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Numerical analysis for the time distributed-order and Riesz space fractional diffusions on bounded domains

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Sub-diffusion equations with distributed-order fractional derivatives describe some important physical phenomena. In this paper, we consider the time distributed-order and Riesz space fractional diffusions on bounded domains with Dirichlet boundary conditions. Here, the time derivative is defined as the distributed-order fractional derivative in the Caputo sense, and the space derivative is defined as the Riesz fractional derivative. Firstly, we discretize the integral term in the time distributed-order and Riesz space fractional diffusions using numerical approximation. Then the given equation can be written as a multi-term time-space fractional diffusion. Secondly, we propose an implicit difference method for the multi-term time-space fractional diffusion. Thirdly, using mathematical induction, we prove the implicit difference method is unconditionally stable and convergent. Also, the solvability for our method is discussed. Finally, two numerical examples are given to show that the numerical results are in good agreement with our theoretical analysis.

Keywords: fractional diffusion; distributed-order fractional derivative; multi-term time-space fractional diffusion; Riesz fractional derivative; implicit difference method; stability and convergence.

1 Introduction

Time-fractional derivatives can be used to model time delays in a diffusion process. When the order of the fractional derivative is distributed over the unit interval, it is useful for modeling a mixture of delay sources (see Meerschaert et al. (2011)). Distributed-order diffusions are also used to model ultraslow diffusion where a plume of particles spreads at a logarithmic rate (see Sinai (1982); Kochubei (2008)). There were many very interesting developments concerning fractional diffusion equations, such as fractional advection dispersion equation (see Benson et al. (2000a,b)), fractional Pearson diffusions (see Leonenko et al. (2013)), fractional diffusion equations with random initial condition (see Anh and Leonenko (2001)). A more extensive development on fractional diffusions presented in the monograph of Meerschaert and Sikorskii (2012). Recently, with the applications arising in distributed-order diffusions, some attention has been paid to the time-fractional equations with distributed-order (see Naber (2004); Eab and Lim (2011); Jiao et al. (2012)). Chechkin et al. (2002) proposed diffusionlike equations with time and space fractional derivatives of the distributed order for the kinetic description of anomalous diffusion and relaxation phenomena and demonstrated that retarding subdiffusion and accelerating superdiffusion were governed by distributed-order fractional diffusion equation. The fundamental solutions for the one-dimensional time fractional diffusion equation and multi-dimensional diffusion-wave equation of distributed order were obtained by Mainardi et al. (2007, 2008) and Atanackovic et al. (2009b), respectively. Atanackovic et al. (2009a) also proved the existence of the solution to the Cauchy problem for the time distributed order diffusion equation and calculated it by the use of Fourier and Laplace transformations. Furthermore, they studied waves in a viscoelastic rod of finite length, where

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viscoelastic material was described by a constitutive equation of fractional distributed-order type (see Atanackovic et al. (2011)). Luchko (2009) proved the uniqueness and continuous dependence on initial conditions for the generalized time-fractional diffusion equation of distributed order on bounded domains. Meerschaert et al. (2011) provided explicit strong solutions and stochastic analogues for distributed-order time-fractional diffusion equations on bounded domains, with Dirichlet boundary conditions.

On the other hand, many numerical methods for fractional partial differential equations have proposed (see Liu et al. (2004, 2007, 2012); Zhuang et al. (2009)). There are also some papers discussing numerical methods of the distributed-order equations. For example, Diethelm and Ford (2009) developed a numerical scheme for the solution of a distributed-order ordinary differential equation and gave a convergence theory for their method. Based on the matrix form representation of discretized fractional operators (see Podlubny (2000)), Podlubny et al. (2013) extended the range of applicability of the matrix approach to discretization of distributed-order derivatives and integrals, and to numerical solution of distributed-order differential equations (both ordinary and partial). As to the multi-term fractional partial differential equations, Liu et al. (2013) proposed some computationally effective numerical methods for simulating the multi-term time-fractional wave-diffusion equations. Jiang et al. (2013) derived the fundamental solutions for the multi-term modified power law wave equations in a finite domain. But there seemed to be little concern about multi-term time-space fractional wave-diffusion equations.

Our attention in this paper is focused on the numerical analysis for the time distributed-order and Riesz space fractional diffusions on bounded domains. Here, the time derivative is defined as the distributed-order fractional derivative in the Caputo sense, and the space derivative is defined as the Riesz fractional derivative. Firstly, we approximate the integral term in the time distributed-order and Riesz space fractional diffusions using numerical approximation. Then the time distributed-order and Riesz space fractional diffusion can be written as a multi-term time-space fractional diffusion. Secondly, we propose an implicit difference method which is uniquely solvable for the multi-term time-space fractional diffusion. Thirdly, using mathematical induction, we prove the implicit difference method is unconditionally stable and convergent. Finally, two numerical examples are provided to show the effectiveness of our method.

The rest of the paper is organized as follows. We present an implicit difference method in Section 2. Section 3 gives some relevant lemmas. In Section 4, we derive the solvability, stability and convergence for the implicit difference method. Two examples are given in Section 5 and some conclusions are drawn in Section 6.

