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Numerical method and analytical technique of the modified anomalous subdiffusion equation with a nonlinear source term ⋆

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Abstract

In this paper, we consider a modified anomalous subdiffusion equation with a nonlinear source term for describing processes that become less anomalous as time progresses by the inclusion of a second fractional time derivative acting on the diffusion term. A new implicit difference method is constructed. The stability and convergence are discussed using a new energy method. Finally, some numerical examples are given. The numerical results demonstrate the effectiveness of theoretical analysis.

Key words: Implicit difference method, modified anomalous subdiffusion equation, nonlinear source terms, energy method, stability and convergence.
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1 Introduction

In recent years, it has been reported that, in numerous physical and biological systems many diffusion rates of species cannot be characterized by the single

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parameter of the diffusion constant [40]. Instead, the (anomalous) diffusion is characterized by a scaling parameter γ as well as a diffusion constant K, and the mean square displacement of diffusing species $\langle x^2(t)\rangle$ scales as a nonlinear power-law in time, i.e.,

$$\langle x^2(t)\rangle \sim \frac{2K_\gamma}{\Gamma(1+\gamma)}t^\gamma, \quad t \to \infty,$$

where γ (with $0 < \gamma < 1$) is the anomalous diffusion exponent and K_γ is the generalized diffusion coefficient. Ordinary (or Brownian) diffusion corresponds to $\gamma = 1$ with $K_1 = D$ (the ordinary diffusion coefficient). For example, single particle tracking experiments and photo-bleaching recovery experiments have revealed sub-diffusion ($0 < \gamma < 1$) of proteins and lipids in a variety of cell membranes [2,7–9,35,37,38]. Anomalous subdiffusion has also been observed in neural cell adhesion molecules [36]. Indeed anomalous subdiffusion (the case with $0 < \gamma < 1$) is generic in media with obstacles [24,25] or binding sites [26]. For anomalous subdiffusive random walks, the continuum description via the ordinary diffusion equation is replaced by the fractional diffusion equation. It has been suggested that the probability density function (pdf) $u(x,t)$ that describes anomalous subdiffusion particles follows the anomalous sub-diffusion equation [21]:

$$\frac{\partial u}{\partial t} = \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\left[K_\gamma \frac{\partial^2 u}{\partial x^2}\right], 0 \le x \le a, 0 < t \le T, \tag{1}$$

where $u(x,t)$ is the probability density that the particle that started at 0 at time 0 is at x at time t, $\partial^{1-\gamma}u/\partial t^{1-\gamma}$ denotes the Riemann-Liouville fractional derivative of order $1-\gamma$ defined by

$$\frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}u(x,t) = {}_0D_t^{1-\gamma}u(x,t) = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\eta)}{(t-\eta)^{1-\gamma}}d\eta, \tag{2}$$

with $0 \le \gamma \le 1$. For $\gamma = 1$ one recovers the identity operator, and for $\gamma = 0$ the ordinary first-order derivative.

Yuste and Lindenberg considered a combination of these two phenomena and proposed to solve the A+A reaction-subdiffusion problem in one dimension [43]. Further, Yuste, Acedo and Lidenberg [44] proposed the $A+B$ reaction-subdiffusion equations:

$$\begin{aligned}
\frac{\partial}{\partial t}a(x,t) &= K_\gamma {}_0D_t^{1-\gamma}\frac{\partial^2}{\partial x^2}a(x,t) - R_\gamma(x,t),\\
\frac{\partial}{\partial t}b(x,t) &= K_\gamma {}_0D_t^{1-\gamma}\frac{\partial^2}{\partial x^2}b(x,t) - R_\gamma(x,t).
\end{aligned}$$

The reaction term has many different forms. Seki, Wojcik and Tachiya [34] proposed a reaction-subdiffusion equation which at long times corresponds to choosing a reaction term of the form $R_\gamma(x,t) = k_0 D_t^{1-\gamma} a(x,t)b(x,t)$.

Tan et al. [41] and Chen et al. [5] considered Stokes' first problem for a heated generalized second grade fluid with fractional derivative with a non-homogeneous forcing term:

$$\frac{\partial u(x,t)}{\partial t} =_0 D_t^{1-\gamma}\left[\kappa_1 \frac{\partial^2 u(x,t)}{\partial x^2}\right] + \kappa_2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t). \tag{3}$$

There are numerous approaches to modeling anomalous diffusive behaviour, such as, continuous time random walks, Monte Carlo simulations [25], Langevin equations and fractional diffusion equations [21]. The fractional diffusion equation is characterised by the presence of either a fractional temporal derivative or fractional spatial derivative or both (time-fractional diffusion equations were introduced by Zaslavsky [46], and also references [21,22] for a recent review). Other fractional variants are the fractional Fokker-Planck equation for anomalous diffusion due to an external force and fractional reaction-diffusion equations [6,10,32,33,44] for reactions where the products and reactants diffuse anomalously. These equations involve only a single temporal fractional derivative acting on the diffusion term.

Recently a model has been proposed [3,12,39,40] for describing processes that become less anomalous as time progresses by the inclusion of a secondary fractional time derivative acting on a diffusion operator, $\mathscr{L}_x = K\partial^2/\partial x^2$,

$$\frac{\partial u(x,t)}{\partial t} = \left(A\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + B\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right)\mathscr{L}_x u(x,t), \tag{4}$$

where $0 < \alpha < \beta \leqslant 1$, K is diffusion coefficient, A and B are positive constants. The subdiffusive motion is characterized by an asymptotic longtime behavior of the mean square displacement of the form

$$\langle x^2(t)\rangle = \frac{2A}{\Gamma(1+\alpha)}t^\alpha + \frac{2B}{\Gamma(1+\beta)}t^\beta. \tag{5}$$

A possible application of this equation is in econophysics where there is an increasing interest in modelling using continuous time random walks [16,23,27–31]. In particular the crossover between more and less anomalous behaviour has been observed in the volatility of some share prices [17–19].

In this paper, we consider the following modified anomalous subdiffusion equation with a nonlinear source term:

$$\frac{\partial u(x,t)}{\partial t} = \left(A\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + B\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right)\left[\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\right] + g(u,x,t). \tag{6}$$

Recently, many researchers have proposed various numerical methods to solve the space or time fractional partial differential equations. Liu, Anh and Turner [13] proposed a computationally effective method of lines for the space fractional partial differential equation. They transformed the space fractional partial differential equation into a system of ordinary differential equations that was then solved using backward differentiation formulas. Meerschaert and Tadjeran [20] developed finite difference approximations for fractional advection-dispersion flow equations. Roop [14] investigated the computational aspects of the Galerkin approximation using continuous piecewise polynomial basis functions on a regular triangulation of a bounded domain in $\mathbb{R}^2$. Liu *et al.* [15] also investigated the stability and convergence of difference methods for the space-time fractional advection-diffusion equation. Yu et al. [42] developed a reliable algorithm of the Adomian decomposition method to solve the linear and nonlinear space-time fractional reaction-diffusion equations in the form of a rapidly convergent series with easily computable components.

Yuste and Acedo [45] proposed an explicit finite difference method and a new Von Neumann-type stability analysis for the anomalous sub-diffusion equation (1). However, they did not give a convergence analysis and pointed out the difficulty of this task when implicit methods are considered. Langlands and Henry [11] also investigated this problem and proposed an implicit numerical scheme (L_1 approximation), and discussed the accuracy and stability of this scheme. However, the global accuracy of the implicit numerical scheme has not been derived and it seems that the unconditional stability for all γ in the range $0 < \gamma \leq 1$ has not been established. Recently, Chen and Liu et al. [4] presented a Fourier method for the anomalous sub-diffusion equation (1), and they gave the stability analysis and the global accuracy analysis of the difference approximation scheme. Zhuang and Liu et al. [47] also proposed new solution and analytical techniques of implicit numerical methods for the anomalous sub-diffusion equation (1). Chen and Liu et al. [5] proposed implicit and explicit numerical approximation schemes for the Stokes' first problem for a heated generalized second grade fluid with fractional derivative (3). The stability and convergence of the numerical schemes are discussed using a Fourier method. A Richardson extrapolation technique for improving the order of convergence of the implicit scheme is presented. However, effective numerical methods and error analysis for the modified anomalous subdiffusion equation with a nonlinear source term are still in their infancy and are open problems. The main purpose of this paper is to solve and analyze this problem by introducing an implicit difference method and new analytical techniques.

The structure of the remainder of this paper as follows. In Section 2, an implicit numerical method for the modified anomalous subdiffusion equation with a nonlinear source term is proposed. In Sections 3 and 4, the stability and convergence of the implicit numerical method are discussed, respectively. Finally, some numerical results for the modified anomalous subdiffusion equation with a nonlinear source term are given.

