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# A numerical method for solving two-dimensional distributed order space-fractional diffusion equation on an irregular domain

Wenping Fan[a], Fawang Liu[b,*], Xiaoyun Jiang[c]

[a]*School of Science, Jiangnan University, Wuxi 214122, China*
[b]*School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, QLD 4001, Australia*
[c]*School of Mathematics, Shandong University, Jinan 250100, China*

**Abstract**

In this paper, the two-dimensional distributed order space-fractional diffusion equation on an irregular domain is considered. The finite element method using unstructured mesh adapted to the irregular domain is proposed. To testify the efficiency of the proposed method, two numerical examples are given. By the the error analysis and the comparison between the numerical solution and the exact solution, the finite element method applied in this paper is shown to be valid in solving the two-dimensional distributed order space-fractional diffusion equation on an irregular domain.

*Keywords:* distributed order space-fractional diffusion equation, finite element method, irregular domain

## 1. Introduction

Fractional calculus has been widely used in various fields, such as physics, chemical, biology, medical science, control system, finance, and so on. Research on fractional partial differential equation (FPDE) with time- or space-fractional derivative and its applications have become the focus of extensive researches. The related literature is rich. Most recently, distributed order FPDEs, analyzed by Chechkin et al. in the year of 2002 [1], has been successfully used to describe several problems in mathematical physics and engineering. Many researchers have payed much attention to this topic. Atanackovic et al. [2] studied the distributed-order time fractional diffusion-wave equation using analytical method. Katsikadelis [3] approximated the distributed order FDE with a multi-term FDE and then developed a numerical method for solving distributed order time-fractional differential equations of general form. Mashayekhi et al. [4] presented a numerical method based upon hybrid functions approximation for solving the distributed order time-fractional differential equations. Li and Wu [5] proposed a numerical method based on the reproducing kernel method for solving distributed order time-fractional diffusion equations.

In most existing literatures, attention is mainly focused on the analytical or numerical analysis for distributed order time-fractional PDEs. However, as for the distributed order space-fractional diffusion equations, the development for numerical methods to solve distributed order space-fractional diffusion equations is still an important issue.

In this paper, we consider the following two-dimensional distributed order space-fractional diffusion equation (2D-DO-SFDE) on an irregular convex domain:

$$\frac{\partial u}{\partial t} = \int_1^2 P(\alpha)\frac{\partial^\alpha u}{\partial |x|^\alpha} + Q(\alpha)\frac{\partial^\alpha u}{\partial |y|^\alpha} d\alpha + f(x, y, t), \ (x, y, t) \in \Omega \times [0, T], \tag{1.1}$$

subject to the initial condition

$$u(x, y, 0) = \psi(x, y), \ (x, y) \in \Omega, \tag{1.2}$$

*Corresponding author.
*Email addresses:* wenping0218@126.com (Wenping Fan), f.liu@qut.edu.au (Fawang Liu)
*URL:* wqjxyf@sdu.edu.cn (Xiaoyun Jiang)

and zero Dirichlet boundary condition

$$u(x, y, t) = 0, \ (x, y) \in \partial\Omega, \tag{1.3}$$

where an irregular convex domain $\Omega$ is defined as $\Omega = \{(x, y)|a(y) \le x \le b(y), c(x) \le y \le d(x)\}$, where $a(y), b(y)$ are the left and right boundaries of $\Omega$, and $c(x), d(x)$ are the lower and upper boundaries of $\Omega$. $P(\alpha)$ and $Q(\alpha)$ are two non-negative weight functions satisfying the conditions

$$P(\alpha), Q(\alpha) \ge 0, P(\alpha) \not\equiv 0, Q(\alpha) \not\equiv 0, \tag{1.4}$$

$$\alpha \in (1, 2), 0 < \int_1^2 P(\alpha)d\alpha < \infty, 0 < \int_1^2 Q(\alpha)d\alpha < \infty. \tag{1.5}$$

