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A Crank-Nicolson ADI Galerkin-Legendre spectral method for the two-dimensional Riesz space distributed-order advection-diffusion equation

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Abstract

In the paper, a Crank-Nicolson alternating direction implicit (ADI) Galerkin-Legendre spectral scheme is presented for the two-dimensional Riesz space distributed-order advection-diffusion equation. The Gauss quadrature has a higher computational accuracy than the mid-point quadrature rule, which is proposed to approximate the distributed order Riesz space derivative so that the considered equation is transformed into a multi-term fractional equation. Moreover, the transformed equation is solved by discretizing in space by the ADI Galerkin-Legendre spectral scheme and in time using the Crank-Nicolson difference method. Stability and convergence analysis are verified for the numerical approximation. A lot of numerical results are demonstrated to justify the theoretical analysis.

Keywords: Two-dimensional Riesz space distributed-order advection-diffusion equation, ADI Galerkin-Legendre spectral method, Gauss quadrature, Stability and convergence analysis

1. Introduction

In recent years, as the foundation of fractal geometry and fractional dimension dynamics, fractional calculus theory has proved to be a valuable tool in modeling many physical phenomena. There is a vast literature on the theoretical research of fractional differential equations [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Chen et al. [16] developed a fractal derivative model of anomalous diffusion and derived the fundamental solution of the fractal derivative equation for anomalous diffusion, which characterizes a clear power law. A new method

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based on the Legendre wavelet expansion together with operational matrices for the fractional integration and derivative of these basis functions was proposed in [17] to solve the time-fractional seventh order KdV (sKdV) equation. In general, these models can be simulated by single and multi-term fractional equations. However, both single and multi-term fractional equations are not suitable for depicting the diffusion processes in multi-fractal media due to lack of fixed scaling exponent. For processes lacking such scaling the corresponding description may be given by distributed-order fractional partial differential equations [18]. It has been reported that the dynamics systems depicting and solving the real world properties have been undergoing two stages. One is from integer-order dynamic systems to fractional-order dynamic systems, and the other is from fractional-order dynamic systems to distributed-order dynamic systems [19].

Furthermore, distributed-order differential equations have recently been investigated for complex dynamical systems, namely, distributed-order dynamic systems, which have been explored to describe some important physical phenomena. Distributed-order differential models are more powerful tools to describe complex dynamical systems than classical and fractional-order models because of their nonlocal properties. Caputo [20, 21] first discussed distributed-order fractional differential equations. The major difference between distributed-order time fractional differential equations and distributed-order space fractional differential equations is that the former represents local variations and is particularly valid when considering local phenomena [22, 23, 24], while in an infinite medium it is more appropriate to introduce the space fractional order derivative to represent the effect of the medium and its space interaction with the fluid [18]. There are many papers that studied how to solve distributed-order fractional equations. The second-order accurate implicit numerical methods for the Riesz space distributed-order advection dispersion equations (RSDO-ADE) in one-dimensional (1D) and two-dimensional (2D) cases were derived and analyzed in [25]. Ye et al. presented an implicit numerical method for a new time distributed-order and two-sided space-fractional advection-dispersion equation [26] and an compact difference scheme for distributed-order time-fractional diffusion-wave equation on bounded domains [27]. An implicit numerical method for the time distributed-order and two-sided space-fractional advection-dispersion equation [28] was developed by Hu et al. Li et al. proposed the finite volume method for a distributed-order space fractional diffusion equation [29] and the Riesz space distributed-order advection-diffusion equation [30]. A numerical method for solving the two-dimensional distributed order space-fractional diffusion equation on an irregular convex domain was given in [31].

However, it seems that no other published research takes into account the spectral method with detailed theoretical analysis for the space distributed-order advection-diffusion equation.

In [18], Sokolov et al. discussed space distributed-order diffusion equation

$$\frac{\partial u}{\partial t} = \int_0^2 P(\alpha) \frac{\partial^\alpha u}{\partial |x|^\alpha} d\alpha, \tag{1}$$

where $P(\alpha)$ is a dimensional function of the order of the derivative α. $\frac{\partial^\alpha}{\partial|x|^\alpha}$ denotes the Riesz space fractional derivative. In the general case $P(\alpha) = H^{\alpha-2}Kw(\alpha)$, H and K are dimensional positive constants, $[H] = \text{cm}$, $[K] = \text{cm}^2/\text{sec}$ and $w(\alpha) = B_1\delta(\alpha-\alpha_1) + B_2\delta(\alpha-\alpha_2)$, $0 < \alpha_1 < \alpha_2 \le 2$, $B_1 > 0$, $B_2 > 0$. The equation for the characteristic function of Eq. (1) has the solution

$$g(h,t) = \exp\left(-a_1|h|^{\alpha_1}t - a_2|h|^{\alpha_2}t\right), \tag{2}$$

with $a_1 = B_1K/H^{2-\alpha_1}$, $a_2 = B_2K/H^{2-\alpha_2}$. Eq. (2) is a product of two characteristic functions of Lévy stable probability density functions with Lévy indices α_1, α_2 and scale parameters a_1^{1/α_1} and a_2^{1/α_2}, respectively. Through a series of analysis, Sokolov et al. [18] concluded that at small times the characteristic displacement grew as t^{1/α_2}, whereas at large times it grew as t^{1/α_1}. This means that the process was an accelerated superdiffusion. Based on this model, the following general two-dimensional Riesz space distributed-order advection-diffusion equation is developed:

$$\begin{aligned}\frac{\partial u}{\partial t} =&\lambda_1\int_1^2 P_1(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}d\alpha + \lambda_2\int_1^2 P_2(\beta)\frac{\partial^\beta u}{\partial|y|^\beta}d\beta + \lambda_3\int_0^1 P_3(\gamma)\frac{\partial^\gamma u}{\partial|x|^\gamma}d\gamma \\ &+ \lambda_4\int_0^1 P_4(\eta)\frac{\partial^\eta}{\partial|y|^\eta}d\eta + f(x,y,t), \quad (x,y,t)\in\Omega\times I,\end{aligned} \tag{3}$$

with boundary condition

$$u = 0, \quad (x,y,t)\in\partial\Omega\times I, \tag{4}$$

and initial condition

$$u(x,y,0) = u_0(x,y), \quad (x,y)\in\Omega, \tag{5}$$

where $I = (0,T]$, $\Omega = (0,L)\times(0,L)$, $f(x,y,t)$ can be used to represent sources and sinks. $\lambda_1,\lambda_2 > 0$ are the diffusion coefficient, $\lambda_3,\lambda_4 > 0$ are the average velocity, $P_1(\alpha)$, $P_2(\beta)$, $P_3(\gamma)$, $P_4(\eta)$ are non-negative and bounded weight functions that satisfy the conditions [25],

$$\begin{aligned}&P_1(\alpha)\ge 0, \quad P_1(\alpha)\not\equiv 0, \quad \alpha\in(1,2), \quad 0<\int_1^2 P_1(\alpha)d\alpha<\infty,\\ &P_2(\beta)\ge 0, \quad P_2(\beta)\not\equiv 0, \quad \beta\in(1,2), \quad 0<\int_1^2 P_2(\beta)d\beta<\infty,\\ &P_3(\gamma)\ge 0, \quad P_3(\gamma)\not\equiv 0, \quad \gamma\in(0,1), \quad 0<\int_0^1 P_3(\gamma)d\gamma<\infty,\\ &P_4(\eta)\ge 0, \quad P_4(\eta)\not\equiv 0, \quad \eta\in(0,1), \quad 0<\int_0^1 P_4(\eta)d\eta<\infty.\end{aligned} \tag{6}$$

The Riesz fractional derivatives on a finite domain $[0,L]$ are defined [11, 32] as follows:

$$
\begin{aligned}
\frac{\partial^\alpha u}{\partial|x|^\alpha} &= -\frac{1}{2\cos(\alpha\pi/2)}\left({}_0D_x^\alpha u + {}_xD_L^\alpha u\right), \quad 0<\alpha<2, \alpha\neq 1,\\
\frac{\partial^\beta u}{\partial|y|^\beta} &= -\frac{1}{2\cos(\beta\pi/2)}\left({}_0D_y^\beta u + {}_yD_L^\beta u\right), \quad 0<\beta<2, \beta\neq 1.
\end{aligned} \tag{7}
$$

For $n-1<\gamma<n$, $n\in\mathbb{N}$, the operators ${}_0D_x^\gamma u$, ${}_xD_L^\gamma u$, ${}_0D_y^\gamma u$ and ${}_yD_L^\gamma u$ are defined as

$$
\begin{aligned}
{}_0D_x^\gamma u &= \frac{1}{\Gamma(n-\gamma)}\frac{\partial^n}{\partial x^n}\int_0^x (x-s)^{n-\gamma-1}u(s,y,t)ds,\\
{}_xD_L^\gamma u &= \frac{(-1)^n}{\Gamma(n-\gamma)}\frac{\partial^n}{\partial x^n}\int_x^L (s-x)^{n-\gamma-1}u(s,y,t)ds,\\
{}_0D_y^\gamma u &= \frac{1}{\Gamma(n-\gamma)}\frac{\partial^n}{\partial y^n}\int_0^y (y-s)^{n-\gamma-1}u(x,s,t)ds,\\
{}_yD_L^\gamma u &= \frac{(-1)^n}{\Gamma(n-\gamma)}\frac{\partial^n}{\partial y^n}\int_y^L (s-y)^{n-\gamma-1}u(x,s,t)ds.
\end{aligned}
$$

Nowadays, some numerical methods have been applied to classical equations [6, 7, 8, 33, 34, 35, 36]. The spectral method is one of these important methods and has been widely applied to fractional integral and differential equations [37, 38, 39, 40] because of its high-order accuracy. In paper [41], the authors presented a space-time spectral method for the numerical solution of the time fractional Fokker-Planck initial-boundary value problem. An unconditional energy stable Fourier spectral scheme for the fractional equation with periodic or Neumann boundary conditions was developed in [42]. In this paper, we propose a Crank-Nicolson ADI Galerkin-Legendre spectral method for the two-dimensional Riesz space distributed-order advection-diffusion equation. The stability and convergence analysis of the numerical method are discussed. The error in spatial attains spectral accuracy.