2 An implicit numerical method for the modified anomalous subdiffusion equation

In this paper, we consider the function describing processes that get less anomalous in the course of time, namely

$$\frac{\partial u(x,t)}{\partial t} = \left(A\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + B\frac{\partial^{1-\beta}}{\partial^{1-\beta}}\right)\left[\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\right] + g(u,x,t),$$
$$0 \le x \le L, \quad 0 \le t \le T, \tag{7}$$

with initial and boundary conditions:

$$u(x,0) = \phi(x), \qquad 0 \le x \le L, \tag{8}$$
$$u(0,t) = \varphi_1(t), u(L,t) = \varphi_2(t), 0 \le t \le T, \tag{9}$$

where $0 \le \alpha \le \beta \le 1$, $u(x,t)$ is the probability density of the particle that started at time 0 is at x at time t. In Eq. (7), the operators $\partial^{1-\alpha}/\partial t^{1-\alpha}$ and $\partial^{1-\beta}/\partial t^{1-\beta}$ denote the Riemann-Liouville fractional derivative operators.

Let $\Omega = [0,L] \times [0,T]$. We define the function space

$$G(\Omega) = \{w(x,t) | \frac{\partial^2 w}{\partial x^2} \in C^2(\Omega) \text{ and } \frac{\partial^5 w}{\partial x^4 \partial t} \in C(\Omega)\}.$$

In this paper, we suppose the continuous problem (7-9) has a smooth solution $u(x,t) \in G(\Omega)$.

Now we construct an implicit difference method using a new solution technique. We define $t_k = k\tau, k = 0,1,\ldots,n; x_i = ih, i = 0,1,\ldots,m$, where $\tau = T/n$ and $h = L/m$ are space and time step sizes, respectively. We introduce the following notations:

$$\mathcal{L}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{10}$$
$$\delta_x^2 u(x,t) = u(x+h,t) - 2u(x,t) + u(x-h,t), \tag{11}$$
$$\mathcal{L}_h v(x,t) = \frac{\delta_x^2 u(x,t)}{h^2}. \tag{12}$$

Integrating both sides of Eq. (7) from t_k to t_{k+1}, we have

$$
\begin{aligned}
& u(x_i, t_{k+1}) - u(x_i, t_k) \\
= & \frac{A}{\Gamma(\alpha)} \int_0^{t_{k+1}} \frac{\mathcal{L}u(x_i, \eta) + f(x_i, \eta)}{(t_{k+1} - \eta)^{1-\alpha}} d\eta - \frac{A}{\Gamma(\alpha)} \int_0^{t_k} \frac{\mathcal{L}u(x_i, \eta) + f(x_i, \eta)}{(t_k - \eta)^{1-\alpha}} d\eta \\
& + \frac{B}{\Gamma(\beta)} \int_0^{t_{k+1}} \frac{\mathcal{L}u(x_i, \eta) + f(x_i, \eta)}{(t_{k+1} - \eta)^{1-\beta}} d\eta - \frac{B}{\Gamma(\beta)} \int_0^{t_k} \frac{\mathcal{L}u(x_i, \eta) + f(x_i, \eta)}{(t_k - \eta)^{1-\beta}} d\eta \\
& + \int_{t_k}^{t_{k+1}} g(u(x_i, \eta), x_i, \eta) d\eta.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& u(x_i, t_{k+1}) \\
= & u(x_i, t_k) \\
& + \frac{A}{\Gamma(\alpha)} \int_0^{\tau} \frac{\mathcal{L}u(x_i, \eta) + f(x_i, \eta)}{(t_{k+1} - \eta)^{1-\alpha}} d\eta + \frac{A}{\Gamma(\alpha)} \int_0^{t_k} \frac{\mathcal{L}v(x_i, \eta) + p(x_i, \eta)}{(t_k - \eta)^{1-\alpha}} d\eta \\
& + \frac{B}{\Gamma(\beta)} \int_0^{\tau} \frac{\mathcal{L}u(x_i, \eta) + f(x_i, \eta)}{(t_{k+1} - \eta)^{1-\beta}} d\eta + \frac{B}{\Gamma(\beta)} \int_0^{t_k} \frac{\mathcal{L}v(x_i, \eta) + p(x_i, \eta)}{(t_k - \eta)^{1-\beta}} d\eta \\
& + \int_{t_k}^{t_{k+1}} g(u(x_i, \eta), x_i, \eta) d\eta, \qquad (13)
\end{aligned}
$$

where

$$
v(x, t) = u(x, t + \tau) - u(x, t), \quad p(x, t) = f(x, t + \tau) - f(x, t).
$$

Let

$$
\begin{aligned}
I_1 = & \frac{A}{\Gamma(\alpha)} \int_0^{\tau} \frac{\mathcal{L}u(x_i, \eta) + f(x_i, \eta)}{(t_{k+1} - \eta)^{1-\alpha}} d\eta + \frac{B}{\Gamma(\beta)} \int_0^{\tau} \frac{\mathcal{L}u(x_i, \eta) + f(x_i, \eta)}{(t_{k+1} - \eta)^{1-\beta}} d\eta, \\
I_2 = & \frac{A}{\Gamma(\alpha)} \int_0^{t_k} \frac{\mathcal{L}v(x_i, \eta) + p(x_i, \eta)}{(t_k - \eta)^{1-\alpha}} d\eta + \frac{B}{\Gamma(\beta)} \int_0^{t_k} \frac{\mathcal{L}v(x_i, \eta) + p(x_i, \eta)}{(t_k - \eta)^{1-\beta}} d\eta, \\
I_3 = & \int_{t_k}^{t_{k+1}} g(u(x_i, \eta), x_i, \eta) d\eta.
\end{aligned}
$$

Eq. (13) can be written as

$$
u(x_i, t_{k+1}) = u(x_i, t_k) + I_1 + I_2 + I_3.
$$

For I_1 , we can get approximation as below:

$$
\begin{aligned}
I_1 =& \frac{A}{\Gamma(\alpha)}\int_0^\tau \frac{\mathcal{L}u(x_i,\tau)+f(x_i,\tau)}{(t_{k+1}-\eta)^{1-\alpha}}d\eta \\
&+\frac{B}{\Gamma(\beta)}\int_0^\tau \frac{\mathcal{L}u(x_i,\tau)+f(x_i,\tau)}{(t_{k+1}-\eta)^{1-\beta}}d\eta + R_{11} \\
=& \frac{A\tau^\alpha}{\Gamma(\alpha+1)}\left[(k+1)^\alpha - k^\alpha\right]\left[\mathcal{L}u(x_i,\tau)+f(x_i,\tau)\right] \\
&+\frac{B\tau^\beta}{\Gamma(\beta+1)}\left[(k+1)^\beta - k^\beta\right]\left[\mathcal{L}u(x_i,\tau)+f(x_i,\tau)\right] + R_{11} \\
=& \frac{A\tau^\alpha}{\Gamma(\alpha+1)}a_k\left[\frac{1}{h^2}\delta_x^2 u(x_i,\tau)+f(x_i,\tau)+R_{12}\right] \\
&+\frac{B\tau^\beta}{\Gamma(\beta+1)}b_k\left[\frac{1}{h^2}\delta_x^2 u(x_i,\tau)+f(x_i,\tau)+R_{12}\right] + R_{11} \\
=& \frac{A\tau^\alpha}{\Gamma(\alpha+1)}a_k\left[\frac{1}{h^2}\delta_x^2 u(x_i,\tau)+f(x_i,\tau)\right] \\
&+\frac{B\tau^\beta}{\Gamma(\beta+1)}b_k\left[\frac{1}{h^2}\delta_x^2 u(x_i,\tau)+f(x_i,\tau)\right] + R_1 \\
=& \left[\frac{A\tau^\alpha a_k}{\Gamma(\alpha+1)}+\frac{B\tau^\beta b_k}{\Gamma(\beta+1)}\right]\left[\frac{1}{h^2}\delta_x^2 u(x_i,\tau)+f(x_i,\tau)\right] + R_1, \qquad (14)
\end{aligned}
$$

where

$$
a_k = (k+1)^\alpha - k^\alpha, b_k = (k+1)^\beta - k^\beta,
$$

$$
R_1 = R_{11} + \frac{A\tau^\alpha a_k}{\Gamma(\alpha+1)}R_{12} + \frac{B\tau^\beta b_k}{\Gamma(\beta+1)}R_{12}.
$$

$$
\begin{aligned}
R_{11} =& \frac{A}{\Gamma(\alpha)}\int_0^\tau \frac{\mathcal{L}u(x_i,\eta)-\mathcal{L}u(x_i,\tau)+f(x_i,\eta)-f(x_i,\tau)}{(t_{k+1}-\eta)^{1-\alpha}}d\eta \\
&+\frac{B}{\Gamma(\beta)}\int_0^\tau \frac{\mathcal{L}u(x_i,\eta)-\mathcal{L}u(x_i,\tau)+f(x_i,\eta)-f(x_i,\tau)}{(t_{k+1}-\eta)^{1-\beta}}d\eta, \\
R_{12} =& \frac{\partial^2 u(x_i,\tau)}{\partial x^2} - \frac{1}{h^2}\delta_x^2 u(x_i,\tau).
\end{aligned}
$$