The Riesz space fractional derivatives $\frac{\partial^\alpha u}{\partial |x|^\alpha}$ and $\frac{\partial^\alpha u}{\partial |y|^\alpha}$ are defined by [6]

$$\frac{\partial^\alpha u(x, y)}{\partial |x|^\alpha} = -c_\alpha \left( {}_xD_L^\alpha u(x, y) + {}_xD_R^\alpha u(x, y)\right), \tag{1.6}$$

$$\frac{\partial^\alpha u(x, y)}{\partial |y|^\alpha} = -c_\alpha \left( {}_yD_L^\alpha u(x, y) + {}_yD_R^\alpha u(x, y)\right), \tag{1.7}$$

where $c_\alpha = \frac{1}{2\cos(\frac{\alpha\pi}{2})}$, and the Riemann-Liouville fractional derivative operators with $n - 1 < \alpha < n$ are defined as

$${}_xD_L^\alpha u(x, y) = \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_{a(y)}^{x} (x - s)^{n-\alpha-1} u(s, y)ds, \tag{1.8}$$

$${}_xD_R^\alpha u(x, y) = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_{x}^{b(y)} (s - x)^{n-\alpha-1} u(s, y)ds. \tag{1.9}$$

With respect to $y$, the fractional derivative operators can be defined similarly.

## 2. Preliminaries

In this section, we need to recall some theories that has been studied previously by Ervin and Roop [7, 8], Zhu *et al.* [9] and Bu *et al.* [10, 11]. As for a convex domain $\Omega \subset \mathbb{R}^2$, resulting from its irregularity, with $x_{min} = \min\limits_{(x,y)\in\Omega} a(y)$, $x_{max} = \max\limits_{(x,y)\in\Omega} b(y)$, $y_{min} = \min\limits_{(x,y)\in\Omega} c(x)$ and $y_{max} = \max\limits_{(x,y)\in\Omega} d(x)$, we denote the inner product and $L^2$-norm as

$$\begin{aligned}(u, v)_{L^2(\Omega)} := \int_\Omega uv d\Omega &= \int_{y_{min}}^{y_{max}} \int_{a(y)}^{b(y)} u(x, y)v(x, y)dxdy, \\ &= \int_{x_{min}}^{x_{max}} \int_{c(x)}^{d(x)} u(x, y)v(x, y)dydx,\end{aligned} \tag{2.10}$$

$$||u||_{L^2(\Omega)} = ((u, u)_{L^2(\Omega)})^{1/2}. \tag{2.11}$$

**Definition 1.** *([9, 8, 10]) (Left fractional derivative space). For $\mu > 0$, we define the semi-norm*

$$|u|_{J_L^\mu(\Omega)} := \left( \left\| {}_xD_L^\mu u \right\|_{L^2(\Omega)}^2 + \left\| {}_yD_L^\mu u \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \tag{2.12}$$

*and norm*

$$||u||_{J_L^\mu(\Omega)} := \left( ||u||_{L^2(\Omega)}^2 + |u|_{J_L^\mu(\Omega)}^2 \right)^{\frac{1}{2}}, \tag{2.13}$$

*where $J_L^\mu(\Omega), J_{L,0}^\mu(\Omega)$ denote the closure of $C^\infty(\Omega), C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{J_L^\mu(\Omega)}$.*

The Right fractional derivative space $(|u|_{J_R^\mu(\Omega)}, ||u||_{J_R^\mu(\Omega)})$, the Fractional Sobolev space $(|u|_{H^\mu(\Omega)}, ||u||_{H^\mu(\Omega)})$ and the Symmetric fractional derivative space $(|u|_{J_S^\mu(\Omega)}, ||u||_{J_S^\mu(\Omega)})$ can also be defined similarly.

