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Compact difference scheme for distributed-order time-fractional diffusion-wave equation on bounded domains

H. Ye[a], F. Liu[b,*], V. Anh[b]

[a]*Department of Applied Mathematics, Donghua University, Shanghai 201620, P. R. China*
[b]*Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Qld. 4001, Australia*

Abstract

In this paper, we derive and analyse a compact difference scheme for a distributed-order time-fractional diffusion-wave equation. The distributed-order diffusion-wave equation is approximated with a multi-term fractional diffusion-wave equation, which is then solved by a compact difference scheme. The unique solvability of the difference solution is discussed. Using discrete energy method, we prove the compact difference scheme is unconditionally stable and convergent. Finally, numerical results are presented to support our theoretical analysis.

Keywords: distributed-order fractional derivative; diffusion-wave equation; compact difference scheme; stability; convergence

1. Introduction

An important application of distributed-order equations is to model ultraslow diffusion where a plume of particles spreads at a logarithmic rate [1–3]. When the order of the fractional derivative is distributed over the unit interval, it is useful for modeling a mixture of delay sources [4]. Also, distributed-order equations may be viewed as consisting of viscoelastic and visco-inertial elements when the order of the fractional derivative varies

*Corresponding author
Email address: f.liu@qut.edu.au (F. Liu)

from zero to two [5, 6]. Motivated by these applications, some attention has been paid to the fractional partial differential equations (FPDEs) with distributed-order [7–10].

Chechkin et al. [11] proposed diffusionlike equations with time fractional derivatives of the distributed order for the kinetic description of anomalous diffusion and relaxation phenomena and proved the positivity of the solutions of their proposed equations. They demonstrated that retarding subdiffusion and accelerating superdiffusion were governed by distributed-order fractional diffusion equation [12]. The fundamental solutions for the one-dimensional time fractional diffusion equation and multi-dimensional diffusion-wave equation of distributed order were obtained by Mainardi et al. [13, 14] and Atanackovic et al. [15], respectively. Atanackovic et al. [16] also proved the existence of the solution to the Cauchy problem for the time distributed order diffusion equation and calculated it by the use of Fourier and Laplace transformations. Furthermore, they studied waves in a viscoelastic rod of finite length, where viscoelastic material was described by a constitutive equation of fractional distributed-order type (see [17]). Luchko [18] proved the uniqueness and continuous dependence on initial conditions for the generalized time-fractional diffusion equation of distributed order on bounded domains. Meerschaert et al. [4] provided explicit strong solutions and stochastic analogues for distributed-order time-fractional diffusion equations on bounded domains, with Dirichlet boundary conditions.

On the other hand, different numerical methods for solving FPDEs have been proposed [19–22]. Recently, Liu et al. [23] proposed some computationally effective numerical methods for simulating the multi-term time-fractional wave-diffusion equations. There are also some papers discussing numerical methods of the distributed-order equations. For example, Diethelm and Ford [24] developed a numerical scheme for the solution of a distributed-order FODE and gave a convergence theory for their method. Based on the matrix form representation of discretized fractional operators (see [25]), Podlubny et al. [26] extended the range of applicability of the matrix approach to discretization of distributed-order derivatives and integrals, and to numerical solution of distributed-order differential equations (both ordinary and partial). Katsikadelis [27] presented an efficient numerical method to solve linear and nonlinear distributed-order FODEs. However, published papers on the numerical methods of the distributed-order FPDEs are sparse. This motivates us to consider effective numerical methods for distributed-order time-fractional diffusion-wave equations.

In this paper, we first approximate the integral term in the distributed-order diffusion-wave equation using numerical approximation. Then the given distributed-order equation can be written as a multi-term time fractional diffusion-wave equation. We derive a compact difference scheme which is uniquely solvable for the multi-term fractional diffusion-wave equation. Using the discrete energy method, we prove the compact difference scheme is unconditionally stable and convergent. Finally, two numerical examples are provided to show the effectiveness of our method.

The rest of the paper is organized as follows. In Section 2, a compact difference scheme is derived. Section 3 presents the solvability, stability and convergence for the compact difference scheme. Two examples are given in Section 4 and some conclusions are drawn in Section 5.

2. Compact difference scheme

Consider the following distributed-order time-fractional diffusion-wave equations

$$\mathbb{D}_t^{\varpi(\alpha)} u(x,t) = K\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \tag{2.1}$$

in an open bounded domain $0 < x < L, 0 < t < T$. Here $K > 0$, x and t are the space and time variables. The time-fractional derivative $\mathbb{D}_t^{\varpi(\alpha)}$ of distributed order is defined by

$$\mathbb{D}_t^{\varpi(\alpha)} u(x,t) = \int_0^2 {}_0^cD_t^\alpha u(x,t)\varpi(\alpha)d\alpha \tag{2.2}$$

with the left-side Caputo fractional derivative ${}_0^cD_t^\alpha$ defined as (see [28])

$$ {}_0^cD_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{\partial^n u}{\partial\tau^n}(x,\tau)d\tau, & n-1<\alpha<n, \\ \frac{\partial^n u}{\partial t^n}(x,t) & , \quad \alpha = n. \end{cases} \tag{2.3}$$

and with a continuous non-negative weight function $\varpi : [0,2] \to \mathcal{R}$ that is not identically equal to zero on the interval $[0,2]$, such that the conditions

$$0 \le \varpi(\alpha), \varpi \ne 0, \alpha \in [0,2], \int_1^2 \varpi(\alpha)d\alpha = W > 0 \tag{2.4}$$

hold true, where W is a positive constant.

