ldots,$ satisfy*
(1) $\omega_0^\alpha = 1,\ \omega_1^\alpha = -\alpha < 0,$ *and* $\omega_j^\alpha > 0, (j \neq 1)$*;*
(2) $\sum_{j=0}^{\infty}\omega_j^\alpha = 0,$ *and for* $n = 1, 2, \ldots, \sum_{j=0}^{n}\omega_j^\alpha < 0.$

Let u_i^k be the numerical solution to U_i^k. By omitting the local truncation error term r_i^{k+1} in (14) and considering the discretization of the initial and boundary value conditions, we obtain the following implicit numerical scheme for the TDO-TSSFADE (1) – (3) :

$$\sum_{m=1}^{S}\frac{P(\beta_m)}{\mu_m S}\left[u_i^{k+1}-\sum_{j=1}^{k}(b_{k-j}^{\beta_m}-b_{k-j+1}^{\beta_m})u_i^j-b_k^{\beta_m}u_i^0\right]=-\frac{V}{h}(u_i^{k+1}-u_{i-1}^{k+1})$$
$$+(\frac{1}{2}+\frac{q}{2})\frac{D}{h^\alpha}\sum_{j=0}^{i+1}\omega_j^\alpha u_{i+1-j}^{k+1}+(\frac{1}{2}-\frac{q}{2})\frac{D}{h^\alpha}\sum_{j=0}^{M-i+1}\omega_j^\alpha u_{i-1+j}^{k+1}+f_i^{k+1},$$
$$i=1,2,\cdots,M-1,\ k=1,2,\cdots,N-1, \tag{16}$$
$$u_0^k=0,\ u_M^k=0,\ k=1,2,\cdots,N, \tag{17}$$
$$u_i^0=\phi_0(x_i),\ i=0,1,\cdots,M. \tag{18}$$

For the convenience of the following theoretical analysis, we rewrite the above scheme as follows:

$$(1+Gr_1+Gr_2\alpha)u_i^{k+1}-Gr_1u_{i-1}^{k+1}-(\frac{1}{2}+\frac{q}{2})Gr_2\sum_{j=0,j\neq 1}^{i+1}\omega_j^\alpha u_{i+1-j}^{k+1}$$
$$-(\frac{1}{2}-\frac{q}{2})Gr_2\sum_{j=0,j\neq 1}^{M-i+1}\omega_j^\alpha u_{i-1+j}^{k+1}=\sum_{m=1}^{S}\frac{P(\beta_m)}{\mu_m S}G\sum_{j=1}^{k}(b_{k-j}^{\beta_m}-b_{k-j+1}^{\beta_m})u_i^j$$
$$+\sum_{m=1}^{S}\frac{P(\beta_m)}{\mu_m S}Gb_k^{\beta_m}u_i^0+Gf_i^{k+1},\ i=1,\cdots,M-1,\ k=1,\cdots,N-1, \tag{19}$$
$$u_0^k=0,\ u_M^k=0,\ k=1,2,\cdots,N, \tag{20}$$
$$u_i^0=\phi_0(x_i),\ i=0,1,\cdots,M, \tag{21}$$

where $G=1/\sum_{m=1}^{S}\frac{P(\beta_m)}{\mu_m S},\ r_1=V/h,\ r_2=D/h^\alpha$ and Lemma 3 (1) is used.

3 Stability and convergence

In this section, we analyze the implicit numerical method (19) – (21) obtained in Section 2.

Lemma 4 *For the coefficients* $G,\ r_1,\ r_2,\ \omega_j^\alpha$ *defined in* (19), *we have*

$$Gr_1+Gr_2\alpha-\frac{1+q}{2}Gr_2\sum_{j=0,j\neq 1}^{i+1}\omega_j^\alpha-\frac{1-q}{2}Gr_2\sum_{j=0,j\neq 1}^{M-i+1}\omega_j^\alpha>0.$$