**Lemma 1.** *([12]) If $\mu \in (1,2), u, v \in J^{\mu}_{L,0}(\Omega)$ (or $J^{\mu}_{R,0}(\Omega)$), then*

$$({}_xD^{\mu}_L u, v)_{L^2(\Omega)} = ({}_xD^{\mu/2}_L u,\ {}_xD^{\mu/2}_R v)_{L^2(\Omega)},\ ({}_yD^{\mu}_L u, v)_{L^2(\Omega)} = ({}_yD^{\mu/2}_L u,\ {}_yD^{\mu/2}_R v)_{L^2(\Omega)},$$
$$({}_xD^{\mu}_R u, v)_{L^2(\Omega)} = ({}_xD^{\mu/2}_R u,\ {}_xD^{\mu/2}_L v)_{L^2(\Omega)},\ ({}_yD^{\mu}_R u, v)_{L^2(\Omega)} = ({}_yD^{\mu/2}_R u,\ {}_yD^{\mu/2}_L v)_{L^2(\Omega)}.$$

**Lemma 2.** *([7]) If $\mu > 0$, then*

$$({}_xD^{\mu}_L u,\ {}_xD^{\mu}_R u)_{L^2(\Omega)} = \cos(\pi\mu)\|\,{}_xD^{\mu}_L u\|^2_{L^2(\Omega)}, \tag{2.14}$$
$$({}_yD^{\mu}_L u,\ {}_yD^{\mu}_R u)_{L^2(\Omega)} = \cos(\pi\mu)\|\,{}_yD^{\mu}_L u\|^2_{L^2(\Omega)}. \tag{2.15}$$

The proofs of the lemmas can be found in the corresponding references by considering $u$ to be a zero-extension outside the domain $\Omega$. Throughout the proceeding sections, we denote $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$, $\|\cdot\|_0 = \|\cdot\|_{L^2(\Omega)}$.

## 3. Numerical method

The interval of fractional order $\alpha \in (1,2)$ can be discretized by the grid $1 = \xi_0 < \xi_1 < \cdots < \xi_S = 2$, $S \in \mathbb{N}$. We denote $\Delta\xi_m = \xi_m - \xi_{m-1} = 1/S = \sigma$, and $\alpha_m = \frac{\xi_m+\xi_{m-1}}{2} = 1 + \frac{2m-1}{2S}$. Then the space distributed-order fractional item of the Eq.(1.1) can be rewrote by using the mid-point quadrature rule as

$$\begin{aligned}\int_1^2 P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha} + Q(\alpha)\frac{\partial^\alpha u}{\partial|y|^\alpha}d\alpha &= \sum_{m=1}^{S}\int_{\xi_{m-1}}^{\xi_m} P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha} + Q(\alpha)\frac{\partial^\alpha u}{\partial|y|^\alpha}d\alpha \\ &= \sum_{m=1}^{S}\left[P(\alpha_m)\frac{\partial^{\alpha_m} u}{\partial|x|^{\alpha_m}} + Q(\alpha_m)\frac{\partial^{\alpha_m} u}{\partial|y|^{\alpha_m}}\right]\cdot\Delta\xi_m + O(\sigma^2),\end{aligned} \tag{3.16}$$

Then Eq.(1.1) can be approximated as

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{S}\left[P(\alpha_m)\frac{\partial^{\alpha_m} u}{\partial|x|^{\alpha_m}} + Q(\alpha_m)\frac{\partial^{\alpha_m} u}{\partial|y|^{\alpha_m}}\right]\cdot\Delta\xi_m + f(x,y,t) + O(\sigma^2), \tag{3.17}$$

By Eq.(1.6) and Eq.(7), we have

$$\frac{\partial u}{\partial t} + \sum_{m=1}^{S} c_{\alpha_m}\Big[P(\alpha_m)\Big({}_xD^{\alpha_m}_L u + {}_xD^{\alpha_m}_R u\Big) + Q(\alpha_m)({}_yD^{\alpha_m}_L u + {}_yD^{\alpha_m}_R u)\Big]\sigma = f(x,y,t) + O(\sigma^2), \tag{3.18}$$

Let $\tau = T/N$ is the time step, $t_n = n\tau, n = 0,1,...,N$. Denote $u(x,y,t_n) = u^n$, $u^{n-\frac{1}{2}} = \frac{u^n+u^{n-1}}{2}$. By using central difference scheme,

$$\frac{\partial u}{\partial t}\Big|_{t_{n-\frac{1}{2}}} = \frac{u^n - u^{n-1}}{\tau} + O(\tau^2) \tag{3.19}$$