In this paper, the initial-boundary conditions

$$u(x,0)=\phi_1(x),\quad u_t(x,0)=\phi_2(x),\qquad 0\le x\le L, \tag{2.5}$$

$$u(0,t)=\psi_1(t),\quad u(L,t)=\psi_2(t),\qquad 0\le t\le T \tag{2.6}$$

for Eq. (2.1) are considered.

Now, we state our numerical method as follows.

Step 1: Discretize the integral term in the distributed-order equation.

Let us discretize the interval [0,2], in which the order α is changing, using the grid $0=\xi_0<\xi_1<\cdots<\xi_q=1<\xi_{q+1}<\xi_{q+2}<\cdots<\xi_{2q}=2(q\in\mathcal{N})$, with the steps $\Delta\xi_s$ not necessarily equidistant. We obtain

$$\begin{aligned}\mathbb{D}_t^{\varpi(\alpha)}u(x,t) &\approx \sum_{s=1}^{2q}\varpi(\alpha_s)\left({}_0^cD_t^{\alpha_s}u(x,t)\right)\Delta\xi_s\\ &= \sum_{s=1}^{2q}d_s\, {}_0^cD_t^{\alpha_s}u(x,t),\end{aligned} \tag{2.7}$$

where $\alpha_s\in(\xi_{s-1},\xi_s]$, $d_s=\varpi(\alpha_s)\Delta\xi_s$, $\Delta\xi_s=\xi_s-\xi_{s-1}$, $s=1,2,\cdots,2q$.

For simplicity of the presentation, but without loss of generality, we take $\Delta\xi_s=\frac{1}{q}=\sigma(q\in\mathcal{N})$ and $d_s=\frac{\varpi(\alpha_s)}{q}$. We can use the mid-point quadrature rule for approximating the integral (2.2). Let $\alpha_s=\frac{\xi_{s-1}+\xi_s}{2}=\frac{2s-1}{2q}$, $s=1,2,\cdots,2q$. Then,

$$\mathbb{D}_t^{\varpi(\alpha)}u(x,t)=\sum_{s=1}^{2q}d_s\, {}_0^cD_t^{\alpha_s}u(x,t)+R_1, \tag{2.8}$$

where $R_1=O(\sigma^2)$ (see [29]). Consider the following multi-term fractional diffusion-wave equation

$$\sum_{s=1}^{2q}d_s\left({}_0^cD_t^{\alpha_s}u(x,t)\right)+R_1=K\frac{\partial^2u(x,t)}{\partial x^2}+f(x,t), \tag{2.9}$$

with the initial-boundary conditions (2.5)-(2.6).

Step 2: Solve the multi-term equation.

We assume that we are working on a uniform grid $x_i=ih, i=0,1,\cdots,M;$ $Mh=L; t_k=k\tau, k=0,1,\cdots,N; N\tau=T$. The domain $[0,L]\times[0,T]$ is covered by $\Omega_h\times\Omega_\tau$, where $\Omega_h=\{x_i|x_i=ih, 0\le i\le M\}$ and $\Omega_\tau=\{t_k|t_k=$

$k\tau, 0 \le k \le N\}$. Suppose $u = \{u_i^k | 0 \le i \le M, 0 \le k \le N\}$ is a grid function on $\Omega_h \times \Omega_\tau$. Introduce the following notations:

$$u_i^{k-\frac{1}{2}} = \frac{1}{2}(u_i^k + u_i^{k-1}), \qquad \delta_t u_i^{k-\frac{1}{2}} = \frac{1}{\tau}(u_i^k - u_i^{k-1}),$$

$$\delta_x u_{i-\frac{1}{2}}^k = \frac{1}{h}(u_i^k - u_{i-1}^k), \qquad \delta_x^2 u_i^k = \frac{1}{h}(\delta_x u_{i+\frac{1}{2}}^k - \delta_x u_{i-\frac{1}{2}}^k).$$

Now we show a compact difference scheme for solving the multi-term equation (2.9) with the initial-boundary conditions (2.5)-(2.6).

Define, on $\Omega_h \times \Omega_\tau$, the following grid functions $U_i^k = u(x_i, t_k), f_i^k = f(x_i, t_k), 0 \le i \le M, 0 \le k \le N$. Suppose $u(x,t) \in \mathcal{C}_{x,t}^{6,3}([0,L] \times [0,T])$.

