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# The unstructured mesh finite element method for the two-dimensional multi-term time-space fractional diffusion-wave equation on an irregular convex domain

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**Abstract** In this paper, the two-dimensional multi-term time-space fractional diffusion-wave equation on an irregular convex domain is considered as a much more general case for wider applications in fluid mechanics. A novel unstructured mesh finite element method is proposed for the considered equation. In most existing works, the finite element method is applied on regular domains using uniform meshes. The case of irregular convex domains, which would require subdivision using unstructured meshes, is mostly still open. Furthermore, the orders of the multi-term time-fractional derivatives have been considered to belong to $(0, 1]$ or $(1, 2]$ separately in existing models. In this paper, we consider two-dimensional multi-term time-space fractional diffusion-wave equations with the time fractional orders belonging to the whole interval $(0, 2)$ on an irregular convex domain. We propose to use a mixed difference scheme in time and an unstructured mesh finite element method in space. Detailed implementation and the stability and convergence analyses of the proposed numerical scheme are given. Numerical examples are conducted to evaluate the theoretical analysis.

**Keywords** multi-term time-space fractional diffusion-wave equation · irregular convex domain · unstructured mesh · stability and convergence analysis

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## 1 Introduction

Over the past decades, fractional partial differential equations have attracted great attention as a useful approach for modelling a range of anomalous phenomena and processes with memory and hereditary properties (see [25, 16, 14, 18, 9, 22]).Among these, multi-term time fractional partial differential equation have been found useful to describe more complicated real processes, especially in the field of non-Newtonian fluids, such as fractional Maxwell viscoelastic fluids [12, 26], fractional Oldroyd-B fluids [24, 20] and fractional Burgers fluids [24, 32].

With its wide applications, analytical and numerical solutions have been studied for different kinds of multi-term time or time-space fractional partial differential equations. Luchko [1] considered the initial-boundary-value problems for linear and non-linear multi-term fractional diffusion equations with the Riemann-Liouville time-fractional derivatives based on the maximum principle. Jiang et al. [17] explored the analytical solutions for the multi-term time-space Caputo-Riesz fractional advection-diffusion equations on a finite domain. Jin et al. [19] studied the Galerkin finite element method for a multi-term time-fractional diffusion equation. Gao et al. [13] constructed a second order difference schemes based on the interpolation approximation for the time multi-term and distributed-order fractional sub-diffusion equations. Ren and Sun [27, 28] considered efficient and stable numerical methods separately for the multi-term time fractional sub-diffusion equation with the fractional orders belonging to $(0, 1)$, and the multi-term time fractional diffusion-wave equation with the fractional orders belonging to $(1, 2)$. Bu et al. [2] considered the finite element multigrid method for one-dimensional multi-term time-space fractional advection diffusion equations with the time fractional orders belonging to $(0, 2]$, but the discussion was also divided into two cases when the time fractional orders belong to $(0, 1]$ and $(1, 2]$ separately.

As for the multi-term time-space fractional diffusion-wave equation, there are two problems that needed to be addressed, which are also the two innovation points of this paper. Firstly, most of existing research is limited to regular domains. Given that many practical problems involve irregular convex domains, research on numerical methods designed for an irregular convex domain is of great significance. Secondly, most of the published works have been concerned with one dimension, and the time fractional orders are considered to belong to $(0, 1]$ and $(1, 2]$ to cover separately fractional diffusion and fractional wave propagation. To the best of the authors' knowledge, there is no work being published on the two-dimensional multi-term time-space fractional diffusion-wave equation defined on an irregular convex domain with the time fractional orders randomly belonging to the whole interval $(0, 2)$.

In this paper, we consider the following two-dimensional multi-term time-space fractional diffusion-wave equation (2D MT-TSFDWE) on an irregular convex domain $\Omega$:

$$P({}_0^C D_t)u(x,y,t) = k_x \frac{\partial^{2\beta_1} u}{\partial |x|^{2\beta_1}} + k_y \frac{\partial^{2\beta_2} u}{\partial |y|^{2\beta_2}} + f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \quad (1)$$

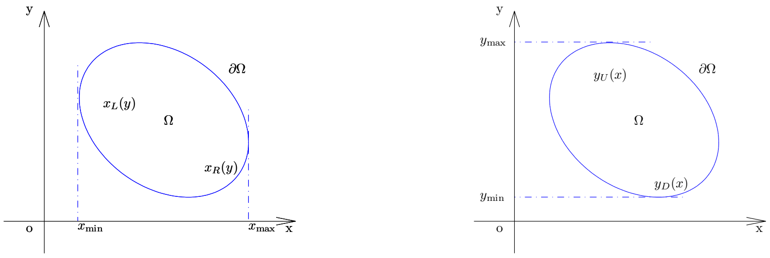


**Fig. 1** A convex domain $\Omega$ with boundaries $x_L(y), x_R(y), y_D(x), y_U(x)$.

with the initial conditions

$$u(x, y, 0) = \psi_0(x, y), \quad (x, y) \in \Omega, \tag{2}$$

$$u_t(x, y, 0) = \psi_1(x, y), \quad (x, y) \in \Omega, \tag{3}$$

and the boundary condition

$$u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T], \tag{4}$$

where

$$\begin{aligned} P({}_0^C D_t)u &= \sum_{i=0}^{s} a_i \, {}_0^C D_t^{\alpha_i} u \\ &= a_0 \, {}_0^C D_t^{\alpha_0} u + ... + a_{s_0-1} \, {}_0^C D_t^{\alpha_{s_0-1}} u + a_{s_0} \frac{\partial u}{\partial t} + a_{s_0+1} \, {}_0^C D_t^{\alpha_{s_0+1}} u + ... + a_s \, {}_0^C D_t^{\alpha_s} u, \end{aligned} \tag{5}$$

the time fractional orders satisfy $0 < \alpha_s < \cdots < \alpha_{s_0+1} < \alpha_{s_0} = 1 < \alpha_{s_0-1} < \cdots < \alpha_1 < \alpha_0 < 2$, $1/2 < \beta_1, \beta_2 < 1$, $k_x > 0, k_y > 0$, $a_i \geq 0$ and $a_i \in \mathbb{R}, (i = 0, 1, ..., s)$. As is shown in Fig.1, the irregular convex domain $\Omega$ is defined as $\Omega = \{(x, y) | x_L(y) \leq x \leq x_R(y), y_D(x) \leq y \leq y_U(x)\}$, where $x_L(y), x_R(y)$ are the left and right boundaries of $\Omega$, and $y_D(x), y_U(x)$ are the lower and upper boundaries of $\Omega$. We denote $x_{\min} = \min\limits_{(x,y)\in\Omega} x_L(y)$, $x_{\max} = \max\limits_{(x,y)\in\Omega} x_R(y)$, $y_{\min} = \min\limits_{(x,y)\in\Omega} y_D(x)$ and $y_{\max} = \max\limits_{(x,y)\in\Omega} y_U(x)$.

In particular, as for the model (1), (i) if there are only three terms in time, that is $a_{s_0} = 1, a_0 = \lambda_2^{\alpha}, a_1 = \lambda_1^{\alpha}$, with $\alpha_0 = 2\alpha + 1, \alpha_1 = \alpha + 1 (0 \leq \alpha \leq 1)$, and $\beta_2 = 1$, $k_x = 0, k_y = \nu$, $f = 0$, then the equation (1) is simplified to the multi-term fractional model for the incompressible fractional Burgers' fluid occupying the space above a flat plate situated in the $(x, z)$ plane, in the absence of a pressure gradient in the flow direction [21].

(ii) If there are only two terms in time, that is $a_{s_0} = 1, a_0 = \lambda$, and $\beta_1 = \beta_2 = 1$, $k_x = k_y = \nu$, $f = 0$, then the equation (1) is equivalent to the governing equation for an incompressible fractional Maxwell fluid at rest occupying the space above an infinite plate perpendicular to the $y$-axis and between two side walls [30].

(iii) Further, if all the coefficients $a_i$ being zero except $a_{s_0} = 1$, and $k_x = 0, k_y = \nu$, $f = 0$, then the equation (1) can be treated as the space fractional Navier-Stokes equation used in describing motion of fluids, regardless of the pressure gradient in the $x$-direction [31].

The time fractional derivative is defined in the Caputo sense:

$$ {}_0^C D_t^\gamma u = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-1-\gamma} \left(\frac{d}{ds}\right)^n u(x,y,s)ds,\ n-1<\gamma<n, n\in\mathbb{N}. \quad (6)$$

The Riesz space fractional derivatives $\frac{\partial^{2\beta_1} u}{\partial |x|^{2\beta_1}}$ is defined by

$$\frac{\partial^{2\beta_1} u(x,y)}{\partial |x|^{2\beta_1}} = -c_{\beta_1}\Big( {}_x\check{D}_L^{2\beta_1} u(x,y) + {}_x\check{D}_R^{2\beta_1} u(x,y)\Big),$$

where $c_{\beta_1} = \frac{1}{2\cos(\beta_1\pi)}$, and the Riemann-Liouville fractional derivative operators with $n-1<\mu<n$ are defined as

$$ {}_x\check{D}_L^{\mu} u(x,y) = \frac{1}{\Gamma(n-\mu)} \frac{\partial^n}{\partial x^n} \int_{x_L(y)}^{x} (x-s)^{n-\mu-1} u(s,y)ds, \quad (7)$$

$$ {}_x\check{D}_R^{\mu} u(x,y) = \frac{(-1)^n}{\Gamma(n-\mu)} \frac{\partial^n}{\partial x^n} \int_{x}^{x_R(y)} (s-x)^{n-\mu-1} u(s,y)ds. \quad (8)$$

Note that ${}_x\check{D}_L^{\mu} u(x,y)$ and ${}_x\check{D}_R^{\mu} u(x,y)$ are different from the definitions used in the finite element method for regular domains, since the boundaries of the irregular domain are functions of the space variables, rather than fixed constants. The Riesz space fractional derivatives $\frac{\partial^{2\beta_2} u}{\partial |y|^{2\beta_2}}$ with respect to $y$ can be defined similarly based on the Riemann-Liouville fractional derivative operators ${}_y\check{D}_D^{\mu} u(x,y)$ and ${}_y\check{D}_U^{\mu} u(x,y)$ [23] .

The finite element method (FEM) has been demonstrated to be a useful numerical tool for solving fractional differential equations involving time or space fractional operators [3,36,38,35]. However, most of the existing FEM schemes are designed for regular domains, such as $\Omega = [a,b]\times[c,d]$, where $a,b,c,d$ are constants. But different from the regular domain case, an irregular domain will have more complex boundaries, which will require involved partitions using a structured mesh. In view of this, for a fractional partial differential equation on an irregular domain, the development of a finite element method suitable for the irregular domain using an unstructured mesh will be of much significance.

As for the two-dimensional multi-term time-space fractional diffusion-wave equation on an irregular convex domain considered in this paper, taking into account the time fractional orders belonging to the whole interval $(0,2)$, we aim to explore its stable numerical solution using a novel finite element method tailored for the irregular convex domain. The rest of this paper is organized as follows. In Section 2, we first recapture some notations and auxiliary lemmas. In Section 3, we construct a fully discrete numerical scheme for the two-dimensional multi-term time-space fractional diffusion-wave equation. In Section 4, we detail the implementation of the finite element method using an unstructured mesh. In Section 5, we establish the stability and convergence of the fully discrete numerical scheme. In Section 6, numerical examples are given to verify the efficiency of the developed numerical method. Some conclusions are drawn in Section 7.

## 2 Preliminaries

The fundamental definitions and lemmas of the FEM for fractional partial differential equations were due to Ervin and Roop [7,8], who constructed the fractional derivative spaces and presented the theoretical framework of the finite element approximation. Developments of the FEM theory have been continued by a number of researchers, such as Bu et al. [4,2], Zhu et al. [37], and Yang et al. [33]. For a convex domain $\Omega$ shown in Fig. 1, we can define the following inner product and $L^2$-norm on the convex domain:

$$\begin{aligned}(u,v)_\Omega := \int_\Omega uv d\Omega &= \int_{y_{\min}}^{y_{\max}} \int_{x_L(y)}^{x_R(y)} u(x,y)v(x,y)dxdy,\\ &= \int_{x_{\min}}^{x_{\max}} \int_{y_D(x)}^{y_U(x)} u(x,y)v(x,y)dydx,\end{aligned} \tag{9}$$

$$\|u\|_{L^2(\Omega)} = ((u,u)_\Omega)^{1/2}. \tag{10}$$

Similar to the preliminaries used in FEM for regular domains [8,4,37], we can give the following definitions and lemmas for the irregular convex domain.