Denote $\overline{\partial_t}u^{n-\frac{1}{2}} = \frac{u^n-u^{n-1}}{\tau}$, we have

$$\overline{\partial_t}u^{n-\frac{1}{2}} + \sigma\sum_{m=1}^{S} c_{\alpha_m}\Big[P(\alpha_m)\Big({}_xD^{\alpha_m}_L u^{n-\frac{1}{2}} + {}_xD^{\alpha_m}_R u^{n-1/2}\Big) + Q(\alpha_m)({}_yD^{\alpha_m}_L u^{n-\frac{1}{2}} + {}_yD^{\alpha_m}_R u^{n-\frac{1}{2}})\Big] = f^{n-\frac{1}{2}} + O(\sigma^2+\tau^2), \tag{3.20}$$

Then by Lemma 1, we obtain the variational formulation of problem (1.1)-(1.3): to find $u^n \in V$, such that

$$(\overline{\partial_t}u^{n-\frac{1}{2}}, v) + \sigma\sum_{m=1}^{S} B_m(u^{n-\frac{1}{2}}, v) = (f^{n-\frac{1}{2}}, v),\ \forall v \in V, \tag{3.21}$$
$$(u^0, v) = (\psi, v),\ \forall v \in V, \tag{3.22}$$

where $V = H_0^{\lambda}(\Omega)$, $\lambda = \max\limits_{1\le m\le S}\{\frac{\alpha_m}{2}\}$, and the bilinear form

$$\begin{aligned} B_m(u,v) =& c_{\alpha_m} P(\alpha_m)\Big[({}_xD_L^{\frac{\alpha_m}{2}} u,\ {}_xD_R^{\frac{\alpha_m}{2}} v) + ({}_xD_R^{\frac{\alpha_m}{2}} u,\ {}_xD_L^{\frac{\alpha_m}{2}} v)\Big] \\ &+ c_{\alpha_m} Q(\alpha_m)\Big[({}_yD_L^{\frac{\alpha_m}{2}} u,\ {}_yD_R^{\frac{\alpha_m}{2}} v) + ({}_yD_R^{\frac{\alpha_m}{2}} u,\ {}_yD_L^{\frac{\alpha_m}{2}} v)\Big] \end{aligned} \tag{3.23}$$

Assume that $\{\mathcal{T}_h\}$ is a family of unstructured triangulations of domain $\Omega$ and $h$ is the maximum diameter of the triangular elements in $\mathcal{T}_h$. The conforming finite element space $V_h \in V$ is defined as

$$V_h = \{v_h | v_h \in C(\Omega) \cap V, v_h|_E \in P_s(E),\ \forall E \in \mathcal{T}_h\}, \tag{3.24}$$

where $P_s(E)$ is the set of polynomials with degree at most $s$ in element $E$.

Let $u_h^n$ be the finite element solution at time $t = t_n$, then the fully discrete scheme for the two-dimensional distributed-order space-fractional diffusion equation can be expressed as: find $u_h^n \in V_h$ for $(n = 1, 2, ..., N)$ such that

$$(\overline{\partial}_t u_h^{n-\frac{1}{2}}, v_h) + \sigma \sum_{m=1}^{S} B_m(u_h^{n-\frac{1}{2}}, v_h) = (f^{n-\frac{1}{2}}, v_h),\ \forall v_h \in V_h, \tag{3.25}$$

and

$$u_h^0 = \mathcal{P}\psi(x,y), \tag{3.26}$$

where $\mathcal{P} : L^2(\Omega) \to V_h$ is a projection operator.

## 4. Numerical examples

To testify the efficiency of the proposed finite element method, we give two examples in this section. In order to conduct the error analysis, we first define the infinite norm and the $L_2$ norm of the errors as

$$||e||_\infty = \max_{1\le n\le N}\{|u_h^n - u^n|\}, \tag{4.27}$$

$$||e||_0 = ||u_h^n - u^n||_{L_2(\Omega)}, \tag{4.28}$$

where $u^n$ denotes the exact solution, and the $u_h^n$ denotes the corresponding numerical solution.