For $0 < \alpha_s < 1$, adopting the L1 discrete scheme in [30], we discretize the Caputo derivative as (see [31])

$$ {}_0^cD_t^{\alpha_s} U_i^k = \frac{\tau}{\mu_s} \sum_{j=1}^{k} a_{k-j}^{\alpha_s} \delta_t U_i^{j-\frac{1}{2}} + R_2^s(x_i, t_k), \tag{2.10}$$

where

$$a_k^{\alpha_s} = (k+1)^{1-\alpha_s} - k^{1-\alpha_s}, \qquad \mu_s = \tau^{\alpha_s}\Gamma(2-\alpha_s), \tag{2.11}$$

$$|R_2^s(x_i,t_k)| \le \frac{1}{\Gamma(2-\alpha_s)} \left[\frac{1-\alpha_s}{12} + \frac{2^{2-\alpha_s}}{2-\alpha_s} - (1+2^{-\alpha_s}) \right] \max_{0\le t \le t_k} |\frac{\partial^2 u}{\partial t^2}| \tau^{2-\alpha_s}, \tag{2.12}$$

$$s = 1, 2, \cdots, q.$$

For $1 < \alpha_s < 2$, using a fully discrete difference scheme derived by Sun and Wu [32] and noting the initial condition (2.5), we have

$$ {}_0^cD_t^{\alpha_s} U_i^{k-\frac{1}{2}} = \frac{\tau}{\bar{\mu}_s} \left[\delta_t U_i^{k-\frac{1}{2}} - \sum_{j=1}^{k-1} \left(b_{k-j-1}^{\alpha_s} - b_{k-j}^{\alpha_s} \right) \delta_t U_i^{j-\frac{1}{2}} - b_{k-1}^{\alpha_s} \phi_2(x_i) \right] + R_3^s, \tag{2.13}$$

where

$$b_k^{\alpha_s} = (k+1)^{2-\alpha_s} - k^{2-\alpha_s}, \qquad \bar{\mu}_s = \tau^{\alpha_s}\Gamma(3-\alpha_s), \tag{2.14}$$

$$|R_3^s| \le \frac{1}{\Gamma(3-\alpha_s)} \left[\frac{2-\alpha_s}{12} + \frac{2^{3-\alpha_s}}{3-\alpha_s} - (1+2^{1-\alpha_s}) + \frac{1}{12} \right] \max_{0\le t \le t_k} |\frac{\partial^3 u}{\partial t^3}| \tau^{3-\alpha_s}, \tag{2.15}$$

$$s = q+1, q+2, \cdots, 2q.$$

Considering the equation (2.9) at the point $(x_i, t_{k-\frac{1}{2}})$, it is natural to have

$$\begin{aligned}
&\sum_{s=1}^{q}\frac{\tau d_s}{2\mu_s}\left(\sum_{j=1}^{k}a_{k-j}^{\alpha_s}\delta_t U_i^{j-\frac{1}{2}}+\sum_{j=1}^{k-1}a_{k-j-1}^{\alpha_s}\delta_t U_i^{j-\frac{1}{2}}\right)\\
&+\sum_{s=q+1}^{2q}\frac{\tau d_s}{\bar{\mu}_s}\left[\delta_t U_i^{k-\frac{1}{2}}-\sum_{j=1}^{k-1}\left(b_{k-j-1}^{\alpha_s}-b_{k-j}^{\alpha_s}\right)\delta_t U_i^{j-\frac{1}{2}}-b_{k-1}^{\alpha_s}\phi_2(x_i)\right]\\
&+\frac{1}{2}\sum_{s=1}^{q}\left[R_2^s(x_i,t_k)+R_2^s(x_i,t_{k-1})\right]+\sum_{s=q+1}^{2q}R_3^s+R_1\\
=\ &\frac{K}{2}\left[U_{xx}(x_i,t_k)+U_{xx}(x_i,t_{k-1})\right]+\frac{1}{2}\left[f(x_i,t_k)+f(x_i,t_{k-1})\right]. \qquad (2.16)
\end{aligned}$$

Denote an average operator $\mathcal{P}$ as follows:

$$\mathcal{P}u_i^k=\begin{cases}\frac{1}{12}\left(u_{i-1}^k+10u_i^k+u_{i+1}^k\right), & 1\le i\le M-1,\quad 0\le k\le N\\ u_i^k, & i=0,M,\quad 0\le k\le N.\end{cases} \qquad (2.17)$$

Acting the operator $\mathcal{P}$ on both sides of (2.16) and noticing that

$$\frac{1}{2}\left[\mathcal{P}U_{xx}(x_i,t_k)+\mathcal{P}U_{xx}(x_i,t_{k-1})\right]=\delta_x^2U_i^{k-\frac{1}{2}}+O(h^4), \qquad (2.18)$$

we obtain

$$\begin{aligned}
&\sum_{s=1}^{q}\frac{\tau d_s}{2\mu_s}\mathcal{P}\left(\sum_{j=1}^{k}a_{k-j}^{\alpha_s}\delta_t U_i^{j-\frac{1}{2}}+\sum_{j=1}^{k-1}a_{k-j-1}^{\alpha_s}\delta_t U_i^{j-\frac{1}{2}}\right)\\
+\ &\sum_{s=q+1}^{2q}\frac{\tau d_s}{\bar{\mu}_s}\mathcal{P}\left[\delta_t U_i^{k-\frac{1}{2}}-\sum_{j=1}^{k-1}\left(b_{k-j-1}^{\alpha_s}-b_{k-j}^{\alpha_s}\right)\delta_t U_i^{j-\frac{1}{2}}-b_{k-1}^{\alpha_s}\phi_2(x_i)\right]\\
=\ &K\delta_x^2U_i^{k-\frac{1}{2}}+\mathcal{P}f_i^{k-\frac{1}{2}}+R_i^{k-\frac{1}{2}},\quad 1\le i\le M-1, 1\le k\le N, \qquad (2.19)
\end{aligned}$$

where

$$R_i^{k-\frac{1}{2}}=-\frac{1}{2}\sum_{s=1}^{q}\mathcal{P}\left[R_2^s(x_i,t_k)+R_2^s(x_i,t_{k-1})\right]-\sum_{s=q+1}^{2q}\mathcal{P}R_3^s+O(h^4)+O(\sigma^2),$$

and there exists a positive constant C_1 such that

$$|R_i^{k-\frac{1}{2}}|\le C_1(\tau^{2-\frac{1}{2}\sigma}/\sigma+h^4+\sigma^2). \qquad (2.20)$$