2 Implicit difference method

Consider the following distributed-order diffusion equations

$$\mathbb{D}_t^{\varpi(\alpha)} u(x,t) = K_\beta \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t) \tag{2.1}$$

in an open bounded domain $0 < x < L, 0 < t < T$. Here $K_\beta > 0$, x and t are the space and time variables. The time fractional derivative $\mathbb{D}_t^{\varpi(\alpha)}$ of distributed order is defined by (see Luchko (2009))

$$\mathbb{D}_t^{\varpi(\alpha)} u(x,t) = \int_0^1 {}_0^cD_t^\alpha u(x,t) \varpi(\alpha) d\alpha \tag{2.2}$$

with the left-side Caputo fractional derivative ${}_0^cD_t^\alpha$ defined as (see Podlubny (1999))

$$ {}_0^cD_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial u}{\partial \tau}(x,\tau)d\tau, & 0<\alpha<1, \\ \frac{\partial u}{\partial t}(x,t) & , \quad \alpha = 1, \end{cases} \tag{2.3} $$

and with a continuous non-negative weight function $\varpi : [0,1] \to \mathscr{R}$ that is not identically equal to zero on the interval $[0,1]$, such that the conditions

$$ 0 \leqslant \varpi(\alpha), \varpi \neq 0, \alpha \in [0,1], \int_0^1 \varpi(\alpha)d\alpha = W > 0 \tag{2.4} $$

hold true, where W is a positive constant. The space fractional derivative $\frac{\partial^\beta u(x,t)}{\partial |x|^\beta}$ is the Riesz fractional derivative operator for $1<\beta<2$ defined by (see Çelik and Duman (2012))

$$ \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} = -\frac{1}{2\cos(\frac{\beta\pi}{2})\Gamma(2-\beta)}\frac{d^2}{dx^2}\int_0^L |x-\xi|^{1-\beta}u(\xi,t)d\xi. \tag{2.5} $$

When $\beta = 2$, $\frac{\partial^\beta u(x,t)}{\partial |x|^\beta} = \frac{\partial^2 u(x,t)}{\partial x^2}$.

In this paper, the initial-boundary conditions

$$ u(x,0) = \phi(x), \quad 0 \leqslant x \leqslant L, \tag{2.6} $$

$$ u(0,t) = 0, \quad u(L,t) = 0, \qquad 0 \leqslant t \leqslant T \tag{2.7} $$

for Eq. (2.1) is considered.

Now, we state our numerical method as follows.

Step 1: Discretize the integral term in the distributed-order equation.

Let us discretize the interval [0,1], in which the order α is changing, using the grid $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_q = 1 (q \in \mathscr{N})$, with the steps $\Delta\xi_s$ not necessarily equidistant. We obtain

$$ \begin{aligned} \mathbb{D}_t^{\varpi(\alpha)} u(x,t) &\approx \sum_{s=1}^{q} \varpi(\alpha_s)\left({}_0^cD_t^{\alpha_s} u(x,t)\right)\Delta\xi_s \\ &= \sum_{s=1}^{q} d_s \, {}_0^cD_t^{\alpha_s} u(x,t), \end{aligned} \tag{2.8} $$

where $\alpha_s \in (\xi_{s-1}, \xi_s]$, $d_s = \varpi(\alpha_s)\Delta\xi_s$, $\Delta\xi_s = \xi_s - \xi_{s-1}, s = 1,2,\cdots,q$.

For the simplicity of the presentation, but without loss of the generality, we take $\Delta\xi_s = \frac{1}{q} = \sigma(q \in \mathscr{N})$ and $d_s = \frac{\varpi(\alpha_s)}{q}$. We can use the mid-point quadrature rule for approximating the integral (2.2). Let $\alpha_s = \frac{\xi_{s-1}+\xi_s}{2} = \frac{2s-1}{2q}, s = 1,2,\cdots,q$. Consider the following multi-term fractional diffusion equation

$$ \sum_{s=1}^{q} d_s \left({}_0^cD_t^{\alpha_s} u(x,t)\right) = K_\beta \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t), \tag{2.9} $$

with the initial-boundary conditions (2.6)-(2.7).

Step 2: Solve the multi-term equation.

We assume that we are working on a uniform grid $x_i = ih, i = 0, 1, \cdots, M$; $Mh = L; t_k = k\tau, k = 0, 1, \cdots, N; N\tau = T$. Let $u_i^k = u(x_i, t_k)$, $f_i^k = f(x_i, t_k), 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant N$.

