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# Stateful-Failure Reactive Designs in Isabelle/UTP

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## Abstract

Stateful-Failure Reactive Designs specialise reactive design contracts with failures traces, as present in languages like CSP and Circus. A failure trace consists of a sequence of events and a refusal set. It intuitively represents a quiescent observation, where certain events have previously occurred, and others are currently being accepted. Following the UTP book, we add an observational variable to represent refusal sets, and healthiness conditions that ensure their well-formedness. Using these, we also specialise our theory of reactive relations with operators to characterise both completed and quiescent interactions, and an accompanying equational theory. We use these to define the core operators — including assignment, event occurrence, and external choice — and specialise our proof strategy to support these. We also demonstrate a link with the CSP failures-divergences semantic model.

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## 1 Introduction

This document contains a mechanisation in Isabelle/UTP [1] of an specialisation of stateful reactive designs with refusal information, as present in languages like Circus [2].

## 2 Stateful-Failure Core Types

```
theory utp-sfrd-core
  imports UTP-Reactive-Designs.utp-rea-designs
begin
```

## 2.1 SFRD Alphabet

**alphabet**  $'\varphi$  *csp-vars* =  $'\sigma$  *rsp-vars* +  
*ref* ::  $'\varphi$  *set*

**declare** *csp-vars.defs* [*lens-defs*]  
**declare** *csp-vars.splits* [*alpha-splits*]

The following two locale interpretations are a technicality to improve the behaviour of the automatic tactics. They enable (re)interpretation of state spaces in order to remove any occurrences of lens types, replacing them by tuple types after the tactics *pred-simp* and *rel-simp* are applied. Eventually, it would be desirable to automate preform these interpretations automatically as part of the **alphabet** command.

**interpretation** *alphabet-csp-prd*:  
*lens-interp*  $\lambda(ok, wait, tr, m). (ok, wait, tr, ref_v m, more m)$   
**apply** (*unfold-locales*)  
**apply** (*rule injI*)  
**apply** (*clarsimp*)  
**done**

**interpretation** *alphabet-csp-rel*:  
*lens-interp*  $\lambda(ok, ok', wait, wait', tr, tr', m, m').$   
 $(ok, ok', wait, wait', tr, tr', ref_v m, ref_v m', more m, more m')$   
**apply** (*unfold-locales*)  
**apply** (*rule injI*)  
**apply** (*clarsimp*)  
**done**

**lemma** *circus-var-ords* [*usubst*]:  
 $\$ref \prec_v \$ref'$   
 $\$ok \prec_v \$ref \ \$ok' \prec_v \$ref' \ \$ok \prec_v \$ref' \ \$ok' \prec_v \$ref$   
 $\$ref \prec_v \$wait \ \$ref' \prec_v \$wait' \ \$ref \prec_v \$wait' \ \$ref' \prec_v \$wait$   
 $\$ref \prec_v \$st \ \$ref' \prec_v \$st' \ \$ref \prec_v \$st' \ \$ref' \prec_v \$st$   
 $\$ref \prec_v \$tr \ \$ref' \prec_v \$tr' \ \$ref \prec_v \$tr' \ \$ref' \prec_v \$tr$   
**by** (*simp-all add: var-name-ord-def*)

**type-synonym**  $('σ, 'ϕ)$  *st-csp* =  $('σ, 'ϕ$  *list*,  $('ϕ, unit)$  *csp-vars-scheme*) *rsp*  
**type-synonym**  $('σ, 'ϕ)$  *action* =  $('σ, 'ϕ)$  *st-csp hrel*  
**type-synonym**  $'ϕ$  *csp* =  $(unit, 'ϕ)$  *st-csp*  
**type-synonym**  $'ϕ$  *rel-csp* =  $'ϕ$  *csp hrel*

There is some slight imprecision with the translations, in that we don't bother to check if the trace event type and refusal set event types are the same. Essentially this is because its very difficult to construct processes where this would be the case. However, it may be better to add a proper ML print translation in the future.

**translations**  
 $(type) ('σ, 'ϕ)$  *st-csp* <=  $(type) ('σ, 'ϕ$  *list*,  $'ϕ1$  *csp-vars*) *rsp*  
 $(type) ('σ, 'ϕ)$  *action* <=  $(type) ('σ, 'ϕ)$  *st-csp hrel*

**notation** *csp-vars-child-lens<sub>a</sub>* ( $\Sigma_c$ )  
**notation** *csp-vars-child-lens* ( $\Sigma_C$ )

## 2.2 Basic laws

**lemma** *R2c-tr-ext*:  $R2c (\$tr' =_u \$tr \hat{=} \langle [a]_{S<} \rangle) = (\$tr' =_u \$tr \hat{=} \langle [a]_{S<} \rangle)$

by (*rel-auto*)

**lemma** *circus-alpha-bij-lens*:

*bij-lens* ( $\{\$ok, \$ok', \$wait, \$wait', \$str, \$str', \$st, \$st', \$ref, \$ref'\}_\alpha :: - \implies ('s, 'e) \text{ st-csp} \times ('s, 'e) \text{ st-csp}$ )  
 by (*unfold-locales, lens-simp+*)

### 2.3 Unrestriction laws

**lemma** *pre-unrest-ref* [*unrest*]:  $\$ref \# P \implies \$ref \# pre_R(P)$   
 by (*simp add: pre\_R-def unrest*)

**lemma** *peri-unrest-ref* [*unrest*]:  $\$ref \# P \implies \$ref \# peri_R(P)$   
 by (*simp add: peri\_R-def unrest*)

**lemma** *post-unrest-ref* [*unrest*]:  $\$ref \# P \implies \$ref \# post_R(P)$   
 by (*simp add: post\_R-def unrest*)

**lemma** *cmt-unrest-ref* [*unrest*]:  $\$ref \# P \implies \$ref \# cmt_R(P)$   
 by (*simp add: cmt\_R-def unrest*)

**lemma** *st-lift-unrest-ref'* [*unrest*]:  $\$ref' \# \lceil b \rceil_{S<}$   
 by (*rel-auto*)

**lemma** *RHS-design-ref-unrest* [*unrest*]:

$\llbracket \$ref \# P; \$ref \# Q \rrbracket \implies \$ref \# (\mathbf{R}_s(P \vdash Q)) \llbracket false / \$wait \rrbracket$   
 by (*simp add: RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest*)