In addition, it follows from the initial and boundary value conditions that

$$U_i^0 = \phi_1(x_i), \qquad 0 \le i \le M, \tag{2.21}$$

$$U_0^k = \psi_1(t_k), \quad U_M^k = \psi_2(t_k), \qquad 0 \le k \le N. \tag{2.22}$$

Let u_i^k be the numerical approximation to $u(x_i, t_k)$. We can derive the following compact difference numerical scheme

$$\begin{aligned}
&\sum_{s=1}^{q} \frac{\tau d_s}{2\mu_s} \mathcal{P} \left(\sum_{j=1}^{k} a_{k-j}^{\alpha_s} \delta_t u_i^{j-\frac{1}{2}} + \sum_{j=1}^{k-1} a_{k-j-1}^{\alpha_s} \delta_t u_i^{j-\frac{1}{2}} \right) \\
&\quad + \sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} \mathcal{P} \left[\delta_t u_i^{k-\frac{1}{2}} - \sum_{j=1}^{k-1} \left(b_{k-j-1}^{\alpha_s} - b_{k-j}^{\alpha_s} \right) \delta_t u_i^{j-\frac{1}{2}} - b_{k-1}^{\alpha_s} \phi_2(x_i) \right] \\
&\quad = K \delta_x^2 u_i^{k-\frac{1}{2}} + \mathcal{P} f_i^{k-\frac{1}{2}}, \quad 1 \le i \le M-1, \ \ 1 \le k \le N, \qquad (2.23) \\
&u_i^0 = \phi_1(x_i), \qquad 0 \le i \le M, \qquad (2.24) \\
&u_0^k = \psi_1(t_k), \quad u_M^k = \psi_2(t_k), \qquad 0 \le k \le N. \qquad (2.25)
\end{aligned}$$

3. Analysis of the compact difference scheme

3.1. Solvability

It is clear to see that at each time level, (2.23)-(2.25) is a linear tridiagonal system that need to be solved. Since the coefficient matrix is strictly diagonally dominant, the difference scheme (2.23)-(2.25) has a unique solution. This can be written as the following result.

Theorem 3.1. *The compact difference scheme (2.23)-(2.25) is uniquely solvable.*

3.2. Stability

Denote the grid function space on Ω_h by $\mathcal{V}_h = \{v | v = (v_0, v_1, \cdots, v_{M-1}, v_M), v_0 = v_M = 0\}$. For any $u, v \in \mathcal{V}_h$, we define the discrete inner product

$$(u, v) = h \sum_{i=1}^{M-1} u_i v_i$$

and denote L_2 norm $\|u\| = \sqrt{(u,u)}$. The H^1 seminorms $|\cdot|_1, \|\cdot\|_A$ and the maximum norm $\|\cdot\|_\infty$ are as follows:

$$\langle \delta_x u, \delta_x v\rangle = h\sum_{i=0}^{M-1}\left(\delta_x u_{i+\frac{1}{2}}\right)\left(\delta_x v_{i+\frac{1}{2}}\right), |u|_1 = \sqrt{\langle \delta_x u, \delta_x u\rangle}, \|u\|_\infty = \max_{0\le i\le M}|u_i|$$

$$\langle \delta_x u, \delta_x v\rangle_A = \langle \delta_x u, \delta_x v\rangle - \frac{h^2}{12}(\delta_x^2 u, \delta_x^2 v), \qquad \|\delta_x u\|_A = \sqrt{\langle \delta_x u, \delta_x u\rangle_A}.$$

Lemma 3.1. *If the grid function* $\{v_i^k | 0 \le i \le M, 0 \le k \le N\}$ *satisfies* $v_0^k = 0, v_M^k = 0, 0 \le k \le N$, *then*

$$-\left(\delta_x^2 v^{k-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}}\right) = \frac{1}{2\tau}\left(\|\delta_x v^k\|_A^2 - \|\delta_x v^{k-1}\|_A^2\right), \quad 1 \le k \le N.$$

PROOF. See Lemma 4.2 in [33].