**Definition 1** (Left fractional derivative space). For $\mu > 0$, we define the semi-norm

$$|u|_{\check{J}_L^\mu(\Omega)} := \left(\left\|{}_x\check{D}_L^\mu u\right\|_{L^2(\Omega)}^2 + \left\|{}_y\check{D}_D^\mu u\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}, \tag{11}$$

and norm

$$\|u\|_{\check{J}_L^\mu(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + |u|_{\check{J}_L^\mu(\Omega)}^2\right)^{\frac{1}{2}}, \tag{12}$$

where $\check{J}_L^\mu(\Omega), \check{J}_{L,0}^\mu(\Omega)$ denote the closure of $C^\infty(\Omega), C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{\check{J}_L^\mu(\Omega)}$.

**Definition 2** (Right fractional derivative space). For $\mu > 0$, we define the semi-norm

$$|u|_{\check{J}_R^\mu(\Omega)} := \left(\left\|{}_x\check{D}_R^\mu u\right\|_{L^2(\Omega)}^2 + \left\|{}_y\check{D}_U^\mu u\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}, \tag{13}$$

and norm

$$\|u\|_{\check{J}_R^\mu(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + |u|_{\check{J}_R^\mu(\Omega)}^2\right)^{\frac{1}{2}}, \tag{14}$$

where $\check{J}_R^\mu(\Omega), \check{J}_{R,0}^\mu(\Omega)$ denote the closure of $C^\infty(\Omega), C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{\check{J}_R^\mu(\Omega)}$.

**Definition 3** (Fractional Sobolev space). For $\mu > 0$, we define the semi-norm

$$|u|_{H^\mu(\Omega)} := \||\xi|^\mu \mathcal{F}(\hat{u})(\xi)\|_{L^2(\mathbb{R}^2)} \tag{15}$$

and norm

$$\|u\|_{H^\mu(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + |u|_{H^\mu(\Omega)}^2\right)^{\frac{1}{2}}, \tag{16}$$

where $\mathcal{F}(\hat{u})(\xi)$ is the Fourier transformation of the function $\hat{u}$, $\hat{u}$ is the zero extension of function $u$ outside of $\Omega$, and $H^\mu(\Omega)$, $H_0^\mu(\Omega)$ denote the closure of $C^\infty(\Omega), C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{H^\mu(\Omega)}$.

**Definition 4** (Symmetric fractional derivative space). For $\mu > 0, \mu \neq n - \frac{1}{2}, n \in \mathbb{N}$, we define the semi-norm

$$|u|_{\check{J}_S^\mu(\Omega)} := \left(\left|\left({}_x\check{D}_L^\mu u,\ {}_x\check{D}_R^\mu u\right)_\Omega\right| + \left|\left({}_y\check{D}_D^\mu u,\ {}_y\check{D}_U^\mu u\right)_\Omega\right|\right)^{\frac{1}{2}}, \tag{17}$$

and norm

$$\|u\|_{\check{J}_S^\mu(\Omega)} := \left(\|u\|^2_{L^2(\Omega)} + |u|^2_{\check{J}_S^\mu(\Omega)}\right)^{\frac{1}{2}}, \tag{18}$$

where $\check{J}_S^\mu(\Omega), \check{J}_{S,0}^\mu(\Omega)$ denote the closure of $C^\infty(\Omega), C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{\check{J}_S^\mu(\Omega)}$.

**Lemma 1** *(see [8, 2]) If $u \in \check{J}_{L,0}^\mu(\Omega)$, $0 < \eta < \mu$, then we have*

$$\|u\|_{L^2(\Omega)} \leq c|u|_{\check{J}_L^\mu(\Omega)}, \quad |u|_{\check{J}_L^\eta(\Omega)} \leq C|u|_{\check{J}_L^\mu(\Omega)}. \tag{19}$$

*If $u \in \check{J}_{R,0}^\mu(\Omega)$, $0 < \eta < \mu$, then we have*

$$\|u\|_{L^2(\Omega)} \leq c|u|_{\check{J}_R^\mu(\Omega)}, \quad |u|_{\check{J}_R^\eta(\Omega)} \leq C|u|_{\check{J}_R^\mu(\Omega)}. \tag{20}$$

*Similar results can be obtained for the fractional Sobolev space $H_0^\mu(\Omega)$ with $\eta \neq n - 1/2$, $n \in \mathbb{N}$.*

**Lemma 2** *(see [8]) If $\mu > 0, \mu \neq n - \dfrac{1}{2}, n \in \mathbb{N}$, $u \in \check{J}_{L,0}^\mu(\Omega) \cap \check{J}_{R,0}^\mu(\Omega) \cap H_0^\mu(\Omega)$, then there exist positive constants $C_1$, $C_1'$ independent of $u$ such that*

$$C_1|u|_{H^\mu(\Omega)} \leq \max\{|u|_{\check{J}_L^\mu(\Omega)}, |u|_{\check{J}_R^\mu(\Omega)}\} \leq C_1'|u|_{H^\mu(\Omega)}, \tag{21}$$

**Lemma 3** *(see [29]) For $u \in H_0^\mu(\Omega)$, $0 < \eta < \mu$, then there exist positive constants $C_1, C_2, C_3, C_4$ independent of $u$ such that*

$$\|u\|_{L^2(\Omega)} \leq C_1\|{}_x\check{D}_L^\eta u\|_{L^2(\Omega)} \leq C_2\|{}_x\check{D}_L^\mu u\|_{L^2(\Omega)}, \tag{22}$$

$$\|u\|_{L^2(\Omega)} \leq C_3\|{}_y\check{D}_D^\eta u\|_{L^2(\Omega)} \leq C_4\|{}_y\check{D}_D^\mu u\|_{L^2(\Omega)}. \tag{23}$$

**Lemma 4** *(see [34, 33]) If $\mu \in (1, 2), u, v \in \check{J}_{L,0}^\mu(\Omega)$ (or $\check{J}_{R,0}^\mu(\Omega)$), then*

$$({}_x\check{D}_L^\mu u, v)_\Omega = ({}_x\check{D}_L^{\mu/2} u,\ {}_x\check{D}_R^{\mu/2} v)_\Omega, ({}_y\check{D}_D^\mu u, v)_\Omega = ({}_y\check{D}_D^{\mu/2} u,\ {}_y\check{D}_U^{\mu/2} v)_\Omega,$$

$$({}_x\check{D}_R^\mu u, v)_\Omega = ({}_x\check{D}_R^{\mu/2} u,\ {}_x\check{D}_L^{\mu/2} v)_\Omega, ({}_y\check{D}_U^\mu u, v)_\Omega = ({}_y\check{D}_U^{\mu/2} u,\ {}_y\check{D}_D^{\mu/2} v)_\Omega.$$

**Lemma 5** *(see [8]) If $\mu > 0, \mu \neq n - \dfrac{1}{2}, n \in \mathbb{N}$, then $\check{J}_{L,0}^\mu(\Omega), \check{J}_{R,0}^\mu(\Omega), \check{J}_{S,0}^\mu(\Omega)$ and $H_0^\mu(\Omega)$ are equivalent with equivalent norms and semi-norms.*

More details on the proofs of the lemmas are given in the corresponding references by considering $u$ to be a zero extension outside the domain $\Omega$. In what follows, for the sake of simplicity, we denote $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$, $\|\cdot\|_0 = \|\cdot\|_{L^2(\Omega)}$.

## 3 The fully discrete numerical scheme

Let $\tau = T/N$ be the time step, $t_k = k\tau, (k = 0, 1, ..., N)$. Denote $u(x, y, t_k) = u^k$ be the exact solution at time $t = t_k$, and $u^{k-1/2} = \frac{u^k + u^{k-1}}{2}$, $\delta_t u^{k-1/2} = \frac{u^k - u^{k-1}}{\tau}$, $f^{k-1/2} = \frac{f^k + f^{k-1}}{2}$. As for the Caputo time fractional operators with different orders, we use the corresponding finite difference schemes.

For $0 < \alpha_i < 1$, we have ${}_0^C D_t^{\alpha_i} u^k = \overline{D}_t^{\alpha_i} u^k + \overline{R}_k^{\alpha_i}$,where [2]

$$\begin{aligned}
\overline{D}_t^{\alpha_i} u^k &= \frac{\tau^{-\alpha_i}}{\Gamma(2-\alpha_i)} \Big[ u^k - \sum_{j=1}^{k-1} (a_{k-j-1}^{\alpha_i} - a_{k-j}^{\alpha_i}) u^j - a_{k-1}^{\alpha_i} u^0 \Big], \\
&= \frac{\tau^{1-\alpha_i}}{\Gamma(2-\alpha_i)} \sum_{j=1}^{k} a_{k-j}^{\alpha_i} \delta_t u^{j-\frac{1}{2}},
\end{aligned} \tag{24}$$

the coefficients $a_j^{\alpha_i} = (j+1)^{1-\alpha_i} - j^{1-\alpha_i}, j = 0, 1, ..., k-1$ satisfy $a_0^{\alpha_i} = 1$, and the truncation error

$$|\overline{R}_k^{\alpha_i}| \le C \max_{0 \le t \le T} |\frac{\partial^2 u}{\partial t^2}| \tau^{2-\alpha_i}, \quad 0 < \alpha_i < 1. \tag{25}$$

For $1 < \alpha_i < 2$, we have ${}_0^C D_t^{\alpha_i} u^{k-\frac{1}{2}} = \overline{\nabla}_t^{\alpha_i} u^{k-\frac{1}{2}} + R_k^{\alpha_i}$, where [10]

$$\overline{\nabla}_t^{\alpha_i} u^{k-\frac{1}{2}} = \frac{\tau^{1-\alpha_i}}{\Gamma(3-\alpha_i)} \Big[ b_0^{\alpha_i} \delta_t u^{k-1/2} - \sum_{j=1}^{k-1} (b_{k-1-j}^{\alpha_i} - b_{k-j}^{\alpha_i}) \delta_t u^{j-1/2} - b_{k-1}^{\alpha_i} u_t^0 \Big], \tag{26}$$

the coefficients $b_j^{\alpha_i} = (j+1)^{2-\alpha_i} - j^{2-\alpha_i}, j = 0, 1, ..., k-1$ satisfy $b_0^{\alpha_i} = 1$, $\sum\limits_{j=1}^{k} b_{k-j}^{\alpha_i} = k^{2-\alpha_i}$, $\sum\limits_{j=1}^{k-1} (b_{k-1-j}^{\alpha_i} - b_{k-j}^{\alpha_i}) + b_{k-1}^{\alpha_i} = 1$. The truncation error

$$|R_k^{\alpha_i}| \le C \max_{0 \le t \le T} |\frac{\partial^3 u}{\partial t^3}| \tau^{3-\alpha_i}, \quad 1 < \alpha_i < 2. \tag{27}$$

Besides, for $\alpha_i = 1$, using the center difference scheme, we have

$$\frac{\partial u(x, y, t_{k-1/2})}{\partial t} = \frac{u(x, y, t_k) - u(x, y, t_{k-1})}{\tau} + R_0, \tag{28}$$

where $R_0 \le C\tau^2$.

Let $u_h^k$ be the numerical solution at time $t = t_k$, then at time $t = t_{k-\frac{1}{2}}$, for the two-dimensional multi-term time-space fractional diffusion-wave equation (1), we have

$$\begin{aligned}
&\sum_{i=0}^{s_0-1} a_i \overline{\nabla}_t^{\alpha_i} u_h^{k-\frac{1}{2}} + a_{s_0} \delta_t u_h^{k-\frac{1}{2}} + \sum_{i=s_0+1}^{s} \frac{a_i}{2} (\overline{D}_t^{\alpha_i} u_h^k + \overline{D}_t^{\alpha_i} u_h^{k-1}) \\
&\qquad = k_x \frac{\partial^{2\beta_1} u_h^{k-\frac{1}{2}}}{\partial |x|^{2\beta_1}} + k_y \frac{\partial^{2\beta_2} u_h^{k-\frac{1}{2}}}{\partial |y|^{2\beta_2}} + f^{k-\frac{1}{2}},
\end{aligned} \tag{29}$$

Define $V = H_0^{\beta_1}(\Omega) \cap H_0^{\beta_2}(\Omega)$ to be the numerical solution space. Assume that $\{\mathcal{T}_h\}$ is a family of unstructured triangulations of domain $\Omega$ and $h$ is the maximum diameter of the triangular elements in $\mathcal{T}_h$. We define the conforming, finite dimensional subspace $V_h \subset V$ as

$$V_h = \{v_h | v_h \in C(\overline{\Omega}) \cap V, v_h|_E \in P_{\hat{s}}(E),\ \forall E \in \mathcal{T}_h\}, \tag{30}$$

where $P_{\hat{s}}(E)$ is the set of polynomials with degree at most $\hat{s}$ in element $E$.