**lemma** *R1-ref-unrest* [*unrest*]:  $\$ref \# P \implies \$ref \# R1(P)$   
 by (*simp add: R1-def unrest*)

**lemma** *R2c-ref-unrest* [*unrest*]:  $\$ref \# P \implies \$ref \# R2c(P)$   
 by (*simp add: R2c-def unrest*)

**lemma** *R1-ref'-unrest* [*unrest*]:  $\$ref' \# P \implies \$ref' \# R1(P)$   
 by (*simp add: R1-def unrest*)

**lemma** *R2c-ref'-unrest* [*unrest*]:  $\$ref' \# P \implies \$ref' \# R2c(P)$   
 by (*simp add: R2c-def unrest*)

**lemma** *R2s-notin-ref'*:  $R2s(\lceil \ll x \gg \rceil_{S<} \notin_u \$ref') = (\lceil \ll x \gg \rceil_{S<} \notin_u \$ref')$   
 by (*pred-auto*)

**lemma** *unrest-circus-alpha*:

**fixes**  $P :: ('e, 't) \text{ action}$

**assumes**

$\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$str \# P$   
 $\$str' \# P \ \$st \# P \ \$st' \# P \ \$ref \# P \ \$ref' \# P$

**shows**  $\Sigma \# P$

by (*rule bij-lens-unrest-all[OF circus-alpha-bij-lens], simp add: unrest assms*)

**lemma** *unrest-all-circus-vars*:

**fixes**  $P :: ('s, 'e) \text{ action}$

**assumes**  $\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$ref \# P \ \Sigma \# r' \ \Sigma \# s \ \Sigma \# s' \ \Sigma \# t \ \Sigma \# t'$   
**shows**  $\Sigma \# \llbracket \$ref' \mapsto_s r', \$st \mapsto_s s, \$st' \mapsto_s s', \$str \mapsto_s t, \$str' \mapsto_s t' \rrbracket \dagger P$

**using** *assms*

by (*simp add: bij-lens-unrest-all-eq[OF circus-alpha-bij-lens] unrest-plus-split plus-vwb-lens*)

(simp add: unrest usubst closure)

**lemma** *unrest-all-circus-vars-st-st'*:

**fixes**  $P :: ('s, 'e)$  action

**assumes**  $\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$ref \# P \ \$ref' \# P \ \Sigma \# s \ \Sigma \# s' \ \Sigma \# t \ \Sigma \# t'$

**shows**  $\Sigma \# [\$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \dagger P$

**using** *assms*

**by** (simp add: bij-lens-unrest-all-eq[OF circus-alpha-bij-lens] unrest-plus-split plus-vwb-lens)

(simp add: unrest usubst closure)

**lemma** *unrest-all-circus-vars-st*:

**fixes**  $P :: ('s, 'e)$  action

**assumes**  $\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$ref \# P \ \$ref' \# P \ \$st' \# P \ \Sigma \# s \ \Sigma \# t \ \Sigma \# t'$

**shows**  $\Sigma \# [\$st \mapsto_s s, \$tr \mapsto_s t, \$tr' \mapsto_s t'] \dagger P$

**using** *assms*

**by** (simp add: bij-lens-unrest-all-eq[OF circus-alpha-bij-lens] unrest-plus-split plus-vwb-lens)

(simp add: unrest usubst closure)

**lemma** *unrest-any-circus-var*:

**fixes**  $P :: ('s, 'e)$  action

**assumes**  $\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$ref \# P \ \$ref' \# P \ \Sigma \# s \ \Sigma \# s' \ \Sigma \# t \ \Sigma \# t'$

**shows**  $x \# [\$st \mapsto_s s, \$st' \mapsto_s s', \$tr \mapsto_s t, \$tr' \mapsto_s t'] \dagger P$

**by** (simp add: unrest-all-var unrest-all-circus-vars-st-st' assms)

**lemma** *unrest-any-circus-var-st*:

**fixes**  $P :: ('s, 'e)$  action

**assumes**  $\$ok \# P \ \$ok' \# P \ \$wait \# P \ \$wait' \# P \ \$ref \# P \ \$ref' \# P \ \$st' \# P \ \Sigma \# s \ \Sigma \# t \ \Sigma \# t'$

**shows**  $x \# [\$st \mapsto_s s, \$tr \mapsto_s t, \$tr' \mapsto_s t'] \dagger P$

**by** (simp add: unrest-all-var unrest-all-circus-vars-st assms)

end

### 3 Stateful-Failure Reactive Relations

**theory** *utp-sfrd-rel*

**imports** *utp-sfrd-core*

**begin**

#### 3.1 Healthiness Conditions

CSP Reactive Relations

**definition** *CRR* ::  $('s, 'e)$  action  $\Rightarrow$   $('s, 'e)$  action **where**

[upred-defs]:  $CRR(P) = (\exists \ \$ref \cdot RR(P))$

**lemma** *CRR-idem*:  $CRR(CRR(P)) = CRR(P)$

**by** (*rel-auto*)

**lemma** *Idempotent-CRR* [closure]: *Idempotent CRR*

**by** (simp add: *CRR-idem Idempotent-def*)

**lemma** *CRR-intro*:

**assumes**  $\$ref \# P$   $P$  is *RR*

**shows**  $P$  is *CRR*

**by** (simp add: *CRR-def Healthy-def*, simp add: *Healthy-if assms ex-unrest*)

## CSP Reactive Conditions

**definition**  $CRC :: ('s, 'e) \text{ action} \Rightarrow ('s, 'e) \text{ action}$  **where**  
[upred-defs]:  $CRC(P) = (\exists \$ref \cdot RC(P))$

**lemma**  $CRC\text{-intro}$ :

**assumes**  $\$ref \# P$   $P$  is  $RC$

**shows**  $P$  is  $CRC$

**by** (*simp add: CRC-def Healthy-def, simp add: Healthy-if assms ex-unrest*)

**lemma**  $ref\text{-unrest-RR}$  [unrest]:  $\$ref \# P \Longrightarrow \$ref \# RR P$   
**by** (*rel-auto, blast+*)

**lemma**  $ref\text{-unrest-RC1}$  [unrest]:  $\$ref \# P \Longrightarrow \$ref \# RC1 P$   
**by** (*rel-auto, blast+*)