Lemma 3.2. *[34] Let* $\{a_0, a_1, \cdots, a_n, \cdots\}$ *be a sequence of real numbers with the properties*

$$a_n \ge 0, \quad a_n - a_{n-1} \le 0, \quad a_{n+1} - 2a_n + a_{n-1} \ge 0.$$

Then for any positive integer M, *and for each vector* $(V_1, V_2, \cdots, V_M)$ *with* M *real entries,*

$$\sum_{n=1}^{M}\left(\sum_{p=0}^{n-1} a_p V_{n-p}\right) V_n \ge 0.$$

Theorem 3.2. *Let* $\{v_i^k | 0 \le i \le M, 0 \le k \le N\}$ *be the solution of the following difference system*

$$\sum_{s=1}^{q}\frac{\tau d_s}{2\mu_s}\mathcal{P}\left(\sum_{j=1}^{k} a_{k-j}^{\alpha_s}\delta_t v_i^{j-\frac{1}{2}} + \sum_{j=1}^{k-1} a_{k-j-1}^{\alpha_s}\delta_t v_i^{j-\frac{1}{2}}\right)$$
$$+ \sum_{s=q+1}^{2q}\frac{\tau d_s}{\bar{\mu}_s}\mathcal{P}\left[\delta_t v_i^{k-\frac{1}{2}} - \sum_{j=1}^{k-1}\left(b_{k-j-1}^{\alpha_s} - b_{k-j}^{\alpha_s}\right)\delta_t v_i^{j-\frac{1}{2}} - b_{k-1}^{\alpha_s}\phi_2(x_i)\right]$$
$$= K\delta_x^2 v_i^{k-\frac{1}{2}} + g_i^{k-\frac{1}{2}}, \quad 1 \le i \le M-1, \quad 1 \le k \le N, \tag{3.1}$$

$$v_i^0 = \phi_1(x_i), \qquad 0 \le i \le M, \tag{3.2}$$

$$v_0^k = 0, \quad v_M^k = 0, \qquad 0 \le k \le N. \tag{3.3}$$

Then it holds that

$$
\begin{aligned}
\|\delta_x v^k\|_A^2 &\le \|\delta_x \phi_1\|_A^2 + \frac{\tau}{K} t_k^{1-\frac{1}{2q}} \frac{1}{\sum_{s=q+1}^{2q} \frac{\varpi(\alpha_s)}{q\Gamma(2-\alpha_s)}} \sum_{l=1}^{k} \|g^{l-\frac{1}{2}}\|^2 \\
&+ \frac{1}{K} t_k^{1-\frac{1}{2q}} \sum_{s=q+1}^{2q} \frac{\varpi(\alpha_s)}{q\Gamma(3-\alpha_s)} \|\mathcal{P}\phi_2\|^2, \qquad 1 \le k \le N. \quad (3.4)
\end{aligned}
$$

Proof. Taking the inner product of (3.1) with $\mathcal{P}\delta_t v^{k-\frac{1}{2}}$, we obtain

$$
\begin{aligned}
&\sum_{s=1}^{q} \frac{\tau d_s}{2\mu_s} \left[\sum_{j=1}^{k} a_{k-j}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) + \sum_{j=1}^{k-1} a_{k-j-1}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) \right] \\
+ &\sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} \Big[\left(\mathcal{P}\delta_t v^{k-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) - \sum_{j=1}^{k-1} \left(b_{k-j-1}^{\alpha_s} - b_{k-j}^{\alpha_s} \right) \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) \\
- &b_{k-1}^{\alpha_s} \left(\mathcal{P}\phi_2(x), \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) \Big] = K(\delta_x^2 v^{k-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}}) + (g^{k-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}}).
\end{aligned}
$$

Applying Lemma 3.1 and Cauchy-Schwarz inequality, noticing that both $b_{k-1}^{\alpha_s}$ and $b_{k-j-1}^{\alpha_s} - b_{k-j}^{\alpha_s}$ are positive, we have

$$
\begin{aligned}
&\sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} \|\mathcal{P}\delta_t v^{k-\frac{1}{2}}\|^2 + \frac{K}{2\tau} \left(\|\delta_x v^k\|_A^2 - \|\delta_x v^{k-1}\|_A^2 \right) \\
= &\sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} \left[\sum_{j=1}^{k-1} \left(b_{k-j-1}^{\alpha_s} - b_{k-j}^{\alpha_s} \right) \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) \right] \\
+ &\sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} b_{k-1}^{\alpha_s} \left(\mathcal{P}\phi_2(x), \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) + (g^{k-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}}) \\
- &\sum_{s=1}^{q} \frac{\tau d_s}{2\mu_s} \left[\sum_{j=1}^{k} a_{k-j}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) + \sum_{j=1}^{k-1} a_{k-j-1}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) \right] \\
\le &\sum_{s=q+1}^{2q} \frac{\tau d_s}{2\bar{\mu}_s} \left[\sum_{j=1}^{k-1} \left(b_{k-j-1}^{\alpha_s} - b_{k-j}^{\alpha_s} \right) \left(\|\mathcal{P}\delta_t v^{j-\frac{1}{2}}\|^2 + \|\mathcal{P}\delta_t v^{k-\frac{1}{2}}\|^2 \right) \right] \\
+ &\sum_{s=q+1}^{2q} \frac{\tau d_s}{2\bar{\mu}_s} b_{k-1}^{\alpha_s} \left(\|\mathcal{P}\phi_2\|^2 + \|\mathcal{P}\delta_t v^{k-\frac{1}{2}}\|^2 \right) + |(g^{k-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}})|
\end{aligned}
$$

$$- \sum_{s=1}^{q} \frac{\tau d_s}{2\mu_s} \left[\sum_{j=1}^{k} a_{k-j}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) + \sum_{j=1}^{k-1} a_{k-j-1}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) \right].$$