Proof By applying Lemma 3, we obtain

$$\frac{1+q}{2}Gr_2\sum_{j=0}^{i+1}\omega_j^\alpha = -\frac{1+q}{2}Gr_2\alpha + \frac{1+q}{2}Gr_2\sum_{j=0,j\neq 1}^{i+1}\omega_j^\alpha < 0, \tag{22}$$

$$\frac{1-q}{2}Gr_2\sum_{j=0}^{M-i+1}\omega_j^\alpha = -\frac{1-q}{2}Gr_2\alpha + \frac{1-q}{2}Gr_2\sum_{j=0,j\neq 1}^{M-i+1}\omega_j^\alpha < 0. \tag{23}$$

By adding (22) and (23) together, we have

$$Gr_2\alpha - \tfrac{1+q}{2}Gr_2\textstyle\sum_{j=0,j\neq 1}^{i+1}\omega_j^\alpha - \tfrac{1-q}{2}Gr_2\sum_{j=0,j\neq 1}^{M-i+1}\omega_j^\alpha > 0,$$

which implies Lemma 4 holds.

The equation (19) can be written in the following matrix form:

$$A\mathbf{u}^{k+1} = \sum_{m=1}^{S}\frac{P(\beta_m)}{\mu_m S}G\sum_{j=1}^{k}(b_{k-j}^{\beta_m} - b_{k-j+1}^{\beta_m})\mathbf{u}^j + \sum_{m=1}^{S}\frac{P(\beta_m)}{\mu_m S}Gb_k^{\beta_m}\mathbf{u}^0 + G\mathbf{f}^{k+1},$$

$$i = 1,\cdots,M-1,\ k = 1,\cdots,N-1, \tag{24}$$

where

$$\mathbf{A} = \left[I + Gr_1\mathbf{B} - (\frac{1}{2}+\frac{q}{2})Gr_2\mathbf{C} - (\frac{1}{2}-\frac{q}{2})Gr_2\mathbf{C}^T\right],$$

$$\mathbf{B} = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & 0 & -1 & 1 \end{pmatrix}_{(M-1)\times(M-1)},$$

$$\mathbf{C} = \begin{pmatrix} \omega_1 & \omega_0 & & \\ \omega_2 & \omega_1 & \omega_0 & \\ & & \ddots & \\ \omega_{M-1} & \omega_{M-2} & \omega_{M-3} & \cdots & \omega_1 \end{pmatrix}_{(M-1)\times(M-1)},$$

and $\mathbf{u}^k = (u_1^k, u_2^k, \cdots, u_{M-1}^k)^T,\ \mathbf{f}^k = (f_1^k, f_2^k, \cdots, f_{M-1}^k)^T$. Lemma 4 implies that the coefficient matrix $\mathbf{A}$ of equation (24) is strictly diagonally dominant. Thus the coefficient matrix $\mathbf{A}$ is reversible and therefore, the difference method (19) – (21) is uniquely solvable.

Now, let us consider the stability and convergence of the numerical method.

Theorem 1 *Let* v_i^k $(0 \le i \le M,\ 0 \le k \le N)$ *be the solution of the implicit numerical method* (19) – (21), *it holds that*

$$\|v^{k+1}\|_\infty \le \|v^0\|_\infty + G(b_k^{\beta_m})^{-1}\max_{1\le l\le N}|f^l|.$$

Proof We use mathematical induction to prove the result. Let $\max_{1\le i\le M-1}|v_i^j| = |v_{i_0}^j|,\ j=0,1,2,\cdots,N$. For $k=0$, from Lemma 4, we have