Given that

$$
\begin{aligned}
&\mathcal{L}u(x_i,\eta)+f(x_i,\eta) \\
=& \frac{\partial^2 u(x_i,\tau)}{\partial x^2} + f(x_i,\tau) + \left[\frac{\partial^3 u(x_i,\xi_1)}{\partial x^2\partial t} + \frac{\partial f(x_i,\xi_2)}{\partial t}\right](\eta-\tau),
\end{aligned}
$$

where $0 \le \tau \le \xi_1 \le \eta; 0 \le \tau \le \xi_2 \le \eta$, we obtain

$$
\begin{aligned}
|R_{11}| \leq & \frac{A}{\Gamma(\alpha)} C_1 \tau \int_0^{\tau} \frac{1}{(t_{k+1}-\eta)^{1-\alpha}} d\eta + \frac{B}{\Gamma(\beta)} C_2 \tau \int_0^{\tau} \frac{1}{(t_{k+1}-\eta)^{1-\beta}} d\eta \\
\leq & \frac{A C_1 \tau^{1+\alpha}}{\Gamma(\alpha+1)} a_k + \frac{B C_2 \tau^{1+\beta}}{\Gamma(\beta+1)} b_k.
\end{aligned}
\tag{15}
$$

Again, it is apparent that $|R_{12}| \leq C_3 h^2$. So, we have

$$
\begin{aligned}
|R_1| \leq & \left| \frac{A C_1 \tau^{1+\alpha}}{\Gamma(\alpha+1)} a_k + \frac{B C_2 \tau^{1+\alpha}}{\Gamma(\beta+1)} b_k + \frac{A \tau^{\alpha}}{\Gamma(\alpha+1)} a_k C_3 h^2 + \frac{B \tau^{\beta}}{\Gamma(\beta+1)} b_k C_3 h^2 \right| \\
\leq & C a_k \tau^{\alpha} \left(\tau + h^2\right) + C b_k \tau^{\beta} \left(\tau + h^2\right) \\
\leq & C \left(a_k \tau^{\alpha} + b_k \tau^{\beta}\right) \left(\tau + h^2\right).
\end{aligned}
\tag{16}
$$

For I_2 , we get the following approximation

$$
\begin{aligned}
I_2 = & \frac{A}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{\mathcal{L} v(x_i, \eta) + p(x_i, \eta)}{(t_k - \eta)^{1-\alpha}} d\eta \\
& + \frac{B}{\Gamma(\beta)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{\mathcal{L} v(x_i, \eta) + p(x_i, \eta)}{(t_k - \eta)^{1-\beta}} d\eta \\
= & \frac{A}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{\mathcal{L} v(x_i, t_{j+1}) + p\left(x_i, t_{j+1}\right)}{(t_k - \eta)^{1-\alpha}} d\eta \\
& + \frac{B}{\Gamma(\beta)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{\mathcal{L} v(x_i, t_{j+1}) + p\left(x_i, t_{j+1}\right)}{(t_k - \eta)^{1-\beta}} d\eta + R_{21} \\
= & \frac{A \tau^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{k-1} a_{k-j-1} \left[\mathcal{L}_h v(x_i, t_{j+1}) + p\left(x_i, t_{j+1}\right) + R_{22}\right] \\
& + \frac{B \tau^{\beta}}{\Gamma(\beta+1)} \sum_{j=0}^{k-1} b_{k-j-1} \left[\mathcal{L}_h v(x_i, t_{j+1}) + p\left(x_i, t_{j+1}\right) + R_{22}\right] + R_{21} \\
= & \left[\frac{A \tau^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{k-1} a_{k-j-1} + \frac{B \tau^{\beta}}{\Gamma(\beta+1)} \sum_{j=0}^{k-1} b_{k-j-1} \right] \\
& \times \left[\mathcal{L}_h v(x_i, t_{j+1}) + p\left(x_i, t_{j+1}\right)\right] + R_2,
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
R_2 = & R_{21} + \frac{A\tau^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{k-1} a_{k-j-1}R_{22} + \frac{B\tau^\beta}{\Gamma(\beta+1)}\sum_{j=0}^{k-1} b_{k-j-1}R_{22}, \\
R_{21} = & \frac{A}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} \frac{\mathcal{L}v(x_i,\eta)-\mathcal{L}v(x_i,t_{j+1})+p(x_i,\eta)-p(x_i,t_{j+1})}{(t_{k+1}-\eta)^{1-\alpha}}d\eta \\
& + \frac{B}{\Gamma(\beta)}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} \frac{\mathcal{L}v(x_i,\eta)-\mathcal{L}v(x_i,t_{j+1})+p(x_i,\eta)-p(x_i,t_{j+1})}{(t_{k+1}-\eta)^{1-\beta}}d\eta, \\
R_{22} = & \mathcal{L}v(x_i,t_{j+1}) - \mathcal{L}_h v(x_i,t_{j+1}). \qquad (18)
\end{aligned}
$$

Because

$$
\begin{aligned}
& \mathcal{L}v(x_i,\eta) + p(x_i,\eta) \\
= & \frac{\partial^2 v(x_i,t_{j+1})}{\partial x^2} + p\left(x_i,t_{j+1}\right) + \left[\frac{\partial^3 v(x_i,\eta_1)}{\partial x^2\partial t} + \frac{\partial p(x_i,\eta_2)}{\partial t}\right](\eta - t_{j+1}) \\
= & \frac{\partial^2 v(x_i,t_{j+1})}{\partial x^2} + p\left(x_i,t_{j+1}\right) + \left[\frac{\partial^4 v(x_i,\tilde{\eta_1})}{\partial x^2\partial t^2} + \frac{\partial^2 p(x_i,\tilde{\eta_2})}{\partial t^2}\right]\tau(\eta - t_{j+1}),
\end{aligned}
$$

where $\eta \le \eta_1 \le t_{j+1}, \quad \eta_1 \le \tilde{\eta_1} \le \eta_1 + \tau, \quad \eta \le \eta_2 \le t_{j+1}, \quad \eta_2 \le \tilde{\eta_2} \le \eta_2 + \tau$, we have

$$
\begin{aligned}
|R_{21}| \le & \frac{A}{\Gamma(\alpha)}\tau^2\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} \frac{1}{(t_k-\eta)^{1-\alpha}}d\eta + \frac{B}{\Gamma(\beta)}\tau^2\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} \frac{1}{(t_k-\eta)^{1-\beta}}d\eta \\
\le & \left(\frac{A}{\Gamma(\alpha+1)}k^\alpha + \frac{B}{\Gamma(\beta+1)}k^\beta\right)\tau^2. \qquad (19)
\end{aligned}
$$

And, using Taylor's formula, we can obtain

$$
\begin{aligned}
\mathcal{L}v(x_i,t_{j+1}) = & \mathcal{L}_h v(x_i,t_{j+1}) + \frac{h^2}{12}\frac{\partial^4 v(\xi_2,t_{j+1})}{\partial x^4} \\
= & \mathcal{L}_h v(x_i,t_{j+1}) + \frac{h^2}{12}\left[\frac{\partial^4 u(\xi_2,t_{j+2})}{\partial x^4} - \frac{\partial^4 u(\xi_2,t_{j+1})}{\partial x^4}\right] \\
= & \mathcal{L}_h v(x_i,t_{j+1}) + \frac{h^2\tau}{12}\frac{\partial^5 u(\xi_2,\tilde{\eta_3})}{\partial x^4\partial t}.
\end{aligned}
$$

Hence, we have

$$
|R_{22}| \le |\mathcal{L}v(x_i,t_{j+1}) - \mathcal{L}_h v(x_i,t_{j+1})| \le C\tau h^2.
$$

$$
\begin{aligned}
|R_2| &\leq \left| R_{21} + \frac{A\tau^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{k-1} a_{k-j-1} R_{22} + \frac{B\tau^{\beta}}{\Gamma(\beta+1)} \sum_{j=0}^{k-1} b_{k-j-1} R_{22} \right| \\
&\leq \frac{A\tau^{\alpha+1}}{\Gamma(\alpha+1)} k^{\alpha} \left(\tau + h^2\right) + \frac{B\tau^{\beta+1}}{\Gamma(\beta+1)} k^{\beta} \left(\tau + h^2\right) \\
&\leq \frac{A\tau}{\Gamma(\alpha+1)} T^{\alpha} \left(\tau + h^2\right) + \frac{B\tau}{\Gamma(\beta+1)} T^{\beta} \left(\tau + h^2\right) \\
&\leq C\tau \left(\tau + h^2\right). \qquad (20)
\end{aligned}
$$

For I_3, we can get the approximation as below:

$$
\begin{aligned}
I_3 &= \int_{t_k}^{t_{k+1}} g(u(x_i, \eta), x_i, \eta) d\eta \\
&= \frac{\tau}{2} \left[g(u(x_i, t_{k+1}), x_i, t_{k+1}) + g(u(x_i, t_k), x_i, t_k) \right] + O(\tau^2). \qquad (21)
\end{aligned}
$$