The method to carry out the distributed-order Riesz space derivative used in most articles [25, 29, 30] is based on the mid-point quadrature rule. In this paper, we propose the Gauss quadrature to approximate the distributed-order Riesz space derivative. The Gauss quadrature is stable and has a higher computational accuracy than the mid-point quadrature rule. For the general distributed-order Riesz space derivative $\int_{\alpha_{\min}}^{\alpha_{\max}} P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}d\alpha$, when the weight function $P(\alpha)$ is smooth, we can use the Legendre-Gauss quadrature to approximate. When the weight function $P(\alpha)$ has a weak singularityis, we can rewrite the distributed-order Riesz space derivative

$$
\int_{\alpha_{\min}}^{\alpha_{\max}} P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}\omega\omega^{-1}d\alpha, \quad \omega=(\alpha-\alpha_{\min})^a(\alpha_{\max}-\alpha)^b, (a,b>-1),
$$

where ω is a suitable weight function such that $P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}\omega$ is more smooth than $P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}$. We can apply the Jacobi-Gauss quadrature to carry out the above integral. In other words, the Gauss-quadrature can achieve a high accuracy for smooth and nonsmooth solutions.

The rest of this paper is organized as follows. The definitions and properties of fractional Sobolev spaces and distributed-order Sobolev spaces are introduced in Section 2. In Section 3, we provide the Gauss quadrature and the Crank-Nicolson ADI Galerkin-Legendre spectral method for solving the two-dimensional Riesz space distributed-order advection-diffusion equation. The stability and convergence analysis are verified for the numerical approximation in Section 4. Section 5 presents some numerical results to show the effectiveness of our numerical method. Finally, some conclusions are made in Section 6.

2. Preliminaries and notations

By $H^\sigma(\mathbb{R}^2)$, $\sigma \geq 0$, we denote the fractional Sobolev space on $\mathbb{R}^2$, defined as

$$H^\sigma(\mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2) | (1+|\mu|^2)^{\sigma/2}\mathcal{F}(u)(\mu) \in L^2(\mathbb{R}^2)\}, \tag{8}$$

which is endowed with the seminorm and norm

$$\begin{aligned} |u|_{H^\sigma(\mathbb{R}^2)} &= \| \, |\mu|^\sigma \mathcal{F}(u)(\mu)\|_{L^2(\mathbb{R}^2)}, \\ \|u\|_{H^\sigma(\mathbb{R}^2)} &= \|(1+|\mu|^2)^{\sigma/2}\mathcal{F}(u)(\mu)\|_{L^2(\mathbb{R}^2)}, \end{aligned} \tag{9}$$

where $\mathcal{F}(u)$ represents the Fourier transform of u. Subsequently, we denote by the fractional Sobolev space on Ω.

$$H^\sigma(\Omega) = \{u \in L^2(\Omega) | \exists \tilde{u} \in H^\sigma(\mathbb{R}^2) \ s.t. \ \tilde{u}|_\Omega = u\}, \tag{10}$$

with the norm

$$\|u\|_{H^\sigma(\Omega)} = \inf_{\tilde{u}\in H^\sigma(\mathbb{R}^2), \tilde{u}|_\Omega = u} \|\tilde{u}\|_{H^\sigma(\mathbb{R}^2)}, \tag{11}$$

and $H_0^\sigma(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with repect to $\|u\|_{H^\sigma(\Omega)}$. Other useful seminorms and norms associated with $H^\sigma(\Omega)$ $(H_0^\sigma(\Omega))$ have been developed in [43]. The seminorm $|u|_{l,\sigma}$ and the norm $\|u\|_{l,\sigma}$ are presented as

$$\begin{aligned} |u|_{l,\sigma} &= \left(\|{}_0D_x^\sigma u\|^2_{L^2(\Omega)} + \|{}_0D_y^\sigma u\|^2_{L^2(\Omega)} \right)^{1/2}, \\ \|u\|_{l,\sigma} &= \left(\|u\|^2_{L^2(\Omega)} + |u|^2_{l,\sigma} \right)^{1/2}. \end{aligned} \tag{12}$$

The seminorm $|u|_{r,\sigma}$ and the norm $\|u\|_{r,\sigma}$ are given by

$$\begin{aligned} |u|_{r,\sigma} &= \left(\|{}_xD_L^\sigma u\|^2_{L^2(\Omega)} + \|{}_yD_L^\sigma u\|^2_{L^2(\Omega)} \right)^{1/2}, \\ \|u\|_{r,\sigma} &= \left(\|u\|^2_{L^2(\Omega)} + |u|^2_{r,\sigma} \right)^{1/2}. \end{aligned} \tag{13}$$

For the distributed-order Riesz space derivative, we obtain that using the Gauss quadrature,

$$
\begin{aligned}
\int_1^2 P_1(\alpha)\frac{\partial^\alpha u}{\partial |x|^\alpha}d\alpha &= \frac{1}{2}\sum_{k=0}^{M_1}\tilde{\omega}_k^{(1)}P_1(\alpha_k)\frac{\partial^{\alpha_k} u}{\partial |x|^{\alpha_k}} + O(M_1^{-r_1}),\\
\int_1^2 P_2(\beta)\frac{\partial^\beta u}{\partial |y|^\beta}d\beta &= \frac{1}{2}\sum_{k=0}^{M_2}\tilde{\omega}_k^{(2)}P_2(\beta_k)\frac{\partial^{\beta_k} u}{\partial |y|^{\beta_k}} + O(M_2^{-r_1}),\\
\int_0^1 P_3(\gamma)\frac{\partial^\gamma u}{\partial |x|^\gamma}d\gamma &= \frac{1}{2}\sum_{k=0}^{M_3}\tilde{\omega}_k^{(3)}P_3(\gamma_k)\frac{\partial^{\gamma_k} u}{\partial |x|^{\gamma_k}} + O(M_3^{-r_1}),\\
\int_0^1 P_4(\eta)\frac{\partial^\eta u}{\partial |y|^\eta}d\eta &= \frac{1}{2}\sum_{k=0}^{M_4}\tilde{\omega}_k^{(4)}P_4(\eta_k)\frac{\partial^{\eta_k} u}{\partial |y|^{\eta_k}} + O(M_4^{-r_1}),
\end{aligned}
\tag{14}
$$

where α_k, β_k are the Legendre-Gauss points on the interval $(1,2)$, γ_k, η_k are the Legendre-Gauss points on the interval $(0,1)$, $\tilde{\omega}_k^{(m)}$ are the Legendre-Gauss weights [44] and $r_1 \leq \max(M_m)+1$, $m=1,2,3,4$. Taking $\omega_k^{(1)} = \frac{1}{2}\tilde{\omega}_k^{(1)}P_1(\alpha_k)$, $\omega_k^{(2)} = \frac{1}{2}\tilde{\omega}_k^{(2)}P_2(\beta_k)$, $\omega_k^{(3)} = \frac{1}{2}\tilde{\omega}_k^{(3)}P_3(\gamma_k)$ and $\omega_k^{(4)} = \frac{1}{2}\tilde{\omega}_k^{(4)}P_4(\eta_k)$, we can rewrite (14) into the following form

$$
\begin{aligned}
\int_1^2 P_1(\alpha)\frac{\partial^\alpha u}{\partial |x|^\alpha}d\alpha &= \sum_{k=0}^{M_1}\omega_k^{(1)}\frac{\partial^{\alpha_k} u}{\partial |x|^{\alpha_k}} + O(M_1^{-r_1}),\\
\int_1^2 P_2(\beta)\frac{\partial^\beta u}{\partial |y|^\beta}d\beta &= \sum_{k=0}^{M_2}\omega_k^{(2)}\frac{\partial^{\beta_k} u}{\partial |y|^{\beta_k}} + O(M_2^{-r_1}),\\
\int_0^1 P_3(\gamma)\frac{\partial^\gamma u}{\partial |x|^\gamma}d\gamma &= \sum_{k=0}^{M_3}\omega_k^{(3)}\frac{\partial^{\gamma_k} u}{\partial |x|^{\gamma_k}} + O(M_3^{-r_1}),\\
\int_0^1 P_4(\eta)\frac{\partial^\eta u}{\partial |y|^\eta}d\eta &= \sum_{k=0}^{M_4}\omega_k^{(4)}\frac{\partial^{\eta_k} u}{\partial |y|^{\eta_k}} + O(M_4^{-r_1}),
\end{aligned}
\tag{15}
$$

We introduce the following weighted seminorm $|u|_{l,P,Q}$ and weighted norm $\|u\|_{l,P,Q}$

$$
\begin{aligned}
|u|_{l,P,Q} &= \Bigg(\sum_{k=0}^{M_1}\omega_k^{(1)}\|{}_0D_x^{\alpha_k/2}u\|^2 + \sum_{k=0}^{M_2}\omega_k^{(2)}\|{}_0D_y^{\beta_k/2}u\|^2\\
&\quad + \sum_{k=0}^{M_3}\omega_k^{(3)}\|{}_0D_x^{\gamma_k/2}u\|^2 + \sum_{k=0}^{M_4}\omega_k^{(4)}\|{}_0D_y^{\eta_k/2}u\|^2\Bigg)^{1/2},\\
\|u\|_{l,P,Q} &= \left(\|u\|^2_{L^2(\Omega)} + |u|^2_{l,P,Q}\right)^{1/2}.
\end{aligned}
\tag{16}
$$

The weighted seminorm $|u|_{r,P,Q}$ and the weighted norm $\|u\|_{r,P,Q}$ are expressed as follows

$$
\begin{aligned}
|u|_{r,P,Q} = \Bigg(& \sum_{k=0}^{M_1} \omega_k^{(1)} \| {}_xD_L^{\alpha_k/2} u\|^2 + \sum_{k=0}^{M_2} \omega_k^{(2)} \| {}_yD_L^{\beta_k/2} u\|^2 \\
& + \sum_{k=0}^{M_3} \omega_k^{(3)} \| {}_xD_L^{\gamma_k/2} u\|^2 + \sum_{k=0}^{M_4} \omega_k^{(4)} \| {}_yD_L^{\eta_k/2} u\|^2 \Bigg)^{1/2}, \qquad (17) \\
\|u\|_{r,P,Q} = \Big(& \|u\|^2_{L^2(\Omega)} + |u|^2_{r,P,Q} \Big)^{1/2}.
\end{aligned}
$$

The following Lemmas used in the later sections are now provided.