$$
\begin{aligned}
&\|v^1\|_\infty = |v_{i_0}^1| \\
&\le |v_{i_0}^1|\left(1+Gr_1-Gr_1+Gr_2\alpha-\frac{1+q}{2}Gr_2\sum_{j=0,j\neq1}^{i+1}\omega_j^\alpha-\frac{1-q}{2}Gr_2\sum_{j=0,j\neq1}^{M-i+1}\omega_j^\alpha\right) \\
&\le |v_{i_0}^1|+Gr_1|v_{i_0}^1|-Gr_1|v_{i_0-1}^1|+Gr_2\alpha|v_{i_0}^1|-\frac{1+q}{2}Gr_2\sum_{j=0,j\neq1}^{i+1}\omega_j^\alpha\left|v_{i_0+1-j}^1\right| \\
&\quad -\frac{1-q}{2}Gr_2\sum_{j=0,j\neq1}^{M-i+1}\omega_j^\alpha\left|v_{i_0-1+j}^1\right| \\
&\le |v_{i_0}^1+Gr_1v_{i_0}^1-Gr_1v_{i_0-1}^1+Gr_2\alpha v_{i_0}^1 \\
&\quad -\frac{1+q}{2}Gr_2\sum_{j=0,j\neq1}^{i+1}\omega_j^\alpha v_{i_0+1-j}^1-\frac{1-q}{2}Gr_2\sum_{j=0,j\neq1}^{M-i+1}\omega_j^\alpha v_{i_0-1+j}^1| \\
&= \left|\sum_{m=1}^{S}\frac{P(\beta_m)}{\mu_m S}Gb_0^{\beta_m}v_{i_0}^0+Gf_{i_0}^1\right| \\
&\le \|v^0\|_\infty+G\max_{1\le l\le N}|f^l| \\
&\le \|v^0\|_\infty+G(b_0^{\beta_m})^{-1}\max_{1\le l\le N}|f^l|.
\end{aligned}
$$

Suppose that $\|v^j\|_\infty\le\|v^0\|_\infty+G(b_{j-1}^{\beta_m})^{-1}\max_{1\le l\le N}|f^l|$ for $j=1,2,\cdots,k$. Then for $j=k+1$, according to the inductive assumption, Lemma 3 and

Lemma 4, and using a similar argument as for the case $k = 0$, we have

$$
\begin{aligned}
&\|v^{k+1}\|_\infty = |v_{i_0}^{k+1}| \\
&\le |v_{i_0}^{k+1}| \left(1 + Gr_1 + Gr_2\alpha - \frac{1+q}{2} Gr_2 \sum_{j=0,j\neq 1}^{i+1} \omega_j^\alpha - \frac{1-q}{2} Gr_2 \sum_{j=0,j\neq 1}^{M-i+1} \omega_j^\alpha \right) \\
&\le |v_{i_0}^{k+1}| + Gr_1|v_{i_0}^{k+1}| + Gr_2\alpha|v_{i_0}^{k+1}| - \frac{1+q}{2} Gr_2 \sum_{j=0,j\neq 1}^{i+1} \omega_j^\alpha \left|v_{i_0}^{k+1}\right| \\
&\quad - \frac{1-q}{2} Gr_2 \sum_{j=0,j\neq 1}^{M-i+1} \omega_j^\alpha \left|v_{i_0}^{k+1}\right| \\
&\le |v_{i_0}^{k+1} + Gr_1 v_{i_0}^{k+1} - Gr_1 v_{i_0-1}^{k+1} + Gr_2\alpha v_{i_0}^{k+1} \\
&\quad - \frac{1+q}{2} Gr_2 \sum_{j=0,j\neq 1}^{i+1} \omega_j^\alpha v_{i_0+1-j}^{k+1} - \frac{1-q}{2} Gr_2 \sum_{j=0,j\neq 1}^{M-i+1} \omega_j^\alpha v_{i_0-1+j}^{k+1}| \\
&= \left| \sum_{m=1}^{S} \frac{P(\beta_m)}{\mu_m S} G \sum_{j=1}^{k} (b_{k-j}^{\beta_m} - b_{k-j+1}^{\beta_m}) v_{i_0}^j + \sum_{m=1}^{S} \frac{P(\beta_m)}{\mu_m S} G b_k^{\beta_m} v_{i_0}^0 + G f_{i_0}^{k+1} \right| \\
&\le \sum_{m=1}^{S} \frac{P(\beta_m)}{\mu_m S} G \sum_{j=1}^{k} (b_{k-j}^{\beta_m} - b_{k-j+1}^{\beta_m}) \left| v_{i_0}^j \right| + \sum_{m=1}^{S} \frac{P(\beta_m)}{\mu_m S} G b_k^{\beta_m} \left| v_{i_0}^0 \right| + G |f_{i_0}^{k+1}| \\
&\le \sum_{m=1}^{S} \frac{P(\beta_m)}{\mu_m S} G \sum_{j=1}^{k} (b_{k-j}^{\beta_m} - b_{k-j+1}^{\beta_m}) \left(\|v^0\|_\infty + G(b_k^{\beta_m})^{-1} \max_{1\le l\le N} |f^l| \right) \\
&\quad + \sum_{m=1}^{S} \frac{P(\beta_m)}{\mu_m S} G b_k^{\beta_m} \|v^0\|_\infty + G \max_{1\le l\le N} |f^l| \\
&\le \sum_{m=1}^{S} \frac{P(\beta_m)}{\mu_m S} G (1 - b_k^{\beta_m}) (\|v^0\|_\infty + G(b_k^{\beta_m})^{-1} \max_{1\le l\le N} |f^l|) + \sum_{m=1}^{S} \frac{P(\beta_m)}{\mu_m S} G b_k^{\beta_m} \|v^0\|_\infty \\
&\quad + G \max_{1\le l\le N} |f^l| \\
&\le \|v^0\|_\infty + G(b_k^{\beta_m})^{-1} \max_{1\le l\le N} |f^l|.
\end{aligned}
$$