By Lemma 4, the fully discrete scheme associated with the variational form of Eq.(1) can be defined as: find $u_h^k \in V_h$ for $(k = 1, 2, ..., N)$ such that

$$\begin{aligned}
&\sum_{i=0}^{s_0-1} \frac{a_i \tau^{1-\alpha_i}}{\Gamma(3-\alpha_i)} \Big[ b_0^{\alpha_i} (\delta_t u_h^{k-1/2}, v_h) - \sum_{j=1}^{k-1} (b_{k-1-j}^{\alpha_i} - b_{k-j}^{\alpha_i})(\delta_t u_h^{j-1/2}, v_h) \\
&- b_{k-1}^{\alpha_i} ((u_h^0)_t, v_h) \Big] + a_{s_0} (\delta_t u_h^{k-\frac{1}{2}}, v_h) \\
&+ \sum_{i=s_0+1}^{s} \frac{a_i \tau^{1-\alpha_i}}{2\Gamma(2-\alpha_i)} \Big[ \sum_{j=1}^{k} a_{k-j}^{\alpha_i} (\delta_t u_h^{j-\frac{1}{2}}, v_h) + \sum_{j=1}^{k-1} a_{k-1-j}^{\alpha_i} (\delta_t u_h^{j-\frac{1}{2}}, v_h) \Big] \\
&+ B(u_h^{k-\frac{1}{2}}, v_h) = (f^{k-\frac{1}{2}}, v_h), \qquad \forall v_h \in V_h.
\end{aligned} \tag{31}$$

with the initial and boundary conditions

$$u_h^0 = u_{0h}, \quad u_h^k|_{\partial\Omega} = 0, \tag{32}$$

where $u_{0h} \in V_h$ is a reasonable approximation for $u^0$, and $(u_h^0)_t \in V_h$ is a reasonable approximation for $u_t^0$. The bilinear form $B(u, v)$ is derived as

$$\begin{aligned}
B(u, v) =& k_x c_{\beta_1} \Big\{ ({}_x\check{D}_L^{\beta_1} u,\ {}_x\check{D}_R^{\beta_1} v) + ({}_x\check{D}_R^{\beta_1} u,\ {}_x\check{D}_L^{\beta_1} v) \Big\} \\
&+ k_y c_{\beta_2} \Big\{ ({}_y\check{D}_D^{\beta_2} u,\ {}_y\check{D}_U^{\beta_2} v) + ({}_y\check{D}_U^{\beta_2} u,\ {}_y\check{D}_D^{\beta_2} v) \Big\}.
\end{aligned} \tag{33}$$

## 4 Implementation of the unstructured mesh FEM

For a general irregular convex domain, the software Gmsh [15] can be applied in the mesh generation using the unstructured triangular or quadrilateral elements. In this paper, we use the unstructured triangulation to deal with the convex domain $\Omega$. As is shown in Fig. 2, the nodes are defined as $\{(x_n, y_n) : n = 1, 2, ..., N_p\}$, where $N_p$ is the total number of the nodes in the mesh. Let $\varphi_n(x_m, y_m) = \delta_{nm}$, $(n, m = 1, 2, ..., N_p)$ be the basis functions, where $\delta_{nm}$ is the Kronecker delta symbol. Then, for each time step $t = t_k$, the finite element solution $u_h^k$ can be expressed as

$$u_h^k = \sum_{n=1}^{N_p} u_n^k \varphi_n(x, y). \tag{34}$$

Let $v_h = \varphi_m(x,y)$, then by combining the equations (31) and (34), we can get that

$$\begin{aligned}
&\sum_{i=0}^{s_0-1} 2\omega_i \sum_{n=1}^{N_p} (\varphi_n,\varphi_m)u_n^k + 2a_{s_0}\sum_{n=1}^{N_p}(\varphi_n,\varphi_m)u_n^k + \sum_{i=s_0+1}^{s} 2r_i \sum_{n=1}^{N_p}(\varphi_n,\varphi_m)u_n^k \\
&+ \tau\sum_{n=1}^{N_p} B(\varphi_n,\varphi_m)u_n^k \\
&= \sum_{i=0}^{s_0-1} 2\omega_i \sum_{n=1}^{N_p} (\varphi_n,\varphi_m)u_n^{k-1} + 2a_{s_0}\sum_{n=1}^{N_p}(\varphi_n,\varphi_m)u_n^{k-1} + \sum_{i=s_0+1}^{s} 2r_i \sum_{n=1}^{N_p}(\varphi_n,\varphi_m)u_n^{k-1} \\
&+ \sum_{i=0}^{s_0-1} 2\omega_i \sum_{j=1}^{k-1}(b_{k-1-j}^{\alpha_i} - b_{k-j}^{\alpha_i})\Big[\sum_{n=1}^{N_p}(\varphi_n,\varphi_m)u_n^j - \sum_{n=1}^{N_p}(\varphi_n,\varphi_m)u_n^{j-1}\Big] \\
&- \sum_{i=s_0+1}^{s} 2r_i \sum_{j=1}^{k-1}(a_{k-j}^{\alpha_i} + a_{k-1-j}^{\alpha_i})\Big[\sum_{n=1}^{N_p}(\varphi_n,\varphi_m)u_n^j - \sum_{n=1}^{N_p}(\varphi_n,\varphi_m)u_n^{j-1}\Big] \\
&+ \sum_{i=0}^{s_0-1} 2\tau\omega_i b_{k-1}^{\alpha_i}((u_h^0)_t,\varphi_m) - \tau\sum_{n=1}^{N_p} B(\varphi_n,\varphi_m)u_n^{k-1} + 2\tau(f^{k-\frac{1}{2}},\varphi_m),
\end{aligned} \tag{35}$$

where $\omega_i = \frac{a_i\tau^{1-\alpha_i}}{\Gamma(3-\alpha_i)}$, $r_i = \frac{a_i\tau^{1-\alpha_i}}{2\Gamma(2-\alpha_i)}$, and $m = 1,2,...,N_p$. The scheme (35) can be rewritten as the following matrix form:

$$\begin{aligned}
&\Big[(\sum_{i=0}^{s_0-1} 2\omega_i + 2a_{s_0} + \sum_{i=s_0+1}^{s} 2r_i)\mathbf{M} + \tau\mathbf{A}\Big]\mathbf{U}^k \\
&= \Big[(\sum_{i=0}^{s_0-1} 2\omega_i + 2a_{s_0} + \sum_{i=s_0+1}^{s} 2r_i)\mathbf{M} - \tau\mathbf{A}\Big]\mathbf{U}^{k-1} \\
&+ \sum_{j=1}^{k-1}\sum_{i=0}^{s_0-1} 2\omega_i(b_{k-1-j}^{\alpha_i} - b_{k-j}^{\alpha_i})\Big[\mathbf{M}\mathbf{U}^j - \mathbf{M}\mathbf{U}^{j-1}\Big] + \sum_{i=0}^{s_0-1} 2\tau\omega_i b_{k-1}^{\alpha_i}\mathbf{W}^0 \\
&- \sum_{j=1}^{k-1}\sum_{i=s_0+1}^{s} 2r_i(a_{k-j}^{\alpha_i} + a_{k-1-j}^{\alpha_i})\Big[\mathbf{M}\mathbf{U}^j - \mathbf{M}\mathbf{U}^{j-1}\Big] + 2\tau\mathbf{F}^k,
\end{aligned} \tag{36}$$

where $\mathbf{U}^k = (u_1^k, u_2^k, ..., u_{N_p}^k)^T$ is the unknown numerical solution vector. The matrix $\mathbf{M} = ((\varphi_n,\varphi_m))_{N_p\times N_p}$ is the mass matrix, and $\mathbf{A} = (B(\varphi_n,\varphi_m))_{N_p\times N_p}$ is the stiffness matrix. We also have the matrices $\mathbf{W}^0 = (w_1^0, w_2^0, ..., w_{N_p}^0)^T$, where $w_m^0 = ((u_h^0)_t,\varphi_m)$, and $\mathbf{F}^k = (F_1^k, F_2^k, ..., F_{N_p}^k)^T$, $F_m^k = \Big(\frac{f^k+f^{k-1}}{2},\varphi_m\Big)$, $n,m = 1,2,...,N_p$, $k = 1,2,...,N$.

To obtain the unknown solution vector $\mathbf{U}^k$ given in Eq. (36), the most critical part is to construct the matrix $\mathbf{A}$. The $(n,m)$ element in the matrix $\mathbf{A}$ is given by

$$\begin{aligned}
B(\varphi_n,\varphi_m) =& k_x c_{\beta_1}\Big\{({}_x\check{D}_L^{\beta_1}\varphi_n,\ {}_x\check{D}_R^{\beta_1}\varphi_m) + ({}_x\check{D}_R^{\beta_1}\varphi_n,\ {}_x\check{D}_L^{\beta_1}\varphi_m)\Big\} \\
&+ k_y c_{\beta_2}\Big\{({}_y\check{D}_D^{\beta_2}\varphi_n,\ {}_y\check{D}_U^{\beta_2}\varphi_m) + ({}_y\check{D}_U^{\beta_2}\varphi_n,\ {}_y\check{D}_D^{\beta_2}\varphi_m)\Big\}.
\end{aligned} \tag{37}$$

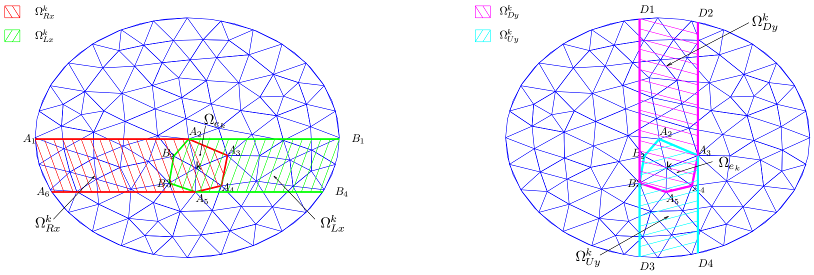


**Fig. 2** The unstructured triangular mesh of domain $\Omega$ with four nonzero support domains of the fractional derivative for node $k$. $\Omega^k_{Lx}$ is the nonzero support domain for ${}_x\check{D}^{\beta_1}_L\varphi_k(x,y)$; $\Omega^k_{Rx}$ is for ${}_x\check{D}^{\beta_1}_R\varphi_k(x,y)$, $\Omega^k_{Dy}$ is for ${}_y\check{D}^{\beta_2}_D\varphi_k(x,y)$, and $\Omega^k_{Uy}$ is for ${}_y\check{D}^{\beta_2}_U\varphi_k(x,y)$.

There are four inner products to be calculated. Taking the first inner product $({}_x\check{D}^{\beta_1}_L\varphi_n,\ {}_x\check{D}^{\beta_1}_R\varphi_m)$ as an example. By using the Gauss quadrature [5,6], we have

$$
\begin{aligned}
({}_x\check{D}^{\beta_1}_L\varphi_n,\ {}_x\check{D}^{\beta_1}_R\varphi_m) &= \sum_{E\in\mathcal{T}_h}\int_E {}_x\check{D}^{\beta_1}_L\varphi_n\cdot\ {}_x\check{D}^{\beta_1}_R\varphi_m dxdy \\
&= \sum_{E\in\mathcal{T}_h}\sum_{(x_{ci},y_{ci})\in G_E} {}_x\check{D}^{\beta_1}_L\varphi_n|_{(x_{ci},y_{ci})}\cdot\ {}_x\check{D}^{\beta_1}_R\varphi_m|_{(x_{ci},y_{ci})}\cdot\omega_i,
\end{aligned}
\tag{38}
$$

where $G_E$ is the set of the Gauss points in a certain element $E$ and $\omega_i$ are the weights corresponding to the Gauss points $(x_{ci},y_{ci})$. To calculate the non-local fractional derivatives ${}_x\check{D}^{\beta_1}_L\varphi_n(x,y)|_{(x_{ci},y_{ci})}$, the piecewise continuous basis functions $\varphi_n(x,y)$ should be formulated first.

As shown in Fig. 3, noting that the support domain $\Omega_{e_n}$ is composed of six triangular elements $E_1,E_2,...,E_6$, by combining the local element shape function of each triangle, we can construct the piecewise continuous basis function $\varphi_n(x,y)$. Since $\forall(x,y)\in\partial\Omega_{e_n}$, $\varphi_n(x,y)=0$, the domain of definition of the basis function $\varphi_n(x,y)$ can be extended from $\Omega_{e_n}$ to the whole domain $\Omega$.

By the definition given in Eq. (7), only when the Gauss point $P(x_c,y_c)$ is within the region $\Omega^n_{Lx}$ that the value of ${}_x\check{D}^{\beta_1}_L\varphi_n(x_c,y_c)$ is nonzero. Thus, $\forall P(x_c,y_c)\in\Omega^n_{Lx}$, we have

$$
\begin{aligned}
{}_x\check{D}^{\beta_1}_L\varphi_n(x_c,y_c) &= \left({}_{x_L(y_c)}\check{D}^{\beta_1}_x\varphi_n(x,y_c)\right)|_{x=x_c} \\
&= \left(\frac{1}{\Gamma(1-\beta_1)}\frac{d}{dx}\int_{x_L(y_c)}^{x}(x-\xi)^{-\beta_1}\varphi_n(\xi,y_c)d\xi\right)_{x=x_c},
\end{aligned}
\tag{39}
$$

which requires an integral from the left boundary $x_L(y_c)$ to $x_c$. As long as we have found the intersection points, we can compute the value of the ${}_xD^{\beta_1}_L\varphi_n(x_c,y_c)$.