Lemma 2.1 ([45])**.** *The norms* $\|\cdot\|_{l,P,Q}$ *and* $\|\cdot\|_{r,P,Q}$ *are equivalent.*

Lemma 2.2 ([43])**.** *Let* $\sigma > 0$*, if* $u \in H_0^\sigma(\Omega)$*, there exists a positive constant* $C_1 < 1$*, such that*

$$C_1 \|u\|_{H_0^\sigma(\Omega)} \le |u|_{H_0^\sigma(\Omega)} \le \|u\|_{H_0^\sigma(\Omega)}.$$

Lemma 2.3 ([46])**.** *For any* $u \in H_0^\sigma(\Omega)$*,* $\nu \in H_0^{\sigma/2}(\Omega)$*, we have*

$$({}_0D_x^\sigma u, \nu) = ({}_0D_x^{\sigma/2} u, {}_xD_L^{\sigma/2} \nu), \qquad ({}_xD_L^\sigma u, \nu) = ({}_xD_L^{\sigma/2} u, {}_0D_x^{\sigma/2} \nu).$$

Lemma 2.4 ([47])**.** *For* $\sigma \ge 0$*,* $u \in C_0^\infty(\Omega)$*. Then*

$$
\begin{aligned}
({}_0D_x^\sigma u,\ {}_xD_L^\sigma u) &= \cos(\sigma\pi) \| {}_{-\infty}D_x^\sigma \hat{u}\|^2_{L^2(\Omega)} = \cos(\sigma\pi) \| {}_xD_\infty^\sigma \hat{u}\|^2_{L^2(\Omega)}, \\
({}_0D_y^\sigma u,\ {}_yD_L^\sigma u) &= \cos(\sigma\pi) \| {}_{-\infty}D_y^\sigma \hat{u}\|^2_{L^2(\Omega)} = \cos(\sigma\pi) \| {}_yD_\infty^\sigma \hat{u}\|^2_{L^2(\Omega)},
\end{aligned}
$$

where $\hat{u}$ *is the extension of* u *by zero outside* Ω*.*

Lemma 2.5 ([43])**.** *For* $\sigma_1, \sigma_2 \ge 0$*,* $u \in C_0^\infty(\Omega)$*. Then*

$$
\begin{aligned}
({}_0D_x^{\sigma_1}\ {}_0D_y^{\sigma_2} u,\ {}_xD_L^{\sigma_1}\ {}_yD_L^{\sigma_2} u) &= \cos(\sigma_1\pi)\cos(\sigma_2\pi) \| {}_{-\infty}D_x^{\sigma_1}\ {}_{-\infty}D_y^{\sigma_2} \hat{u}\|^2_{L^2(\Omega)}, \\
({}_0D_x^{\sigma_1}\ {}_yD_L^{\sigma_2} u,\ {}_xD_L^{\sigma_1}\ {}_0D_y^{\sigma_2} u) &= \cos(\sigma_1\pi)\cos(\sigma_2\pi) \| {}_{-\infty}D_x^{\sigma_1}\ {}_{-\infty}D_y^{\sigma_2} \hat{u}\|^2_{L^2(\Omega)},
\end{aligned}
$$

where $\hat{u}$ *is the extension of* u *by zero outside* Ω*.*

3. The Crank-Nicolson ADI Galerkin-Legendre spectral method

Some function spaces on $I \times \Omega$ are introduced. Assume that X is a Banach space on Ω with norm $\|\cdot\|_X$. For $\mu \ge 0$, the space $H^\mu(I;X)$ is defined as

$$
\begin{aligned}
H^\mu(I;X) &:= \left\{ y \in L^2(I;X) \big| \|y(\cdot,t)\|_X \in H^\mu(I) \right\}, \\
H_0^\mu(I;X) &:= \left\{ y \in L^2(I;X) \big| \|y(\cdot,t)\|_X \in H_0^\mu(I) \right\}.
\end{aligned}
$$

Let

$$F_x u = \lambda_1 \sum_{k=0}^{M_1} \omega_k^{(1)} \frac{\partial^{\alpha_k} u}{\partial |x|^{\alpha_k}} + \lambda_3 \sum_{k=0}^{M_3} \omega_k^{(3)} \frac{\partial^{\gamma_k} u}{\partial |x|^{\gamma_k}},$$
$$F_y u = \lambda_2 \sum_{k=0}^{M_2} \omega_k^{(2)} \frac{\partial^{\beta_k} u}{\partial |y|^{\beta_k}} + \lambda_4 \sum_{k=0}^{M_4} \omega_k^{(4)} \frac{\partial^{\eta_k} u}{\partial |y|^{\eta_k}}.$$

Then (3) can be rewritten as

$$\frac{\partial u}{\partial t} = F_x u + F_y u + f(x, y, t) + O(M^{-r_1}), \tag{18}$$

where $M = \max(M_m)$, $m = 1, 2, 3, 4$. We discretize the time domain $[0, T]$ by $t_n = n\tau$, $n = 0, 1, \ldots, L$ with $\tau = T/L$. Denote $t_{n+1/2} = (t_{n+1} + t_n)/2$ for $n = 0, 1, \ldots, L-1$. For the function $u(x, y, t) \in C(\Omega \times [0, T])$, take $u^n = u^n(\cdot) = u(\cdot, t_n)$. For convenience, we introduce the following notations,

$$\delta_t u^{n+1/2} = \frac{u^{n+1} - u^n}{\tau}, \quad u^{n+1/2} = \frac{u^{n+1} + u^n}{2}.$$

The temporal derivative of (18) is discretized using the Crank-Nicolson method which leads to

$$\delta_t u^{n+1/2} = (F_x + F_y) u^{n+1/2} + f^{n+1/2} + O(M^{-r_1} + \tau^2), \tag{19}$$

In other words,

$$u^{n+1} - u^n = \tau (F_x + F_y) u^{n+1/2} + \tau f^{n+1/2} + O(\tau M^{-r_1} + \tau^3), \tag{20}$$

We add the perturbation term $\frac{\tau^3}{4} F_x F_y \delta_t u^{n+1/2} = O(\tau^3)$ to the left side of (20) to obtain the following equivalent form

$$\begin{aligned}\left(1 - \frac{\tau}{2} F_x\right)\left(1 - \frac{\tau}{2} F_y\right) u^{n+1} = \left(1 + \frac{\tau}{2} F_x\right)\left(1 + \frac{\tau}{2} F_y\right) u^n + \tau f^{n+1/2} \\ + O(\tau M^{-r_1} + \tau^3).\end{aligned} \tag{21}$$

For the spatial discretization, we use the following basis function [43] in the x and y directions:

$$\begin{aligned}\phi_i(x) &= L_i(\hat{x}) - L_{i+2}(\hat{x}), \quad \hat{x} \in [-1, 1], \quad x = \frac{L\hat{x} + L}{2} \in [0, L], \\ \varphi_i(y) &= L_i(\hat{y}) - L_{i+2}(\hat{y}), \quad \hat{y} \in [-1, 1], \quad y = \frac{L\hat{y} + L}{2} \in [0, L],\end{aligned} \tag{22}$$

where $i = 0, 1, \ldots, N-2$ and $L_i(\hat{z})$ is the Legendre polynomial [44]. The function space S_N can be specified as follows:

$$S_N = \mathrm{span}\{\phi_i(x)\varphi_j(y), \quad i, j = 0, 1, \ldots, N-2\}.$$

Therefore, we can obtain the fully discrete form of (18) as follows: Find $u_N^{n+1} \in S_N$ for $n = 0, 1, \ldots, L-1$, such that

$$\begin{cases} \left(\left(1 - \frac{\tau}{2}F_x\right)\left(1 - \frac{\tau}{2}F_y\right) u_N^{n+1}, v\right) = & \left(\left(1 + \frac{\tau}{2}F_x\right)\left(1 + \frac{\tau}{2}F_y\right) u_N^{n}, v\right) \\ & +\tau(f^{n+1/2}, v), \quad \forall v \in S_N, \\ u_N^0 = \Pi_N^{1,0} u_0, \end{cases} \tag{23}$$

where $\Pi_N^{1,0}$ is a suitable projection operator [48] satisfying

$$(\Pi_N^{1,0} u - u, v) + (\partial_x(\Pi_N^{1,0} u - u), \partial_x v) = 0 \quad \forall v \in S_N. \tag{24}$$

The approximated solution u_N^{n+1} has the following form,

$$u_N^{n+1} = \sum_{i=0}^{N-2} \sum_{j=0}^{N-2} d_{i,j}^{n+1} \phi_i(x) \varphi_j(y). \tag{25}$$