It follows that

$$\begin{aligned}
& \sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} \|\mathcal{P}\delta_t v^{k-\frac{1}{2}}\|^2 + \frac{K}{\tau} \left(\|\delta_x v^k\|_A^2 - \|\delta_x v^{k-1}\|_A^2 \right) \\
\le & \sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} \left[\sum_{j=1}^{k-1} \left(b_{k-j-1}^{\alpha_s} - b_{k-j}^{\alpha_s} \right) \|\mathcal{P}\delta_t v^{j-\frac{1}{2}}\|^2 \right] \\
+ & \sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} b_{k-1}^{\alpha_s} \|\mathcal{P}\phi_2\|^2 + 2|(g^{k-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}})| \\
- & \sum_{s=1}^{q} \frac{\tau d_s}{\mu_s} \left[\sum_{j=1}^{k} a_{k-j}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) + \sum_{j=1}^{k-1} a_{k-j-1}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) \right].
\end{aligned}$$

Denote $F^0 = \frac{K}{\tau} \|\delta_x v^0\|_A^2$ and

$$F^k = \frac{K}{\tau} \|\delta_x v^k\|_A^2 + \sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} \left(\sum_{j=1}^{k} b_{k-j}^{\alpha_s} \|\mathcal{P}\delta_t v^{j-\frac{1}{2}}\|^2 \right).$$

Then,

$$\begin{aligned}
F^k \le & \; F^{k-1} + \sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} b_{k-1}^{\alpha_s} \|\mathcal{P}\phi_2\|^2 + 2|(g^{k-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}})| \\
- & \sum_{s=1}^{q} \frac{\tau d_s}{\mu_s} \left[\sum_{j=1}^{k} a_{k-j}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) + \sum_{j=1}^{k-1} a_{k-j-1}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{k-\frac{1}{2}} \right) \right].
\end{aligned}$$

Replacing k by l and summing up for l from 1 to k, we obtain

$$\begin{aligned}
F^k \le & \; F^0 + \sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} \sum_{l=1}^{k} b_{l-1}^{\alpha_s} \|\mathcal{P}\phi_2\|^2 \\
+ & \sum_{l=1}^{k} \frac{1}{\sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} b_{k-l}^{\alpha_s}} \|g^{l-\frac{1}{2}}\|^2 + \sum_{l=1}^{k} \left(\sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} b_{k-l}^{\alpha_s} \right) \|\mathcal{P}\delta_t v^{l-\frac{1}{2}}\|^2 \\
- & \sum_{s=1}^{q} \frac{\tau d_s}{\mu_s} \sum_{l=1}^{k} \left[\sum_{j=1}^{l} a_{l-j}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{l-\frac{1}{2}} \right) + \sum_{j=1}^{l-1} a_{l-j-1}^{\alpha_s} \left(\mathcal{P}\delta_t v^{j-\frac{1}{2}}, \mathcal{P}\delta_t v^{l-\frac{1}{2}} \right) \right].
\end{aligned}$$

An application of Lemma 3.2 yields

$$\sum_{s=1}^{q}\frac{\tau d_s}{\mu_s}\sum_{l=1}^{k}\left[\sum_{j=1}^{l}a_{l-j}^{\alpha_s}\left(\mathcal{P}\delta_t v^{j-\frac{1}{2}},\mathcal{P}\delta_t v^{l-\frac{1}{2}}\right)+\sum_{j=1}^{l-1}a_{l-j-1}^{\alpha_s}\left(\mathcal{P}\delta_t v^{j-\frac{1}{2}},\mathcal{P}\delta_t v^{l-\frac{1}{2}}\right)\right]\geq 0.$$

Therefore,

$$\begin{aligned}\|\delta_x v^k\|_A^2 &\leq \|\delta_x\phi_1\|_A^2+\frac{1}{K}\sum_{l=1}^{k}\frac{\tau}{\sum_{s=q+1}^{2q}\frac{\tau d_s}{\bar{\mu}_s}b_{k-l}^{\alpha_s}}\|g^{l-\frac{1}{2}}\|^2\\ &+\frac{1}{K}\sum_{s=q+1}^{2q}\frac{\tau^2 d_s}{\bar{\mu}_s}\sum_{l=1}^{k}b_{l-1}^{\alpha_s}\|\mathcal{P}\phi_2\|^2, \qquad 1\leq k\leq N.\end{aligned}$$

Observing that $b_{k-l}^{\alpha_s}\geq(2-\alpha_s)(k-l+1)^{1-\alpha_s}\geq(2-\alpha_s)k^{1-\alpha_s}$, we have

$$\sum_{s=q+1}^{2q}\frac{\tau d_s}{\bar{\mu}_s}b_{k-l}^{\alpha_s}\geq\sum_{s=q+1}^{2q}t_k^{1-\alpha_s}\frac{\varpi(\alpha_s)}{q\Gamma(2-\alpha_s)}\geq t_k^{\frac{1}{2q}-1}\sum_{s=q+1}^{2q}\frac{\varpi(\alpha_s)}{q\Gamma(2-\alpha_s)}.$$

Also, since $\sum_{l=1}^{k}b_{l-1}^{\alpha_s}=k^{2-\alpha_s}$, it follows that

$$\begin{aligned}&\frac{1}{K}\sum_{s=q+1}^{2q}\frac{\tau^2 d_s}{\bar{\mu}_s}\sum_{l=1}^{k}b_{l-1}^{\alpha_s}\|\mathcal{P}\phi_2\|^2=\frac{1}{K}\sum_{s=q+1}^{2q}t_k^{2-\alpha_s}\frac{\varpi(\alpha_s)}{q\Gamma(3-\alpha_s)}\|\mathcal{P}\phi_2\|^2\\ \leq\ &\frac{1}{K}t_k^{1-\frac{1}{2q}}\sum_{s=q+1}^{2q}\frac{\varpi(\alpha_s)}{q\Gamma(3-\alpha_s)}\|\mathcal{P}\phi_2\|^2.\end{aligned}$$

Finally, we obtain the inequality (3.4) and the theorem is proved.