Remark. In fact, since $(1-\beta_m)(k+1)^{-\beta_m} < b_k^{\beta_m} < (1-\beta_m)k^{-\beta_m},\ k = 0, 1, \cdots, N+1$ (see [28] for further deatails), we have

$$
\begin{aligned}
G(b_k^{\beta_m})^{-1} &= \frac{1}{\sum_{m=1}^{S} \frac{P(\beta_m) b_k^{\beta_m}}{\mu_m S}} \le \frac{1}{\sum_{m=1}^{S} \frac{P(\beta_m)(1-\beta_m)}{(k+1)^{\beta_m} \mu_m S}} \\
&\le \frac{1}{\sum_{m=1}^{S} \frac{(1-\beta_m) P(\beta_m)}{T^{\beta_m} \Gamma(2-\beta_m) S}} \le \frac{1}{\int_0^1 \frac{(1-\beta) P(\beta)}{T^\beta \Gamma(2-\beta)} d\beta} \le C.
\end{aligned}
$$

Hence, the conclusion of Theorem 1 can actually be written as

$$\|v^{k+1}\|_\infty \le \|v^0\|_\infty + C \max_{1\le l\le N} |f^l|.$$

Since the equation is linear, the implicit numerical method $(19)-(21)$ is unconditionally stable to the initial data and the right source term.

Theorem 2 *Suppose $u(x,t)$ satisfies the smooth conditions of Lemma 1, Lemma 2 and is the smooth solution of problem* (1)–(3) *and $\{u_i^k|\ 0\le i\le M,\ 0\le k\le N\}$ be the numerical solution of the scheme* $(19)-(21)$. *Let $e_i^k = u(x_i,t_k)-u_i^k,\ 0\le j\le M,\ 0\le k\le N$. Then for $k\tau\le T$, it holds that*

$$\|e^{k+1}\|_\infty \le C\left(\tau^{1+\frac{\sigma}{2}} + h + \sigma^2\right),\ k=1,\cdots,N-1.$$

Proof By subtracting (14) from (19), we obtain the error equation

$$\begin{aligned}(1+Gr_1+Gr_2\alpha)e_i^{k+1} - Gr_1e_{i-1}^{k+1} - (\frac{1}{2}+\frac{q}{2})Gr_2\sum_{j=0,j\ne1}^{i+1}\omega_j^\alpha e_{i+1-j}^{k+1}\\ -(\frac{1}{2}-\frac{q}{2})Gr_2\sum_{j=0,j\ne1}^{M-i+1}\omega_j^\alpha e_{i-1+j}^{k+1} = \sum_{m=1}^{S}\frac{P(\beta_m)}{\mu_m S}G\sum_{j=1}^{k}(b_{k-j}^{\beta_m}-b_{k-j+1}^{\beta_m})e_i^j\\ +\sum_{m=1}^{S}\frac{P(\beta_m)}{\mu_m S}Gb_k^{\beta_m}e_i^0 + Gr_i^{k+1},\ i=1,\cdots,M-1,\ k=1,\cdots,N-1,\quad (25)\end{aligned}$$

where r_i^k satisfy (15). By using a similar argument to that of Theorem 1, we obtain

$$\|e^{k+1}\|_\infty \le G(b_k^{\beta_m})^{-1}\max\nolimits_{1\le l\le N}|r^l| \le C(\tau^{1+\frac{\sigma}{2}}+h+\sigma^2),\ k=1,2,\cdots,N-1.$$

Therefore, Theorem 2 holds.