**lemma**  $ref\text{-unrest-RC}$  [unrest]:  $\$ref \# P \Longrightarrow \$ref \# RC P$   
**by** (*simp add: RC-R2-def ref-unrest-RC1 ref-unrest-RR*)

**lemma**  $RR\text{-ex-ref}$ :  $RR (\exists \$ref \cdot RR P) = (\exists \$ref \cdot RR P)$   
**by** (*rel-auto*)

**lemma**  $RC1\text{-ex-ref}$ :  $RC1 (\exists \$ref \cdot RC1 P) = (\exists \$ref \cdot RC1 P)$   
**by** (*rel-auto, meson dual-order.trans*)

**lemma**  $ex\text{-ref}'\text{-RR-closed}$  [closure]:

**assumes**  $P$  is  $RR$

**shows**  $(\exists \$ref' \cdot P)$  is  $RR$

**proof** –

**have**  $RR (\exists \$ref' \cdot RR(P)) = (\exists \$ref' \cdot RR(P))$

**by** (*rel-auto*)

**thus** *?thesis*

**by** (*metis Healthy-def assms*)

**qed**

**lemma**  $CRC\text{-idem}$ :  $CRC(CRC(P)) = CRC(P)$

**apply** (*simp add: CRC-def ex-unrest unrest*)

**apply** (*simp add: RC-def RR-ex-ref*)

**apply** (*metis (no-types, hide-lams) Healthy-def RC1-RR-closed RC1-ex-ref RR-ex-ref RR-idem*)

**done**

**lemma**  $Idempotent\text{-CRC}$  [closure]:  $Idempotent\ CRC$   
**by** (*simp add: CRC-idem Idempotent-def*)

## 3.2 Closure Properties

**lemma**  $CRR\text{-implies-RR}$  [closure]:

**assumes**  $P$  is  $CRR$

**shows**  $P$  is  $RR$

**proof** –

**have**  $RR(CRR(P)) = CRR(P)$

**by** (*rel-auto*)

**thus** *?thesis*

**by** (*metis Healthy-def' assms*)

**qed**

**lemma** *CRC-implies-RR* [*closure*]:  
**assumes** *P is CRC*  
**shows** *P is RR*  
**proof** –  
**have**  $RR(CRC(P)) = CRC(P)$   
**by** (*rel-auto*)  
(*metis (no-types, lifting) Prefix-Order.prefixE Prefix-Order.prefixI append.assoc append-minus*) +  
**thus** *?thesis*  
**by** (*metis Healthy-def assms*)  
**qed**

**lemma** *CRC-implies-RC* [*closure*]:  
**assumes** *P is CRC*  
**shows** *P is RC*  
**proof** –  
**have**  $RC1(CRC(P)) = CRC(P)$   
**by** (*rel-auto, meson dual-order.trans*)  
**thus** *?thesis*  
**by** (*simp add: CRC-implies-RR Healthy-if RC1-def RC-intro assms*)  
**qed**

**lemma** *CRR-unrest-ref* [*unrest*]:  $P \text{ is } CRR \implies \$ref \# P$   
**by** (*metis CRR-def CRR-implies-RR Healthy-def in-var-uvar ref-vwb-lens unrest-as-exists*)

**lemma** *CRC-implies-CRR* [*closure*]:  
**assumes** *P is CRC*  
**shows** *P is CRR*  
**apply** (*rule CRR-intro*)  
**apply** (*simp-all add: unrest assms closure*)  
**apply** (*metis CRC-def CRC-implies-RC Healthy-def assms in-var-uvar ref-vwb-lens unrest-as-exists*)  
**done**

**lemma** *unrest-ref'-neg-RC* [*unrest*]:  
**assumes** *P is RR P is RC*  
**shows**  $\$ref' \# P$   
**proof** –  
**have**  $P = (\neg_r \neg_r P)$   
**by** (*simp add: closure rpred assms*)  
**also have**  $\dots = (\neg_r (\neg_r P)) ;; true_r$   
**by** (*metis Healthy-if RC1-def RC-implies-RC1 assms(2) calculation*)  
**also have**  $\$ref' \# \dots$   
**by** (*rel-auto*)  
**finally show** *?thesis* .  
**qed**

**lemma** *rea-true-CRR* [*closure*]: *true<sub>r</sub> is CRR*  
**by** (*rel-auto*)

**lemma** *rea-true-CRC* [*closure*]: *true<sub>r</sub> is CRC*  
**by** (*rel-auto*)

**lemma** *false-CRR* [*closure*]: *false is CRR*  
**by** (*rel-auto*)

**lemma** *false-CRC* [*closure*]: *false is CRC*



by (*rel-auto*)

**lemma** *st-pred-CRR* [*closure*]:  $[P]_{S<} \text{ is CRR}$   
by (*rel-auto*)

**lemma** *st-cond-CRC* [*closure*]:  $[P]_{S<} \text{ is CRC}$   
by (*rel-auto*)

**lemma** *conj-CRC-closed* [*closure*]:  
 $\llbracket P \text{ is CRC}; Q \text{ is CRC} \rrbracket \implies (P \wedge Q) \text{ is CRC}$   
by (*rule CRC-intro, simp-all add: unrest closure*)

**lemma** *disj-CRC-closed* [*closure*]:  
 $\llbracket P \text{ is CRC}; Q \text{ is CRC} \rrbracket \implies (P \vee Q) \text{ is CRC}$   
by (*rule CRC-intro, simp-all add: unrest closure*)

**lemma** *shEx-CRR-closed* [*closure*]:

assumes  $\bigwedge x. P\ x \text{ is CRR}$   
shows  $(\exists x. P(x)) \text{ is CRR}$

**proof** –

have  $CRR(\exists x. CRR(P(x))) = (\exists x. CRR(P(x)))$   
by (*rel-auto*)

thus *?thesis*

by (*metis Healthy-def assms shEx-cong*)