For $0 < \alpha_s < 1$, adopting the L1 discrete scheme in Oldham and Spanier (1974), we discretize the Caputo time fractional derivative as

$$ {}_0^cD_t^{\alpha_s} u_i^{k+1} \approx \frac{1}{\mu_s}\left[u_i^{k+1} - \sum_{j=1}^{k}\left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)u_i^j - a_k^{\alpha_s} u_i^0\right], \tag{2.10} $$

where

$$ a_k^{\alpha_s} = (k+1)^{1-\alpha_s} - k^{1-\alpha_s}, \quad \mu_s = \tau^{\alpha_s}\Gamma(2-\alpha_s), \qquad s = 1, 2, \cdots, q. $$

Using the fractional centered difference (see Çelik and Duman (2012); Ortigueira (2006)) and noticing the boundary-value condition (2.7), we can obtain the following numerical discretization scheme for space-fractional derivative:

$$ \frac{\partial^\beta}{\partial |x|^\beta} u_i^{k+1} \approx -h^{-\beta} \sum_{\rho=1}^{M-1} g_{i-\rho} u_\rho^{k+1}, \tag{2.11} $$

where

$$ g_\rho = \frac{(-1)^\rho \Gamma(\beta+1)}{\Gamma(\frac{\beta}{2} - \rho + 1)\Gamma(\frac{\beta}{2} + \rho + 1)}, \quad 1 < \beta < 2. \tag{2.12} $$

Let U_i^k be the numerical approximation to $u(x_i, t_k)$. We can derive the implicit numerical scheme

$$ \begin{aligned} & \sum_{s=1}^{q} \frac{d_s}{\mu_s}\left[U_i^{k+1} - \sum_{j=1}^{k}\left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)U_i^j - a_k^{\alpha_s} U_i^0\right] \\ = \quad & -K_\beta h^{-\beta} \sum_{\rho=1}^{M-1} g_{i-\rho} U_\rho^{k+1} + f_i^{k+1}, \quad 1 \leqslant i \leqslant M-1, \ \ 0 \leqslant k \leqslant N-1. \end{aligned} \tag{2.13} $$

Denote

$$ D = \frac{K_\beta h^{-\beta}}{\sum_{r=1}^{q} \frac{d_r}{\mu_r}}, \ \ \overline{D} = \frac{1}{\sum_{r=1}^{q} \frac{d_r}{\mu_r}}, \ \ D_s = \frac{d_s}{\mu_s \sum_{r=1}^{q} \frac{d_r}{\mu_r}}, \ \ s = 1, 2, \cdots, q. \tag{2.14} $$

Thus we have the following implicit difference approximation

$$ \begin{aligned} & U_i^{k+1} + D \sum_{\rho=1}^{M-1} g_{i-\rho} U_\rho^{k+1} \\ = \quad & \sum_{j=1}^{k}\left[\sum_{s=1}^{q} D_s\left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)\right] U_i^j + \sum_{s=1}^{q} D_s a_k^{\alpha_s} U_i^0 + \overline{D} f_i^{k+1}, \\ & \qquad\qquad i = 1, 2, \cdots, M-1, \quad k = 0, 1, \cdots, N-1, \end{aligned} \tag{2.15} $$

$$ U_i^0 = \phi_i^0 = \phi(x_i), \qquad 0 \leqslant i \leqslant M, \tag{2.16} $$

$$ U_0^k = U_M^k = 0, \qquad 0 \leqslant k \leqslant N. \tag{2.17} $$

3 Some lemmas

To analyze the difference scheme, we need the following lemmas.

LEMMA 3.1 (See Çelik and Duman (2012).) Let $g_k = \frac{(-1)^k \Gamma(\beta+1)}{\Gamma(\frac{\beta}{2}-k+1)\Gamma(\frac{\beta}{2}+k+1)}$ be the coefficients of the centered finite difference approximation (2.11) for $k = 0, \pm 1, \pm 2, \cdots$, and $1 < \beta < 2$. Then

(1) $g_0 \geqslant 0$; (2) $g_{-k} = g_k \leqslant 0$, for all $|k| \geqslant 1$;
(3) $\sum_{k=-\infty}^{\infty} g_k = 0$; (4) $g_0 = \sum_{k=-\infty, k\neq 0}^{\infty} |g_k|$.

LEMMA 3.2 (See Gao and Sun (2011).) Suppose $0 < \alpha < 1$, u is absolutely continuous in t on $[0,T]$ and $\frac{\partial^2 u}{\partial t^2} \in C([0,L]\times[0,t_k])$. Then

$$ {}_0^cD_t^\alpha u_i^k = \frac{1}{\mu}\left[u_i^k - \sum_{j=1}^{k-1}\left(a_{k-j-1}^\alpha - a_{k-j}^\alpha\right)u_i^j - a_{k-1}^\alpha u_i^0\right] + O(\tau^{2-\alpha}), \tag{3.1} $$

where $a_k^\alpha = (k+1)^{1-\alpha} - k^{1-\alpha},\ \mu = \tau^\alpha \Gamma(2-\alpha),\ 0 \leqslant t_k \leqslant T$

LEMMA 3.3 (See Çelik and Duman (2012).) Let $\frac{\partial^5 u}{\partial x^5} \in C([0,L]\times[0,T])$ and u satisfies the boundary condition (2.7). Then

$$ \frac{\partial^\beta}{\partial |x|^\beta} u_i^k = -h^{-\beta}\sum_{\rho=1}^{M-1} g_{i-\rho} u_\rho^k + O(h^2), \tag{3.2} $$

when $h \to 0$, $\frac{\partial^\beta}{\partial |x|^\beta} u_i^k$ is the Riesz fractional derivative for $1 < \beta < 2$ and g_ρ is as in the expression (2.12).