As for the Gauss point $P(x_c,y_c)$, the line $y=y_c$ intersects the support domain $\Omega_{e_n}$ at three points $P_1(x_1,y_1),P_2(x_2,y_2)$, and $P_3(x_3,y_3)$. $P_0(x_L(y_c),y_c)$ is the boundary point intersected with the domain $\Omega$, as is shown is Fig. 3. To calculate

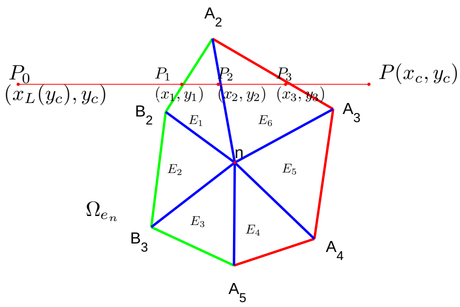


**Fig. 3** The support domain $\Omega_{e_n}$ composed of six triangular elements $E_1, E_2, ..., E_6$, inserted by line $y = y_c$.

the left fractional derivative operator with respect to $x$, the basis function $\varphi_n(x, y_c)$ can be constructed in the following form:

$$\varphi_n(x, y_c) = \begin{cases} 0, \ x < x_1 \ \texttt{or} \ x > x_3, \\ \varphi_{k1}(x, y_c), \ x_1 \le x < x_2, \\ \varphi_{k2}(x, y_c), \ x_2 \le x \le x_3, \end{cases} \tag{40}$$

where $\varphi_{k1}(x, y_c)$ and $\varphi_{k2}(x, y_c)$ can be easily constructed within the single triangle to which $x$ belongs. Then we have

$$ {}_x\check{D}_L^{\beta_1}\varphi_n(x, y_c) = \begin{cases} 0, x < x_1, \\ {}_{x_1}\check{D}_x^{\beta_1}\varphi_{k1}(x, y_c), x_1 \le x < x_2, \\ {}_{x_1}\check{D}_{x_2}^{\beta_1}\varphi_{k1}(x, y_c) + {}_{x_2}\check{D}_x^{\beta_1}\varphi_{k2}(x, y_c), x_2 \le x < x_3, \\ {}_{x_1}\check{D}_{x_2}^{\beta_1}\varphi_{k1}(x, y_c) + {}_{x_2}\check{D}_{x_3}^{\beta_1}\varphi_{k2}(x, y_c), x_3 \le x. \end{cases} \tag{41}$$

Finding the interval to which $x_c$ belongs, and replacing $x$ with $x_c$, then we can obtain ${}_x\check{D}_L^{\beta_1}\varphi_n(x_c, y_c)$.

Similarly, we can obtain the fractional derivatives ${}_x\check{D}_R^{\beta_1}\varphi_n(x_c, y_c)$, ${}_y\check{D}_D^{\beta_2}\varphi_n(x_c, y_c)$, and ${}_y\check{D}_U^{\beta_2}\varphi_n(x_c, y_c)$, with support domains being $\Omega_{Rx}^n$, $\Omega_{Dy}^n$, $\Omega_{Uy}^n$ (shown in Fig. 2), respectively. Then the matrix $\mathbf{A}$ can be formed, which allows the numerical solution $\mathbf{U}^k$ to be obtained.

For each node $n$ $(n = 1, 2, ..., N_p)$, the algorithm for calculating ${}_x\check{D}_L^{\beta_1}\varphi_n(x_c, y_c)$ is summarized in Algorithm 1.

## 5 Stability and Convergence

To analyse the stability and convergence of the fully discrete scheme on the irregular convex domain, similar to that defined on regular domains, we first define the

**Algorithm 1** Calculate ${}_x\check{D}_L^{\beta_1}\varphi_n(x_c, y_c)$ at Gauss point $P(x_c, y_c)$.

```
Partition the convex domain Ω using an unstructured triangular mesh via a suitable
  mesh generation software; Save the element information (nodes number and coordinates
  (x_n, y_n), (n = 1, 2, ..., N_p), the triangular elements E_j ∈ T_h (j = 1, 2, ..., N_e) characterized
  by three vertices;)
for j = 1, 2, ..., N_e do
  Generate the Gauss points G_{E_j} = {(x_ci, y_ci) | i = 1, ..., N_g} and the corresponding
    weights ω_i for each element E_j ;
end for
Find the support domain Ω_{e_n};
Calculate the underline(y_n) = min{y | (x, y) ∈ Ω_{e_n}}, overline(y_n) = max{y | (x, y) ∈ Ω_{e_n}}, then construct the
  nonzero support domain Ω^n_{Lx}, as shown in Fig. 2;
for all Gauss point (x_c, y_c) ∈ G_{E_1} ∪ G_{E_2} ··· ∪ G_{E_{N_e}} do
  if (x_c, y_c) ∈ Ω^n_{Lx} then
    Intersect Ω_{e_n} with line y = y_c, find the coordinates of the intersection points and
      calculate the piecewise continuous basis function φ_n(x, y_c) ;
    Calculate the _x Ď_L^{β_1} φ_n(x_c, y_c);
  else
    _x Ď_L^{β_1} φ_n(x_c, y_c) = 0.
  end if
end for
```

new semi-norm $|\cdot|_{(\beta_1,\beta_2)}$ and the norm $\|\cdot\|_{(\beta_1,\beta_2)}$ as

$$|u|_{(\beta_1,\beta_2)} = \left(k_x\|{}_x\check{D}_L^{\beta_1}u\|_0^2 + k_y\|{}_y\check{D}_D^{\beta_2}u\|_0^2\right)^{1/2}, \tag{42}$$

$$\|u\|_{(\beta_1,\beta_2)} = \left(\|u\|_0^2 + |u|^2_{(\beta_1,\beta_2)}\right)^{1/2}. \tag{43}$$

Throughout the following sections, we suppose $C, C_1, C_2, C_3$ are positive constants that may be different depending on the discussed context.

### 5.1 Stability

Before giving the proof, we first introduce some lemmas.

**Lemma 6** *(see [3]) The bilinear form $B(u, v)$ is symmetrical, continuous and coercive. Therefore, $\forall u, v \in H_0^{\beta_1}(\Omega) \cap H_0^{\beta_2}(\Omega)$, there $\exists\, C_1, C_2$ satisfying*

$$B(u, v) \le C_1\|u\|_{(\beta_1,\beta_2)}\|v\|_{(\beta_1,\beta_2)},\, B(u, u) \ge C_2\|u\|^2_{(\beta_1,\beta_2)}. \tag{44}$$

**Lemma 7** *For $0 < \alpha < 1$, define $a_j^\alpha = (j + 1)^{1-\alpha} - j^{1-\alpha}, j = 0, 1, 2, ...,$ then for any positive integer $M$ and vector $(v_1, v_2, ..., v_M) \in \mathbb{R}^M$, we have*

$$\sum_{l=1}^{M}\sum_{j=1}^{l} a_{l-j}^{\alpha}(v_j, v_l) \ge 0, \tag{45}$$

$$\sum_{l=1}^{M}\sum_{j=1}^{l} a_{l-j}^{\alpha}(v_j, v_l) + \sum_{l=1}^{M}\sum_{j=1}^{l-1} a_{l-1-j}^{\alpha}(v_j, v_l) \ge 0. \tag{46}$$

*Proof* To proof the inequality (45), we have

$$\begin{aligned}\sum_{l=1}^{M}\sum_{j=1}^{l} a_{l-j}^{\alpha}(v_j,v_l) &= \sum_{l=1}^{M}\sum_{j=1}^{l} a_{l-j}^{\alpha}\int_{\Omega} v_j v_l dxdy\\ &= \int_{\Omega}(\sum_{l=1}^{M}\sum_{j=1}^{l} a_{l-j}^{\alpha} v_j v_l)dxdy.\end{aligned} \tag{47}$$

Since $\sum\limits_{l=1}^{M}\sum\limits_{j=1}^{l} a_{l-j}^{\alpha} v_j v_l \geq 0$ (see Lemma 5 in [11]), then we can obtain the inequality $\sum\limits_{l=1}^{M}\sum\limits_{j=1}^{l} a_{l-j}^{\alpha}(v_j,v_l) \geq 0$.

Since $\sum\limits_{l=1}^{M}\sum\limits_{j=1}^{l} a_{l-j}^{\alpha} v_j v_l + \sum\limits_{l=1}^{M}\sum\limits_{j=1}^{l-1} a_{l-1-j}^{\alpha} v_j v_l \geq 0$ (see Lemma 5 in [11]), the similar proof of (46) can be obtained.

**Theorem 1** *(Stability) The fully discrete scheme (31) is unconditionally stable.*

*Proof* In scheme (31), let $v_h = \delta_t u_h^{k-1/2}$, then

$$\begin{aligned}&\sum_{i=0}^{s_0-1}\frac{a_i\tau^{1-\alpha_i}}{\Gamma(3-\alpha_i)}\Big[b_0^{\alpha_i}\|\delta_t u_h^{k-1/2}\|_0^2 - \sum_{j=1}^{k-1}(b_{k-1-j}^{\alpha_i}-b_{k-j}^{\alpha_i})(\delta_t u_h^{j-1/2},\delta_t u_h^{k-1/2})\\ &- b_{k-1}^{\alpha_i}((u_h^0)_t,\delta_t u_h^{k-1/2})\Big] + a_{s_0}\|\delta_t u_h^{k-1/2}\|_0^2 + \sum_{i=s_0+1}^{s}\frac{a_i\tau^{1-\alpha_i}}{2\Gamma(2-\alpha_i)}\times\\ &\Big[\sum_{j=1}^{k} a_{k-j}^{\alpha_i}(\delta_t u_h^{j-1/2},\delta_t u_h^{k-1/2}) + \sum_{j=1}^{k-1} a_{k-1-j}^{\alpha_i}(\delta_t u_h^{j-1/2},\delta_t u_h^{k-1/2})\Big]\\ &+ B(u_h^{k-1/2},\delta_t u_h^{k-1/2}) = (f^{k-1/2},\delta_t u_h^{k-1/2}).\end{aligned} \tag{48}$$

Note that $B(u_h^{k-1/2},\delta_t u_h^{k-1/2}) = \frac{B(u_h^k,u_h^k)-B(u_h^{k-1},u_h^{k-1})}{2\tau}$, denoting

$$\begin{aligned}Q_{s,k} = \sum_{i=s_0+1}^{s}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(2-\alpha_i)}\Big[&\sum_{j=1}^{k} a_{k-j}^{\alpha_i}(\delta_t u_h^{j-1/2},\delta_t u_h^{k-1/2})\\ &+ \sum_{j=1}^{k-1} a_{k-1-j}^{\alpha_i}(\delta_t u_h^{j-1/2},\delta_t u_h^{k-1/2})\Big],\end{aligned} \tag{49}$$

then we have

$$\begin{aligned}&\sum_{i=0}^{s_0-1}\frac{2a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\|\delta_t u_h^{k-1/2}\|_0^2 + 2\tau a_{s_0}\|\delta_t u_h^{k-1/2}\|_0^2 + B(u_h^k,u_h^k)\\ &= \sum_{i=0}^{s_0-1}\frac{2a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\Big[\sum_{j=1}^{k-1}(b_{k-1-j}^{\alpha_i}-b_{k-j}^{\alpha_i})(\delta_t u_h^{j-1/2},\delta_t u_h^{k-1/2}) + b_{k-1}^{\alpha_i}((u_h^0)_t,\delta_t u_h^{k-1/2})\Big]\\ &+ B(u_h^{k-1},u_h^{k-1}) + 2\tau(f^{k-1/2},u_h^{k-1}) - Q_{s,k}.\end{aligned} \tag{50}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\sum_{i=0}^{s_0-1}\frac{2a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\|\delta_t u_h^{k-1/2}\|_0^2+2\tau a_{s_0}\|\delta_t u_h^{k-1/2}\|_0^2+B(u_h^k,u_h^k)\\
&\le\sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\Big[\sum_{j=1}^{k-1}(b_{k-1-j}^{\alpha_i}-b_{k-j}^{\alpha_i})(\|\delta_t u_h^{j-1/2}\|_0^2+\|\delta_t u_h^{k-1/2}\|_0^2)\\
&+b_{k-1}^{\alpha_i}(\|(u_h^0)_t\|_0^2+\|\delta_t u_h^{k-1/2}\|_0^2)\Big]+\tau(\frac{1}{2a_{s_0}}\|f^{k-1/2}\|_0^2+2a_{s_0}\|\delta_t u_h^{k-1/2}\|_0^2)\\
&+B(u_h^{k-1},u_h^{k-1})-Q_{s,k}.
\end{aligned}\tag{51}$$