Let $c(\alpha) = -\frac{1}{2\cos(\pi\alpha/2)}$. Taking $v = \phi_i \varphi_j$ $(i, j = 0, 1, \ldots, N-2)$, we obtain the matrix representation,

$$(M_x - \frac{\tau}{2} S_x) D^{n+1} (M_y - \frac{\tau}{2} S_y)^T = (M_x + \frac{\tau}{2} S_x) D^n (M_y + \frac{\tau}{2} S_y)^T + \tau G^n, \tag{26}$$

where M_x, S_x, G^n, $D^{n+1} \in \mathbb{R}^{(N-1)\times(N-1)}$, satisfy

$$\begin{aligned}
&(M_x)_{i,j} = (\phi_i, \phi_j), \quad (M_y)_{i,j} = (\varphi_i, \varphi_j), \\
&(S_x)_{i,j} = \lambda_1 \sum_{k=0}^{M_1} \omega_k^{(1)} c(\alpha_k) \left(S_x^{(\alpha_k)} + (S_x^{(\alpha_k)})^T \right) + \lambda_3 \sum_{k=0}^{M_3} \omega_k^{(3)} c(\gamma_k) \left(S_x^{(\gamma_k)} + (S_x^{(\gamma_k)})^T \right), \\
&(S_y)_{i,j} = \lambda_2 \sum_{k=0}^{M_2} \omega_k^{(2)} c(\beta_k) \left(S_y^{(\beta_k)} + (S_y^{(\beta_k)})^T \right) + \lambda_4 \sum_{k=0}^{M_4} \omega_k^{(4)} c(\eta_k) \left(S_y^{(\eta_k)} + (S_y^{(\eta_k)})^T \right), \\
&(D^{n+1})_{i,j} = d_{i,j}^{n+1}, \quad (G^n)_{i,j} = (f^{n+1/2}, \phi_i \varphi_j), \quad i, j = 0, 1, \ldots, N-2,
\end{aligned}$$

and

$$\begin{aligned}
&(S_x^{(\alpha_k)})_{i,j} = ({}_0D_x^{\alpha_k} \phi_i,\ {}_xD_L^{\alpha_k} \phi_j), \quad (S_x^{(\gamma_k)})_{i,j} = ({}_0D_x^{\gamma_k} \phi_i,\ {}_xD_L^{\gamma_k} \phi_j), \\
&(S_y^{(\beta_k)})_{i,j} = ({}_0D_y^{\beta_k} \varphi_i,\ {}_yD_L^{\beta_k} \varphi_j), \quad (S_y^{(\eta_k)})_{i,j} = ({}_0D_y^{\eta_k} \varphi_i,\ {}_yD_L^{\eta_k} \varphi_j).
\end{aligned}$$

The matrices M_x and M_y can be easily obtained by the orthogonality of the Legendre polynomial [44], so we ignore the details. The matrices $S_x^{(\alpha_k)}$, $S_x^{(\gamma_k)}$, $S_y^{(\beta_k)}$ and $S_y^{(\eta_k)}$ can be obtained using the Jacobi-Gauss quadrature, see the detailed implementation in [43]. Therefore, the computation of the matrices S_x and S_y is obtained easily.

4. Stability and convergence analysis

In this section, we demonstrate the stability and convergence analysis of the Crank-Nicolson ADI Galerkin-Legendre spectral method for the two-dimensional Riesz space distributed-order advection-diffusion equation.

Taking

$$\begin{aligned}
A_x^{\alpha_k}(u,v) &= ({}_0D_x^{\alpha_k/2}u,\ {}_xD_L^{\alpha_k/2}v) + ({}_xD_L^{\alpha_k/2}u,\ {}_0D_x^{\alpha_k/2}v),\\
A_y^{\beta_k}(u,v) &= ({}_0D_y^{\beta_k/2}u,\ {}_yD_L^{\beta_k/2}v) + ({}_yD_L^{\beta_k/2}u,\ {}_0D_y^{\beta_k/2}v),\\
A_x^{\gamma_k}(u,v) &= ({}_0D_x^{\gamma_k/2}u,\ {}_xD_L^{\gamma_k/2}v) + ({}_xD_L^{\gamma_k/2}u,\ {}_0D_x^{\gamma_k/2}v),\\
A_y^{\eta_k}(u,v) &= ({}_0D_y^{\eta_k/2}u,\ {}_yD_L^{\eta_k/2}v) + ({}_yD_L^{\eta_k/2}u,\ {}_0D_y^{\eta_k/2}v).
\end{aligned}$$

Let

$$\begin{aligned}
A(u,v) =&\lambda_1 \sum_{k=0}^{M_1} \omega_k^{(1)} c(\alpha_k) A_x^{\alpha_k}(u,v) + \lambda_2 \sum_{k=0}^{M_2} \omega_k^{(2)} c(\beta_k) A_y^{\beta_k}(u,v)\\
&+ \lambda_3 \sum_{k=0}^{M_3} \omega_k^{(3)} c(\gamma_k) A_x^{\gamma_k}(u,v) + \lambda_4 \sum_{k=0}^{M_4} \omega_k^{(4)} c(\eta_k) A_y^{\eta_k}(u,v).
\end{aligned} \tag{27}$$

The orthogonal projection operator $\Pi_N^{P,Q} : H_0^{\alpha_{M_1}/2}(\Omega) \cap H_0^{\beta_{M_2}/2}(\Omega) \to S_N$ is defined as

$$A(u - \Pi_N^{P,Q}u, v) = 0, \quad \forall v \in S_N.$$

For simplicity, we denote $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. We define the new distributed seminorm $|\cdot|_{P,Q}$ and distributed norm $\|\cdot\|_{P,Q}$ as

$$\begin{aligned}
|u|_{P,Q} = &\left(\lambda_1 \sum_{k=0}^{M_1} \omega_k^{(1)} \|{}_0D_x^{\alpha_k/2}u\|^2 + \lambda_2 \sum_{k=0}^{M_2} \omega_k^{(2)} \|{}_0D_y^{\beta_k/2}u\|^2 \right.\\
&\left. + \lambda_3 \sum_{k=0}^{M_3} \omega_k^{(3)} \|{}_0D_x^{\gamma_k/2}u\|^2 + \lambda_4 \sum_{k=0}^{M_4} \omega_k^{(4)} \|{}_0D_y^{\eta_k/2}u\|^2 \right)^{1/2},\\
\|u\|_{P,\Omega} = &\left(\|u\|^2 + |u|_{P,Q}^2\right)^{1/2}.
\end{aligned}$$

Then we investgate the properties of the projectors $\Pi_N^{P,Q}$ and $\Pi_N^{1,0}$.

Lemma 4.1 ([49])**.** *Let s and r be real numbers satisfying $0 \le s \le r$. Then there exists a positive constant C depending on r such that, for any function $u \in H_0^s(\Omega) \cap H^r(\Omega)$, the following estimate holds:*

$$\|u - \Pi_N^{1,0}u\|_s \le CN^{s-r}\|u\|_r.$$

Lemma 4.2. *Let r be real numbers satisfying $0 < \alpha_k/2, \beta_l/2 < 1 < r$, $\alpha_k/2, \beta_l/2 \neq \frac{1}{2}$, $k = 0, \ldots, M_1, l = 0, \ldots, M_2$. Then there exists a positive constant C independent of N such that, for any function $u \in H_0^{\alpha_{M_1}/2}(\Omega) \cap H_0^{\beta_{M_2}/2}(\Omega) \cap H^r(\Omega)$,* *the following estimate holds:*

$$|u-\Pi_N^{P,Q}u|_{P,\Omega} \le C\left(\sum_{k=0}^{M_1}\omega_k^{(1)}N^{\alpha_k/2-r}\|u\|_r+\sum_{k=0}^{M_2}\omega_k^{(2)}N^{\beta_k/2-r}\|u\|_r\right.$$
$$\left.+\sum_{k=0}^{M_3}\omega_k^{(3)}N^{\gamma_k/2-r}\|u\|_r+\sum_{k=0}^{M_4}\omega_k^{(4)}N^{\eta_k/2-r}\|u\|_r\right).$$

Proof. Based on the results in [43], we have

$$\begin{aligned}|u-\Pi_N^{P,Q}u|^2_{P,Q} &= A(u-\Pi_N^{P,Q}u,u-\Pi_N^{P,Q}u)=A(u-\Pi_N^{P,Q}u,u-u_N)\\ &\le C|u-\Pi_N^{P,Q}u|_{P,Q}|u-u_N|_{P,Q}.\end{aligned} \tag{28}$$

Letting $u_N=\Pi_N^{1,0}u$ and using Lemma 4.1 gives

$$\begin{aligned}|u-\Pi_N^{P,Q}u|_{P,Q} &\le C|u-\Pi_N^{1,0}u|_{P,Q}\\ &\le C\left(\sum_{k=0}^{M_1}\omega_k^{(1)}N^{\alpha_k/2-r}\|u\|_r+\sum_{k=0}^{M_2}\omega_k^{(2)}N^{\beta_k/2-r}\|u\|_r\right.\\ &\quad\left.+\sum_{k=0}^{M_3}\omega_k^{(3)}N^{\gamma_k/2-r}\|u\|_r+\sum_{k=0}^{M_4}\omega_k^{(4)}N^{\eta_k/2-r}\|u\|_r\right).\end{aligned} \tag{29}$$

□

4.1. Stability and convergence analysis for the semi-discrete format

The stability analysis of the semi-discrete form (21) can be presented.