LEMMA 3.4 (See Diethelm and Ford (2009).) Suppose u is absolutely continuous in t on $[0,T]$ and $\frac{\partial u}{\partial t} \in C([0,L]\times[0,T])$. For every fixed $(x,t) \in [0,L]\times(0,T]$, consider ${}_0^cD_t^\alpha u(x,t) =: z(\alpha)$ as a function of α. Then z is a C^∞ function on $(0,1]$.

LEMMA 3.5 (See Faires and Burden (2013).) If $z(\alpha) \in C^2[0,1], \triangle\alpha = \frac{1}{q} = \sigma\ (q \in \mathscr{N})$, then

$$ \int_0^1 z(\alpha)d\alpha = \sum_{s=1}^{q} z\left(\frac{2s-1}{2q}\right)\frac{1}{q} + O(\sigma^2). \tag{3.3} $$

4 Analysis of the implicit difference scheme

4.1 *Solvability*

The difference scheme (2.15)-(2.17) can be written in the following matrix form:

$$ AU^1 = b_0 I U^0 + \overline{D} f^1, \tag{4.1} $$

$$ AU^{k+1} = \sum_{j=1}^{k} c_{k,j} I U^j + b_k I U^0 + \overline{D} f^{k+1}, \quad k = 1,2,\cdots,N-1, \tag{4.2} $$

where

$$ A = \begin{pmatrix} 1+Dg_0 & Dg_{-1} & \cdots & Dg_{-M+2} \\ Dg_1 & 1+Dg_0 & \cdots & Dg_{-M+3} \\ \cdots & \cdots & \cdots & \cdots \\ Dg_{M-2} & Dg_{M-3} & \cdots & 1+Dg_0 \end{pmatrix}_{(M-1)\times(M-1)}, \tag{4.3} $$

$U^k = (U_1^k, U_2^k, \cdots, U_{M-1}^k)^T$, $f^k = (f_1^k, f_2^k, \cdots, f_{M-1}^k)^T$, $b_k = \sum_{s=1}^q D_s a_k^{\alpha_s}$, $c_{k,j} = \sum_{s=1}^q D_s(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s})$. Lemma 3.1 implies that matrix A is strictly diagonally dominant; thus U^1 can be obtained from (4.1) and $U^2, U^3, \cdots, U^N$ can be obtained from (4.2). This can be written as the following result.

THEOREM 4.1 The difference scheme (2.15)-(2.17) is uniquely solvable.

4.2 *Stability*

In this subsection, we consider the stability of the implicit difference approximation (2.15)-(2.17). We assume that the initial data have errors $\varepsilon_i^0 (i = 1, 2, \cdots, M-1)$. Let $\tilde{\phi}_i^0 = \phi_i^0 + \varepsilon_i^0$, U_i^k and $\tilde{U}_i^k (i = 1, 2, \cdots, M-1)$ be the numerical solutions of Eq. (2.15) corresponding to the initial data ϕ_i^0 and $\tilde{\phi}_i^0 (i = 1, 2, \cdots, M-1)$, respectively. Then $\varepsilon_i^k = U_i^k - \tilde{U}_i^k$ satisfies

$$\varepsilon_i^1 + D\sum_{\rho=1}^{M-1} g_{i-\rho}\varepsilon_\rho^1 = \sum_{s=1}^q D_s a_0^{\alpha_s}\varepsilon_i^0 = \varepsilon_i^0 \tag{4.4}$$

$$\varepsilon_i^{k+1} + D\sum_{\rho=1}^{M-1} g_{i-\rho}\varepsilon_\rho^{k+1} = \sum_{j=1}^k \left[\sum_{s=1}^q D_s\left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)\right]\varepsilon_i^j + \sum_{s=1}^q D_s a_k^{\alpha_s}\varepsilon_i^0, \quad k = 1, 2, \cdots, N-1. \tag{4.5}$$

In the following theorem, we denote $E^k = [\varepsilon_1^k, \varepsilon_2^k, \cdots, \varepsilon_{M-1}^k]^T$.

THEOREM 4.2 The implicit difference approximation defined by (2.15)-(2.17) for distributed-order fractional diffusions is unconditionally stable, where $1 < \beta < 2$.