**qed**

**lemma** *USUP-ind-CRR-closed* [*closure*]:

assumes  $\bigwedge i. P\ i \text{ is CRR}$

shows  $(\bigsqcup i. P(i)) \text{ is CRR}$

by (*rule CRR-intro, simp-all add: assms unrest closure*)

**lemma** *UINF-ind-CRR-closed* [*closure*]:

assumes  $\bigwedge i. P\ i \text{ is CRR}$

shows  $(\bigsqcap i. P(i)) \text{ is CRR}$

by (*rule CRR-intro, simp-all add: assms unrest closure*)

**lemma** *cond-tt-CRR-closed* [*closure*]:

assumes  $P \text{ is CRR } Q \text{ is CRR}$

shows  $P \triangleleft \$tr' =_u \$tr \triangleright Q \text{ is CRR}$

by (*rule CRR-intro, simp-all add: unrest assms closure*)

**lemma** *rea-implies-CRR-closed* [*closure*]:

$\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \Rightarrow_r Q) \text{ is CRR}$

by (*simp-all add: CRR-intro closure unrest*)

**lemma** *conj-CRR-closed* [*closure*]:

$\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \wedge Q) \text{ is CRR}$

by (*simp-all add: CRR-intro closure unrest*)

**lemma** *disj-CRR-closed* [*closure*]:

$\llbracket P \text{ is CRR}; Q \text{ is CRR} \rrbracket \implies (P \vee Q) \text{ is CRR}$

by (*rule CRR-intro, simp-all add: unrest closure*)

**lemma** *rea-not-CRR-closed* [*closure*]:

$P \text{ is CRR} \implies (\neg_r P) \text{ is CRR}$

using *false-CRR rea-implies-CRR-closed* by *fastforce*

**lemma** *disj-R1-closed* [closure]:  $\llbracket P \text{ is } R1; Q \text{ is } R1 \rrbracket \implies (P \vee Q) \text{ is } R1$   
by (*rel-blast*)

**lemma** *st-cond-R1-closed* [closure]:  $\llbracket P \text{ is } R1; Q \text{ is } R1 \rrbracket \implies (P \triangleleft b \triangleright_R Q) \text{ is } R1$   
by (*rel-blast*)

**lemma** *cond-st-RR-closed* [closure]:  
assumes  $P \text{ is } RR \ Q \text{ is } RR$   
shows  $(P \triangleleft b \triangleright_R Q) \text{ is } RR$   
apply (*rule RR-intro, simp-all add: unrest closure assms, simp add: Healthy-def R2c-condr*)  
apply (*simp add: Healthy-if assms RR-implies-R2c*)  
apply (*rel-auto*)  
done

**lemma** *cond-st-CRR-closed* [closure]:  
 $\llbracket P \text{ is } CRR; Q \text{ is } CRR \rrbracket \implies (P \triangleleft b \triangleright_R Q) \text{ is } CRR$   
by (*simp-all add: CRR-intro closure unrest*)

**lemma** *seq-CRR-closed* [closure]:  
assumes  $P \text{ is } CRR \ Q \text{ is } RR$   
shows  $(P ;; Q) \text{ is } CRR$   
by (*rule CRR-intro, simp-all add: unrest assms closure*)

**lemma** *tr-extend-seqr-lit* [rdes]:  
fixes  $P :: ('s, 'e) \text{ action}$   
assumes  $\$ok \# P \ \$wait \# P \ \$ref \# P$   
shows  $(\$tr' =_u \$tr \hat{\ }_u \langle \langle a \rangle \rangle \wedge \$st' =_u \$st) ;; P = P[\$tr \hat{\ }_u \langle \langle a \rangle \rangle / \$tr]$   
using *assms* by (*rel-auto, meson*)

**lemma** *tr-assign-comp* [rdes]:  
fixes  $P :: ('s, 'e) \text{ action}$   
assumes  $\$ok \# P \ \$wait \# P \ \$ref \# P$   
shows  $(\$tr' =_u \$tr \wedge [(\sigma)_a]_S) ;; P = [\sigma]_{S\sigma} \dagger P$   
using *assms* by (*rel-auto, meson*)

**lemma** *RR-msubst-tt*:  $RR((P \ t)[[t \rightarrow \&tt]]) = (RR \ (P \ t))[[t \rightarrow \&tt]]$   
by (*rel-auto*)

**lemma** *RR-msubst-ref'*:  $RR((P \ r)[[r \rightarrow \$ref']]) = (RR \ (P \ r))[[r \rightarrow \$ref']]$   
by (*rel-auto*)

**lemma** *msubst-tt-RR* [closure]:  $\llbracket \bigwedge t. P \ t \text{ is } RR \rrbracket \implies (P \ t)[[t \rightarrow \&tt]] \text{ is } RR$   
by (*simp add: Healthy-def RR-msubst-tt*)

**lemma** *msubst-ref'-RR* [closure]:  $\llbracket \bigwedge r. P \ r \text{ is } RR \rrbracket \implies (P \ r)[[r \rightarrow \$ref']] \text{ is } RR$   
by (*simp add: Healthy-def RR-msubst-ref'*)

**lemma** *conj-less-tr-RR-closed* [closure]:  
assumes  $P \text{ is } CRR$   
shows  $(P \wedge \$tr <_u \$tr') \text{ is } CRR$   
**proof** –  
have  $CRR(CRR(P) \wedge \$tr <_u \$tr') = (CRR(P) \wedge \$tr <_u \$tr')$   
apply (*rel-auto, blast+*)

**using** *less-le* **apply** *fastforce+*  
**done**  
**thus** *?thesis*  
**by** (*metis Healthy-def assms*)  
**qed**

**lemma** *conj-eq-tr-RR-closed* [*closure*]:

**assumes** *P is CRR*  
**shows** ( $P \wedge \$tr' =_u \$tr$ ) *is CRR*  
**proof** –  
**have**  $CRR(CRR(P) \wedge \$tr' =_u \$tr) = (CRR(P) \wedge \$tr' =_u \$tr)$   
**by** (*rel-auto, blast+*)  
**thus** *?thesis*  
**by** (*metis Healthy-def assms*)  
**qed**

### 3.3 Introduction laws

Extensionality principles for introducing refinement and equality of Circus reactive relations. It is necessary only to consider a subset of the variables that are present.