4 Numerical results

A combined space-time nonlocal model can be used to describe the multiple anomalous behaviors [27]. Furthermore, it has been demonstrated that by using the distributed-order concept, we can depict the dynamical process of the real world more accurately [12]. The TDO-TSSFADE model investigated in this paper has more potential in describing the multiple anomalous dispersion in heterogeneous soils, aquifers, and rivers than the model discussed in paper [**?**].

In this section, we demonstrate the effectiveness of our numerical scheme by applying it to the following two TDO-TSSFADE examples.

Example 1. Consider the following problem

$$\int_0^1 \Gamma(3-\beta){}_0^C\mathcal{D}_t^\beta u(x,t)d\beta = -\frac{\partial u(x,t)}{\partial x} + \frac{1}{2}\frac{\partial u^\alpha(x,t)}{\partial x^\alpha} + \frac{1}{2}\frac{\partial u^\alpha(x,t)}{\partial(-x)^\alpha} + f(x,t),\ (x,t)\in[0,1]\times[0,T], \quad (26)$$

$$u(0,t)=0,\ u(1,t)=0,\ t\in[0,T], \quad (27)$$

$$u(x,0)=0,\ x\in[0,1], \quad (28)$$

with $1<\beta<2$ and

$$\begin{aligned} f(x,t) &= f_1(x,t)+f_2(x,t)+f_3(x,t)+f_4(x,t),\\ f_1(x,t) &= 2x^2(1-x)^2(t^2-t)/\ln t,\\ f_2(x,t) &= 2x(1-x)(1-2x)t^2,\\ f_3(x,t) &= -x^{2-\alpha}t^2[(3-\alpha)(4-\alpha)-6(4-\alpha)x+12x^2]/\Gamma(5-\alpha),\\ f_4(x,t) &= -(1-x)^{2-\alpha}t^2\times f_5(x,t)/\Gamma(5-\alpha),\\ f_5(x,t) &= (3-\alpha)(4-\alpha)-6(4-\alpha)(1-x)+12(1-x)^2. \end{aligned}$$

The exact solution is $u(x,t)=t^2x^2(1-x)^2$.

To solve problem (26)–(28) numerically, we first transform the integral distributed-order equation (26) into the following multi-term equation

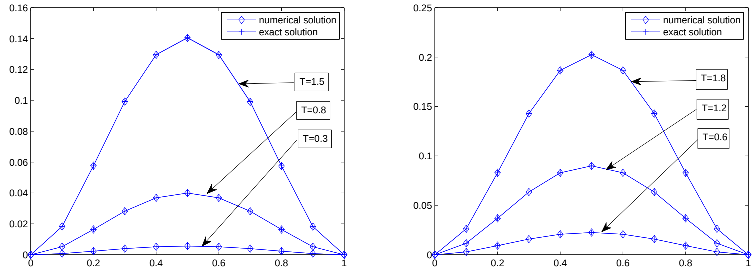
$$\sum_{m=1}^{S}\frac{\Gamma(3-\beta_m)}{S}\ {}_0^C\mathcal{D}_t^{\beta_m}u(x,t) = -\frac{\partial u(x,t)}{\partial x} + \frac{1}{2}\frac{\partial u^\alpha(x,t)}{\partial x^\alpha} + \frac{1}{2}\frac{\partial u^\alpha(x,t)}{\partial(-x)^\alpha} + f(x,t). \quad (29)$$

Then we use the method (19)-(21) to compute the numerical solution of equation (29). Take $h=1/1000,\ \tau=T/1000,\ \sigma=1/10$. Figure 1 exhibits a comparison of the exact and numerical solutions for this example with different α and T. We can see that the numerical solutions are in good agreement with the exact solutions.