Note that $\sum\limits_{j=1}^{k-1}(b_{k-1-j}^{\alpha_i}-b_{k-j}^{\alpha_i})+b_{k-1}^{\alpha_i}=1$, and $b_0^{\alpha_i}=1$, then we have

$$\begin{aligned}
&\sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\|\delta_t u_h^{k-1/2}\|_0^2+B(u_h^k,u_h^k)+\sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\sum_{j=1}^{k-1}b_{k-j}^{\alpha_i}\|\delta_t u_h^{j-1/2}\|_0^2\\
&\le B(u_h^{k-1},u_h^{k-1})+\sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\sum_{j=1}^{k-1}b_{k-1-j}^{\alpha_i}\|\delta_t u_h^{j-1/2}\|_0^2\\
&+\sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}b_{k-1}^{\alpha_i}\|(u_h^0)_t\|_0^2+\frac{\tau}{2a_{s_0}}\|f^{k-1/2}\|_0^2-Q_{s,k}.
\end{aligned}\tag{52}$$

Let

$$G_k=B(u_h^k,u_h^k)+\sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\sum_{j=1}^{k}b_{k-j}^{\alpha_i}\|\delta_t u_h^{j-1/2}\|_0^2,\tag{53}$$

then (52) can be rewritten as

$$G_k\le G_{k-1}+\sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}b_{k-1}^{\alpha_i}\|(u_h^0)_t\|_0^2+\frac{\tau}{2a_{s_0}}\|f^{k-1/2}\|_0^2-Q_{s,k}.\tag{54}$$

For the both side of the eq.(54), summing $k$ from 1 to $N$, the equality (52) can be estimated as

$$G_N\le G_0+\sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\sum_{k=1}^{N}b_{k-1}^{\alpha_i}\|(u_h^0)_t\|_0^2+\frac{\tau}{2a_{s_0}}\sum_{k=1}^{N}\|f^{k-1/2}\|_0^2-\sum_{k=1}^{N}Q_{s,k},\tag{55}$$

where

$$\begin{aligned}
&\sum_{k=1}^{N}Q_{s,k}\\
&=\sum_{i=s_0+1}^{s}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(2-\alpha_i)}\sum_{k=1}^{N}\Big[\sum_{j=1}^{k}a_{k-j}^{\alpha_i}(\delta_t u_h^{j-1/2},\delta_t u_h^{k-1/2})+\sum_{j=1}^{k-1}a_{k-1-j}^{\alpha_i}(\delta_t u_h^{j-1/2},\delta_t u_h^{k-1/2})\Big].
\end{aligned}\tag{56}$$

By Lemma 7, the term

$$\sum_{k=1}^{N} Q_{s,k} > 0, \tag{57}$$

then

$$G_N \leq G_0 + \sum_{i=0}^{s_0-1} \frac{a_i \tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)} \sum_{k=1}^{N} b_{k-1}^{\alpha_i} \|(u_h^0)_t\|_0^2 + \frac{\tau}{2a_{s_0}} \sum_{k=1}^{N} \|f^{k-1/2}\|_0^2. \tag{58}$$

Since $\sum_{k=1}^{N} b_{k-1}^{\alpha_i} = N^{2-\alpha_i}$, then (53) and (58) yield

$$B(u_h^N, u_h^N) \leq B(u_h^0, u_h^0) + \sum_{i=0}^{s_0-1} \frac{a_i \tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)} N^{2-\alpha_i} \|(u_h^0)_t\|_0^2 + \frac{\tau N}{2a_{s_0}} \max_{1\leq k\leq N} \|f^{k-1/2}\|_0^2. \tag{59}$$

Based on Lemma 6 and $\tau N \leq T$, the inequality (59) can be rewritten as

$$\|u_h^N\|_{(\beta_1,\beta_2)}^2 \leq C_1 \|u_h^0\|_{(\beta_1,\beta_2)}^2 + C_2 \|(u_h^0)_t\|_0^2 + C_3 \max_{1\leq k\leq N} \|f^{k-1/2}\|_0^2. \tag{60}$$

Hence the fully discrete scheme (31) is unconditionally stable.

## 5.2 Convergence

To proceed with the convergence analysis for the fully discrete scheme (31), we suppose the interpolation operator $\coprod_h : H^{\hat{s}+1}(\Omega) \to V_h$ satisfies (see [7])

$$\|u - \Pi_h u\|_{H^\nu(\Omega)} \leq C h^{\mu-\nu} \|u\|_{H^\mu(\Omega)}, 0 \leq \nu < \mu \leq \hat{s}+1. \tag{61}$$

For $u \in V$, we define a projection operator $P_h : V \to V_h$ characterized by

$$B(P_h u, v_h) = B(u, v_h), \ \forall v_h \in V_h. \tag{62}$$

Then we can obtain the following lemma for the operator $P_h$.

**Lemma 8** *(see [4]) If $u \in H^\mu(\Omega) \cap V$, $\lambda < \mu \leq \hat{s}+1$, then there exists a constant $C$ independent of $h$ and $u$ such that*

$$|u - P_h u|_{(\beta_1,\beta_2)} \leq C h^{\mu-\lambda} \|u\|_{H^\mu(\Omega)}, \tag{63}$$

*where $\lambda = \max\{\beta_1, \beta_2\}$.*

**Theorem 2** *(Convergence) Assume that $u^N = u(x, y, t_N)$ is the exact solution of problem (1)-(5) with $u, u_{ttt}$, ${}_0^C D_t^{\alpha_i} u \in L^\infty(H^\mu(\Omega); 0, T)$, $\lambda < \mu \leq \hat{s}+1$, then the numerical solution $u_h^N$ satisfies*

$$\begin{aligned} \|u_h^N - u^N\|_{(\beta_1,\beta_2)}^2 &\leq C\tau^{\min\{2(3-\alpha_0), 2(2-\alpha_{s_0+1}), 4\}} \\ &+ Ch^{2(\mu-\lambda)} \left( \|u^N\|_\mu^2 + \|\psi_0\|_\mu^2 + C_1 \|\psi_1\|_\mu^2 + C_2 \max_{1\leq k\leq N} \|{}_0^C D_t^{\alpha_i} u^{k-\frac{1}{2}}\|_\mu^2 \right) \\ &+ Ch^{2(\mu-\lambda)} \left( a_{s_0} \max_{1\leq k\leq N} \|\frac{\partial}{\partial t} u^{k-\frac{1}{2}}\|_\mu^2 + C_3 \max_{1\leq k\leq N} \|{}_0^C D_t^{\alpha_i} u^k\|_\mu^2 \right), \end{aligned} \tag{64}$$

*where* $\alpha_0 = \max\{\alpha_i | 1 < \alpha_i < 2\}$, $\alpha_{s_0+1} = \max\{\alpha_i | 0 < \alpha_i < 1\}$, $\lambda = \max\{\beta_1, \beta_2\}$,*and the constants* $C_1, C_2, C_3$ *are related to the coefficients* $a_i$ *in the model (1)-(5), that is* $C_1 = \sum\limits_{i=0}^{s_0-1} \frac{a_i T^{2-\alpha_i}}{\Gamma(3-\alpha_i)}$, $C_2 = \sum\limits_{i=0}^{s_0-1} 2a_i T^{\alpha_i} \Gamma(2-\alpha_i)$, $C_3 = \frac{1}{2}\Sigma'_{ij} T^2 \sum\limits_{i=s_0+1}^{s} a_i$.

*Proof* Let $e^k = u_h^k - u^k$, based on the equalities (1) and (29), $e^{k-\frac{1}{2}}$ satisfies

$$\begin{aligned}
&\sum_{i=0}^{s_0-1} a_i(\overline{\nabla}_t^{\alpha_i} e^{k-\frac{1}{2}} - R_{k-1/2}^{\alpha_i}, v_h) + a_{s_0}(\delta_t e^{k-\frac{1}{2}} - R_0, v_h) \\
&+ \sum_{i=s_0+1}^{s} \frac{a_i}{2}(\overline{D}_t^{\alpha_i} e^k + \overline{D}_t^{\alpha_i} e^{k-1} - \overline{R}_k^{\alpha_i} - \overline{R}_{k-1}^{\alpha_i}, v_h) + B(e^{k-\frac{1}{2}}, v_h) = 0.
\end{aligned} \tag{65}$$

Denote $e^{k-\frac{1}{2}} = \theta^{k-\frac{1}{2}} + \rho^{k-\frac{1}{2}}$, where $\rho^{k-\frac{1}{2}} = P_h u^{k-\frac{1}{2}} - u^{k-\frac{1}{2}}$, $\theta^{k-\frac{1}{2}} = u_h^{k-\frac{1}{2}} - P_h u^{k-\frac{1}{2}}$. Noting that $B(\rho^{k-\frac{1}{2}}, v_h) = 0$, then

$$\begin{aligned}
&\sum_{i=0}^{s_0-1} a_i(\overline{\nabla}_t^{\alpha_i} \theta^{k-\frac{1}{2}}, v_h) + a_{s_0}(\delta_t \theta^{k-\frac{1}{2}}, v_h) + \sum_{i=s_0+1}^{s} \frac{a_i}{2}(\overline{D}_t^{\alpha_i} \theta^k + \overline{D}_t^{\alpha_i} \theta^{k-1}, v_h) \\
&+ B(\theta^{k-\frac{1}{2}}, v_h) \\
&= \sum_{i=0}^{s_0-1} a_i(R_{k-1/2}^{\alpha_i}, v_h) + a_{s_0}(R_0 - \delta_t \rho^{k-\frac{1}{2}}, v_h) + \sum_{i=s_0+1}^{s} \frac{a_i}{2}(\overline{R}_k^{\alpha_i} + \overline{R}_{k-1}^{\alpha_i}, v_h) \\
&- \sum_{i=0}^{s_0-1} a_i(\overline{\nabla}_t^{\alpha_i} \rho^{k-\frac{1}{2}}, v_h) - \sum_{i=s_0+1}^{s} \frac{a_i}{2}(\overline{D}_t^{\alpha_i} \rho^k + \overline{D}_t^{\alpha_i} \rho^{k-1}, v_h)
\end{aligned} \tag{66}$$

Based on (24) and (26), taking $v_h = \delta_t \theta^{k-\frac{1}{2}}$, then we have

$$\begin{aligned}
&\sum_{i=0}^{s_0-1} \frac{a_i \tau^{1-\alpha_i}}{\Gamma(3-\alpha_i)} \Big[ b_0^{\alpha_i} \|\delta_t \theta^{k-\frac{1}{2}}\|_0^2 - \sum_{j=1}^{k-1} (b_{k-1-j}^{\alpha_i} - b_{k-j}^{\alpha_i})(\delta_t \theta^{j-\frac{1}{2}}, \delta_t \theta^{k-\frac{1}{2}}) \\
&- b_{k-1}^{\alpha_i}(\theta_t^0, \delta_t \theta^{k-\frac{1}{2}}) \Big] + a_{s_0} \|\delta_t \theta^{k-\frac{1}{2}}\|_0^2 \\
&+ \sum_{i=s_0+1}^{s} \frac{a_i \tau^{1-\alpha_i}}{2\Gamma(2-\alpha_i)} \Big[ \sum_{j=1}^{k} a_{k-j}^{\alpha_i}(\delta_t \theta^{j-\frac{1}{2}}, \delta_t \theta^{k-\frac{1}{2}}) + \sum_{j=1}^{k-1} a_{k-1-j}^{\alpha_i}(\delta_t \theta^{j-\frac{1}{2}}, \delta_t \theta^{k-\frac{1}{2}}) \Big] \\
&+ B(\theta^{k-\frac{1}{2}}, \delta_t \theta^{k-\frac{1}{2}}) \\
&= \sum_{i=0}^{s_0-1} a_i(R_{k-\frac{1}{2}}^{\alpha_i} - \overline{\nabla}_t^{\alpha_i} \rho^{k-\frac{1}{2}}, \delta_t \theta^{k-\frac{1}{2}}) + a_{s_0}(R_0 - \delta_t \rho^{k-\frac{1}{2}}, \delta_t \theta^{k-\frac{1}{2}}) \\
&+ \sum_{i=s_0+1}^{s} \frac{a_i}{2}(\overline{R}_k^{\alpha_i} + \overline{R}_{k-1}^{\alpha_i} - \overline{D}_t^{\alpha_i} \rho^k - \overline{D}_t^{\alpha_i} \rho^{k-1}, \delta_t \theta^{k-\frac{1}{2}}).
\end{aligned} \tag{67}$$