Theorem 4.1. *Assume that u^{n+1} is the solution of problem (21), then there exists a positive constant C independent of N such that*

$$\|u^{n+1}\|_{P,Q} \le C\left(|u^0|_{P,Q}+\tau\sum_{k=0}^{n}\|f^{k+1/2}\|\right).$$

Proof. We give the semi-discrete format

$$(\delta_t u^{k+1/2},v)+\frac{\tau^2}{4}(F_xF_y\delta_t u^{k+1/2},v)=((F_x+F_y)u^{k+1/2},v)+(f^{k+1/2},v). \tag{30}$$

Letting $v=\delta_t u^{k+1/2}$ in (30)

$$\begin{aligned}(\delta_t u^{k+1/2},\delta_t u^{k+1/2})+\frac{\tau^2}{4}(F_xF_y\delta_t u^{k+1/2},\delta_t u^{k+1/2}) =&((F_x+F_y)u^{k+1/2},\delta_t u^{k+1/2})\\ &+(f^{k+1/2},\delta_t u^{k+1/2}).\end{aligned} \tag{31}$$

For the term (F_xF_yv, v), we use Lemma 2.3 to obtain

$$\begin{aligned}(F_xF_yv,v) =&2\lambda_1\lambda_2\sum_{k=0}^{M_1}\sum_{l=0}^{M_2}\omega_k^{(1)}c(\alpha_k)\omega_l^{(2)}c(\beta_l)A^{(\alpha_k,\beta_l)}(v,v)\\ &+2\lambda_1\lambda_4\sum_{k=0}^{M_1}\sum_{l=0}^{M_4}\omega_k^{(1)}c(\alpha_k)\omega_l^{(4)}c(\eta_l)A^{(\alpha_k,\eta_l)}(v,v)\\ &+2\lambda_2\lambda_3\sum_{k=0}^{M_2}\sum_{l=0}^{M_3}\omega_k^{(2)}c(\beta_k)\omega_l^{(3)}c(\gamma_l)A^{(\beta_k,\gamma_l)}(v,v)\\ &+2\lambda_3\lambda_4\sum_{k=0}^{M_3}\sum_{l=0}^{M_4}\omega_k^{(3)}c(\gamma_k)\omega_l^{(4)}c(\eta_l)A^{(\gamma_k,\eta_l)}(v,v),\end{aligned} \tag{32}$$

where

$$\begin{aligned}A^{(\alpha_k,\beta_l)}(v,v) =&({}_0D_x^{\alpha_k/2}\ {}_yD_L^{\beta_l/2}v,\ {}_xD_L^{\alpha_k/2}\ {}_0D_y^{\beta_l/2}v)\\ &+({}_0D_x^{\alpha_k/2}\ {}_0D_y^{\beta_l/2}v,\ {}_xD_L^{\alpha_k/2}\ {}_yD_L^{\beta_l/2}v).\end{aligned}$$

Lemma 2.5 then gives

$$\begin{aligned}2c(\alpha_k)c(\beta_l)A^{(\alpha_k,\beta_l)}(v,v) &= 2c(\alpha_k)c(\beta_l)\{({}_0D_x^{\alpha_k/2}\ {}_yD_L^{\beta_l/2}v,\ {}_xD_L^{\alpha_k/2}\ {}_0D_y^{\beta_l/2}v)\\ &\quad+({}_0D_x^{\alpha_k/2}\ {}_0D_y^{\beta_l/2}v,\ {}_xD_L^{\alpha_k/2}\ {}_yD_L^{\beta_l/2}v)\}\\ &=4c(\alpha_k)c(\beta_l)\cos(\alpha_k\pi/2)\cos(\beta_l\pi/2)\|{}_{-\infty}D_x^{\alpha_k/2}\ {}_{-\infty}D_y^{\beta_l/2}\hat{v}\|^2\\ &=\|{}_{-\infty}D_x^{\alpha_k/2}\ {}_{-\infty}D_y^{\beta_l/2}\hat{v}\|^2.\end{aligned} \tag{33}$$

We have

$$\begin{aligned}(F_xF_yv,v) =&\lambda_1\lambda_2\sum_{k=0}^{M_1}\sum_{l=0}^{M_2}\omega_k^{(1)}\omega_l^{(2)}\|{}_{-\infty}D_x^{\alpha_k/2}\ {}_{-\infty}D_y^{\beta_l/2}\hat{v}\|^2\\ &+\lambda_1\lambda_4\sum_{k=0}^{M_1}\sum_{l=0}^{M_4}\omega_k^{(1)}\omega_l^{(4)}\|{}_{-\infty}D_x^{\alpha_k/2}\ {}_{-\infty}D_y^{\eta_l/2}\hat{v}\|^2\\ &+\lambda_2\lambda_3\sum_{k=0}^{M_2}\sum_{l=0}^{M_3}\omega_k^{(2)}\omega_l^{(3)}\|{}_{-\infty}D_x^{\gamma_l/2}\ {}_{-\infty}D_y^{\beta_k/2}\hat{v}\|^2\\ &+\lambda_3\lambda_4\sum_{k=0}^{M_3}\sum_{l=0}^{M_4}\omega_k^{(3)}\omega_l^{(4)}\|{}_{-\infty}D_x^{\gamma_k/2}\ {}_{-\infty}D_y^{\eta_l/2}\hat{v}\|^2\geq 0,\end{aligned} \tag{34}$$

where $\hat{v}$ is the extension of v by zero outside Ω. Therefore,

$$(F_xF_y\delta_tu^{k+1/2},\delta_tu^{k+1/2})\geq 0. \tag{35}$$

In addition, using Lemma 2.3 and Lemma 2.4

$$
\begin{aligned}
&(F_x u^{k+1/2}, \delta_t u^{k+1/2}) \\
&= \lambda_1 \left(\sum_{k=0}^{M_1} \omega_k^{(1)} \frac{\partial^{\alpha_k} u^{k+1/2}}{\partial |x|^{\alpha_k}}, \delta_t u^{k+1/2} \right) + \lambda_3 \left(\sum_{k=0}^{M_3} \omega_k^{(3)} \frac{\partial^{\gamma_k} u^{k+1/2}}{\partial |x|^{\gamma_k}}, \delta_t u^{k+1/2} \right) \\
&= \frac{\lambda_1}{2\tau} \sum_{k=0}^{M_1} \omega_k^{(1)} c(\alpha_k) [A_x^{\alpha_k}(u^{k+1}, u^{k+1}) - A_x^{\alpha_k}(u^k, u^k)] \\
&\quad + \frac{\lambda_3}{2\tau} \sum_{k=0}^{M_3} \omega_k^{(3)} c(\gamma_k) [A_x^{\gamma_k}(u^{k+1}, u^{k+1}) - A_x^{\gamma_k}(u^k, u^k)] \\
&= \frac{\lambda_1}{2\tau} \sum_{k=0}^{M_1} \omega_k^{(1)} \left(\|{}_{-\infty}D_x^{\alpha_k/2} \hat{u}^k\|^2 - \|{}_{-\infty}D_x^{\alpha_k/2} \hat{u}^{k+1}\|^2 \right) \\
&\quad + \frac{\lambda_3}{2\tau} \sum_{k=0}^{M_3} \omega_k^{(3)} \left(\|{}_{-\infty}D_x^{\gamma_k/2} \hat{u}^k\|^2 - \|{}_{-\infty}D_x^{\gamma_k/2} \hat{u}^{k+1}\|^2 \right),
\end{aligned} \tag{36}
$$

where $\hat{u}^n$ is the extension of u^n by zero outside Ω. Similar to the above calculation process, the following equality can be given

$$
\begin{aligned}
(F_y u^{k+1/2}, \delta_t u^{k+1/2}) =& \frac{\lambda_2}{2\tau} \sum_{k=0}^{M_2} \omega_k^{(2)} \left(\|{}_{-\infty}D_y^{\beta_k/2} \hat{u}^k\|^2 - \|{}_{-\infty}D_y^{\beta_k/2} \hat{u}^{k+1}\|^2 \right) \\
&+ \frac{\lambda_4}{2\tau} \sum_{k=0}^{M_4} \omega_k^{(4)} \left(\|{}_{-\infty}D_y^{\eta_k/2} \hat{u}^k\|^2 - \|{}_{-\infty}D_y^{\eta_k/2} \hat{u}^{k+1}\|^2 \right).
\end{aligned} \tag{37}
$$

According to the above results, we have

$$
|u^{k+1}|_{P,Q}^2 \le |u^k|_{P,Q}^2 + 2\tau \|f^{k+1/2}\|^2. \tag{38}
$$

Summing over k in (47) from 0 to n and using Lemma 2.2 lead to

$$
\|u^{n+1}\|_{P,Q} \le C \left(|u^0|_{P,Q} + \tau \sum_{k=0}^{n} \|f^{k+1/2}\| \right). \tag{39}
$$

The proof is completed. □

Theorem 4.2. *Let $r_1 \le M+1$. Assume that u and u^k are the solutions of problem (3) and (21), respectively, satisfying $u \in H^3(I; H^r(\Omega))$. Then there exists a positive constant C independent of k and τ such that*

$$
\|u^k - u(x, y, t_k)\|_{P,Q} \le C(\tau^2 + M^{-r_1}).
$$

Proof. Let $\theta^k = u^k - u(x, y, t_k)$. We can derive the following error equation,

$$
(\delta_t \theta^{n+1/2}, v) + \frac{\tau^2}{4} (F_x F_y \delta_t \theta^{n+1/2}, v) = ((F_x + F_y)\theta^{n+1/2}, v) + (H_1^n, v) \quad \forall v \in S_N, \tag{40}
$$

where

$$H_1^n = O(\tau^2 + M^{-r_1}). \tag{41}$$

Similar to Theorem 4.1 with $v = \delta_t \theta^{n+1/2}$, we have

$$\|u^k - u(x, y, t_k)\|_{P,Q} \leq C(\tau^2 + M^{-r_1}).$$

The proof is completed. □

4.2. Stability and convergence analysis for the fully discretization

Firstly, we give the stability analysis of the numerical scheme (23).