From Theorem 2, we see that the convergence order of the scheme constructed here is $O(\tau+h+\sigma^2)$ when σ is small enough. We take $\tau=T/1000,\ \sigma=1/100$. Table 1 provides some numerical results of the maximum errors and the spatial convergence orders with $\alpha=1.2,\ 1.8$ respectively at $T=1.5$ computed by the formula $\log_2\frac{e_\infty(\tau,2h,\sigma)}{e_\infty(\tau,h,\sigma)}$.

Then we take $h=\tau$ and $\sigma=\tau^{\frac{1}{2}}$. Table 2 provides some numerical results of the max errors and the temporal convergence orders with $\alpha=1.2,\ 1.8$ respectively at $T=1.5$ computed by the formula $\log_2\frac{e_\infty(2\tau,2h,2^{\frac{1}{2}}\sigma)}{e_\infty(\tau,h,\sigma)}$.

Finally, we take $\tau=h=\sigma^{\frac{1}{2}}$. Table 3 provides some numerical results of the max errors and the numerical integration convergence orders with $\alpha=1.2,\ 1.8$ respectively at $T=1.5$ computed by the formula $\log_2\frac{e_\infty(4\tau,4h,2\sigma)}{e_\infty(\tau,h,\sigma)}$. The data



(a) (b)

Fig. 1 Exact and numerical solutions: (a) $\alpha = 1.4,\ T = 1.5,\ 0.8,\ 0.3$; (b) $\alpha = 1.8,\ T = 1.8,\ 1.2,\ 0.6$.

Table 1 Maximum errors and spatial convergence orders with $\alpha = 1.2,\ 1.8$ at $T = 1.5$.

h	$\alpha = 1.2$		$\alpha = 1.8$	
	$e_\infty(\tau, h, \sigma)$	Order	$e_\infty(\tau, h, \sigma)$	Order
1/20	1.8367e-2	-	3.8700e-3	-
1/40	9.5767e-3	0.9395	2.1868e-3	0.8235
1/80	4.8640e-3	0.9774	1.1552e-3	0.9207
1/160	2.4407e-3	0.9947	5.9241e-4	0.9635

Table 2 Maximum errors and temporal convergence orders with $\alpha = 1.2,\ 1.8$ at $T = 1.5$.

τ	$\alpha = 1.2$		$\alpha = 1.8$	
	$e_\infty(\tau, h, \sigma)$	Order	$e_\infty(\tau, h, \sigma)$	Order
T/100	3.8475e-3	-	9.1494e-4	-
T/200	1.9313e-3	0.9943	4.6833e-4	0.9661
T/400	9.6663e-4	0.9985	2.3699e-4	0.9762
T/800	4.8339e-4	0.9997	1.19262e-4	0.9907

Table 3 Maximum errors and numerical integration convergence orders with $\alpha = 1.2,\ 1.8$ at $T = 1.5$.

σ	$\alpha = 1.2$		$\alpha = 1.8$	
	$e_\infty(\tau, h, \sigma)$	Order	$e_\infty(\tau, h, \sigma)$	Order
1/20	1.8040e-2	-	3.7372e-3	-
1/40	4.7960e-3	1.9113	1.1300e-3	1.7257
1/80	1.2080e-3	1.9892	2.9533e-4	1.9359
1/160	3.0212e-4	1.9994	7.4736e-5	1.9825

of these three tables are computed for case 1. From these tables, we can see that the convergence order of the scheme is $O(\tau + h + \sigma^2)$ as anticipated.