Note that $B(\theta^{k-\frac{1}{2}}, \delta_t\theta^{k-\frac{1}{2}}) = \frac{B(\theta^k,\theta^k)-B(\theta^{k-1},\theta^{k-1})}{2\tau}$, denoting

$$\tilde{Q}_{s,k} = \sum_{i=s_0+1}^{s} \frac{a_i\tau^{2-\alpha_i}}{\Gamma(2-\alpha_i)}\Big[\sum_{j=1}^{k} a_{k-j}^{\alpha_i}(\delta_t\theta^{j-\frac{1}{2}}, \delta_t\theta^{k-\frac{1}{2}}) + \sum_{j=1}^{k-1} a_{k-1-j}^{\alpha_i}(\delta_t\theta^{j-\frac{1}{2}}, \delta_t\theta^{k-\frac{1}{2}})\Big], \tag{68}$$

then we have

$$\begin{aligned}
&\sum_{i=0}^{s_0-1} \frac{2a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)} b_0^{\alpha_i}\|\delta_t\theta^{k-\frac{1}{2}}\|_0^2 + 2\tau a_{s_0}\|\delta_t\theta^{k-\frac{1}{2}}\|_0^2 + B(\theta^k,\theta^k)\\
&= B(\theta^{k-1},\theta^{k-1}) + \sum_{i=0}^{s_0-1}\frac{2a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\Big[\sum_{j=1}^{k-1}(b_{k-1-j}^{\alpha_i} - b_{k-j}^{\alpha_i})(\delta_t\theta^{j-\frac{1}{2}}, \delta_t\theta^{k-\frac{1}{2}})\\
&+ b_{k-1}^{\alpha_i}(\theta_t^0, \delta_t\theta^{k-\frac{1}{2}})\Big] - 2\tau\tilde{Q}_{s,k} \qquad (69)\\
&+ \sum_{i=0}^{s_0-1} 2\tau a_i(R_{k-\frac{1}{2}}^{\alpha_i} - \overline{\nabla}_t^{\alpha_i}\rho^{k-\frac{1}{2}}, \delta_t\theta^{k-\frac{1}{2}}) + 2\tau a_{s_0}(R_0 - \delta_t\rho^{k-\frac{1}{2}}, \delta_t\theta^{k-\frac{1}{2}})\\
&+ \sum_{i=s_0+1}^{s} \tau a_i(\overline{R}_k^{\alpha_i} + \overline{R}_{k-1}^{\alpha_i} - \overline{D}_t^{\alpha_i}\rho^k - \overline{D}_t^{\alpha_i}\rho^{k-1}, \delta_t\theta^{k-\frac{1}{2}}).
\end{aligned}$$

By the Cauchy-Schwartz inequality, that is $(u,v) \le \frac{1}{2}(\|u\|_0^2 + \|v\|_0^2)$, and $b_0^{\alpha_i} = 1$, $\sum_{j=1}^{k-1}(b_{k-1-j}^{\alpha_i} - b_{k-j}^{\alpha_i}) + b_{k-1}^{\alpha_i} = 1$, we have

$$\begin{aligned}
&\sum_{i=0}^{s_0-1} \frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)} \sum_{j=1}^{k} b_{k-j}^{\alpha_i}\|\delta_t\theta^{j-\frac{1}{2}}\|_0^2 + 2\tau a_{s_0}\|\delta_t\theta^{k-\frac{1}{2}}\|_0^2 + B(\theta^k,\theta^k)\\
&\le B(\theta^{k-1},\theta^{k-1}) + \sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\sum_{j=1}^{k-1} b_{k-1-j}^{\alpha_i}\|\delta_t\theta^{j-\frac{1}{2}}\|_0^2\\
&+ \sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)} b_{k-1}^{\alpha_i}\|\theta_t^0\|_0^2 - 2\tau\tilde{Q}_{s,k}\\
&+ \sum_{i=0}^{s_0-1} 2\tau a_i(R_{k-\frac{1}{2}}^{\alpha_i} - \overline{\nabla}_t^{\alpha_i}\rho^{k-\frac{1}{2}}, \delta_t\theta^{k-\frac{1}{2}}) + 2\tau a_{s_0}\|\delta_t\theta^{k-\frac{1}{2}}\|_0^2 + \frac{\tau a_{s_0}}{2}\|R_0 - \delta_t\rho^{k-\frac{1}{2}}\|_0^2\\
&+ \sum_{i=s_0+1}^{s} \tau a_i(\overline{R}_k^{\alpha_i} + \overline{R}_{k-1}^{\alpha_i} - \overline{D}_t^{\alpha_i}\rho^k - \overline{D}_t^{\alpha_i}\rho^{k-1}, \delta_t\theta^{k-\frac{1}{2}}).
\end{aligned} \tag{70}$$

Denote

$$E^k = B(\theta^k,\theta^k) + \sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\sum_{j=1}^{k} b_{k-j}^{\alpha_i}\|\delta_t\theta^{j-1/2}\|_0^2, \tag{71}$$

then we have

$$E^k - E^{k-1} \leq \sum_{i=0}^{s_0-1} \frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)} b_{k-1}^{\alpha_i}\|\theta_t^0\|_0^2 - 2\tau\tilde{Q}_{s,k} + \sum_{i=0}^{s_0-1} 2\tau a_i(J_{k-\frac{1}{2}}^{\alpha_i}, \delta_t\theta^{k-\frac{1}{2}}) + \frac{\tau a_{s_0}}{2}\|Z_{k-\frac{1}{2}}\|_0^2 + \sum_{i=s_0+1}^{s} \tau a_i(L_k^{\alpha_i}, \delta_t\theta^{k-\frac{1}{2}}). \tag{72}$$

where $J_{k-\frac{1}{2}}^{\alpha_i} = R_{k-\frac{1}{2}}^{\alpha_i} - \overline{\nabla}_t^{\alpha_i}\rho^{k-\frac{1}{2}}$, $Z_{k-\frac{1}{2}} = R_0 - \delta_t\rho^{k-\frac{1}{2}}$ and $L_k^{\alpha_i} = \overline{R}_k^{\alpha_i} + \overline{R}_{k-1}^{\alpha_i} - \overline{D}_t^{\alpha_i}\rho^k - \overline{D}_t^{\alpha_i}\rho^{k-1}$.

For the both side of the Eq.(72), summing $k$ from 1 to $N$, then we have then we have

$$E^N \leq E^0 + \sum_{i=0}^{s_0-1} \frac{a_i\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)} \sum_{k=1}^{N} b_{k-1}^{\alpha_i}\|\theta_t^0\|_0^2 - 2\tau\sum_{k=1}^{N}\tilde{Q}_{s,k} + \frac{\tau a_{s_0}}{2}\sum_{k=1}^{N}\|Z_{k-\frac{1}{2}}\|_0^2 + \sum_{i=0}^{s_0-1} 2\tau a_i \sum_{k=1}^{N}(J_{k-\frac{1}{2}}^{\alpha_i}, \delta_t\theta^{k-\frac{1}{2}}) + \sum_{i=s_0+1}^{s}\tau a_i\sum_{k=1}^{N}(L_k^{\alpha_i}, \delta_t\theta^{k-\frac{1}{2}}). \tag{73}$$

For the last two terms of Eq.(73), using the Cauchy-Schwartz inequality again, we have

$$\begin{aligned}
&\sum_{i=0}^{s_0-1} 2\tau a_i \sum_{k=1}^{N}(J_{k-\frac{1}{2}}^{\alpha_i}, \delta_t\theta^{k-\frac{1}{2}}) \\
&= \sum_{i=0}^{s_0-1} 2\tau a_i \sum_{k=1}^{N}\Big(\frac{1}{\sqrt{\frac{\tau^{1-\alpha_i}}{2\Gamma(3-\alpha_i)}b_{N-k}^{\alpha_i}}}J_{k-\frac{1}{2}}^{\alpha_i}, \sqrt{\frac{\tau^{1-\alpha_i}}{2\Gamma(3-\alpha_i)}b_{N-k}^{\alpha_i}}\,\delta_t\theta^{k-\frac{1}{2}}\Big) \\
&\leq \sum_{i=0}^{s_0-1} 2a_i\tau^{\alpha_i}\Gamma(3-\alpha_i)\sum_{k=1}^{N}\frac{1}{b_{N-k}^{\alpha_i}}\|J_{k-\frac{1}{2}}^{\alpha_i}\|_0^2 + \sum_{i=0}^{s_0-1}\frac{a_i\tau^{2-\alpha_i}}{2\Gamma(3-\alpha_i)}\sum_{k=1}^{N} b_{N-k}^{\alpha_i}\|\delta_t\theta^{k-\frac{1}{2}}\|_0^2.
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
&\sum_{i=s_0+1}^{s}\tau a_i\sum_{k=1}^{N}(L_k^{\alpha_i}, \delta_t\theta^{k-\frac{1}{2}}) \\
&= \sum_{i=s_0+1}^{s}\tau a_i\sum_{k=1}^{N}\Big(\frac{1}{\sqrt{\frac{\sum_{j=0}^{s_0-1}\frac{a_j\tau^{2-\alpha_j}}{\Gamma(3-\alpha_j)}b_{N-k}^{\alpha_j}}{\sum_{i=s_0+1}^{s}\tau a_i}}}L_k^{\alpha_i}, \sqrt{\frac{\sum_{j=0}^{s_0-1}\frac{a_j\tau^{2-\alpha_j}}{\Gamma(3-\alpha_j)}b_{N-k}^{\alpha_j}}{\sum_{i=s_0+1}^{s}\tau a_i}}\,\delta_t\theta^{k-\frac{1}{2}}\Big) \\
&\leq \frac{(\sum_{i=s_0+1}^{s}\tau a_i)^2}{2\sum_{j=0}^{s_0-1}\frac{a_j\tau^{2-\alpha_j}}{\Gamma(3-\alpha_j)}}\sum_{k=1}^{N}\frac{1}{b_{N-k}^{\alpha_j}}\|L_k^{\alpha_i}\|_0^2 + \sum_{j=0}^{s_0-1}\frac{a_j\tau^{2-\alpha_j}}{2\Gamma(3-\alpha_j)}\sum_{k=1}^{N} b_{N-k}^{\alpha_j}\|\delta_t\theta^{k-\frac{1}{2}}\|_0^2.
\end{aligned} \tag{75}$$

Besides, by Lemma 7, the term

$$2\tau \sum_{k=1}^{N} \tilde{Q}_{s,k} > 0. \tag{76}$$

Then by Eqs.(5.2.9)-(5.2.12), we have

$$\begin{aligned}B(\theta^N, \theta^N) \leq B(\theta^0, \theta^0) + \sum_{i=0}^{s_0-1} \frac{a_i \tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)} \sum_{k=1}^{N} b_{k-1}^{\alpha_i} \|\theta_t^0\|_0^2 + \frac{\tau a_{s_0}}{2} \sum_{k=1}^{N} \|Z_{k-\frac{1}{2}}\|_0^2 \\ + \sum_{i=0}^{s_0-1} 2a_i \tau^{\alpha_i} \Gamma(3-\alpha_i) \sum_{k=1}^{N} \frac{1}{b_{N-k}^{\alpha_i}} \|J_{k-\frac{1}{2}}^{\alpha_i}\|_0^2 + \frac{1}{2} \Sigma_{ij} \sum_{i=s_0+1}^{s} a_i \tau^{\alpha_j} \sum_{k=1}^{N} \frac{1}{b_{N-k}^{\alpha_j}} \|L_k^{\alpha_i}\|_0^2.\end{aligned} \tag{77}$$

where

$$\Sigma_{ij} = \frac{\sum\limits_{i=s_0+1}^{s} a_i}{\sum\limits_{j=0}^{s_0-1} \frac{a_j}{\Gamma(3-\alpha_j)}}, \Sigma'_{ij} = \frac{\sum\limits_{i=s_0+1}^{s} a_i}{\sum\limits_{j=0}^{s_0-1} \frac{a_j}{\Gamma(2-\alpha_j)}}. \tag{78}$$

Note that $\sum\limits_{k=1}^{N} b_{k-1}^{\alpha_j} = N^{2-\alpha_j}$ and $b_{N-k}^{\alpha_j} \geq (2-\alpha_j)N^{1-\alpha_j}$, $\sum\limits_{k=1}^{N} \frac{1}{b_{N-k}^{\alpha_j}} \leq \frac{N^{\alpha_j}}{2-\alpha_j}$, $(j = 0, 1, ..., s_0 - 1)$, the Eq.(77) can be rewritten as

$$\begin{aligned}B(\theta^N, \theta^N) \leq B(\theta^0, \theta^0) + \sum_{i=0}^{s_0-1} \frac{a_i T^{2-\alpha_i}}{\Gamma(3-\alpha_i)} \|\theta_t^0\|_0^2 + \frac{T a_{s_0}}{2} \max_{1 \leq k \leq N} \|Z_{k-\frac{1}{2}}\|_0^2 \\ + \sum_{i=0}^{s_0-1} 2a_i T^{\alpha_i} \Gamma(2-\alpha_i) \max_{1 \leq k \leq N} \|J_{k-\frac{1}{2}}^{\alpha_i}\|_0^2 + \frac{1}{2} \Sigma'_{ij} T^2 \sum_{i=s_0+1}^{s} a_i \max_{1 \leq k \leq N} \|L_k^{\alpha_i}\|_0^2.\end{aligned} \tag{79}$$