From the above result, we obtain

$$
\begin{aligned}
& u(x_i, t_{k+1}) \\
&= u(x_i, t_k) + \left[\frac{A\tau^{\alpha} a_k}{\Gamma(\alpha+1)} + \frac{B\tau^{\beta} b_k}{\Gamma(\beta+1)} \right] \left[\frac{1}{h^2} \delta_x^2 u(x_i, \tau) + f\left(x_i, \tau\right) \right] \\
&+ \left[\frac{A\tau^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{k-1} a_{k-j-1} + \frac{B\tau^{\beta}}{\Gamma(\beta+1)} \sum_{j=0}^{k-1} b_{k-j-1} \right] \left[\frac{1}{h^2} \delta_x^2 v(x_i, t_{j+1}) + p\left(x_i, t_{j+1}\right) \right] \\
&+ \frac{\tau}{2} \left[g(u(x_i, t_{k+1}), x_i, t_{k+1}) + g(u(x_i, t_k), x_i, t_k) \right] + R_i^{k+1} \\
&= u(x_i, t_k) + \left[\frac{A\tau^{\alpha} a_k}{\Gamma(\alpha+1)} + \frac{B\tau^{\beta} b_k}{\Gamma(\beta+1)} \right] \left[\frac{1}{h^2} \delta_x^2 u(x_i, \tau) + f\left(x_i, \tau\right) \right] \\
&+ \sum_{j=0}^{k-1} \left[\frac{A\tau^{\alpha}}{\Gamma(\alpha+1)} a_{k-j-1} + \frac{B\tau^{\beta}}{\Gamma(\beta+1)} b_{k-j-1} \right] \left[\frac{1}{h^2} \delta_x^2 v(x_i, t_{j+1}) + p\left(x_i, t_{j+1}\right) \right] \\
&+ \frac{\tau}{2} \left[g(u(x_i, t_{k+1}), x_i, t_{k+1}) + g(u(x_i, t_k), x_i, t_k) \right] + R_i^{k+1}. \qquad (22)
\end{aligned}
$$

Let

$$
\frac{A\tau^{\alpha}}{\Gamma(\alpha+1)} = r_1, \frac{B\tau^{\beta}}{\Gamma(\beta+1)} = r_2.
$$

We have

$$
\begin{aligned}
&u(x_i, t_{k+1})\\
=&u(x_i, t_k) + (r_1 a_k + r_2 b_k)\left[\delta_x^2 u(x_i, \tau)/h^2 + f(x_i, \tau)\right]\\
&+\sum_{j=0}^{k-1}(r_1 a_{k-j-1} + r_2 b_{k-j-1}) \times \left[\left(\delta_x^2 u(x_i, t_{j+2}) - \delta_x^2 u(x_i, t_{j+1})\right)/h^2\right.\\
&\left.+f\left(x_i, t_{j+2}\right) - f\left(x_i, t_{j+1}\right)\right]\\
&+\frac{\tau}{2}\left[g\left(u(x_i, t_{k+1}), x_i, t_{k+1}\right) + g(u(x_i, t_k), x_i, t_k)\right] + R_i^{k+1},
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
\left|R_i^{k+1}\right| &\le C\left(a_k \tau^{\alpha} + b_k \tau^{\beta}\right)\left(\tau + h^2\right) + C\tau\left(\tau + h^2\right)\\
&\le \tilde{C}\left(a_k \tau^{\alpha} + b_k \tau^{\beta} + \tau\right)\left(\tau + h^2\right).
\end{aligned}
\tag{24}
$$

From [47] we have following lemma.

Lemma 1: In (24), the coefficient a_k, b_k $(k = 0, 1, 2\ldots)$ satisfy:

(1) $a_0 = 1, a_k > 0, b_0 = 1, b_k > 0, k = 0, 1, 2, \ldots;$

(2) $a_k > a_{k+1}, b_k > b_{k+1}, k = 0, 1, 2, \ldots;$

(3) there exists a positive constant $C > 0,$ such that

$$\tau \le C a_k \tau^{\alpha}, \tau \le C b_k \tau^{\beta}, k = 1, 2, \ldots.$$

Let

$$u = (u_1, u_2, \ldots, u_{m-1})^T, \qquad v = (v_1, v_2, \ldots, v_{m-1})^T.$$

We define

$$(u, v) = \sum_{i=1}^{m-1} u_i v_i h, \qquad \|u_2\| = \sqrt{(u, u)} = \left(\sum_{i=1}^{m-1} u_i^2 h\right)^{\frac{1}{2}}.$$

We have

$$R^k = (R_1^k, R_2^k, \ldots, R_{m-1}^k)^T, \|R^k\|_2 = \sqrt{h\sum_{i=1}^{m-1}|R_i^k|^2}.$$

We obtain

$$\left\|R^k\right\|_2 \le C\left(a_k \tau^{\alpha} + b_k \tau^{\beta}\right)\left(\tau + h^2\right).$$

We denote u_i^k for the numerical approximation to $u(x_i, t_k)$, $f_i^k = f\left(x_i, t_k\right)$, $g_i^k = (u(x_i, t_k), x_i, t_k)$ and write $\delta_x^2 u_i^k = u_{i+1}^k - 2u_i^k + u_{i-1}^k$. We obtain the following implicit difference scheme:

$$
\begin{aligned}
u_i^{k+1} = & u_i^k + (r_1 a_k + r_2 b_k)\left[\delta_x^2 u_i^1/h^2 + f_i^1\right] \\
& + \sum_{j=0}^{k-1}(r_1 a_{k-j-1} + r_2 b_{k-j-1})\left[(\delta_x^2 u_i^{j+2} - \delta_x^2 u_i^{j+1})/h^2 + f_i^{j+2} - f_i^{j+1}\right] \\
& + \frac{\tau}{2}\left(g_i^{k+1} + g_i^k\right). \qquad (25)
\end{aligned}
$$

The implicit numerical method can be rewritten as the following form:

$$
\begin{aligned}
u_i^{k+1} = & u_i^k + r_1 a_0(\delta_x^2 u_i^{k+1}/h^2 + f_i^{k+1}) + r_1 \sum_{j=0}^{k-1}(a_{j+1} - a_j)(\delta_x^2 u_i^{k-j}/h^2 + f_i^{k-j}) \\
& + r_2 b_0(\delta_x^2 u_i^{k+1}/h^2 + f_i^{k+1}) + r_2 \sum_{j=0}^{k-1}(b_{j+1} - b_j)(\delta_x^2 u_i^{k-j}/h^2 + f_i^{k-j}) \\
& + \frac{\tau}{2}(g_i^{k+1} + g_i^k) \\
= & u_i^k + (r_1 + r_2)(\delta_x^2 u_i^{k+1}/h^2 + f_i^{k+1}) \\
& + \sum_{j=0}^{k-1}[r_1(a_{j+1} - a_j) + r_2(b_{j+1} - b_j)](\delta_x^2 u_i^{k-j}/h^2 + f_i^{k-j}) \\
& + \frac{\tau}{2}(g_i^{k+1} + g_i^k). \qquad (26)
\end{aligned}
$$

The initial and boundary conditions are

$$
u_i^0 = \phi(ih), \quad i = 0, 1, 2, \ldots, m; \qquad (27)
$$
$$
u_0^k = \varphi_1(k\tau), \quad u_m^k = \varphi_2(k\tau), \quad k = 1, 2, \ldots, n. \qquad (28)
$$

We simplify the equation as

$$
\begin{aligned}
u_i^{k+1} - u_i^k = & (r_1 + r_2)\left(\delta_x^2 u_i^{k+1}/h^2 + f_i^{k+1}\right) \\
& + \sum_{j=0}^{k-1}\left[r_1(a_{j+1} - a_j) + r_2(b_{j+1} - b_j)\right]\left(\delta_x^2 u_i^{k-j}/h^2 + f_i^{k-j}\right) \\
& + \frac{\tau}{2}\left(g_i^{k+1} + g_i^k\right). \qquad (29)
\end{aligned}
$$

We sum from $u_i^1 - u_i^0$ to $u_i^{k+1} - u_i^k$, and use the definition of a_j, b_j to obtain

$$\begin{aligned} u_i^{k+1} - u_i^0 = & \sum_{j=-1}^{k-1} (r_1 a_{j+1} + r_2 b_{j+1}) \left(\delta_x^2 u_i^{k-j}/h^2 + f_i^{k-j} \right) \\ & + \frac{\tau}{2} \sum_{j=1}^{k+1} \left(g_i^j + g_i^{j-1} \right) \\ = & \sum_{j=0}^{k} (r_1 a_j + r_2 b_j) \left(\delta_x^2 u_i^{k+1-j}/h^2 + f_i^{k+1-j} \right) + \tau \sum_{j=1}^{k} g_i^j \\ & + \frac{\tau}{2} \left(g_i^0 + g_i^{k+1} \right). \end{aligned} \tag{30}$$

So that, we have:

$$u_i^k = u_i^0 + \sum_{j=0}^{k-1} (r_1 a_j + r_2 b_j) \left(\delta_x^2 u_i^{k-j}/h^2 + f_i^{k-j} \right) + \tau \sum_{j=1}^{k-1} g_i^j + \frac{\tau}{2}(g_i^0 + g_i^k), \tag{31}$$

where

$$\delta_x^2 u_i^k = u_{i+1}^k - 2u_i^k + u_{i-1}^k, r_1 = \frac{A\tau^\alpha}{\Gamma(\alpha+1)}, r_2 = \frac{B\tau^\beta}{\Gamma(\beta+1)},$$

$$a_k = (k+1)^\alpha - k^\alpha, b_k = (k+1)^\beta - k^\beta, i = 1, 2, \ldots, m-1, k = 1, 2, \ldots, n.$$

The initial and boundary conditions are (27),(28).

3 Stability of the implicit numerical method

We give a stability analysis as follows.