**lemma** *CRR-refine-ext*:

**assumes**  
*P is CRR Q is CRR*  
 $\bigwedge t s s' r'. P[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref'] \sqsubseteq Q[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
**shows**  $P \sqsubseteq Q$   
**proof** –  
**have**  $\bigwedge t s s' r'. (CRR P)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
 $\sqsubseteq (CRR Q)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
**by** (*simp add: assms Healthy-if*)  
**hence**  $CRR P \sqsubseteq CRR Q$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*metis Healthy-if assms(1) assms(2)*)  
**qed**

**lemma** *CRR-eq-ext*:

**assumes**  
*P is CRR Q is CRR*  
 $\bigwedge t s s' r'. P[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref'] = Q[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
**shows**  $P = Q$   
**proof** –  
**have**  $\bigwedge t s s' r'. (CRR P)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
 $= (CRR Q)[\langle \rangle, \langle t \rangle, \langle s \rangle, \langle s' \rangle, \langle r' \rangle / \$tr, \$tr', \$st, \$st', \$ref']$   
**by** (*simp add: assms Healthy-if*)  
**hence**  $CRR P = CRR Q$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*metis Healthy-if assms(1) assms(2)*)  
**qed**

**lemma** *CRR-refine-impl-prop*:

**assumes** *P is CRR Q is CRR*  
 $\bigwedge t s s' r'. 'Q[\langle r' \rangle, \langle s \rangle, \langle s' \rangle, \langle \rangle, \langle t \rangle / \$ref', \$st, \$st', \$tr, \$tr']' \implies 'P[\langle r' \rangle, \langle s \rangle, \langle s' \rangle, \langle \rangle, \langle t \rangle / \$ref', \$st, \$st', \$tr, \$tr']'$   
**shows**  $P \sqsubseteq Q$   
**by** (*rule CRR-refine-ext, simp-all add: assms closure unrest usubst*)

(rule refine-prop-intro, simp-all add: unrest unrest-all-circus-vars closure assms)

### 3.4 Weakest Precondition

**lemma** *nil-least* [simp]:

$\langle \rangle \leq_u x = \text{true}$  **by** *rel-auto*

**lemma** *minus-nil* [simp]:

$xs - \langle \rangle = xs$  **by** *rel-auto*

**lemma** *wp-rea-circus-lemma-1*:

**assumes**  $P$  is CRR  $\$ref' \# P$

**shows**  $out\alpha \# P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']$

**proof** –

**have**  $out\alpha \# (CRR (\exists \$ref' \cdot P))[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']$

**by** *(rel-auto)*

**thus** *?thesis*

**by** (*simp add: Healthy-if assms(1) assms(2) ex-unrest*)

**qed**

**lemma** *wp-rea-circus-lemma-2*:

**assumes**  $P$  is CRR

**shows**  $in\alpha \# P[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr]$

**proof** –

**have**  $in\alpha \# (CRR P)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr]$

**by** *(rel-auto)*

**thus** *?thesis*

**by** (*simp add: Healthy-if assms ex-unrest*)

**qed**

The meaning of reactive weakest precondition for Circus.  $P \text{ wp}_r Q$  means that, whenever  $P$  terminates in a state  $s_0$  having done the interaction trace  $t_0$ , which is a prefix of the overall trace, then  $Q$  must be satisfied. This in particular means that the remainder of the trace after  $t_0$  must not be a divergent behaviour of  $Q$ .

**lemma** *wp-rea-circus-form*:

**assumes**  $P$  is CRR  $\$ref' \# P$   $Q$  is CRC

**shows**  $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \ll t_0 \gg \leq_u \$tr' \wedge P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \Rightarrow_r Q[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr]$

**proof** –

**have**  $(P \text{ wp}_r Q) = (\neg_r (\exists t_0 \cdot P[\ll t_0 \gg / \$tr']) \;; (\neg_r Q)[\ll t_0 \gg / \$tr] \wedge \ll t_0 \gg \leq_u \$tr')$

**by** (*simp-all add: wp-rea-def R2-tr-middle closure assms*)

**also have**  $\dots = (\neg_r (\exists (s_0, t_0) \cdot P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \;; (\neg_r Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \wedge \ll t_0 \gg \leq_u \$tr')$

**by** *(rel-blast)*

**also have**  $\dots = (\neg_r (\exists (s_0, t_0) \cdot P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \wedge (\neg_r Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \wedge \ll t_0 \gg \leq_u \$tr')$

**by** (*simp add: seqr-to-conj add: wp-rea-circus-lemma-1 wp-rea-circus-lemma-2 assms closure conj-assoc*)

**also have**  $\dots = (\forall (s_0, t_0) \cdot \neg_r P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \vee \neg_r (\neg_r Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \vee \neg_r \ll t_0 \gg \leq_u \$tr')$

**by** *(rel-auto)*

**also have**  $\dots = (\forall (s_0, t_0) \cdot \neg_r P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \vee \neg_r (\neg_r RR Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \vee \neg_r \ll t_0 \gg \leq_u \$tr')$

**by** (*simp add: Healthy-if assms closure*)

**also have**  $\dots = (\forall (s_0, t_0) \cdot \neg_r P[\ll s_0 \gg, \ll t_0 \gg / \$st', \$tr']) \vee (RR Q)[\ll s_0 \gg, \ll t_0 \gg / \$st, \$tr] \vee \neg_r \ll t_0 \gg \leq_u \$tr')$