As for the terms in Eq.(79), we also have the following approximation.

$$\begin{aligned}&\|J_{k-\frac{1}{2}}^{\alpha_i}\|_0^2 \leq \|R_{k-\frac{1}{2}}^{\alpha_i}\|_0^2 + \|\overline{\nabla}_t^{\alpha_i} \rho^{k-\frac{1}{2}}\|_0^2 \\ &\leq C\tau^{2(3-\alpha_i)} \max_{1 \leq k \leq N} \|u_{ttt}^{k-\frac{1}{2}}\|_0^2 + \|\overline{\nabla}_t^{\alpha_i} \rho^{k-\frac{1}{2}} - {}_0^C D_t^{\alpha_i} \rho^{k-\frac{1}{2}} + {}_0^C D_t^{\alpha_i} \rho^{k-\frac{1}{2}}\|_0^2 \\ &\leq C\tau^{2(3-\alpha_i)} + Ch^{2(\mu-\lambda)} \|{}_0^C D_t^{\alpha_i} u^{k-\frac{1}{2}}\|_\mu^2,\end{aligned} \tag{80}$$

similarly,

$$\begin{aligned}&\|L_k^{\alpha_i}\|_0^2 \leq \|\overline{R}_k^{\alpha_i}\|_0^2 + \|\overline{R}_{k-1}^{\alpha_i}\|_0^2 + \|\overline{D}_t^{\alpha_i} \rho^k\|_0^2 + \|\overline{D}_t^{\alpha_i} \rho^{k-1}\|_0^2 \\ &\leq C\tau^{2(2-\alpha_i)} \max_{1 \leq k \leq N} \|u_{tt}^k\|_0^2 + C\tau^{2(2-\alpha_i)} \max_{1 \leq k \leq N} \|u_{tt}^{k-1}\|_0^2 + Ch^{2(\mu-\lambda)} \|{}_0^C D_t^{\alpha_i} u^k\|_\mu^2 \\ &+ Ch^{2(\mu-\lambda)} \|{}_0^C D_t^{\alpha_i} u^{k-1}\|_\mu^2 \\ &\leq C\tau^{2(2-\alpha_i)} + Ch^{2(\mu-\lambda)} \|{}_0^C D_t^{\alpha_i} u^k\|_\mu^2 + Ch^{2(\mu-\lambda)} \|{}_0^C D_t^{\alpha_i} u^{k-1}\|_\mu^2.\end{aligned} \tag{81}$$

Also,

$$\begin{aligned}\|Z_{k-\frac{1}{2}}\|_0^2 &= \|R_0 - \delta_t\rho^{k-\frac{1}{2}}\|_0^2 \le \|R_0\|_0^2 + \|\delta_t\rho^{k-\frac{1}{2}}\|_0^2 \\ &\le C\tau^4 + \|\delta_t\rho^{k-\frac{1}{2}} - \frac{\partial\rho^{k-\frac{1}{2}}}{\partial t} + \frac{\partial\rho^{k-\frac{1}{2}}}{\partial t}\|_0^2 \\ &\le C\tau^4 + Ch^{2(\mu-\lambda)}\|\frac{\partial}{\partial t}u^{k-\frac{1}{2}}\|_\mu^2.\end{aligned} \tag{82}$$

Then by the Lemma 6 of the bilinear form $B(u,v)$, and combining the Eqs.(79)-(82), we get

$$\begin{aligned}\|\theta^N\|_{(\beta_1,\beta_2)}^2 \le{}& C\|\theta^0\|_{(\beta_1,\beta_2)}^2 + C\sum_{i=0}^{s_0-1}\frac{a_iT^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\|\theta_t^0\|_0^2 \\ &+ C\sum_{i=0}^{s_0-1}2a_iT^{\alpha_i}\Gamma(2-\alpha_i)(\tau^{2(3-\alpha_i)} + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|{}_0^CD_t^{\alpha_i}u^{k-\frac{1}{2}}\|_\mu^2) \\ &+ C\frac{Ta_{s_0}}{2}(\tau^4 + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|\frac{\partial}{\partial t}u^{k-\frac{1}{2}}\|_\mu^2) \\ &+ C\frac{1}{2}\Sigma'_{ij}T^2\sum_{i=s_0+1}^{s}a_i(\tau^{2(2-\alpha_i)} + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|{}_0^CD_t^{\alpha_i}u^k\|_\mu^2)\end{aligned} \tag{83}$$

By the Minkowski inequality,

$$\begin{aligned}\|\theta^0\|_{(\beta_1,\beta_2)}^2 &= \|u_h^0 - u^0 + u^0 - P_hu^0\|_{(\beta_1,\beta_2)}^2 \\ &\le C\Big(\|u_h^0 - u^0\|_{(\beta_1,\beta_2)}^2 + \|u^0 - P_hu^0\|_{(\beta_1,\beta_2)}^2\Big) \\ &\le C\|u_h^0 - \psi_0\|_{(\beta_1,\beta_2)}^2 + Ch^{2\mu-2\lambda}\|\psi_0\|_\mu^2.\end{aligned} \tag{84}$$

Note that $\|\cdot\|_0 \le C\|\cdot\|_{(\alpha,\beta)}$, then

$$\begin{aligned}\|\theta_t^0\|_0 &= \|(u_h^0 - u^0 + u^0 - P_hu^0)_t\|_0^2 \\ &\le C\Big(\|(u_h^0)_t - u_t^0\|_0^2 + \|u_t^0 - P_hu_t^0)\|_0^2\Big) \\ &\le C\|(u_h^0)_t - \psi_1\|_0^2 + Ch^{2\mu-2\lambda}\|\psi_1\|_\mu^2.\end{aligned} \tag{85}$$

Then the estimation (83) can be written as

$$\begin{aligned}\|\theta^N\|_{(\beta_1,\beta_2)}^2 \le{}& C\|u_h^0 - \psi_0\|_{(\beta_1,\beta_2)}^2 + Ch^{2(\mu-\lambda)}\|\psi_0\|_\mu^2 \\ &+ C\sum_{i=0}^{s_0-1}\frac{a_iT^{2-\alpha_i}}{\Gamma(3-\alpha_i)}(\|(u_h^0)_t - \psi_1\|_0^2 + h^{2(\mu-\lambda)}\|\psi_1\|_\mu^2) \\ &+ C\sum_{i=0}^{s_0-1}2a_iT^{\alpha_i}\Gamma(2-\alpha_i)(\tau^{2(3-\alpha_0)} + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|{}_0^CD_t^{\alpha_i}u^{k-\frac{1}{2}}\|_\mu^2) \\ &+ C\frac{Ta_{s_0}}{2}(\tau^4 + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|\frac{\partial}{\partial t}u^{k-\frac{1}{2}}\|_\mu^2) \\ &+ C\frac{1}{2}\Sigma'_{ij}T^2\sum_{i=s_0+1}^{s}a_i(\tau^{2(2-\alpha_{s_0+1})} + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|{}_0^CD_t^{\alpha_i}u^k\|_\mu^2).\end{aligned} \tag{86}$$

By Lemma 8, we have

$$
\begin{aligned}
\|u_h^N - u^N\|^2_{(\beta_1,\beta_2)} &\le \|\theta^N\|^2_{(\beta_1,\beta_2)} + \|\rho^N\|^2_{(\beta_1,\beta_2)} \\
&\le Ch^{2(\mu-\lambda)}\|u^N\|^2_\mu + C\|u_h^0 - \psi_0\|^2_{(\beta_1,\beta_2)} + Ch^{2(\mu-\lambda)}\|\psi_0\|^2_\mu \\
&+ C\sum_{i=0}^{s_0-1}\frac{a_i T^{2-\alpha_i}}{\Gamma(3-\alpha_i)}(\|(u_h^0)_t - \psi_1\|^2_0 + h^{2(\mu-\lambda)}\|\psi_1\|^2_\mu) \\
&+ C\sum_{i=0}^{s_0-1} 2a_i T^{\alpha_i}\Gamma(2-\alpha_i)(\tau^{2(3-\alpha_0)} + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|{}_0^C D_t^{\alpha_i} u^{k-\frac{1}{2}}\|^2_\mu) \\
&+ C\frac{Ta_{s_0}}{2}(\tau^4 + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|\frac{\partial}{\partial t}u^{k-\frac{1}{2}}\|^2_\mu) \\
&+ C\frac{1}{2}\Sigma'_{ij}T^2\sum_{i=s_0+1}^{s} a_i(\tau^{2(2-\alpha_{s_0+1})} + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|{}_0^C D_t^{\alpha_i} u^k\|^2_\mu).
\end{aligned} \tag{87}
$$

Choosing the interpolations as initial values of $u$ and $u_t$ at time $t_0$, i.e. $u_h^0 = \Pi_h\psi_0$, $(u_h^0)_t = \Pi_h\psi_1$, we have

$$
\begin{aligned}
\|u_h^N - u^N\|^2_{(\beta_1,\beta_2)} &\le Ch^{2(\mu-\lambda)}(\|u^N\|^2_\mu + \|\psi_0\|^2_\mu + \sum_{i=0}^{s_0-1}\frac{a_i T^{2-\alpha_i}}{\Gamma(3-\alpha_i)}\|\psi_1\|^2_\mu) \\
&+ C\sum_{i=0}^{s_0-1} 2a_i T^{\alpha_i}\Gamma(2-\alpha_i)(\tau^{2(3-\alpha_0)} + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|{}_0^C D_t^{\alpha_i} u^{k-\frac{1}{2}}\|^2_\mu) \\
&+ C\frac{Ta_{s_0}}{2}(\tau^4 + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|\frac{\partial}{\partial t}u^{k-\frac{1}{2}}\|^2_\mu) \\
&+ C\frac{1}{2}\Sigma'_{ij}T^2\sum_{i=s_0+1}^{s} a_i(\tau^{2(2-\alpha_{s_0+1})} + h^{2(\mu-\lambda)}\max_{1\le k\le N}\|{}_0^C D_t^{\alpha_i} u^k\|^2_\mu).
\end{aligned} \tag{88}
$$

That is,

$$
\begin{aligned}
\|u_h^N - u^N\|^2_{(\beta_1,\beta_2)} &\le C\tau^{\min\{2(3-\alpha_0),2(2-\alpha_{s_0+1}),4\}} \\
&+ Ch^{2(\mu-\lambda)}\left(\|u^N\|^2_\mu + \|\psi_0\|^2_\mu + C_1\|\psi_1\|^2_\mu + C_2\max_{1\le k\le N}\|{}_0^C D_t^{\alpha_i} u^{k-\frac{1}{2}}\|^2_\mu\right) \\
&+ Ch^{2(\mu-\lambda)}\left(a_{s_0}\max_{1\le k\le N}\|\frac{\partial}{\partial t}u^{k-\frac{1}{2}}\|^2_\mu + C_3\max_{1\le k\le N}\|{}_0^C D_t^{\alpha_i} u^k\|^2_\mu\right),
\end{aligned} \tag{89}
$$

where $\alpha_0 = \max\{\alpha_i | 1 < \alpha_i < 2\}$, $\alpha_{s_0+1} = \max\{\alpha_i | 0 < \alpha_i < 1\}$, $\lambda = \max\{\beta_1, \beta_2\}$, and the constants $C_1, C_2, C_3$ are related to the coefficients $a_i$ in the model (1)-(5), that is $C_1 = \sum_{i=0}^{s_0-1}\frac{a_i T^{2-\alpha_i}}{\Gamma(3-\alpha_i)}$, $C_2 = \sum_{i=0}^{s_0-1} 2a_i T^{\alpha_i}\Gamma(2-\alpha_i)$, $C_3 = \frac{1}{2}\Sigma'_{ij}T^2\sum_{i=s_0+1}^{s} a_i$.

## 6 Numerical examples

In this section, we use the linear triangular elements to construct a numerical example to testify the theoretical analysis. Based on Theorem 2, the convergence order in space should be $O(h^2)$ in the $\|\cdot\|_0$ norm, and $O(h^{2-\lambda})$ in the $\|\cdot\|_{(\beta_1,\beta_2)}$ norm. As for time, the convergence order in the $\|\cdot\|_0$ norm should be $O(\tau^{\min\{3-\alpha_0,2-\alpha_{s_0+1},2\}})$, where $\alpha_0=\max\{\alpha_i|1<\alpha_i<2\}$, and $\alpha_{s_0+1}=\max\{\alpha_i|0<\alpha_i<1\}$. The convergence order can be calculated by the following formulation:

$$Order=\begin{cases}\frac{\log(\|error(h_1)\|_\epsilon/\|error(h_2)\|_\epsilon)}{\log(h_1/h_2)}, \text{ in space},\\ \frac{\log(\|error(\tau_1)\|_0/\|error(\tau_2)\|_0)}{\log(\tau_1/\tau_2)}, \text{ in time},\end{cases}$$

where $\|error(h_1)\|_\epsilon$ denotes the error in the $\|\cdot\|_0$ norm or in the $\|\cdot\|_{(\beta_1,\beta_2)}$ norm with space step being $h_1$.