We suppose that $\tilde{u}_i^k, i = 0, 1, \ldots, m; k = 0, 1, \ldots, n$ is the approximate solution of Eq.(7), $\tilde{g}_i^k$ denotes $g\left(\tilde{u}(x_i, t_k), x_i, t_k\right)$, the error $\varepsilon_i^k = u_i^k - \tilde{u}_i^k$, satisfies $\varepsilon_0^k = 0, \varepsilon_m^k = 0, k = 1, 2, \ldots, n$ and

$$\begin{aligned} \varepsilon_i^{k+1} = & \varepsilon_i^k + (r_1 + r_2) \times \left(\delta_x^2 \varepsilon_i^{k+1}/h^2 \right) \\ & + \sum_{j=0}^{k-1} [r_1(a_{j+1} - a_j) + r_2(b_{j+1} - b_j)] \times \left(\delta_x^2 \varepsilon_i^{k-j}/h^2 \right) \\ & + \frac{\tau}{2} \left(g_i^{k+1} - \tilde{g}_i^{k+1} + g_i^k - \tilde{g}_i^k \right). \end{aligned} \tag{32}$$

We also suppose that the function $g(u(x,t), x, t)$ satisfies the Lipschitz condition,

$$|g(u_1, x, t) - g(u_2, x, t)| \leq L|u_1 - u_2|, \forall u_1, u_2,$$

so that

$$\left| g_i^k - \tilde{g}_i^k \right| \leq L \left| \varepsilon_i^k \right|.$$

We can easily prove the following result:

Lemma 2: Let $\Delta v_i = v_{i+1} - v_i, \Delta w_i = w_{i+1} - w_i, \delta^2 v_i = v_{i+1} - 2v_i + v_{i-1}, \delta^2 w_i = w_{i+1} - 2w_i + w_{i-1}$. If $v_0 = w_m = 0$, then

$$\left(\delta^2 v, w\right) = -v_1 w_1 h - (\Delta v, \Delta w)$$

where

$$\delta^2 v = \left(\delta^2 v_1, \delta^2 v_2, \cdots, \delta^2 v_{m-1}\right)^T,$$

$$\Delta v = (\Delta v_1, \Delta v_2, \cdots, \Delta v_{m-1})^T,$$

$$\Delta w = (\Delta w_1, \Delta w_2, \cdots, \Delta w_{m-1})^T.$$

Proof:

$$\begin{aligned}
\left(\delta^2 v, w\right) &= \sum_{i=1}^{m-1} \delta^2 v_i w_i h \\
&= h \sum_{i=1}^{m-1} (v_{i+1} - v_i) w_i - h \sum_{i=1}^{m-1} (v_i - v_{i-1}) w_i \\
&= h \sum_{i=1}^{m-1} (v_{i+1} - v_i) w_i - h \sum_{i=1}^{m-1} (v_{i+1} - v_i) w_{i+1} \\
&\quad + h(v_m - v_{m-1}) w_m - h(v_1 - v_0) w_1 \\
&= -v_1 w_1 h - (\Delta v, \Delta w).
\end{aligned}$$

Now let $E^k = (\varepsilon_1^k, \varepsilon_2^k, \ldots, \varepsilon_{m-1}^k)^T$. Multiplying Eq.(32) by $h\varepsilon_i^{k+1}$, summing for i from 1 to $m-1$, we obtain:

$$\begin{aligned}
\left\|E^{k+1}\right\|_2^2 &= \left(E^{k+1}, E^k\right) + (r_1 + r_2) \times \left[\left(\delta_x^2 E^{k+1}, E^{k+1}\right)/h^2\right] \\
&\quad + \sum_{j=0}^{k-1} \left[r_1 (a_{j+1} - a_j) + r_2 (b_{j+1} - b_j)\right] \times \left[\left(\delta_x^2 E^{k-j}, E^{k+1}\right)/h^2\right] \\
&\quad + \frac{\tau}{2} \sum_{i=1}^{m-1} \left(g_i^{k+1} - \tilde{g}_i^{k+1} + g_i^k - \tilde{g}_i^k\right) \varepsilon_i^{k+1} h \\
&= \left(E^{k+1}, E^k\right) - (r_1 + r_2) \left[\left(\left(\varepsilon_1^{k+1}\right)^2 h + \left\|\Delta_x E^{k+1}\right\|_2^2\right)/h^2\right] \\
&\quad - \sum_{j=0}^{k-1} \left[r_1 (a_{j+1} - a_j) + r_2 (b_{j+1} - b_j)\right] \\
&\quad \times \left[\left(\varepsilon_1^{k-j} \varepsilon_1^{k+1} h + \left(\Delta_x E^{k-j}, \Delta_x E^{k+1}\right)\right)/h^2\right] \\
&\quad + \frac{\tau}{2} \sum_{i=1}^{m-1} \left(g_i^{k+1} - \tilde{g}_i^{k+1} + g_i^k - \tilde{g}_i^k\right) \varepsilon_i^{k+1} h.
\end{aligned} \tag{33}$$

According to the Schwarz inequality, we have:

$$2|\left(\Delta_x E^{k-j}, \Delta_x E^{k+1}\right)| \leq \left\|\Delta_x E^{k-j}\right\|_2^2 + \left\|\Delta_x E^{k+1}\right\|_2^2,$$

$$2|\varepsilon_1^{k-j}\varepsilon_1^{k+1}| \leq \left(\varepsilon_1^{k-j}\right)^2 + \left(\varepsilon_1^{k+1}\right)^2.$$

Because

$$\sum_{j=0}^{k-1}[(a_{j+1}-a_j)] = a_k - a_0 = a_k - 1 \quad \text{and} \quad a_k > 0,$$

we have

$$\begin{aligned}
&\left\|E^{k+1}\right\|_2^2 \\
\leq &\frac{1}{2}\left(\left\|E^{k+1}\right\|_2^2 + \left\|E^k\right\|_2^2\right) - (r_1+r_2)\times\left[\left(\left(\varepsilon_i^{k+1}\right)^2 + \left\|\Delta E^{k+1}\right\|_2^2\right)/h^2\right] \\
&-\frac{1}{2}\sum_{j=0}^{k-1}[r_1(a_{j+1}-a_j) + r_2(b_{j+1}-b_j)] \\
&\times\left[\left(\left(\varepsilon_1^{k-j}\right)^2 h + \left(\varepsilon_1^{k+1}\right)^2 h + \left\|\Delta E^{k-j}\right\|_2^2 + \left\|\Delta E^{K+1}\right\|_2^2\right)/h^2\right] \\
&+\frac{\tau L}{2}\left\|E^{k+1}\right\|_2^2 + \frac{\tau L}{4}\left(\left\|E^{k+1}\right\|_2^2 + \left\|E^k\right\|_2^2\right) \\
\leq &\frac{1}{2}\left(\left\|E^{k+1}\right\|_2^2 + \left\|E^k\right\|_2^2\right) \\
&-\frac{(1+a_k)r_1 + (1+b_k)r_2}{2}\times\left[\left(\left(\varepsilon_1^{k+1}\right)^2 h + \left\|\Delta E^{k+1}\right\|_2^2\right)/h^2\right] \\
&-\frac{1}{2}\sum_{j=0}^{k-1}[r_1(a_{j+1}-a_j) + r_2(b_{j+1}-b_j)]\times\left[\left(\left(\varepsilon_1^{k-j}\right)^2 + \left\|\Delta E^{k-j}\right\|_2^2\right)/h^2\right] \\
&+\frac{3\tau L}{4}\left\|E^{k+1}\right\|_2^2 + \frac{\tau L}{4}\left\|E^k\right\|_2^2 \\
\leq &\frac{1}{2}\left(\left\|E^{k+1}\right\|_2^2 + \left\|E^k\right\|_2^2\right) - \frac{r_1+r_2}{2}\left[\left(\left(\varepsilon_1^{k+1}\right)^2 h + \left\|\Delta E^{k+1}\right\|_2^2\right)/h^2\right] \\
&-\frac{1}{2}\sum_{j=0}^{k-1}[r_1(a_{j+1}-a_j) + r_2(b_{j+1}-b_j)]\times\left[\left(\left(\varepsilon_1^{k-j}\right)^2 h + \left\|\Delta E^{k-j}\right\|_2^2\right)/h^2\right] \\
&+\frac{3\tau L}{4}\left\|E^{k+1}\right\|_2^2 + \frac{\tau L}{4}\left\|E^k\right\|_2^2. \qquad (34)
\end{aligned}$$

Thus

$$
\begin{aligned}
&\left\|E^{k+1}\right\|_2^2 + \sum_{j=0}^{k} [r_1 a_j + r_2 b_j] \left[\left(\left(\varepsilon_1^{k+1-j}\right)^2 h + \left\|\Delta E^{k+1-j}\right\|_2^2\right)/h^2\right] \\
\le &\left\|E^{k}\right\|_2^2 + \sum_{j=0}^{k-1} [r_1 a_j + r_2 b_j] \left[\left(\left(\varepsilon_1^{k-j}\right)^2 h + \left\|\Delta E^{k-j}\right\|_2^2\right)/h^2\right] \\
&+\frac{3\tau L}{2}\left\|E^{k+1}\right\|_2^2 + \frac{\tau L}{2}\left\|E^{k}\right\|_2^2 .
\end{aligned} \tag{35}
$$