Using Theorem 3.1, we obtain the following stability statement.

Theorem 3.3. *The compact difference numerical scheme (2.23)-(2.25) is unconditionally stable to the initial values $\phi_1(x)$ and $\phi_2(x)$ and the forcing term f.*

3.3. Convergence

We now consider the convergence of the difference approximation. Noticing that U_i^k is the exact solution of the system (2.1), (2.5)-(2.6) and u_i^k is the numerical solution of the compact difference approximation (2.23)-(2.25), we denote the error

$$e_i^k = U_i^k - u_i^k, \qquad 0 \le i \le M, \quad 0 \le k \le N.$$

Subtracting (2.23)-(2.25) from (2.19),(2.21)-(2.22), we obtain the error equations

$$\begin{aligned}
&\sum_{s=1}^{q} \frac{\tau d_s}{2\mu_s} \mathcal{P} \left(\sum_{j=1}^{k} a_{k-j}^{\alpha_s} \delta_t e_i^{j-\frac{1}{2}} + \sum_{j=1}^{k-1} a_{k-j-1}^{\alpha_s} \delta_t e_i^{j-\frac{1}{2}} \right) \\
&\quad + \sum_{s=q+1}^{2q} \frac{\tau d_s}{\bar{\mu}_s} \mathcal{P} \left[\delta_t e_i^{k-\frac{1}{2}} - \sum_{j=1}^{k-1} \left(b_{k-j-1}^{\alpha_s} - b_{k-j}^{\alpha_s} \right) \delta_t e_i^{j-\frac{1}{2}} \right] \\
&\quad = K \delta_x^2 e_i^{k-\frac{1}{2}} + R_i^{k-\frac{1}{2}}, \quad 1 \le i \le M-1, 1 \le k \le N, \qquad (3.5)\\
&e_i^0 = 0, \qquad 0 \le i \le M, \qquad (3.6)\\
&e_0^k = 0, \quad e_M^k = 0, \qquad 0 \le k \le N. \qquad (3.7)
\end{aligned}$$

Theorem 3.2 implies the error satisfies

$$\|\delta_x e^k\|_A^2 \le \frac{\tau}{K} t_k^{1-\frac{1}{2q}} \frac{1}{\sum_{s=q+1}^{2q} \frac{\varpi(\alpha_s)}{q\Gamma(2-\alpha_s)}} \sum_{l=1}^{k} \|R^{l-\frac{1}{2}}\|^2, \qquad 1 \le k \le N. \qquad (3.8)$$

Since

$$|R_i^{k-\frac{1}{2}}| \le C_1(\tau^{2-\frac{1}{2}\sigma}/\sigma + h^4 + \sigma^2),$$

and

$$\sum_{s=q+1}^{2q} \frac{\varpi(\alpha_s)}{q\Gamma(2-\alpha_s)} \to \int_1^2 \frac{\varpi(\alpha)}{\Gamma(2-\alpha)} d\alpha = C_2 > 0,$$

there exists a positive C, such that

$$\|\delta_x e^k\|_A^2 \le C T^{2-\frac{1}{2}\sigma} (\tau^{2-\frac{1}{2}\sigma}/\sigma + h^4 + \sigma^2)^2.$$

To proceed further, we need the following lemmas.

Lemma 3.3. *[31] For any mesh function $u \in \mathcal{V}_h$, it holds that*

$$\frac{2}{3}|u|_1^2 \le \|\delta_x u\|_A^2 \le |u|_1^2.$$

Lemma 3.4. *[35] For any mesh function $u \in \mathcal{V}_h$, it holds that*

$$\|u\|_\infty \leq \frac{\sqrt{L}}{2}|u|_1.$$

Now, we can state the following result.

Theorem 3.4. *Suppose that the continuous problem (2.1), (2.5)-(2.6) has a smooth solution $u(x,t) \in C^{6,3}_{x,t}(\Omega)$, and let u_i^k be the solution of the difference scheme (2.23)-(2.25). If $\tau^{2-\frac{1}{2}\sigma} = o(\sigma)$, then the solution u_i^k converges to $u(x_i, t_k)$ as h, τ and σ tend to zero. Furthermore, there is a positive constant C such that the error satisfies*

$$\|e^k\|_\infty \leq \frac{1}{4}\sqrt{6LCT^{2-\frac{1}{2}\sigma}}(\tau^{2-\frac{1}{2}\sigma}/\sigma + h^4 + \sigma^2), \quad 0 \leq k \leq N.$$

4. Numerical results

In order to illustrate the behaviour of our numerical method and demonstrate the effectiveness of our theoretical analysis, two examples are now presented.