Proof. The stability condition is equivalent to

$$\|E^{k+1}\|_\infty \leqslant \|E^0\|_\infty, \quad k = 0, 1, 2, \cdots. \tag{4.6}$$

We will use the mathematical induction to get the above result. For $k = 0$, let $|\varepsilon_l^1| = \max_{1\leqslant i\leqslant M-1}|\varepsilon_i^1|$. Noticing that $\sum_{\rho=1}^{M-1} g_{l-\rho} > 0$ and Lemma 3.1, we have

$$\begin{aligned}
\|E^1\|_\infty &= |\varepsilon_l^1| \leqslant |\varepsilon_l^1| + D\sum_{\rho=1}^{M-1} g_{l-\rho}|\varepsilon_l^1| \\
&= |\varepsilon_l^1| + Dg_0|\varepsilon_l^1| + D\sum_{\rho=1,\rho\neq l}^{M-1} g_{l-\rho}|\varepsilon_l^1| \\
&\leqslant |\varepsilon_l^1| + Dg_0|\varepsilon_l^1| - D\sum_{\rho=1,\rho\neq l}^{M-1} |g_{l-\rho}\varepsilon_\rho^1| \\
&\leqslant |\varepsilon_l^1 + Dg_0\varepsilon_l^1 + D\sum_{\rho=1,\rho\neq l}^{M-1} g_{l-\rho}\varepsilon_\rho^1| \\
&= |\varepsilon_l^1 + D\sum_{\rho=1}^{M-1} g_{l-\rho}\varepsilon_\rho^1| = |\varepsilon_l^0| = \|E^0\|_\infty.
\end{aligned}$$

Suppose that $\|E^j\|_\infty \leqslant \|E^0\|_\infty,\ j=1,2,\cdots,k$. Let $|\varepsilon_l^{k+1}| = \max_{1\leqslant i\leqslant M-1}|\varepsilon_i^{k+1}|$. It follows that

$$\begin{aligned}
\|E^{k+1}\|_\infty &= |\varepsilon_l^{k+1}| \leqslant |\varepsilon_l^{k+1}|\left[1+D\sum_{\rho=1}^{M-1} g_{l-\rho}\right] \\
&= |\varepsilon_l^{k+1}| + Dg_0|\varepsilon_l^{k+1}| + D\sum_{\rho=1,\rho\neq l}^{M-1} g_{l-\rho}|\varepsilon_l^{k+1}| \\
&\leqslant |\varepsilon_l^{k+1}| + Dg_0|\varepsilon_l^{k+1}| - D\sum_{\rho=1,\rho\neq l}^{M-1} |g_{l-\rho}\varepsilon_\rho^{k+1}| \\
&\leqslant |\varepsilon_l^{k+1} + Dg_0\varepsilon_l^{k+1} + D\sum_{\rho=1,\rho\neq l}^{M-1} g_{l-\rho}\varepsilon_\rho^{k+1}| \\
&\leqslant \sum_{j=1}^{k}\left[\sum_{s=1}^{q} D_s\left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)\right]|\varepsilon_l^j| + \sum_{s=1}^{q} D_s a_k^{\alpha_s}|\varepsilon_l^0| \\
&\leqslant \left\{\sum_{j=1}^{k}\left[\sum_{s=1}^{q} D_s\left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)\right] + \sum_{s=1}^{q} D_s a_k^{\alpha_s}\right\}\|E^0\|_\infty \\
&= \left\{\sum_{s=1}^{q} D_s\left[\sum_{j=1}^{k}\left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)\right] + \sum_{s=1}^{q} D_s a_k^{\alpha_s}\right\}\|E^0\|_\infty \\
&= \sum_{s=1}^{q} D_s\|E^0\|_\infty = \|E^0\|_\infty.
\end{aligned}$$

Hence, the proof is completed. □

4.3 *Convergence*

Suppose that the continuous problem (2.1), (2.6)-(2.7) has a smooth solution $u(x,t) \in C_{x,t}^{5,2}(\Omega)$, where $\Omega = [0,L]\times[0,T]$, and

$$C_{x,t}^{5,2}(\Omega) = \left\{u(x,t)\left|\frac{\partial^5 u(x,t)}{\partial x^5}, \frac{\partial^2 u(x,t)}{\partial t^2} \in C(\Omega)\right.\right\}.$$

We now consider the convergence of the implicit difference approximation. Let u be the exact solution of the system (2.1), (2.6)-(2.7), and U be the numerical solution of the implicit difference approximation (2.15)-(2.17). Let the error $e = u - U$, and at the mesh points (x_i,t_k) be defined by $e_i^k = u_i^k - U_i^k$ $(i = 1,2,\cdots,M-1;\ k=0,1,2,\cdots,N)$. We denote $R^k = [e_1^k, e_2^k,\cdots,e_{M-1}^k]^T$. Then $R^0 = [e_1^0,e_2^0,\cdots,e_{M-1}^0]^T = 0$.

Substituting $U_i^k = u_i^k - e_i^k$ into Eq. (2.15) leads to the following two cases. When $k=0$,

$$e_i^1 + D\sum_{\rho=1}^{M-1} g_{i-\rho}e_\rho^1 = u_i^1 + D\sum_{\rho=1}^{M-1} g_{i-\rho}u_\rho^1 - \sum_{s=1}^{q} D_s a_0^{\alpha_s}u_i^0 - \overline{D}f_i^1. \tag{4.7}$$