We define the energy norm

$$
\left\|E^{k}\right\|_E^2 = \left\|E^{k}\right\|_2^2 + \sum_{j=0}^{k-1} (r_1 a_j + r_2 b_j) \left[\left(\left(\varepsilon_1^{k-j}\right)^2 h + \left\|\Delta E^{k-j}\right\|_2^2\right)/h^2\right].
$$

Suppose that $\tau < \frac{2}{3L}$, then we have

$$
\left(1-\frac{3\tau L}{2}\right)\left\|E^{k+1}\right\|_E^2 \le \left(1+\frac{\tau L}{2}\right)\left\|E^{k}\right\|_E^2 .
$$

Thus, we have

$$
\left\|E^{k}\right\|_E^2 \le \left(\frac{1+\frac{1}{2}\tau L}{1-\frac{3}{2}\tau L}\right)^{k-1}\left\|E^{1}\right\|_E^2 \le \left(\frac{1+\frac{1}{2}\tau L}{1-\frac{3}{2}\tau L}\right)^{n}\left\|E^{1}\right\|_E^2 .
$$

Note that $n = T/\tau$, and

$$
\lim_{n\to\infty}\left(\frac{1+\frac{1}{2}L\tau}{1-\frac{3}{2}L\tau}\right)^{n} = \lim_{n\to\infty}\left(\frac{1+\frac{1}{2n}LT}{1-\frac{3}{2n}LT}\right)^{n} = \frac{e^{\frac{1}{2}LT}}{e^{-\frac{3}{2}LT}} = e^{2LT}.
$$

Hence, there is a positive constant $C_1 > 0$, such that

$$
\left(\frac{1+\frac{1}{2}L\tau}{1-\frac{3}{2}L\tau}\right)^{n} \le C_1,
$$

thereby

$$
\left\|E^{k}\right\|_2^2 \le \left\|E^{k}\right\|_E^2 \le C_1\left\|E^{1}\right\|_E^2 .
$$

We have

$$
\varepsilon_i^1 = \varepsilon_i^0 + (r_1+r_2)\times\left(\delta_x^2\varepsilon_i^1/h^2\right) + \frac{\tau}{2}\left(g_i^1 - \tilde{g}_i^1 + g_i^0 - \tilde{g}_i^0\right)
$$

$$
\begin{aligned}
\left\|E^{1}\right\|_2^2 \le &\left\|E^{0}\right\|_2^2 - (r_1+r_2)\times\left[\left(\left(\varepsilon_1^{1}\right)^2 h + \left\|\Delta E^{1}\right\|_2^2\right)/h^2\right] \\
&+\frac{3}{2}\tau L\left\|E^{1}\right\|_2^2 + \frac{1}{2}\tau L\left\|E^{0}\right\|_2^2 .
\end{aligned} \tag{36}
$$

Therefore,

$$\left\|E^1\right\|_E^2 \leq \left(\frac{1+\frac{1}{2}\tau L}{1-\frac{3}{2}\tau L}\right)\left\|E^0\right\|_2^2.$$

Hence, we have

$$\left\|E^1\right\|_2^2 \leq C\left\|E^0\right\|_2^2,$$

thus giving a stability result, i.e.,

$$\|\mathbf{E}^1\|_E^2 = \|\mathbf{E}^1\|_2^2 + r_1\sum_{j=0}^{k} b_j\{|\varepsilon_1^{1-j}|^2 h + \|\Delta_x \mathbf{E}^{1-j}\|_2^2\} \leq \|\mathbf{E}^0\|_2^2, \tag{37}$$

so that $\|\mathbf{E}^{k+1}\|_2^2 \leq \|\mathbf{E}^0\|_2^2$.

Hence, the following theorem of stability is obtained.

Theorem 1. Assuming $\tau < \frac{2}{3L}$, the fractional implicit numerical method defined by (25) is stable.

4 Convergence of the implicit numerical method

We let $u(x_i, t_k), i = 0, 1, \ldots, m, k = 0, 1, \ldots, n$ be the exact solution of equation (7) at mesh point (x_i, t_k). Define $\eta_i^k = u(x_i, t_k) - u_i^k, i = 0, 1, \ldots, m, k = 0, 1, \ldots, n, Y^k = \left(\eta_1^k, \ldots, \eta_{m-1}^k\right)^T$. We get

$$\begin{aligned}
\eta_i^{k+1} =& \eta_i^k + (r_1 + r_2) \times \left(\delta_x^2 \eta_i^{k+1}/h^2\right) \\
&+ \sum_{j=0}^{k-1} [r_1(a_{j+1} - a_j) + r_2(b_{j+1} - b_j)] \times \left(\delta_x^2 \eta_i^{k-j}/h^2\right) \\
&+ \frac{\tau}{2}\left[g\left(u\left(x_i, t_{k+1}\right), x_i, t_{k+1}\right) - g_i^{k+1} + g\left(u\left(x_i, t_k\right), x_i, t_k\right) - g_i^k\right] \\
&+ R_i^{k+1},
\end{aligned} \tag{38}$$

where $\eta_i^0 = \eta_0^k = \eta_m^k = 0, i = 0, 1, \ldots, m$. Multiplying by $h\eta^{k+1}$, and summing for i from 1 to $m-1$, we obtain:

$$
\begin{aligned}
&\left\|Y^{k+1}\right\|_2^2 \\
=&\left(Y^{k+1}, Y^k\right)+\left(r_1+r_2\right) \times\left[\left(\delta_x^2 Y^{k+1}, Y^{k+1}\right) / h^2\right] \\
&+\sum_{j=0}^{k-1}\left[r_1\left(a_{j+1}-a_j\right)+r_2\left(b_{j+1}-b_j\right)\right] \times\left[\left(\delta_x^2 Y^{k-j}, Y^{k+1}\right) / h^2\right] \\
&+\frac{\tau}{2} \sum_{i=1}^{m-1}\left(g_i^{k+1}-\tilde{g}_i^{k+1}+g_i^k-\tilde{g}_i^k\right) \varepsilon_i^{k+1} h+\left(R^{k+1}, Y^{k+1}\right) \\
=&\left(Y^{k+1}, Y^k\right)-\left(r_1+r_2\right) \times\left[\left(\left(\eta_1^{k+1}\right)^2 h+\left\|\Delta_x Y^{k+1}\right\|_2^2\right) / h^2\right] \\
&-\sum_{j=0}^{k-1}\left[r_1\left(a_{j+1}-a_j\right)+r_2\left(b_{j+1}-b_j\right)\right] \\
&\times\left[\left(\eta_1^{k-j} \eta_1^{k+1} h+\left(\Delta_x Y^{k-j}, \Delta_x Y^{k+1}\right)\right) / h^2\right] \\
&+\frac{\tau}{2} \sum_{i=1}^{m-1}\left(g_i^{k+1}-\tilde{g}_i^{k+1}+g_i^k-\tilde{g}_i^k\right) \varepsilon_i^{k+1} h+\left(R^{k+1}, Y^{k+1}\right) .
\end{aligned} \tag{39}
$$

We have

$$
\left(Y^{k+1}, Y^k\right) \leq \frac{1}{2}\left[\left\|Y^{k+1}\right\|^2+\left\|Y^k\right\|^2\right],
$$

$$
\eta_1^{k+1} \eta_1^j \leq \frac{1}{2}\left[\left|\eta_1^{k+1}\right|^2+\left|\eta_i^j\right|^2\right],
$$

$$
\left|\left(R^{k+1}, Y^{k+1}\right)\right| \leq \frac{(r_1 a_k+r_2 b_k) h^2}{L^2}\left\|Y^{k+1}\right\|_2^2+\frac{L^2}{4(r_1 a_k+r_2 b_k) h^2}\left\|R^{k+1}\right\|_2^2 .
$$