**by** *(rel-auto)*

also have ... =  $(\forall (s_0, t_0) \cdot \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle t_0 \rangle / \$st', \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle t_0 \rangle / \$st, \$tr]$   
 by (rel-auto)  
 also have ... =  $(\forall (s_0, t_0) \cdot \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle t_0 \rangle / \$st', \$tr']) \Rightarrow_r Q[\langle s_0 \rangle, \langle t_0 \rangle / \$st, \$tr]$   
 by (simp add: Healthy-if assms closure)  
 finally show ?thesis .  
 qed

lemma wp-rea-circus-form-alt:

assumes  $P$  is CRR  $\$ref' \# P$   $Q$  is CRC

shows  $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \$tr \hat{=} \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle \rangle, \langle t_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1(Q[\langle s_0 \rangle, \langle \rangle, \&tt - \langle t_0 \rangle / \$st, \$tr, \$tr'])$

proof –

have  $(P \text{ wp}_r Q) = R2(P \text{ wp}_r Q)$

by (simp add: CRC-implies-RR CRR-implies-RR Healthy-if RR-implies-R2 assms wp-rea-R2-closed)

also have ... =  $R2(\forall (s_0, tr_0) \cdot \langle tr_0 \rangle \leq_u \$tr' \wedge (RR P)[\langle s_0 \rangle, \langle tr_0 \rangle / \$st', \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle tr_0 \rangle / \$st, \$tr]$

by (simp add: wp-rea-circus-form assms closure Healthy-if)

also have ... =  $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \langle tr_0 \rangle \leq_u \langle tt_0 \rangle \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle tr_0 \rangle, \langle tt_0 \rangle / \$st, \$tr, \$tr']) \wedge \$tr' =_u \$tr \hat{=} \langle tt_0 \rangle$

by (simp add: R2-form, rel-auto)

also have ... =  $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \langle tr_0 \rangle \leq_u \langle tt_0 \rangle \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle \rangle, \langle tt_0 - tr_0 \rangle / \$st, \$tr, \$tr']) \wedge \$tr' =_u \$tr \hat{=} \langle tt_0 \rangle$

by (rel-auto)

also have ... =  $(\exists tt_0 \cdot (\forall (s_0, tr_0) \cdot \$tr \hat{=} \langle tr_0 \rangle \leq_u \$tr' \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r (RR Q)[\langle s_0 \rangle, \langle \rangle, \&tt - \langle tr_0 \rangle / \$st, \$tr, \$tr']) \wedge \$tr' =_u \$tr \hat{=} \langle tt_0 \rangle$

by (rel-auto, (metis list-concat-minus-list-concat)+)

also have ... =  $(\forall (s_0, tr_0) \cdot \$tr \hat{=} \langle tr_0 \rangle \leq_u \$tr' \wedge (RR P)[\langle s_0 \rangle, \langle \rangle, \langle tr_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1((RR Q)[\langle s_0 \rangle, \langle \rangle, \&tt - \langle tr_0 \rangle / \$st, \$tr, \$tr'])$

by (rel-auto, blast+)

also have ... =  $(\forall (s_0, t_0) \cdot \$tr \hat{=} \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle \rangle, \langle t_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1(Q[\langle s_0 \rangle, \langle \rangle, \&tt - \langle t_0 \rangle / \$st, \$tr, \$tr'])$

by (simp add: Healthy-if assms closure)

finally show ?thesis .

qed

lemma wp-rea-circus-form-alt:

assumes  $P$  is CRR  $\$ref' \# P$   $Q$  is CRC

shows  $(P \text{ wp}_r Q) = (\forall (s_0, t_0) \cdot \$tr \hat{=} \langle t_0 \rangle \leq_u \$tr' \wedge P[\langle s_0 \rangle, \langle \rangle, \langle t_0 \rangle / \$st', \$tr, \$tr']) \Rightarrow_r R1(Q[\langle s_0 \rangle, \langle \rangle, \&tt - \langle t_0 \rangle / \$st, \$tr, \$tr'])$

oops

### 3.5 Trace Substitution

definition trace-subst  $(-[\_])_t$  [999, 0] 999

where [upred-defs]:  $P[v]_t = (P[\&tt - [v]_{S<} / \&tt] \wedge \$tr + [v]_{S<} \leq_u \$tr')$

lemma unrest-trace-subst [unrest]:

$\llbracket \text{mwb-lens } x; x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \# P \rrbracket \implies x \# P[v]_t$

by (simp add: trace-subst-def lens-indep-sym unrest)

lemma trace-subst-RR-closed [closure]:

assumes  $P$  is RR

shows  $P[v]_t$  is RR

proof –

```

have (RR P)[v]t is RR
  apply (rel-auto)
  apply (metis diff-add-cancel-left' trace-class.add-left-mono)
  apply (metis le-add minus-cancel-le trace-class.add-diff-cancel-left)
  using le-add order-trans apply blast
done
thus ?thesis
  by (simp add: Healthy-if assms)
qed

```

```

lemma trace-subst-CRR-closed [closure]:
  assumes P is CRR
  shows P[v]t is CRR
  by (rule CRR-intro, simp-all add: closure assms unrest)

```

```

lemma tsubst-nil [usubst]:
  assumes P is CRR
  shows P[⟨⟩]t = P
proof –
  have (CRR P)[⟨⟩]t = CRR P
    by (rel-auto)
  thus ?thesis
    by (simp add: Healthy-if assms)
qed

```

```

lemma tsubst-false [usubst]: false[y]t = false
  by rel-auto

```

```

lemma cond-rea-tt-subst [usubst]:
  (P ◁ b ▷R Q)[v]t = (P[v]t ◁ b ▷R Q[v]t)
  by (rel-auto)

```

```

lemma tsubst-conj [usubst]: (P ∧ Q)[v]t = (P[v]t ∧ Q[v]t)
  by (rel-auto)

```

```

lemma tsubst-disj [usubst]: (P ∨ Q)[v]t = (P[v]t ∨ Q[v]t)
  by (rel-auto)

```

```

lemma rea-subst-R1-closed [closure]: P[v]t is R1
  apply (rel-auto) using le-add order.trans by blast

```

```

lemma tsubst-UINF-ind [usubst]: (∏ i · P(i))[v]t = (∏ i · (P(i))[v]t)
  by (rel-auto)

```

### 3.6 Initial Interaction

**definition** *rea-init* :: 's upred ⇒ ('t::trace, 's) uexpr ⇒ ('s, 't, 'α, 'β) rel-rsp (*I*'(-,-)) **where**  
[upred-defs]:  $\mathcal{I}(s,t) = ([s]_{S<} \wedge \$tr + [t]_{S<} \leq_u \$tr')$