Based on (2.14), Lemma 3.2-Lemma 3.5 and (2.1), we have

$$
\begin{aligned}
& e_i^1 + D\sum_{\rho=1}^{M-1} g_{i-\rho} e_\rho^1 \\
= {} & \overline{D}\left[\sum_{s=1}^{q} d_s \, {}_0^cD_t^{\alpha_s} u_i^1 - K_\beta \frac{\partial^\beta}{\partial |x|^\beta} u_i^1 - f_i^1 + O(\tau^{2-\alpha_q}) + O(h^2)\right] \\
= {} & \overline{D}\left[O(h^2) + O(\tau^{1+\frac{\sigma}{2}}) + O(\sigma^2)\right]. \qquad (4.8)
\end{aligned}
$$

When $k \geqslant 1$,

$$
\begin{aligned}
e_i^{k+1} \quad + \quad & D\sum_{\rho=1}^{M-1} g_{i-\rho} e_\rho^{k+1} - \sum_{j=1}^{k}\left[\sum_{s=1}^{q} D_s \left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)\right] e_i^j - \sum_{s=1}^{q} D_s a_k^{\alpha_s} e_i^0 \\
= \quad & u_i^{k+1} + D\sum_{\rho=1}^{M-1} g_{i-\rho} u_\rho^{k+1} - \sum_{j=1}^{k}\left[\sum_{s=1}^{q} D_s \left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)\right] u_i^j \\
- \quad & \sum_{s=1}^{q} D_s a_k^{\alpha_s} u_i^0 - \overline{D} f_i^{k+1}. \qquad (4.9)
\end{aligned}
$$

Based on (2.14), Lemma 3.2-Lemma 3.5 and (2.1),

$$
\begin{aligned}
& \frac{1}{\overline{D}}\left\{u_i^{k+1} + D\sum_{\rho=1}^{M-1} g_{i-\rho} u_\rho^{k+1} - \sum_{j=1}^{k}\left[\sum_{s=1}^{q} D_s \left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)\right] u_i^j \right. \\
& \left. - \sum_{s=1}^{q} D_s a_k^{\alpha_s} u_i^0 - \overline{D} f_i^{k+1}\right\} \\
= {} & \sum_{s=1}^{q} d_s \, {}_0^cD_t^{\alpha_s} u_i^{k+1} - K_\beta \frac{\partial^\beta}{\partial |x|^\beta} u_i^{k+1} - f_i^{k+1} + O(\tau^{2-\alpha_q}) + O(h^2) \\
= {} & O(\tau^{1+\frac{\sigma}{2}}) + O(h^2) + O(\sigma^2). \qquad (4.10)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& e_i^{k+1} + D\sum_{\rho=1}^{M-1} g_{i-\rho} e_\rho^{k+1} \\
= {} & \sum_{j=1}^{k}\left[\sum_{s=1}^{q} D_s \left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)\right] e_i^j + \sum_{s=1}^{q} D_s a_k^{\alpha_s} e_i^0 \\
& + \overline{D}\left[O(\tau^{1+\frac{\sigma}{2}}) + O(h^2) + O(\sigma^2)\right]. \qquad (4.11)
\end{aligned}
$$

Now we can derive the following result by mathematical induction.

THEOREM 4.3 Suppose that the continuous problem (2.1), (2.6)-(2.7) has a smooth solution $u(x,t) \in C_{x,t}^{5,2}(\Omega)$, and let U be the solution of the difference scheme (2.15)-(2.17) for $1 < \beta < 2$. Then there is a positive constant C such that the error satisfies

$$
\|R^k\|_\infty \leqslant C(h^2 + \tau^{1+\frac{\sigma}{2}} + \sigma^2) \Big/ \sum_{s=1}^{q} \frac{d_s a_{k-1}^{\alpha_s}}{\mu_s}, \quad k = 1, 2, \cdots, N. \qquad (4.12)
$$

Proof. For $k=1$, let $\|R^1\|_\infty = |e_l^1| = \max_{1\leqslant i\leqslant M-1}|e_i^1|$. According to Lemma 3.1, (2.14) and (4.8), we have

$$\begin{aligned}
\|R^1\|_\infty &= |e_l^1| \leqslant |e_l^1| + D\sum_{\rho=1}^{M-1} g_{l-\rho}|e_l^1| \\
&\leqslant |e_l^1| + Dg_0|e_l^1| - D\sum_{\rho=1,\rho\neq l}^{M-1}|g_{l-\rho}e_\rho^1| \\
&\leqslant |e_l^1 + D\sum_{\rho=1}^{M-1} g_{l-\rho}e_\rho^1| \\
&\leqslant C\overline{D}[h^2+\tau^{1+\frac{\sigma}{2}}+\sigma^2] = C(h^2+\tau^{1+\frac{\sigma}{2}}+\sigma^2)\Big/\sum_{s=1}^{q}\frac{d_s a_0^{\alpha_s}}{\mu_s}.
\end{aligned}$$