### *4.1. Example 1*

In this example, we consider the following problem on a rectangular region $\Omega = (0,1)\times(0,1)$:

$$\begin{cases} \frac{\partial u}{\partial t} = \int_1^2 P(\alpha)\frac{\partial^\alpha u}{\partial |x|^\alpha} + Q(\alpha)\frac{\partial^\alpha u}{\partial |y|^\alpha} d\alpha + f(x,y,t), & (x,y,t) \in \Omega\times[0,T], \\ u(x,y,0) = \psi(x,y) = x^2(1-x)^2 y^2(1-y)^2, & (x,y) \in \Omega, \\ u(x,y,t) = 0, & (x,y) \in \partial\Omega, \end{cases} \tag{4.29}$$

with

$$P(\alpha) = Q(\alpha) = -2\Gamma(5-\alpha)\cos(\frac{\pi\alpha}{2}), \tag{4.30}$$

$$f(x,y,t) = e^t x^2(1-x)^2 y^2(1-y)^2 - e^t x^2(1-x)^2[R(x) + R(1-x)] - e^t y^2(1-y)^2[R(y) + R(1-y)], \tag{4.31}$$

where

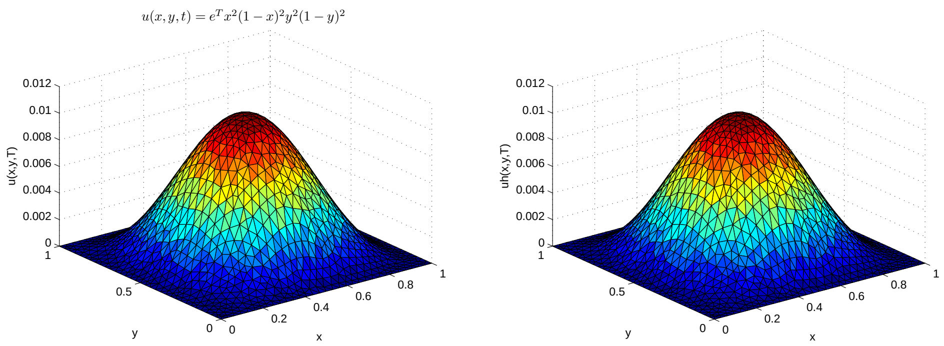
$$R(r) = \Gamma(5)\cdot R_1(r) - 2\Gamma(4)\cdot R_2(r) + \Gamma(3)\cdot R_3(r). \tag{4.32}$$

and

$$R_1(r) = \frac{1}{\ln r}(r^3 - r^2), \tag{4.33}$$

$$R_2(r) = \frac{1}{\ln r}(3r^2 - 2r) + \frac{1}{(\ln r)^2}(r - r^2), \tag{4.34}$$

$$R_3(r) = \frac{1}{\ln r}(6r - 2) + \frac{1}{(\ln r)^2}(3 - 5r) + \frac{2}{(\ln r)^3}(r - 1). \tag{4.35}$$




(a) the exact solution

(b) the numerical solution

Figure 1: Comparison between the exact solution and the numerical solution of Example 2 with $h = 1/64$.

Table 1: Errors analysis for Example 1 with $\tau = 0.01$, $S = 10$, $T = 1$.

| h | $\|\|e\|\|_0$ | $\|\|e\|\|_\infty$ |
|---|---|---|
| 1/4 | 6.9577e-4 | |
| 1/8 | 1.6575e-4 | |
| 1/16 | 4.1401e-5 | |
| 1/24 | 1.8792e-5 | |
| 1/32 | 1.0592e-5 | |

The exact solution for the problem (4.29) can be obtained as $u(x,y,t) = e^t x^2(1-x)^2 y^2(1-y)^2$.

To conduct the comparison of the figures between the exact solution and the numerical solution, we take $\tau = 0.01$, $S = 10, T = 1$. As shown in Fig. 1, the numerical solution is in well accordance with the exact solution, demonstrating that the finite element method proposed to solve the two-dimensional distributed order space-fractional diffusion equation is valid and feasible. Besides, with different choices of the space step $h$, the errors between the exact solution and the numerical solution are given in Tab. 1. The errors in $10^{-4}$ magnitude and the decreasing tendency with $h$ also verify the efficiency of the proposed finite element method in solve the two-dimensional distributed order space-fractional diffusion equation.