Theorem 4.3. *Assume that u_N^{n+1} is the solution of problem (23), then there exists a positive constant C independent of N such that*

$$\|u_N^{n+1}\|_{P,Q} \leq C\left(|u_N^0|_{P,Q} + \tau \sum_{k=0}^{n} \|f^{k+1/2}\| \right).$$

Proof. The following problem can be easily derived

$$(\delta_t u_N^{k+1/2}, v) + \frac{\tau^2}{4}(F_x F_y \delta_t u_N^{k+1/2}, v) = ((F_x + F_y) u_N^{k+1/2}, v) + (f^{k+1/2}, v). \tag{42}$$

Letting $v = \delta_t u_N^{k+1/2}$ in (42) leads to

$$\begin{aligned}(\delta_t u_N^{k+1/2}, \delta_t u_N^{k+1/2}) + \frac{\tau^2}{4}(F_x F_y \delta_t u_N^{k+1/2}, \delta_t u_N^{k+1/2}) =& ((F_x + F_y) u_N^{k+1/2}, \delta_t u_N^{k+1/2}) \\ &+ (f^{k+1/2}, \delta_t u_N^{k+1/2}).\end{aligned} \tag{43}$$

By the Theorem 4.1, we have

$$(F_x F_y \delta_t u_N^{k+1/2}, \delta_t u_N^{k+1/2}) \geq 0, \tag{44}$$

$$\begin{aligned}&(F_x u_N^{k+1/2}, \delta_t u_N^{k+1/2}) \\ &= \frac{\lambda_1}{2\tau} \sum_{k=0}^{M_1} \omega_k^{(1)} \left(\|{}_{-\infty}D_x^{\alpha_k/2} \hat{u}_N^k\|^2 - \|{}_{-\infty}D_x^{\alpha_k/2} \hat{u}_N^{k+1}\|^2 \right) \\ &\quad + \frac{\lambda_3}{2\tau} \sum_{k=0}^{M_3} \omega_k^{(3)} \left(\|{}_{-\infty}D_x^{\gamma_k/2} \hat{u}_N^k\|^2 - \|{}_{-\infty}D_x^{\gamma_k/2} \hat{u}_N^{k+1}\|^2 \right),\end{aligned} \tag{45}$$

and

$$\begin{aligned}(F_y u_N^{k+1/2}, \delta_t u_N^{k+1/2}) =& \frac{\lambda_2}{2\tau} \sum_{k=0}^{M_2} \omega_k^{(2)} \left(\|{}_{-\infty}D_y^{\beta_k/2} \hat{u}_N^k\|^2 - \|{}_{-\infty}D_y^{\beta_k/2} \hat{u}_N^{k+1}\|^2 \right) \\ &+ \frac{\lambda_4}{2\tau} \sum_{k=0}^{M_4} \omega_k^{(4)} \left(\|{}_{-\infty}D_y^{\eta_k/2} \hat{u}_N^k\|^2 - \|{}_{-\infty}D_y^{\eta_k/2} \hat{u}_N^{k+1}\|^2 \right).\end{aligned} \tag{46}$$

where $\hat{u}_N^n$ is the extension of u_N^n by zero outside Ω. According to the above results, we have

$$|u_N^{k+1}|_{P,Q}^2 \leq |u_N^k|_{P,Q}^2 + 2\tau\|f^{k+1/2}\|^2. \tag{47}$$

Summing over k in (47) from 0 to n and using Lemma 2.2 lead to

$$\|u_N^{n+1}\|_{P,Q} \leq C\left(|u_N^0|_{P,Q} + \tau\sum_{k=0}^{n}\|f^{k+1/2}\|\right). \tag{48}$$

The proof is completed. □

Then the convergence analysis of the full discretization can be established.

Theorem 4.4. *Let $r \geqslant 1$ and $r_1 \leq M+1$. Assume that u and u_N^k are the solutions of problem (3) and (23), respectively, satisfying $u \in H^3(I; H^r(\Omega))$. Then there exists a positive constant C independent of k, τ, and N such that*

$$\begin{aligned}\|u_N^k - u(x,y,t_k)\|_{P,Q} \leq & C\Bigg(\tau^2 + M^{-r_1} + \sum_{l=0}^{M_1}\omega_l^{(1)}N^{\alpha_l/2-r} + \sum_{l=0}^{M_2}\omega_l^{(2)}N^{\beta_l/2-r} \\ & + \sum_{l=0}^{M_3}\omega_l^{(3)}N^{\gamma_l/2-r} + \sum_{l=0}^{M_4}\omega_l^{(4)}N^{\eta_l/2-r}\Bigg).\end{aligned}$$

Proof. Let $\tilde{u} = \Pi_N^{P,Q}u$, $\xi = u - \tilde{u}$, $e = \tilde{u} - u_N$. We can present the following error equation,

$$(\delta_t e^{n+1/2}, v) + \frac{\tau^2}{4}(F_xF_y\delta_t e^{n+1/2}, v) = ((F_x+F_y)e^{n+1/2}, v) + (H_2^n, v) \quad \forall v \in S_N, \tag{49}$$

where

$$H_2^n = -\delta_t\xi^{n+1/2} + O(\tau^2 + M^{-r_1}) - \frac{\tau^2}{4}F_xF_y\delta_t\xi^{n+1/2}. \tag{50}$$

Similar to Theorem 4.3 with $v = \delta_t e^{n+1/2}$, we only need to estimate

$$|e^0|_{P,Q} + \|H_2^n\|, \quad n = 1, 2, \ldots, L.$$

For the initial error e^0, we have

$$\begin{aligned}|e^0|_{P,Q} &\leq \|e^0\|_{P,Q} = \|\Pi_N^{P,Q}u_0 - \Pi_N^{1,0}u_0\|_{P,Q} \\ &\leq \|\Pi_N^{P,Q}u_0 - u_0\|_{P,Q} + \|u_0 - \Pi_N^{1,0}u_0\|_{P,Q} \\ &\leq C\Bigg(\sum_{l=0}^{M_1}\omega_l^{(1)}N^{\alpha_l/2-r} + \sum_{l=0}^{M_2}\omega_l^{(2)}N^{\beta_l/2-r} \\ &\quad + \sum_{li=0}^{M_3}\omega_l^{(3)}N^{\gamma_l/2-r} + \sum_{l=0}^{M_4}\omega_l^{(4)}N^{\eta_l/2-r}\Bigg).\end{aligned} \tag{51}$$

Furthermore, the following estimates can be obtained,

$$\begin{aligned}\|\delta_t\xi^{n+1/2}\| \le C|\delta_t\xi^{n+1/2}|_{P,Q} \le& C\Bigg(\sum_{l=0}^{M_1}\omega_l^{(1)}N^{\alpha_l/2-r}+\sum_{l=0}^{M_2}\omega_l^{(2)}N^{\beta_l/2-r}\\ &+\sum_{l=0}^{M_3}\omega_l^{(3)}N^{\gamma_l/2-r}+\sum_{l=0}^{M_4}\omega_l^{(4)}N^{\eta_l/2-r}\Bigg),\end{aligned} \tag{52}$$

$$\|\frac{\tau^2}{4}F_xF_y\delta_t\xi^{n+1/2}\| \le C\tau^2. \tag{53}$$

Combining (51)-(53) and using Lemma 4.2 gives

$$\begin{aligned}\|u_N^k-u(x,y,t_k)\|_{P,Q} \le& C\Bigg(\tau^2+M^{-r_1}+\sum_{l=0}^{M_1}\omega_l^{(1)}N^{\alpha_l/2-r}+\sum_{l=0}^{M_2}\omega_l^{(2)}N^{\beta_l/2-r}\\ &+\sum_{l=0}^{M_3}\omega_l^{(3)}N^{\gamma_l/2-r}+\sum_{l=0}^{M_4}\omega_l^{(4)}N^{\eta_l/2-r}\Bigg).\end{aligned}$$

The proof is completed. □

Remark 4.1. *When the weight function $P(\alpha)$ exists weak singularityis, we can rewrite the distributed-order Riesz space derivative $\int_{\alpha_{\min}}^{\alpha_{\max}}P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}d\alpha$ as follows*

$$\int_{\alpha_{\min}}^{\alpha_{\max}}P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}d\alpha=\int_{\alpha_{\min}}^{\alpha_{\max}}P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}\omega\omega^{-1}d\alpha,$$

where $\omega=(\alpha-\alpha_{\min})^a(\alpha_{\max}-\alpha)^b$ $(a,b>-1)$ is a suitable weight function such that $P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}\omega$ is more smooth than $P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}$. Based on the form of ω^{-1}, we can employ the Jacobi-Gauss quadrature,

$$\begin{aligned}&\int_{\alpha_{\min}}^{\alpha_{\max}}P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}\omega\omega^{-1}d\alpha\\ &=\left(\frac{\alpha_{\max}-\alpha_{\min}}{2}\right)^{1-a-b}\int_{-1}^{1}\hat{P}(\hat{\alpha})(1+\hat{\alpha})^{-a}(1-\hat{\alpha})^{-b}d\hat{\alpha}\\ &=\left(\frac{\alpha_{\max}-\alpha_{\min}}{2}\right)^{1-a-b}\sum_{i=0}^{\tilde{M}}\hat{P}(\hat{\alpha})(1+\hat{\alpha}_i)^{-a}(1-\hat{\alpha}_i)^{-b},\end{aligned}$$

here $\hat{P}(\hat{\alpha})=P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}\omega$ and $\alpha=\frac{(\alpha_{\max}-\alpha_{\min})\hat{\alpha}+\alpha_{\min}+\alpha_{\max}}{2}$, $\{\hat{\alpha}_i\}$ are the Jacobi-Gauss points with respect to the weight function $(1+\hat{\alpha})^{-a}(1-\hat{\alpha})^{-b}$.

5. Numerical results

Space distributed-order fractional equations are more flexible to represent the effect of the medium and its spatial interactions with the fluid, such as accelerated superdiffusion [18]. In this section, some numerical examples are researched for illustrating the theoretical analysis.