$\mathcal{I}(s,t)$  is a predicate stating that, if the initial state satisfies state predicate  $s$ , then the trace  $t$  is an initial trace.

```

lemma unrest-rea-init [unrest]:
  [ x ⊗ ($tr)v; x ⊗ ($tr')v; x ⊗ ($st)v ] ⇒ x #  $\mathcal{I}(s,t)$ 
  by (simp add: rea-init-def unrest lens-indep-sym)

```

**lemma** *rea-init-R1* [closure]:  $\mathcal{I}(s,t)$  is *R1*  
**apply** (*rel-auto*) **using** *dual-order.trans le-add* **by** *blast*

**lemma** *rea-init-R2c* [closure]:  $\mathcal{I}(s,t)$  is *R2c*  
**apply** (*rel-auto*)  
**apply** (*metis diff-add-cancel-left' trace-class.add-left-mono*)  
**apply** (*metis le-add minus-cancel-le trace-class.add-diff-cancel-left*)  
**done**

**lemma** *rea-init-R2* [closure]:  $\mathcal{I}(s,t)$  is *R2*  
**by** (*metis Healthy-def R1-R2c-is-R2 rea-init-R1 rea-init-R2c*)

**lemma** *msp-init-RR* [closure]:  $\mathcal{I}(s,t)$  is *RR*  
**apply** (*rel-auto*)  
**apply** (*metis diff-add-cancel-left' trace-class.add-left-mono*)  
**apply** (*metis le-add minus-cancel-le trace-class.add-diff-cancel-left*)  
**apply** (*metis le-add less-le less-le-trans*)  
**done**

**lemma** *msp-init-CRR* [closure]:  $\mathcal{I}(s,t)$  is *CRR*  
**by** (*rule CRR-intro, simp-all add: unrest closure*)

**lemma** *rea-init-impl-st* [closure]:  $(\mathcal{I}(b,t) \Rightarrow_r [c]_{S<})$  is *RC*  
**apply** (*rule RC-intro*)  
**apply** (*simp add: closure*)  
**apply** (*rel-auto*)  
**using** *order-trans* **by** *auto*

**lemma** *rea-init-RC1*:  
 $\neg_r \mathcal{I}(P,t)$  is *RC1*  
**apply** (*rel-auto*) **using** *dual-order.trans* **by** *blast*

**lemma** *init-acts-empty* [*rpred*]:  $\mathcal{I}(\text{true}, \langle \rangle) = \text{true}_r$   
**by** (*rel-auto*)

**lemma** *rea-not-init* [*rpred*]:  
 $(\neg_r \mathcal{I}(P, \langle \rangle)) = \mathcal{I}(\neg P, \langle \rangle)$   
**by** (*rel-auto*)

**lemma** *rea-init-conj* [*rpred*]:  
 $(\mathcal{I}(P,t) \wedge \mathcal{I}(Q,t)) = \mathcal{I}(P \wedge Q, t)$   
**by** (*rel-auto*)

**lemma** *rea-init-empty-trace* [*rpred*]:  $\mathcal{I}(s, \langle \rangle) = [s]_{S<}$   
**by** (*rel-auto*)

**lemma** *rea-init-disj-same* [*rpred*]:  $(\mathcal{I}(s_1,t) \vee \mathcal{I}(s_2,t)) = \mathcal{I}(s_1 \vee s_2, t)$   
**by** (*rel-auto*)

**lemma** *rea-init-impl-same* [*rpred*]:  $(\mathcal{I}(s_1,t) \Rightarrow_r \mathcal{I}(s_2,t)) = (\mathcal{I}(s_1, t) \Rightarrow_r [s_2]_{S<})$   
**apply** (*rel-auto*) **using** *dual-order.trans le-add* **by** *blast+*

**lemma** *tsubst-st-cond* [*usubst*]:  $[P]_{S<}[\![t]\!]_t = \mathcal{I}(P,t)$   
**by** (*rel-auto*)

**lemma** *tsubst-rea-init* [*usubst*]:  $(\mathcal{I}(s,x))\llbracket y \rrbracket_t = \mathcal{I}(s,y+x)$   
**apply** (*rel-auto*)  
**apply** (*metis add.assoc diff-add-cancel-left' trace-class.add-le-imp-le-left trace-class.add-left-mono*)  
**apply** (*metis add.assoc diff-add-cancel-left' le-add trace-class.add-le-imp-le-left trace-class.add-left-mono*) +  
**done**

**lemma** *tsubst-rea-not* [*usubst*]:  $(\neg_r P)\llbracket v \rrbracket_t = ((\neg_r P)\llbracket v \rrbracket_t) \wedge \mathcal{I}(true,v)$   
**apply** (*rel-auto*)  
**using** *le-add order-trans* **by** *blast*

**lemma** *tsubst-true* [*usubst*]:  $true_r\llbracket v \rrbracket_t = \mathcal{I}(true,v)$   
**by** (*rel-auto*)

**lemma** *R4-csp-init* [*rpred*]:  $R4(\mathcal{I}(s,bop\ Cons\ x\ xs)) = \mathcal{I}(s,bop\ Cons\ x\ xs)$   
**using** *less-list-def* **by** (*rel-blast*)

**lemma** *R5-csp-init* [*rpred*]:  $R5(\mathcal{I}(s,bop\ Cons\ x\ xs)) = false$   
**by** (*rel-auto*)

**lemma** *R4-trace-subst* [*rpred*]:  
 $R4(P\llbracket bop\ Cons\ x\ xs \rrbracket_t) = P\llbracket bop\ Cons\ x\ xs \rrbracket_t$   
**using** *le-imp-less-or-eq* **by** (*rel-blast*)

**lemma** *R5-trace-subst* [*rpred*]:  
 $R5(P\llbracket bop\ Cons\ x\ xs \rrbracket_t) = false$   
**by** (*rel-auto*)

### 3.7 Enabled Events

**definition** *csp-enable* ::  $'s\ upred \Rightarrow ('e\ list, 's)\ uexpr \Rightarrow ('e\ set, 's)\ uexpr \Rightarrow ('s, 'e)\ action\ (\mathcal{E}'(-, -, -))$   
**where**  
[*upred-defs*]:  $\mathcal{E}(s,t,E) = (\llbracket s \rrbracket_{S<} \wedge \$tr' =_u \$tr \hat{=} [t]_{S<} \wedge (\forall e \in [E]_{S<} \cdot \ll e \gg \notin_u \$ref'))$

Predicate  $\mathcal{E}(s,t, E)$  states that, if the initial state satisfies predicate  $s$ , then  $t$  is a possible (failure) trace, such that the events in the set  $E$  are enabled after the given interaction.