### *4.2. Example 2*

In this example, we consider the following problem defined on a region $\Omega = \{(x,y)|\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$:

$$\begin{cases} \frac{\partial u}{\partial t} = \int_1^2 P(\alpha)\frac{\partial^\alpha u}{\partial |x|^\alpha} + Q(\alpha)\frac{\partial^\alpha u}{\partial |y|^\alpha} d\alpha + f(x,y,t), & (x,y,t) \in \Omega \times [0,T], \\ u(x,y,0) = \psi(x,y) = (\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1)^2, & (x,y) \in \Omega, \\ u(x,y,t) = 0, & (x,y) \in \partial\Omega, \end{cases} \tag{4.36}$$

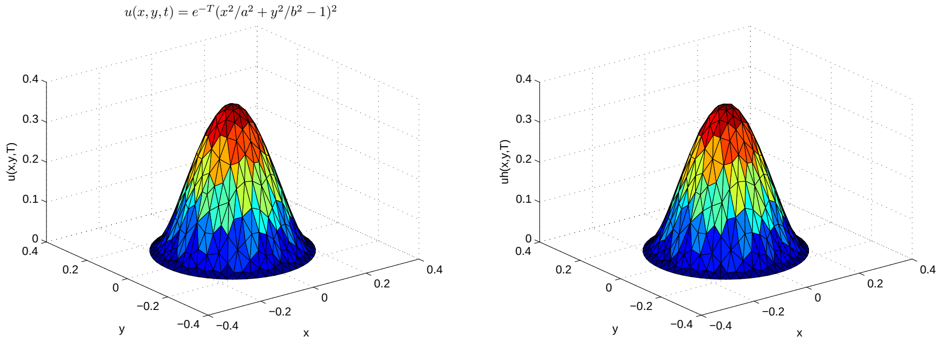
with

$$P(\alpha) = Q(\alpha) = -2\Gamma(5-\alpha)\cos(\frac{\pi\alpha}{2}), \tag{4.37}$$

$$f(x,y,t) = -e^{-t}(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1)^2 - e^{-t}P_1(x,y,t) - e^{-t}Q_1(x,y,t), \tag{4.38}$$

and

$$\begin{aligned} P_1(x,y,t) = & \frac{\Gamma(5)}{a^4}(R_1(x-x_l) + R_1(x_r - x)) + \frac{4x_l\Gamma(4)}{a^4}R_2(x-x_l) \\ & - \frac{4x_r\Gamma(4)}{a^4}R_2(x_r - x) + \Gamma(3)\frac{4x_l^2}{a^4}R_3(x-x_l) + \Gamma(3)\frac{4x_r^2}{a^4}R_3(x_r - x), \end{aligned} \tag{4.39}$$




(a) the exact solution (b) the numerical solution

Figure 2: Comparison between the exact solution and the numerical solution of Example 2 with $h = 1/32$.

$$
\begin{aligned}
Q_1(x,y,t) = {} & \frac{\Gamma(5)}{b^4}(R_1(y-y_l)+R_1(y_r-y)) + \frac{4y_l\Gamma(4)}{b^4}R_2(y-y_l) \\
& - \frac{4y_r\Gamma(4)}{b^4}R_2(y_r-y) + \Gamma(3)\frac{4y_l^2}{b^4}R_3(y-y_l) + \Gamma(3)\frac{4y_r^2}{b^4}R_3(y_r-y),
\end{aligned} \tag{4.40}
$$

where $x_l = -\frac{a}{b}\sqrt{b^2-y^2}$, $x_r = \frac{a}{b}\sqrt{b^2-y^2}$, $y_l = -\frac{b}{a}\sqrt{a^2-x^2}$, $y_r = \frac{b}{a}\sqrt{a^2-x^2}$.

The exact solution for the problem (4.36) can be obtained as $u(x,y,t) = e^{-t}(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)^2$.

We assume $a = b = 1/4$, $\tau = 0.01$, $S = 10$, $T = 1$, then the exact solution and the numerical solution can be obtained as in Fig. 2.

## 5. Conclusions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