Example 1. *Consider the following distributed-order diffusion-wave equation:*

$$\int_0^2 \nu^\alpha \, {}_0^cD_t^\alpha u(x,t) d\alpha = K\frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1, 0 < t \leq T, \tag{4.1}$$

where ν is a positive constant that can be physically interpreted as the relaxation time, K is also a positive constant representing the diffusion coefficient. Here, the initial-boundary conditions

$$u(x,0) = x^2(1-x^2), \quad u_t(x,0) = 0, \quad 0 \leq x \leq 1, \tag{4.2}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad 0 \leq t \leq T \tag{4.3}$$

for Eq. (4.1) are considered.

Using the numerical method described in Sec. 2, we obtain the numerical solutions (Fig.1) of the fractional diffusion equation for $K = 1, T = 1.5$, $\nu = 0.3, 0.6, 0.9$ respectively, with $h = 0.02, \tau = 0.015, \sigma = 0.1$.

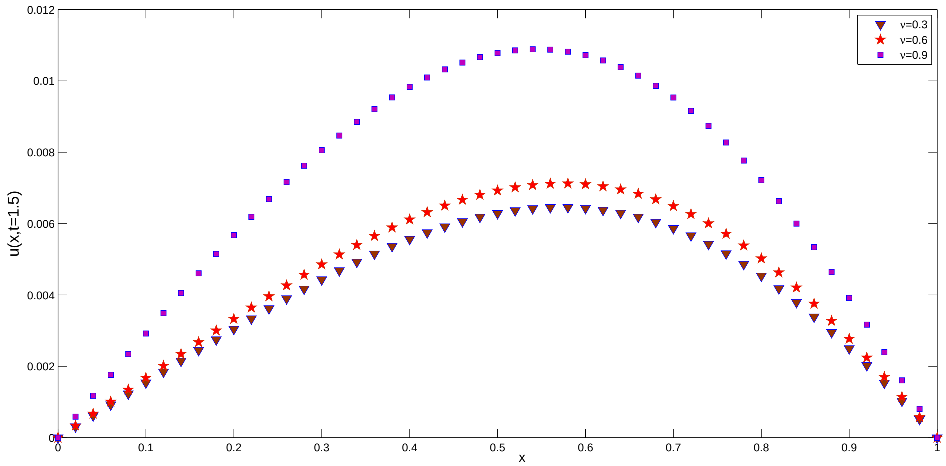

Fig. 1. The numerical approximation of u(x,t) for the system (4.1)−(4.3) when ν=0.3, 0.6, 0.9, respectively.

Example 2. *Consider the following distributed-order time-fractional diffusion-wave equation:*

$$\begin{cases} \int_0^2 \Gamma(4-\alpha)\,{}_0^cD_t^\alpha u(x,t)d\alpha = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ \qquad\qquad\qquad\qquad\qquad 0 < x < \pi, \quad 0 < t \le T, \\ u(x,0) = 4sinx, \quad u_t(x,0) = 2sinx, \quad 0 \le x \le \pi, \\ u(0,t) = u(\pi,t) = 0, \quad 0 \le t \le T, \end{cases} \tag{4.4}$$

where

$$f(x,t) = sinx\left[t^3 + 2t + 4 + \frac{6t^3 + 6t - 4}{logt} + \frac{6 - 10t}{(logt)^2} + \frac{4t - 4}{(logt)^3}\right].$$

The exact solution of the above problem is $u(x,t) = (t^3 + 2t + 4)sinx$.

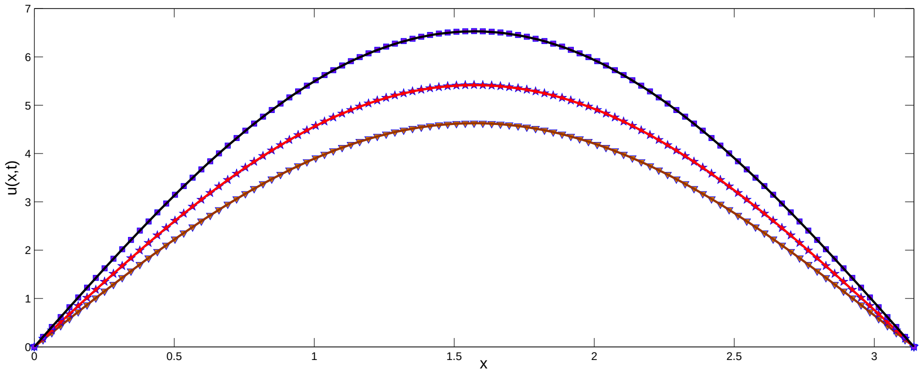

Fig. 2. Exact solutions (lines) and numerical solutions (symbols) at t=0.3 (triangles), t=0.6 (stars) and t=0.9 (squares).

A comparison of the exact solution and the numerical solution with $h = \pi/100, \tau = 0.01, \sigma = 0.1$ at $t = 0.3$ (triangles), $t = 0.6$ (stars) and $t = 0.9$ (squares) is shown in Fig. 2. From Fig. 2, it can be seen that the numerical solution is in good agreement with the exact solution.

5. Conclusion

In this paper, a compact difference scheme for the distributed-order time-fractional diffusion-wave equations on bounded domains has been described. We prove that the compact difference scheme is stable and convergent. Two numerical examples demonstrate the effectiveness of theoretical results.

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