Suppose that $\|R^j\|_\infty \leqslant C(h^2+\tau^{1+\frac{\sigma}{2}}+\sigma^2)\Big/\sum_{s=1}^{q}\frac{d_s a_{j-1}^{\alpha_s}}{\mu_s},\ \ j=1,2,\cdots,k$ and let $|e_l^{k+1}| = \max_{1\leqslant i\leqslant M-1}|e_i^{k+1}|$. Based on Lemma 3.1, (2.14), (4.11) and noticing that the coefficients $a_j^{\alpha_s}$ are decreasing for $j = 0,1,2,\cdots$, we obtain

$$\begin{aligned}
\|R^{k+1}\|_\infty &= |e_l^{k+1}| \leqslant |e_l^{k+1} + D\sum_{\rho=1}^{M-1} g_{l-\rho}e_\rho^{k+1}| \\
&\leqslant \sum_{j=1}^{k}\left[\sum_{s=1}^{q} D_s\left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)\right]|e_i^j| + C\overline{D}[h^2+\tau^{1+\frac{\sigma}{2}}+\sigma^2] \\
&\leqslant \sum_{s=1}^{q} D_s\left[\sum_{j=1}^{k}\left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)C(h^2+\tau^{1+\frac{\sigma}{2}}+\sigma^2)\Big/\sum_{s=1}^{q}\frac{d_s a_{j-1}^{\alpha_s}}{\mu_s}\right] \\
&\quad + C\overline{D}[h^2+\tau^{1+\frac{\sigma}{2}}+\sigma^2] \\
&\leqslant \sum_{s=1}^{q} D_s\left[\sum_{j=1}^{k}\left(a_{k-j}^{\alpha_s} - a_{k-j+1}^{\alpha_s}\right)C(h^2+\tau^{1+\frac{\sigma}{2}}+\sigma^2)\Big/\sum_{s=1}^{q}\frac{d_s a_{k}^{\alpha_s}}{\mu_s}\right] \\
&\quad + C\overline{D}[h^2+\tau^{1+\frac{\sigma}{2}}+\sigma^2] \\
&= C(h^2+\tau^{1+\frac{\sigma}{2}}+\sigma^2)\Big/\sum_{s=1}^{q}\frac{d_s a_{k}^{\alpha_s}}{\mu_s}.
\end{aligned}$$

Thus, the theorem is proved. □

Since

$$\lim_{k\to\infty}\frac{k^{-\alpha_s}}{a_k^{\alpha_s}} = \lim_{k\to\infty}\frac{1}{k\left[(1+\frac{1}{k})^{1-\alpha_s}-1\right]} = \frac{1}{1-\alpha_s}, \tag{4.13}$$

there is a constant C_1 such that $a_k^{\alpha_s} \geqslant C_1 k^{-\alpha_s}(1-\alpha_s)$. It follows that

$$\begin{aligned}
&\sum_{s=1}^{q}\frac{d_s a_k^{\alpha_s}}{\mu_s} \geqslant C_1\sum_{s=1}^{q}\frac{d_s}{(k\tau)^{\alpha_s}\Gamma(1-\alpha_s)} \\
\geqslant\ & C_1\sum_{s=1}^{q}\frac{d_s}{T^{\alpha_s}\Gamma(1-\alpha_s)} \to C_1\int_0^1\frac{\varpi(\alpha)}{T^{\alpha}\Gamma(1-\alpha)}d\alpha = C_2.
\end{aligned}$$

Theorem 4.3 implies that there is a constant $\widetilde{C}$ such that

$$\|R^k\|_\infty \leqslant \widetilde{C}(h^2 + \tau^{1+\frac{\sigma}{2}} + \sigma^2).$$

In fact, we can obtain the following result.

THEOREM 4.4 Suppose that the continuous problem (2.1), (2.6)-(2.7) has a smooth solution $u(x,t) \in C_{x,t}^{5,2}(\Omega)$, and let U be the solution of the difference scheme (2.15)-(2.17). Then the solution U unconditionally converges to u as h,τ and σ tend to zero. Furthermore, there is a positive constant C such that

$$|u_i^k - U_i^k| \leqslant C(h^2 + \tau^{1+\frac{\sigma}{2}} + \sigma^2),\ i = 1,2,\cdots,M-1; k = 1,2,\cdots,N.$$

5 Numerical results

In order to illustrate the behaviour of our numerical method and demonstrate the effectiveness of our theoretical analysis, two examples are now presented.

EXAMPLE 5.1 Consider the following time distributed order and Riesz space fractional diffusion equation:

$$\int_0^1 \nu^{\alpha-1}\, {}_0^cD_t^\alpha u(x,t)d\alpha = K\frac{\partial^\beta u(x,t)}{\partial|x|^\beta},\ \ 0<x<1, 0<t<T, \tag{5.1}$$

where ν is a positive constant that can be physically interpreted as the relaxation time, K is also a positive constant representing the diffusion coefficient, $1 < \beta \leqslant 2$. When $\beta = 2$, Chechkin et al. (2002) showed that the distributed-order time fractional diffusion equation describes the subdiffusion random process that is subordinated to the Wiener process and whose diffusion exponent decreases in time (retarding subdiffusion). This process may lead to ultraslow diffusion, with the mean square displacement growing logarithmically in time.

Here, the initial-boundary conditions

$$u(x,0) = x^2(1-x^2), \quad 0 \leqslant x \leqslant 1, \tag{5.2}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad 0 \leqslant t \leqslant T \tag{5.3}$$

for Eq. (5.1) are considered.