Thus,

$$
\begin{aligned}
&\left\|Y^{k+1}\right\|_2^2 \\
\leq&\frac{1}{2}\left[\left\|Y^{k+1}\right\|_2^2+\|Y\|_2^2\right]-\left(r_1+r_2\right) \times\left[\left(\left(\eta_1^{k+1}\right)^2 h+\left\|\Delta_x Y^{k+1}\right\|_2^2\right) / h^2\right] \\
&-\frac{1}{2} \sum_{j=0}^{k-1}\left[r_1(a_{j+1}-a_j)+r_2(b_{j+1}-b_j)\right] \\
&\times\left[\left(\left(\eta_1^{k-j}\right)^2 h+(\eta_1^{k+1})^2 h+\left\|\Delta Y^{k-j}\right\|_2^2+\left\|\Delta Y^{k+1}\right\|_2^2\right) / h^2\right] \\
&+\frac{\tau L}{2}\left\|Y^{k+1}\right\|_2^2+\frac{\tau L}{4}\left(\left\|Y^{k+1}\right\|_2^2+\left\|Y^k\right\|_2^2\right)+\left(R^{k+1}, Y^{k+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left[\left\|Y^{k+1}\right\|_2^2+\|Y\|_2^2\right] \\
&-\frac{(1+a_k)r_1+(1+b_k)r_2}{2}\times\left[\left(\left(\eta_1^{k+1}\right)^2h+\left\|\Delta_xY^{k+1}\right\|_2^2\right)/h^2\right] \\
&-\frac{1}{2}\sum_{j=0}^{k-1}\left[r_1(a_{j+1}-a_j)+r_2(b_{j+1}-b_j)\right]\times\left[\left(\left(\eta_1^{k-j}\right)^2h+\left\|\Delta Y^{k-j}\right\|_2^2\right)/h^2\right] \\
&+\frac{3\tau L}{4}\left\|Y^{k+1}\right\|_2^2+\frac{\tau L}{4}\left\|Y^k\right\|_2^2 \\
&+\frac{(r_1a_k+r_2b_k)h^2}{L^2}\left\|Y^{k+1}\right\|_2^2+\frac{L^2}{4(r_1a_k+r_2b_k)h^2}\left\|R^{k+1}\right\|_2^2 \\
&\leq \frac{1}{2}\left[\left\|Y^{k+1}\right\|_2^2+\|Y\|_2^2\right] \\
&-\frac{r_1+r_2}{2}\times\left[\left(\left(\eta_1^{k+1}\right)^2h+\left\|\Delta_xY^{k+1}\right\|_2^2\right)/h^2\right] \\
&-\frac{1}{2}\sum_{j=0}^{k-1}\left[r_1(a_{j+1}-a_j)+r_2(b_{j+1}-b_j)\right]\times\left[\left(\left(\eta_1^{k-j}\right)^2h+\left\|\Delta Y^{k-j}\right\|_2^2\right)/h^2\right] \\
&+\frac{3\tau L}{4}\left\|Y^{k+1}\right\|_2^2+\frac{\tau L}{4}\left\|Y^k\right\|_2^2-\frac{r_1a_k+r_2b_k}{2}\left(\left(\eta_1^{k+1}\right)^2h+\left\|\Delta_xY^{k+1}\right\|_2^2\right)/h^2 \\
&+\frac{(r_1a_k+r_2b_k)h^2}{L^2}\left\|Y^{k+1}\right\|_2^2+\frac{L^2}{4(r_1a_k+r_2b_k)h^2}\left\|R^{k+1}\right\|_2^2. \qquad (40)
\end{aligned}
$$

Lemma 3: Suppose

$$
\left\|Y^k\right\|_2=\sqrt{h\sum_{i=1}^{m-1}|\eta_i^k|^2},\left|Y^k\right|_\infty=\max_{1\leq i\leq m-1}\left|\eta_i^k\right|,
$$

then

$$
\left\|Y^k\right\|_2^2\leq L\left\|Y^k\right\|_\infty^2\leq\frac{L^2}{2h^2}\left[h|\eta_1^k|^2+\left\|\Delta_xY^k\right\|_2^2\right].
$$

Proof: The first inequality is apparent.

For the second inequality, let

$$
|\eta_{i_0}^k|=\max_{1\leq i\leq m-1}|\eta_i^k\|,
$$

$$
\eta_{i_0}^k=\eta_1^k+\sum_{j=1}^{i_0-1}\Delta_x\eta_j^k,\eta_{i_0}^k=-\sum_{j=i_0}^{m-1}\Delta_x\eta_j^k.
$$

Thus

$$
2|\eta_{i_0}^k|\leq|\eta_1^k|+\sum_{j=1}^{m-1}|\Delta_x\eta_j^k|.
$$

Using the Cauchy-Schwartz inequality, we have

$$4|\eta_{i_0}^k|^2 \leq 2m \left[|\eta_1^k|^2 + \sum_{j=1}^{m-1} |\Delta_x \eta_j^k|^2 \right] \leq \frac{2L}{h^2} \left[h|v_1|^2 + \|\Delta v\|_2^2 \right].$$

Therefore,

$$\left\|Y^k\right\|_\infty^2 \leq \frac{L^2}{2h^2} \left[h|\eta_1^k|^2 + \left\|\Delta_x Y^k\right\|_2^2 \right].$$

The lemma is proved.

Applying Lemma 3, we obtain

$$\begin{aligned} &\left\|Y^{k+1}\right\|_2^2 + \sum_{j=0}^{k} (r_1 a_j + r_2 b_j) \left[\left(\left|\eta_1^{k+1-j}\right|^2 h + \left\|\Delta_x Y^{k+1-j}\right\|_2^2 \right) / h^2 \right] \\ \leq &\left\|Y^k\right\|_2^2 + \sum_{j=0}^{k} (r_1 a_j + r_2 b_j) \left[\left(\left|\eta_1^{k-j}\right|^2 h + \left\|\Delta_x Y^{k-j}\right\|_2^2 \right) / h^2 \right] \\ &+ \frac{3\tau d_1}{2} \left\|Y^{k+1}\right\|_2^2 + \frac{\tau L}{2} \left\|Y^k\right\|_2^2 + \frac{L^2}{2(r_1 a_k + r_2 b_k)} \left\|R^{k+1}\right\|_2^2. \end{aligned} \tag{41}$$

Let

$$\rho_k = \left\|Y^k\right\|_2^2 + \sum_{j=0}^{k} (r_1 a_j + r_2 b_j) \left[\left(\left|\eta_1^{k-j}\right|^2 h + \left\|\Delta_x Y^{k-j}\right\|_2^2 \right) / h^2 \right] \tag{42}$$

and using Lemma 1

$$\left\|R^k\right\|_2 \leq C \left(a_k \tau^\alpha + b_k \tau^\beta \right) \left(\tau + h^2 \right), r_1 = \frac{A\tau^\alpha}{\Gamma(\alpha+1)}, r_2 = \frac{B\tau^\beta}{\Gamma(\beta+1)},$$

then

$$\left(1 - \frac{3}{2}\tau L\right) \rho_{k+1} \leq \left(1 + \frac{1}{2}\tau L\right) \rho_k + C' \left(a_k \tau^\alpha + b_k \tau^\beta \right) \left(\tau + h^2 \right)^2.$$

Therefore, we obtain

$$\rho_{k+1} \leq \frac{1 + \frac{1}{2}\tau L}{1 - \frac{3}{2}\tau L} \left[\rho_k + C' \left(a_k \tau^\alpha + b_k \tau^\beta \right) \left(\tau + h^2 \right)^2 \right]. \tag{43}$$

$$\rho_{k+1} \leq \left(\frac{1 + \frac{1}{2}\tau L}{1 - \frac{3}{2}\tau L} \right)^{k+1} \left[\rho_0 + \sum_{j=0}^{k} C' \left(a_j \tau^\alpha + b_j \tau^\beta \right) \left(\tau + h^2 \right)^2 \right]. \tag{44}$$

Note that $\rho_0 = 0$, We can therefore conclude that there exists a positive constant $\hat{C}$, such that

$$\rho_{k+1} \leq \hat{C} \sum_{j=0}^{k} \left(a_k \tau^{\alpha} + b_k \tau^{\beta}\right) \left(\tau + h^2\right)^2,$$

and

$$\sum_{j=0}^{k} a_k \tau^{\alpha} = (k+1)^{\alpha} \tau^{\alpha} \leq T^{\alpha}.$$

Thus

$$\rho_{k+1} \leq \hat{C} \left(T^{\alpha} + T^{\beta}\right) \left(\tau + h^2\right)^2.$$

Because $\left\| y^{K+1} \right\|_2^2 \leq \rho_{k+1}$, we have

$$\left\| Y^{k+1} \right\|_2^2 \leq \hat{C} \left(T^{\alpha} + T^{\beta}\right) \left(\tau + h^2\right)^2.$$

Consequently, the following theorem of convergence is obtained.

Theorem 2: Let $u(x,t) \in G(\Omega)$ be the solution of (7-9) and assuming $\tau < \frac{2}{3L}$. Then the fractional implicit difference method defined by (25) is convergent, and there exists a positive constant $C > 0$ such that

$$\left\| Y^{k+1} \right\|_2 \leq C(\tau + h^2).$$

5 Numerical Results

In this Section we illustrate some of the theory through numerical simulations.