Example 2. Consider the following time distributed-order and two-sided space-fractional advection-dispersion problem

$$\int_0^1 P(\beta)\,{}_0^C\mathcal{D}_t^\beta u(x,t)d\beta = -\frac{\partial u(x,t)}{\partial x} + \frac{\partial u^\alpha(x,t)}{\partial x^\alpha} + \frac{\partial u^\alpha(x,t)}{\partial(-x)^\alpha} + \sin x(1+t^2),$$
$$(x,t) \in [0,1] \times [0,5], \quad (30)$$

$$u(0,t) = 0,\ u(1,t) = 0,\ t \in [0,5], \quad (31)$$

$$u(x,0) = 10\delta(x),\ x \in [0,1], \quad (32)$$

It has been proved that when $p(\beta) = \delta(\beta - \alpha),\ 0 < \alpha < 1$, the fundamental solution of a distributed-order time-factional diffusion equation can be interpreted as a probability density with respect to the spatial variable x evolving in time t [10]. Atanackovic et al. [2] considered the solutions of the distributed-order time-factional diffusion-wave equation in the special cases of the derivative weight function $p(\beta) = \delta(\beta - \alpha),\ 0 < \alpha < 2$ and $p(\beta) = \tau^\beta$, where τ is a positive constant.

Here we also take these two cases as examples to investigate the numerical solutions of this problem. We take $P(\beta) = \delta(\beta - 0.5)$ and $P(\beta) = \tau^\beta$ respectively. Figures 2 a, b, c and d illustrate the effect of these two weight function $P(\beta)$ and the spatial fractional order α with $h = 1/40,\ \tau = 1/20,\ \sigma = 1/10$. First, we can see from these four solution profiles that different diffusion phenomena occur under different weight function $P(\beta)$ condition with fixed α. In this example, when $P(\beta) = \delta(\beta - 0.5)$, it leads to a faster diffusion than that of $P(\beta) = \tau^\beta$. This implies that we can model different complex dynamical process by choosing appropriate $P(\beta)$. Then, we can see the effect of the spatial order α with fixed $P(\beta)$. In this example, we see that increasing the value of α leads to a lower diffusion.

5 Conclusions

In view of the potential application of the distributed-order partial differential equations, in this paper, we have investigated an implicit numerical method for the time distributed-order and two-sided space-fractional advection-dispersion equation. We first discretize the integral term in the equation by applying the mid-point quadrature numerical integration method enabling the distributed-order equation to be transformed into a multi-term time-space fractional partial differential equation. Then, we solve the multi-term time-space fractional partial differential equation by an implicit difference method. The method is proved to be stable and convergent with convergence order $O(\tau^{1+\frac{\sigma}{2}} + h + \sigma^2)$. Numerical experiments show that the method is effective and simple to be implemented. And we can select appropriate weight function $P(\beta)$ to simulate different complicated dynamical process which can not be described by the fractional-order system.

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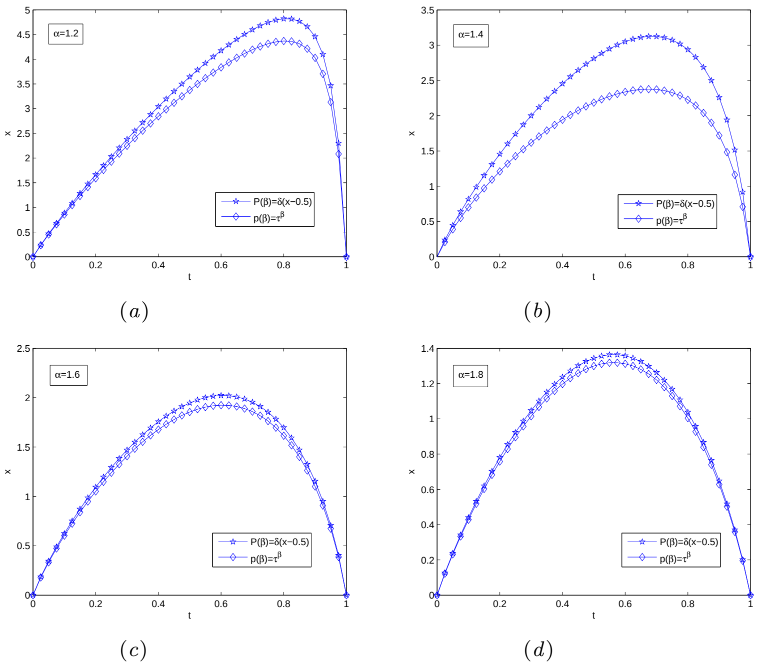

Fig. 2 Numerical solutions of Example 2 at $T = 5$ with different α and $P(\beta)$.

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