Using the numerical method described in Sec. 2, we obtain the numerical solutions (Fig.1) of the fractional diffusion equation for $\nu = 0.5, K = 1$, $\beta = 1.6, 1.8, 2$, respectively, with $h = 0.02, \tau = 0.015, \sigma = 0.1$.

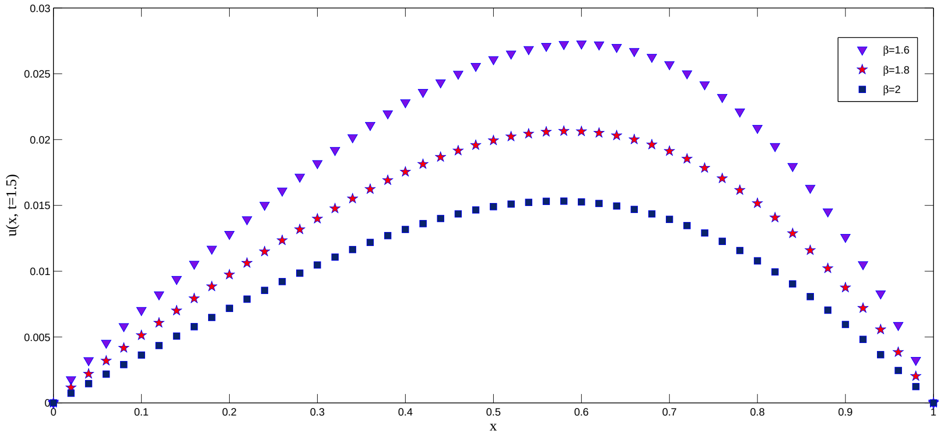

Fig.1. The numerical approximation of u(x,t) for the system (5.1)−(5.3) when t=1.5, ν=0.5 and K=1.

EXAMPLE 5.2 Consider the following time distributed-order and Riesz space fractional diffusion equation:

$$\begin{cases} \int_0^1 \Gamma(3-\alpha)\,{}_0^cD_t^{\alpha}u(x,t)d\alpha = \frac{\partial^{\beta}u(x,t)}{\partial|x|^{\beta}} + f(x,t), \\ \qquad 0 < x < 1, 0 < t \leqslant T, \\ u(x,0) = x^2(1-x)^2, \quad 0 \leqslant x \leqslant 1, \\ u(0,t) = u(1,t) = 0, \quad 0 \leqslant t \leqslant T, \end{cases} \tag{5.4}$$

where $1 < \beta \leqslant 2$,

$$\begin{aligned} f(x,t) &= \frac{1}{2\cos(\frac{\beta\pi}{2})}(1-t^2)\left[\frac{\Gamma(3)}{\Gamma(3-\beta)}\left(x^{2-\beta}+(1-x)^{2-\beta}\right)\right. \\ &- 2\frac{\Gamma(4)}{\Gamma(4-\beta)}\left(x^{3-\beta}+(1-x)^{3-\beta}\right)+\left.\frac{\Gamma(5)}{\Gamma(5-\beta)}\left(x^{4-\beta}+(1-x)^{4-\beta}\right)\right] \\ &- 2x^2(1-x)^2(t^2-t)/lnt. \end{aligned}$$

The exact solution of the above problem is $u(x,t) = x^2(1-x)^2(1-t^2)$.

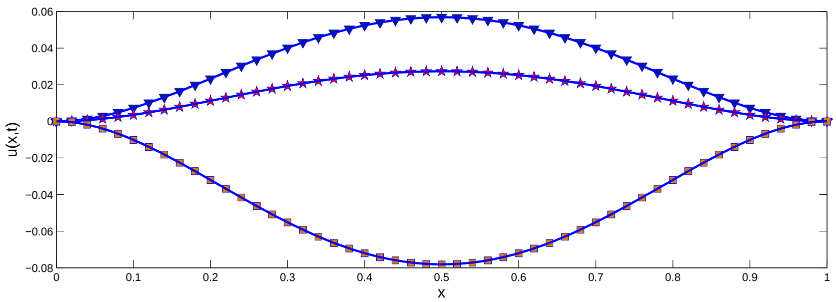

Fig. 2. Exact solutions (lines) and numerical solutions (symbols) with β=1.8 at t=0.3 (triangles), t=0.75 (stars) and t=1.5 (squares).

A comparison of the exact solution and the numerical solution for $\beta = 1.8$ with $h = 0.02, \tau = 0.015, \sigma = 0.1$ at $t = 0.3$ (triangles), $t = 0.75$ (stars) and $t = 1.5$ (squares) is shown in Fig. 2. From Fig. 2, it can be seen that the numerical solution is in good agreement with the exact solution.

6 Conclusion

In this paper, an implicit difference scheme for the time distributed-order and Riesz space fractional diffusions on bounded domains has been described. We prove that the implicit difference scheme is unconditionally stable and convergent. Two numerical examples demonstrate the effectiveness theoretical results.

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