Example 5.1. *We consider the following Riesz space distributed-order equation*

$$\begin{aligned}\frac{\partial u}{\partial t} = &\int_1^2 P(\alpha)\frac{\partial^\alpha u}{\partial |x|^\alpha}d\alpha + \int_1^2 P(\alpha)\frac{\partial^\alpha u}{\partial |y|^\alpha}d\alpha + \int_0^1 Q(\gamma)\frac{\partial^\gamma u}{\partial |x|^\gamma}d\gamma \\ &+ \int_0^1 Q(\gamma)\frac{\partial^\gamma u}{\partial |y|^\gamma}d\gamma + f(x,y,t), \quad (x,y,t)\in\Omega\times I,\end{aligned} \tag{54}$$

with boundary condition

$$u = 0, \quad (x,y,t)\in\partial\Omega\times I, \tag{55}$$

and initial condition

$$u(x,y,0) = x^2(1-x)^2y^2(1-y)^2, \quad (x,y)\in\Omega, \tag{56}$$

where $I = (0,1]$, $\Omega = (0,1)\times(0,1)$,

$$P(\alpha) = -2\Gamma(5-\alpha)\cos(\pi\alpha/2), \quad Q(\gamma) = -2\Gamma(5-\gamma)\cos(\pi\gamma/2).$$
$$\begin{aligned}f(x,y,t) = &e^tx^2(1-x)^2y^2(1-y)^2 - e^ty^2(1-y)^2[g_1(x)+g_1(1-x)+g_2(x) \\ &+ g_2(1-x)] - e^tx^2(1-x)^2[g_1(y)+g_1(1-y)+g_2(y)+g_2(1-y)],\end{aligned}$$

here

$$\begin{aligned}g_1(z) =&\Gamma(5)\frac{z^3-z^2}{\ln z} - 2\Gamma(4)\left[\frac{3z^2-2z}{\ln z} - \frac{z^2-z}{(\ln z)^2}\right] \\ &+ \frac{\Gamma(3)}{\ln z}\left[6z-2-\frac{5z}{\ln z}+\frac{3}{\ln z}+\frac{2z}{(\ln z)^2}-\frac{2}{(\ln z)^2}\right], \\ g_2(z) =&\Gamma(5)\frac{z^4-z^3}{\ln z} - 2\Gamma(4)\left[\frac{4z^3-3z^2}{\ln z} - \frac{z^3-z^2}{(\ln z)^2}\right] \\ &+ \frac{\Gamma(3)}{\ln z}\left[12z^2-6z-\frac{1}{\ln z}\left(7z^2-5z-\frac{2z^2}{\ln z}+\frac{2z}{\ln z}\right)\right].\end{aligned}$$

The exact solution of (54) is $u(x,y,t) = e^tx^2(1-x)^2y^2(1-y)^2$. The convergence orders in time and space in the L^2-norm sense are defined as

$$\text{order} = \begin{cases} \frac{\log(\|error(\tau_1)\|/\|error(\tau_2)\|)}{\log(\tau_1/\tau_2)} & \text{in time,} \\ \frac{\log(\|error(N_1)\|/\|error(N_2)\|)}{\log(N_2/N_1)} & \text{in space,}\end{cases} \tag{57}$$

where error $= \|u_{\text{exact}} - u_N^n\|$ is the error equation, $\tau_1 \neq \tau_2$, and $N_1 \neq N_2$.

In the example, we take $\tau = 0.01$, $N = 50$ and $M_m = M = 10$ ($m = 1,2,3,4$). Figure 1 depicts the numerical solution and the exact solution of Example 5.1. We can observe that the numerical solution is in good agreement with the exact solution. Table 1 demonstrates the relationship of the L^2-errors, $H^{P,Q}$-errors and convergence order with the change of τ for the considered equation when $N = 32$ and $M = 10$. It is obvious that our method has second order convergence in time. The L^2-errors, $H^{P,Q}$-errors and convergence order versus

Table 1: The errors versus τ, and order of L^2-norm and $H^{P,Q}$-norm for Example 5.1.

τ	L^2- error	order	$H^{P,Q}-$ error	order
1/20	5.5639e-05	–	5.8957e-04	–
1/40	1.4026e-05	1.9880	1.4949e-04	1.9796
1/80	3.5131e-06	1.9973	3.7518e-05	1.9944
1/160	8.7790e-07	2.0006	9.4026e-06	1.9965

Table 2: The errors versus N, and order of L^2-norm and $H^{P,Q}$-norm for Example 5.1.

N	L^2- error	order	$H^{P,Q}-$ error	order
8	5.1567e-05	–	2.6448e-04	–
16	1.6841e-05	1.6145	9.5294e-05	1.4727
32	1.0670e-06	3.9803	7.4703e-06	3.6731
64	2.4951e-08	5.4183	2.6190e-07	4.8341

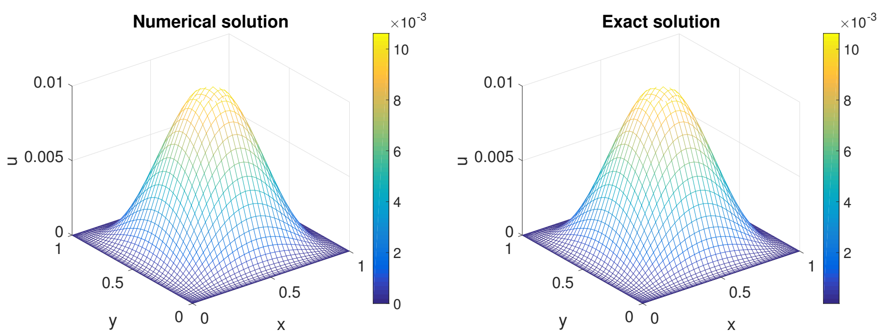

Figure 1: The exact solution and the numerical solution for Example 5.1.

Table 3: Comparison of the L^∞-errors of our method (23) and the finite difference scheme in [25] for Example 5.1 when $M = 1000$.

N	$1/\tau$	$L^\infty-$ error (23)	$L^\infty-$ error [25]
20	20	5.5633e-05	3.592e-02
40	40	1.4026e-05	8.916e-03
80	80	3.5141e-06	2.204e-03
160	160	8.7901e-07	5.388e-04

Table 4: Comparison between CPU time of the ADI scheme and CPU time of the scheme without ADI.

N	τ	CPU(ADI scheme)	CPU(scheme without ADI)
10	1/10	0.4126s	0.4602s
20	1/20	1.0791s	1.2941s
80	1/80	18.7639s	27.5882s
100	1/100	31.1635s	58.1755s

Table 5: The errors (MQ) versus M and order of L^2-norm and $H^{P,Q}$-norm for Example 5.1.

M	$L^2 -$ error (MQ)	order	$H^{P,Q} -$ error (MQ)	order
4	2.2335e-04	–	1.4511e-03	–
8	5.6082e-05	1.9937	3.6378e-04	1.9960
16	1.3797e-05	2.0232	8.7772e-05	2.0512
32	3.1492e-06	2.1313	2.2409e-05	1.9697

Table 6: The errors (GQ) versus M and order of L^2-norm and $H^{P,Q}$-norm for Example 5.1.

M	$L^2 -$ error (GQ)	order	$H^{P,Q} -$ error (GQ)	order
4	1.5572e-04	–	1.0213e-03	–
8	4.5767e-05	1.7666	3.6442e-04	1.4867
16	4.6176e-06	3.3091	4.7858e-05	2.9288
32	1.7143e-07	4.7515	3.1709e-06	3.9158

Table 7: The errors versus τ, and order of L^2-norm and $H^{P,Q}$-norm for Example 5.2.

τ	L^2- error	order	$H^{P,Q}-$ error	order
1/80	2.7713e-03	–	6.7778e-03	–
1/160	6.9947e-04	1.9862	1.7212e-03	1.9774
1/320	1.7549e-04	2.0280	4.3200e-04	1.9943
1/640	4.2493e-05	2.1185	1.0618e-04	2.0245

Table 8: The errors versus N, and order of L^2-norm and $H^{P,Q}$-norm for Example 5.2.

N	L^2- error	order	$H^{P,Q}-$ error	order
16	3.4062e-04	–	1.3757e-03	–
32	9.6235e-05	1.8235	4.9548e-04	1.4733
64	6.5342e-06	3.8805	6.4332e-05	2.9452
128	1.4996e-07	5.4454	2.5178e-06	4.6753

N are shown in Table 2 when fixing $\tau = 0.001$ and $M = 10$. The errors display exponential decay. In other words, the spectral accuracy in space is observed. In [25], the finite difference scheme for the one-dimensional and two-dimensional Riesz space distributed-order advection-diffusion equation is studied. We compare the L^∞-error between our numerical method (23) and the scheme of [25] in Table 3. Obviously, our method presents better numerical results in this example. In other words, the above results describe that the Crank-Nicolson ADI Galerkin-Legendre spectral method has a high order accuracy is superior to the schemes proposed in [25]. Table 4 presents the comparison between CPU time of the ADI scheme and CPU time of the scheme without ADI, the ADI scheme reduces the used CPU time and improves the computational efficiency.

Moreover, some articles [29, 30] applied the midpoint quadrature (MQ) rule to carry out the distributed-order Riesz derivative $\int_{\alpha_{\min}}^{\alpha_{\max}} P(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}d\alpha$. The Gauss quadrature (GQ) is proposed in this paper. In order to compare the computational accuracy between the MQ rule and the GQ, we use the MQ rule for approximating $\int_1^2 P_1(\alpha)\frac{\partial^\alpha u}{\partial|x|^\alpha}d\alpha$, $\int_1^2 P_2(\beta)\frac{\partial^\beta u}{\partial|y|^\beta}d\beta$, $\int_0^1 P_3(\gamma)\frac{\partial^\gamma u}{\partial|x|^\gamma}d\gamma$ and $\int_0^1 P_4(\eta)\frac{\partial^\eta u}{\partial|y|^\eta}d\eta$ similar to the procedure of the above research. Table 5 shows the errors (MQ) and order of L^2-norm and $H^{P,Q}$-norm versus M. The errors (GQ) and order of L^2-norm and $H^{P,Q}$-norm versus M are displayed in Table 6. It is obvious that the GQ has a higher computational accuracy than the MQ rule.