Example 1. We consider the following modified anomalous subdiffusion equation with a nonlinear source term:

$$\frac{\partial u(x,t)}{\partial t} = \left(A \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + B \frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right) \left[\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\right] + g(u(x,t),x,t), \quad (45)$$

where $A = B = 0.5$, $f(u,t) = 0$,

$$g(u(x,t),x,t) = e^x[(1+\alpha)t^{\alpha} - \frac{\Gamma(2+\alpha)}{\Gamma(1+2\alpha)} t^{2\alpha}] + e^x[(1+\beta)t^{\beta} - \frac{\Gamma(2+\beta)}{\Gamma(1+2\beta)} t^{2\beta}], \quad (46)$$

with boundary condition and initial conditions

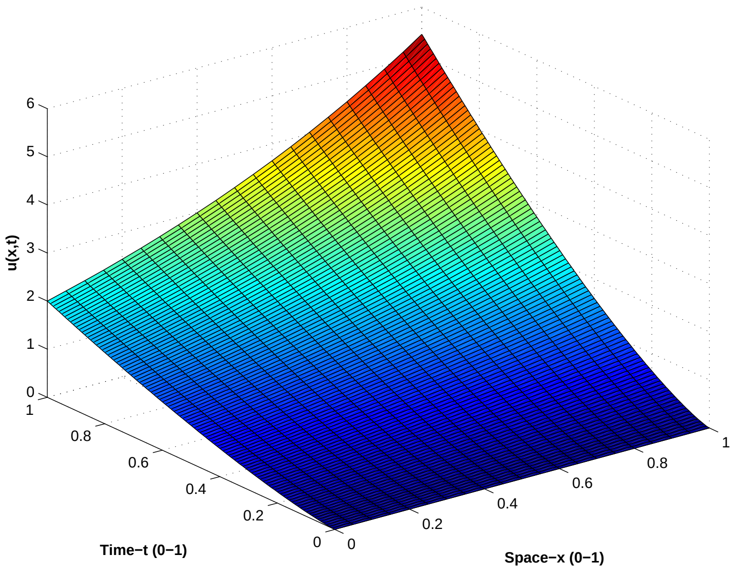

Fig. 1. Numerical solution of problem (45)-(47), when $\alpha = 0.5, \beta = 0.2$

$$\begin{aligned} u(x,0) &= 0, \quad 0 \le x \le 1, \\ u(0,t) &= t^{1+\alpha} + t^{1+\beta}, \\ u(1,t) &= et^{1+\alpha} + et^{1+\beta}. \end{aligned} \tag{47}$$

The exact solution of Equations (45)-(47) is

$$u(x,t) = e^x(t^{1+\alpha} + t^{1+\beta}).$$

We take $\alpha = 0.5, \quad \beta = 0.2, \quad 0 \le t \le 1, \quad 0 \le x \le 1.$

The simulation results with $\alpha = 0.5$, $\beta = 0.2$, $0 \le t \le 1$, $0 \le x \le 1$ are shown in Figure 1. The system exhibits behaviors of the solution and its derivatives of order $\alpha = 0.5, \beta = 0.2$. We can also see that the $u(x,t)$ increases with time.

The comparisons of the numerical solution and exact solution are shown in Figure 2 when $t = 0.2, 1$, respectively. From Figure 2, it can be seen that the numerical solution is in good agreement with the exact solution.

We take $\tau = 0.01, h = 0.1$. The errors between the numerical solution and exact solution are shown in Table 1, when $t = 1$. From Table 1, we can see that the errors satisfy the relation $Error \le C\,(\tau + h^2)$.

Example 2. We consider the following modified anomalous subdiffusion equation with a nonlinear source term:

$$\frac{\partial u(x,t)}{\partial t} = \left(A\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + B\frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right)\left[\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\right] + g(u,x,t), \tag{48}$$

Table 1
The error, numerical solution and exact solution, when $t = 1$, $\tau = 0.01, h = 0.1$

Space	Numerical solution	Exact solution	Error	$\frac{Error}{\tau+h^2}$
0.10	2.212387	2.210342	0.002045	0.10225
0.20	2.446536	2.442806	0.003730	0.18650
0.30	2.704757	2.699718	0.005039	0.25195
0.40	2.989597	2.983649	0.005947	0.29735
0.50	3.303859	3.297442	0.006417	0.32085
0.60	3.650636	3.644238	0.006398	0.31990
0.70	4.033338	4.027505	0.005833	0.29165
0.80	4.455724	4.451082	0.004642	0.23210
0.90	4.921941	4.919207	0.002734	0.13670

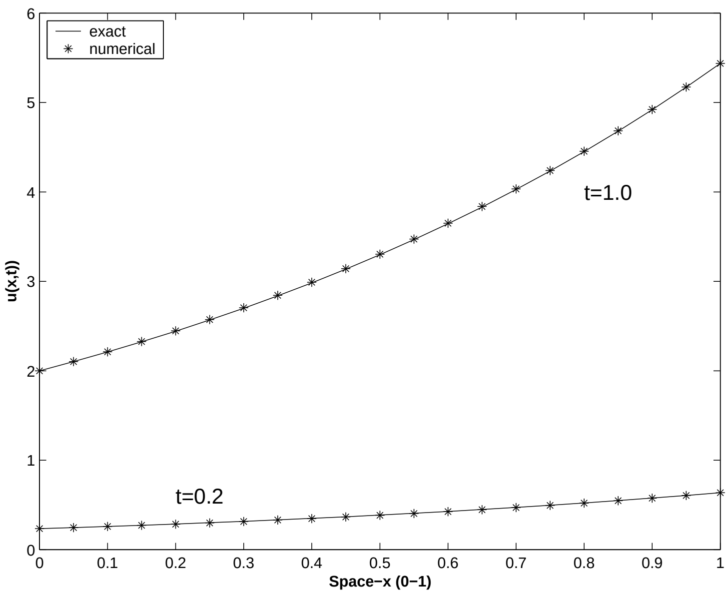

Fig. 2. Numerical solution of problem (45)-(47), when $\alpha = 0.5, \beta = 0.2$ with initial and boundary conditions:

$$u(x,0) = \begin{cases} 8x, & 0 \le x \le 0.5, \\ -4x^2 + \frac{22}{3}x + \frac{4}{3}, & 0.5x \le x \le 2, \end{cases}$$

$$u(0,t) = u(2,t) = 0, A = B = 1.0,$$
$$f(x,t) = e^x, g(u,x,t) = \mu(u - u^2/K), \qquad (49)$$

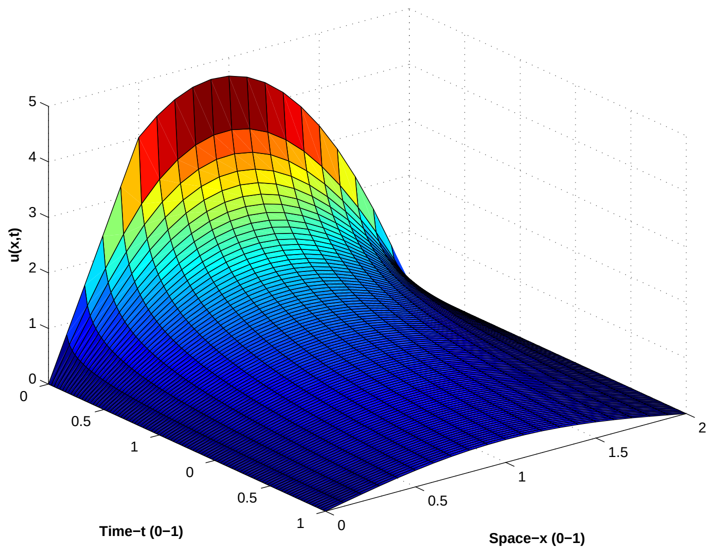

Fig. 3. Numerical solution of (48),(49), when $\alpha = 0.5, \beta = 0.9, t = 0 \sim 1$,

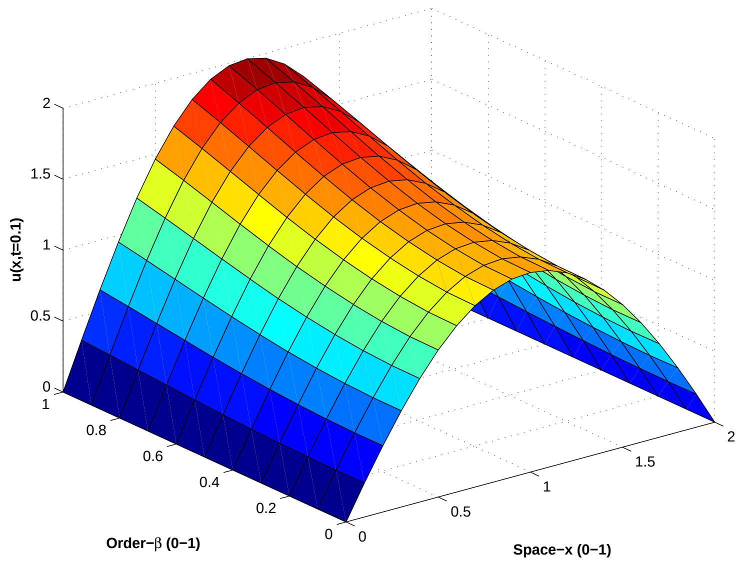

Fig. 4. Numerical solution of (48),(49), when $\alpha = 0.5, 0 \leq \beta \leq 1, t = 0.1$,

where $g(u, x, t)$ is a Fisher nonlinear source term [1]. Here, we take $\mu = 0.5, K = 1$.

Figure 3 shows solution behaviors when $\alpha = 0.5, \beta = 0.9, t = 0 \sim 1$, while Figure 4 shows the response of the diffusion system for different real numbers $0 \leq \beta \leq 1$, $\alpha = 0.5$ at $t = 0.4$ and for different x.

Figures 3-4 show that the system exhibits sub-diffusive behaviors and that the solution continuously depends on the time fractional derivative.

6 Conclusions

In this paper, a new implicit numerical method for a modified anomalous subdiffusion equation with a nonlinear source term in a bounded domain has been described and demonstrated. We prove that the inplicit numerical method is stable and convergent using a new energy method. The implicit numerical method and analytical technique provide computationally effective tools for simulating the behavior of the solution of the modified anomalous subdiffusion equation with a nonlinear source term. This method and analytical technique can also be extended to any fractional integro-differential equations and higher-dimensional problems.

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