### 6.1 Example 1

To verify the theoretical analysis, we consider a two-dimensional multi-term time-space fractional diffusion-wave equation defined on an elliptical domain $\Omega$, where $\Omega=\{(x,y)|\frac{x^2}{a^2}+\frac{y^2}{b^2}<1\}$:

$$\begin{cases}{}_0^CD_t^{\alpha_0}u+\frac{\partial u}{\partial t}+{}_0^CD_t^{\alpha}u=k_x\frac{\partial^{2\beta_1}u}{\partial|x|^{2\beta_1}}+k_y\frac{\partial^{2\beta_2}u}{\partial|y|^{2\beta_2}}+f(x,y,t),\\ u(x,y,0)=\phi(x,y),\quad (x,y)\in\Omega,\\ u_t(x,y,0)=0,\quad (x,y)\in\Omega,\\ u(x,y,t)=0,\quad (x,y,t)\in\partial\Omega\times(0,T],\end{cases}\tag{90}$$

where $(x,y,t)\in\Omega\times(0,T]$, $0<\alpha<1<\alpha_0<2$, $1/2<\beta_1,\beta_2<1$, $\phi(x,y)=(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)^2$, and

$$\begin{aligned}&f(x,y,t)\\&=\frac{2t^{2-\alpha_0}}{\Gamma(3-\alpha_0)}(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)^2+\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)^2+2t(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)^2\\&+C_x\frac{\Gamma(5)(x-x_l)^{4-2\beta_1}}{a^4\Gamma(5-2\beta_1)}+C_x\frac{4x_l\Gamma(4)(x-x_l)^{3-2\beta_1}}{a^4\Gamma(4-2\beta_1)}\\&+C_x(\frac{6x_l^2}{a^4}+\frac{2y^2}{a^2b^2}-\frac{2}{a^2})\frac{\Gamma(3)(x-x_l)^{2-2\beta_1}}{\Gamma(3-2\beta_1)}+C_x\frac{\Gamma(5)(x_r-x)^{4-2\beta_1}}{a^4\Gamma(5-2\beta_1)}\\&-C_x\frac{4x_r\Gamma(4)(x_r-x)^{3-2\beta_1}}{a^4\Gamma(4-2\beta_1)}+C_x(\frac{6x_r^2}{a^4}+\frac{2y^2}{a^2b^2}-\frac{2}{a^2})\frac{\Gamma(3)(x_r-x)^{2-2\beta_1}}{\Gamma(3-2\beta_1)}\\&+C_y\frac{\Gamma(5)(y-y_l)^{4-2\beta_2}}{b^4\Gamma(5-2\beta_2)}+C_y\frac{4y_l\Gamma(4)(y-y_l)^{3-2\beta_2}}{b^4\Gamma(4-2\beta_2)}\\&+C_y(\frac{6y_l^2}{b^4}+\frac{2x^2}{a^2b^2}-\frac{2}{b^2})\frac{\Gamma(3)(y-y_l)^{2-2\beta_2}}{\Gamma(3-2\beta_2)}+C_y\frac{\Gamma(5)(y_r-y)^{4-2\beta_2}}{b^4\Gamma(5-2\beta_2)}\\&-C_y\frac{4y_r\Gamma(4)(y_r-y)^{3-2\beta_2}}{b^4\Gamma(4-2\beta_2)}+C_y(\frac{6y_r^2}{b^4}+\frac{2x^2}{a^2b^2}-\frac{2}{b^2})\frac{\Gamma(3)(y_r-y)^{2-2\beta_2}}{\Gamma(3-2\beta_2)},\end{aligned}\tag{91}$$

**Table 1** The convergence orders in space with $\tau = 1/3200$.

| | h | $\|u_h^N - u(t_N)\|_0$ | order | $\|u_h^N - u(t_N)\|_{(\alpha,\beta)}$ | order |
|---|---|---|---|---|---|
| | 1/8 | 4.293e-2 | | 7.6203e-1 | |
| $\alpha_0 = 1.3$ | 1/16 | 1.0931e-2 | 1.9736 | 3.4203e-1 | 1.1557 |
| $\alpha = 0.8$ | 1/22 | 5.7303e-3 | 2.0279 | 2.2957e-1 | 1.2517 |
| $\beta_1 = 0.65$ | 1/32 | 2.6862e-3 | 2.0221 | 1.4716e-1 | 1.1868 |
| $\beta_2 = 0.8$ | 1/48 | 1.1844e-3 | 2.0182 | 9.0292e-2 | 1.2020 |

**Table 2** The convergence orders in space with $\tau = 1/3200$.

| | h | $\|u_h^N - u(t_N)\|_0$ | order | $\|u_h^N - u(t_N)\|_{(\alpha,\beta)}$ | order |
|---|---|---|---|---|---|
| | 1/8 | 4.2943e-2 | | 5.4253e-1 | |
| $\alpha_0 = 1.5$ | 1/16 | 1.0983e-2 | 1.9661 | 2.2428e-1 | 1.2744 |
| $\alpha = 0.6$ | 1/22 | 5.7967e-3 | 2.0067 | 1.4507e-1 | 1.3681 |
| $\beta_1 = 0.6$ | 1/32 | 2.7901e-3 | 1.9516 | 9.0448e-2 | 1.2609 |
| $\beta_2 = 0.7$ | 1/48 | 1.2799e-3 | 1.9220 | 5.4908e-2 | 1.2310 |

**Table 3** The convergence orders in time with $\tau^{\min\{3-\alpha_0,2-\alpha,2\}} \approx h^2$.

| | $\tau$ | $\|u_h^N - u(t_N)\|_0$ | order |
|---|---|---|---|
| | 1/32 | 4.6567e-2 | |
| $\alpha_0 = 1.6$ | 1/101 | 1.3182e-2 | 1.0981 |
| $\alpha = 0.8$ | 1/173 | 7.3492e-3 | 1.0856 |
| $\beta_1 = 0.8$ | 1/322 | 3.8381e-3 | 1.0456 |
| $\beta_2 = 0.6$ | 1/634 | 1.7508e-3 | 1.1585 |

**Table 4** The convergence orders in time with $\tau^{\min\{3-\alpha_0,2-\alpha,2\}} \approx h^2$.

| | $\tau$ | $\|u_h^N - u(t_N)\|_0$ | order |
|---|---|---|---|
| | 1/25 | 4.4414e-2 | |
| $\alpha_0 = 1.7$ | 1/71 | 1.1425e-2 | 1.3026 |
| $\alpha = 0.2$ | 1/116 | 6.1114e-3 | 1.2745 |
| $\beta_1 = 0.8$ | 1/207 | 2.9466e-3 | 1.2604 |
| $\beta_2 = 0.8$ | 1/386 | 1.2888e-3 | 1.3271 |

where $C_x = \frac{k_x(t^2+1)}{2\cos(\beta_1\pi)}$, $C_y = \frac{k_y(t^2+1)}{2\cos(\beta_2\pi)}$, $x_l = -\frac{a}{b}\sqrt{b^2-y^2}, x_r = \frac{a}{b}\sqrt{b^2-y^2}$, $y_l = -\frac{b}{a}\sqrt{a^2-x^2}, y_r = \frac{b}{a}\sqrt{a^2-x^2}$. The exact solution is $u(x,y,t) = (t^2+1)(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)^2$.

Suppose that $a = 1/2, b = 1/4, T = 1$, and $k_x = 2, k_y = 1$. The convergence orders both in time and in space are given in Tables 6.1-6.4.

For space, the convergence orders in the $\|\cdot\|_0$ norm and the $\|\cdot\|_{(\beta_1,\beta_2)}$ norm are shown in Table 1 and Table 2 respectively. As is shown in the tables, the order in the $\|\cdot\|_0$ norm is about 2, and in the $\|\cdot\|_{(\beta_1,\beta_2)}$ norm is about $2-\max\{\beta_1,\beta_2\}$, which agree with the theoretical results. With different choices of the fractional orders, the numerical results are in agreement with the theoretical analysis, indicating the validity of the proposed method.

For time, based on the theoretical analysis, the convergence order should be $O(\tau^{\min\{3-\alpha_0,2-\alpha,2\}})$. As is shown in Table 3 and Table 4, with different fractional orders, the numerical results coincide with the theoretical analysis. They demonstrate that the proposed unstructured mesh finite element method is efficient in dealing with two-dimensional multi-term time-space fractional diffusion-wave e-quations defined on a convex domain.

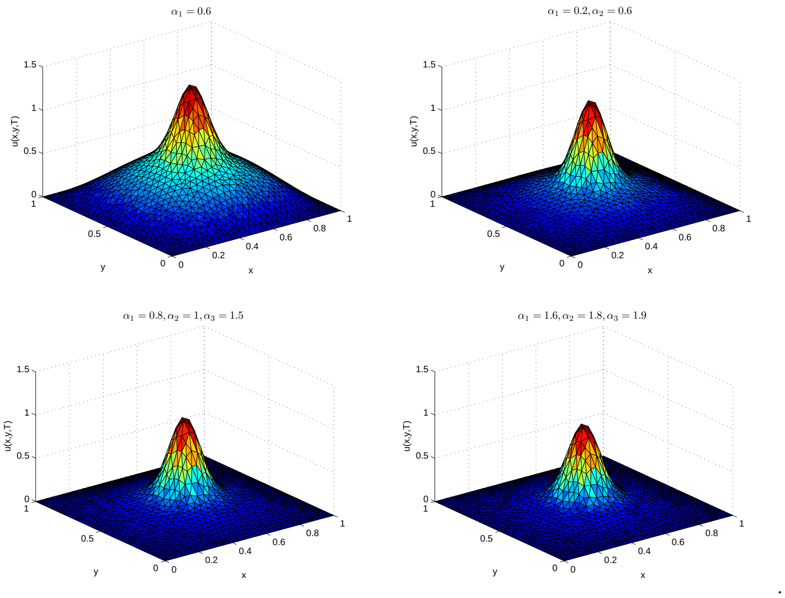


**Fig. 4** The profile of $u(x, y, t)$ with different time fractional orders $\alpha_i$.

## 6.2 Example 2

Next, we try to explore the role of the time fractional orders played in the two-dimensional multi-term time-space fractional partial differential equation. As an example, we consider the following 3-term case on a convex domain:

$$\begin{cases} \sum_{i=1}^{3} a_i \, {}_0^C D_t^{\alpha_i} u = k_x \frac{\partial^{2\beta_1} u}{\partial |x|^{2\beta_1}} + k_y \frac{\partial^{2\beta_2} u}{\partial |y|^{2\beta_2}} + f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \\ u(x, y, 0) = e^{-100((x-0.5)^2+(y-0.5)^2)}, \quad (x, y) \in \Omega, \\ u_t(x, y, 0) = 0, \quad (x, y) \in \Omega, \\ u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T], \end{cases} \tag{92}$$

where $f(x, y, t) = t^2 x(x-1)y(y-1)$.

For the sake of simplicity, the domain is supposed to be $\Omega = [0, 1] \times [0, 1]$, the space fractional orders are fixed as $\beta_1 = \beta_2 = 0.8$, $k_x = 5 \times 10^{-4}, k_y = 1.6 \times 10^{-4}, T = 3$, and $\alpha_i \in (0, 2)$. Since it is hard to obtain the exact solution for this problem, we main pay attention to the numerical solution under different cases of time fractional orders.

Taking $h = 0.03125$, $\tau = 0.01$, the profile of $u(x, y, t)$ with different time fractional orders $\alpha_i$ are shown in Fig.4, with $a_i = 0$ or $1, i = 1, 2, 3$. That is, for the case $\alpha_1 = 0.6$, the coefficients are given as $a_1 = 1, a_2 = a_3 = 0$. It can be seen that, with different time fractional orders, the fractional partial differential equation shows different profile, especially when $\alpha_i$ belongs to different regions. In practical problems, the number of the time fractional term and the corresponding values of the fractional order can be determined based on the relevant physical backgrounds.

## 7 Conclusions

In this paper, the two-dimensional multi-term time-space fractional diffusion-wave equation defined on an irregular convex domain in considered, which is a much more general case that has wide applications in fluid mechanics. A novel unstructured mesh finite element method is proposed to deal with the considered equation. Taking into account the Caputo time fractional orders belonging to the whole interval $(0, 2)$, a mixed difference scheme is used in time. Given the irregular convex domain, which is difficult to be subdivided well using a structured mesh, a novel finite element scheme using unstructured mesh is proposed in space. Implementation of the numerical scheme is detailed, and its stability and convergence are established. To verify the validity of the proposed numerical method, two numerical examples are constructed. Numerical results show consistency with the theoretical analysis. The work demonstrates that the proposed unstructured mesh finite element method is efficient and valid in dealing with two-dimensional multi-term time-space fractional diffusion-wave equations defined on an irregular convex domain, allowing for the time fractional orders belonging to the whole interval $(0, 2)$.

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