Example 5.2. The following Riesz space distributed-order equation is con-

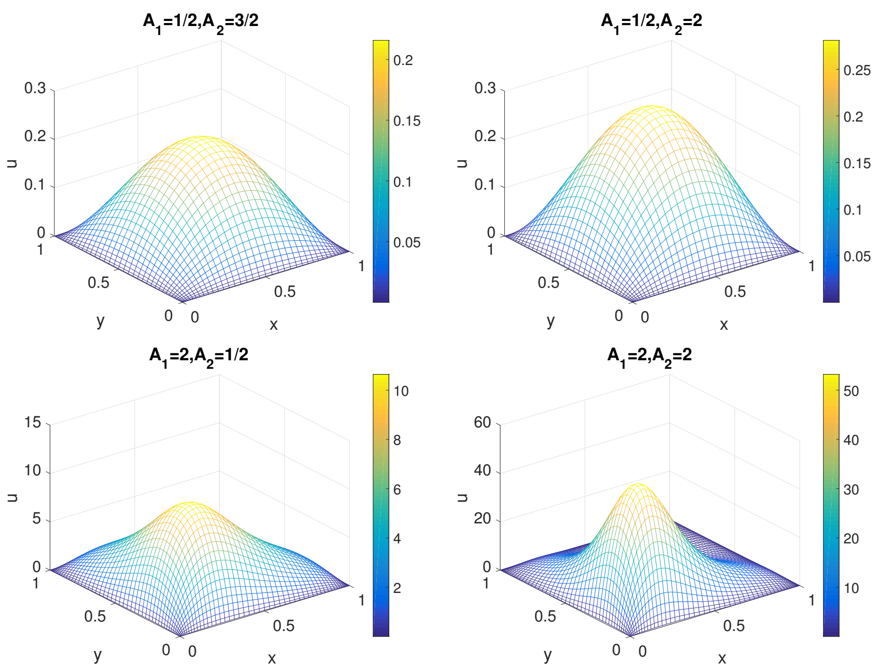

Figure 2: The numerical solution of Example 5.2 for different A_1 and A_2 with $M = 10$, $N = 50$, $\tau = 1/1000$, $t = 1$.

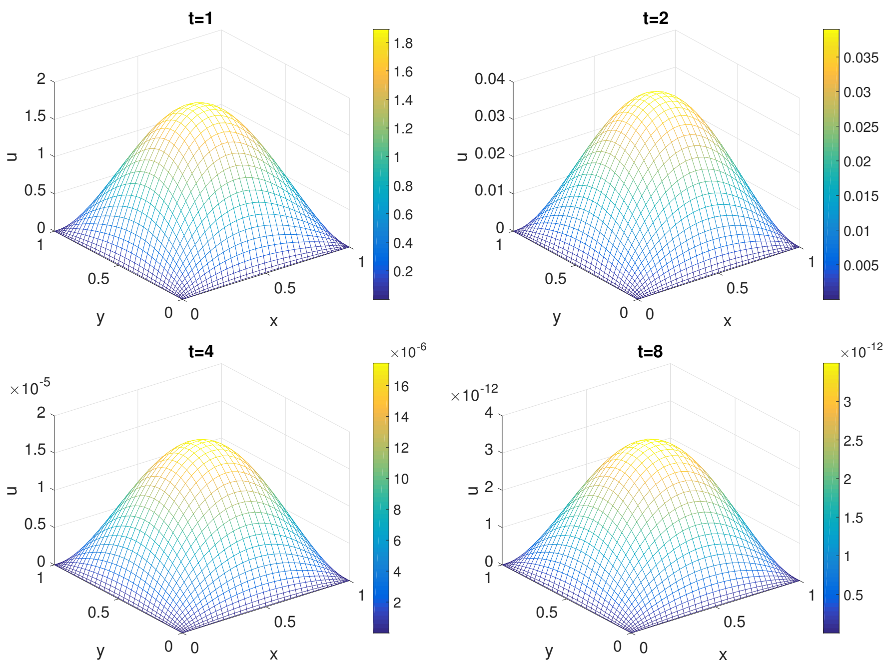

Figure 3: The numerical solution of Example 5.2 for different t with $M = 10$, $N = 50$, $\tau = 1/1000$, $A_1 = 1$, $A_2 = 1$.

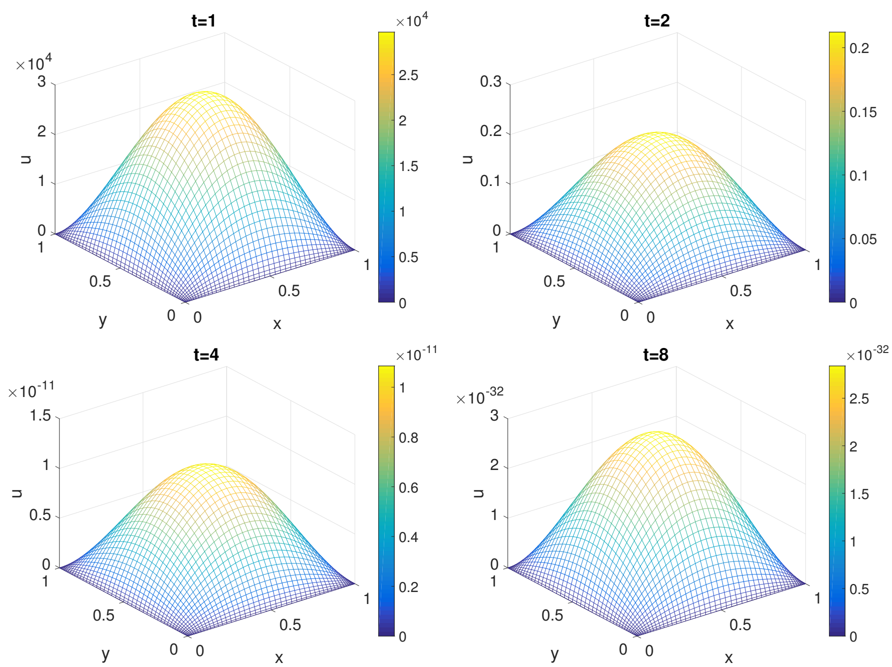

Figure 4: The numerical solution of (62) for different t with $N = 50$, $\tau = 1/1000$.

sidered,

$$\frac{\partial u}{\partial t} = \int_1^2 P(\alpha)\frac{\partial^\alpha u}{\partial |x|^\alpha} d\alpha + \int_1^2 P(\alpha)\frac{\partial^\alpha u}{\partial |y|^\alpha} d\alpha + \int_0^1 Q(\gamma)\frac{\partial^\gamma u}{\partial |x|^\gamma} d\gamma \\ + \int_0^1 Q(\gamma)\frac{\partial^\gamma u}{\partial |y|^\gamma} d\gamma, \quad (x, y, t) \in \Omega \times I, \tag{58}$$

with boundary condition

$$u = 0, \quad (x, y, t) \in \partial\Omega \times I, \tag{59}$$

with initial condition

$$u(x, y, 0) = e^{-10[(x-0.5)+(y-0.5)]}, \quad (x, y) \in \Omega, \tag{60}$$

where $I = (0, 8]$, $\Omega = (0, 1) \times (0, 1)$. $P(\alpha)$ and $Q(\gamma)$ can be chosen as

$$P(\alpha) = e^{-A_1\alpha}, \quad Q(\alpha) = e^{-A_2\alpha} \tag{61}$$

here $A_1, A_2 > 0$. To obtain the error estimate, we approximate the error as $\text{error}(\tau) = \|u(x, y, t, 0.0005) - u(x, y, t, \tau)\|_{L^2}$, or $\text{error}(N) = \|u(x, y, t, 1024) - u(x, y, t, N)\|_{L^2}$ when $u(x, y, t)$ represents the numerical solution.

Taking $A_1 = A_2 = 1$. Table 7 shows the relationship of the L^2-errors, $H^{P,Q}$-errors and convergence order with the change of τ for the considered equation when $N = 50$. The L^2-errors, $H^{P,Q}$-errors with convergence order in space are exhibited in Table 8 when fixing $\tau = 0.001$. It is shown that the proposed scheme is convergent of second-order accuracy in time and spectral accuracy in space. Figure 2 illustrates the impact of A_1 and A_2 on the diffusion behaviour of $u(x, y, t)$. We can observe that with the change of A_1 and A_2, $u(x, y, t)$ can present different shapes. When A_1 and A_2 become larger, there is a decrease in amplitude and more diffusive behaviour, and the maximum peak always occurs at the center of the domain. The numerical solutions of $u(x, y, t)$ at different times $t = 1, 2, 4, 8$ are displayed in Fig. 3. The diffusion behaviour of $u(x, y, t)$ decays with increasing time. The observations are in accordance with the known results. For the Riesz space fractional advection-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^\alpha u}{\partial |x|^\alpha} + \frac{\partial^\alpha u}{\partial |y|^\alpha} + \frac{\partial^\gamma u}{\partial |x|^\gamma} + \frac{\partial^\gamma u}{\partial |y|^\gamma}, \tag{62}$$

with the same initial-boundary conditions as (58). In Fig. 4, the evolution of (62) at different times $t = 1, 2, 4, 8$ is depicted when $\alpha = \frac{3}{2}$ and $\gamma = \frac{1}{2}$, which decays with increasing time. The descent rate of diffusion behaviour in Fig. 4 is faster than diffusion behaviour in Fig. 3.

6. Conclusions

It is well known that distributed-order differential equations are more powerful tools to describe complex dynamical systems than classical and fractional-order models because of their nonlocal properties. Thus, pursuing a numerical method of high order accuracy is a very important issue for solving the distributed-order diffusion equations. In this work, we consider the two-dimensional Riesz space distributed-order advection-diffusion equation. The Gauss quadrature has higher computational accuracy than the general mid-point quadrature rule is applied to approximating the distributed order Riesz space derivative such that the considered equation is transformed into a multi-term fractional equation. Next, we propose the Crank-Nicolson ADI Galerkin-Legendre spectral scheme for the transformed equation. The stability and convergence analysis are proved for the numerical method. It is shown to be convergent of second-order accuracy in time and spectral accuracy in space which is higher than some recently studied schemes. Some numerical examples are given to demonstrate the high efficiency of our method as well as the high order accuracy.

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