**lemma** *csp-enable-R1-closed* [*closure*]:  $\mathcal{E}(s,t,E)$  is *R1*  
**by** (*rel-auto*)

**lemma** *csp-enable-R2-closed* [*closure*]:  $\mathcal{E}(s,t,E)$  is *R2c*  
**by** (*rel-auto*)

**lemma** *csp-enable-RR* [*closure*]:  $\mathcal{E}(s,t,E)$  is *CRR*  
**by** (*rel-auto*)

**lemma** *tsubst-csp-enable* [*usubst*]:  $\mathcal{E}(s,t_2,e)\llbracket t_1 \rrbracket_t = \mathcal{E}(s,t_1 \hat{=} t_2,e)$   
**apply** (*rel-auto*)  
**apply** (*metis append.assoc less-eq-list-def prefix-concat-minus*)  
**apply** (*simp add: list-concat-minus-list-concat*)  
**done**

**lemma** *csp-enable-unrests* [*unrest*]:  
 $\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \bowtie (\$ref')_v \rrbracket \Longrightarrow x \# \mathcal{E}(s,t,e)$   
**by** (*simp add: csp-enable-def R1-def lens-indep-sym unrest*)

**lemma** *csp-enable-tr'-eq-tr* [*rpred*]:



$\mathcal{E}(s, \langle \rangle, r) \triangleleft \$tr' =_u \$tr \triangleright false = \mathcal{E}(s, \langle \rangle, r)$   
**by** (*rel-auto*)

**lemma** *csp-enable-st-pred* [*rpred*]:  
 $([s_1]_{S<} \wedge \mathcal{E}(s_2, t, E)) = \mathcal{E}(s_1 \wedge s_2, t, E)$   
**by** (*rel-auto*)

**lemma** *csp-enable-conj* [*rpred*]:  
 $(\mathcal{E}(s, t, E_1) \wedge \mathcal{E}(s, t, E_2)) = \mathcal{E}(s, t, E_1 \cup_u E_2)$   
**by** (*rel-auto*)

**lemma** *csp-enable-cond* [*rpred*]:  
 $\mathcal{E}(s_1, t_1, E_1) \triangleleft b \triangleright_R \mathcal{E}(s_2, t_2, E_2) = \mathcal{E}(s_1 \triangleleft b \triangleright s_2, t_1 \triangleleft b \triangleright t_2, E_1 \triangleleft b \triangleright E_2)$   
**by** (*rel-auto*)

**lemma** *csp-enable-rea-assm* [*rpred*]:  
 $[b]^\top_r ;; \mathcal{E}(s, t, E) = \mathcal{E}(b \wedge s, t, E)$   
**by** (*rel-auto*)

**lemma** *csp-enable-tr-empty*:  $\mathcal{E}(true, \langle \rangle, \{v\}_u) = (\$tr' =_u \$tr \wedge [v]_{S<} \notin_u \$ref')$   
**by** (*rel-auto*)

**lemma** *csp-enable-nothing*:  $\mathcal{E}(true, \langle \rangle, \{\}_u) = (\$tr' =_u \$tr)$   
**by** (*rel-auto*)

**lemma** *msubst-nil-csp-enable* [*usubst*]:  
 $\mathcal{E}(s(x), t(x), E(x)) \llbracket x \rightarrow \langle \rangle \rrbracket = \mathcal{E}(s(x) \llbracket x \rightarrow \langle \rangle \rrbracket, t(x) \llbracket x \rightarrow \langle \rangle \rrbracket, E(x) \llbracket x \rightarrow \langle \rangle \rrbracket)$   
**by** (*pred-auto*)

**lemma** *msubst-csp-enable* [*usubst*]:  
 $\mathcal{E}(s(x), t(x), E(x)) \llbracket x \rightarrow [v]_{S\leftarrow} \rrbracket = \mathcal{E}(s(x) \llbracket x \rightarrow v \rrbracket, t(x) \llbracket x \rightarrow v \rrbracket, E(x) \llbracket x \rightarrow v \rrbracket)$   
**by** (*rel-auto*)

**lemma** *csp-enable-false* [*rpred*]:  $\mathcal{E}(false, t, E) = false$   
**by** (*rel-auto*)

**lemma** *conj-csp-enable* [*rpred*]:  $(\mathcal{E}(b_1, t, E_1) \wedge \mathcal{E}(b_2, t, E_2)) = \mathcal{E}(b_1 \wedge b_2, t, E_1 \cup_u E_2)$   
**by** (*rel-auto*)

**lemma** *USUP-csp-enable* [*rpred*]:  
 $(\bigsqcup x \cdot \mathcal{E}(s, t, A(x))) = \mathcal{E}(s, t, (\bigvee x \cdot A(x)))$   
**by** (*rel-auto*)

**lemma** *R4-csp-enable-nil* [*rpred*]:  
 $R4(\mathcal{E}(s, \langle \rangle, E)) = false$   
**by** (*rel-auto*)

**lemma** *R5-csp-enable-nil* [*rpred*]:  
 $R5(\mathcal{E}(s, \langle \rangle, E)) = \mathcal{E}(s, \langle \rangle, E)$   
**by** (*rel-auto*)

**lemma** *R4-csp-enable-Cons* [*rpred*]:  
 $R4(\mathcal{E}(s, \text{bop Cons } x \text{ } xs, E)) = \mathcal{E}(s, \text{bop Cons } x \text{ } xs, E)$   
**by** (*rel-auto*, *simp add: Prefix-Order.strict-prefixI'*)

**lemma** *R5-csp-enable-Cons* [*rpred*]:  
 $R5(\mathcal{E}(s, \text{bop } \text{Cons } x \text{ } xs, E)) = \text{false}$   
**by** (*rel-auto*)

### 3.8 Completed Trace Interaction

**definition** *csp-do* ::  $'s \text{ upred} \Rightarrow ('s \Rightarrow 's) \Rightarrow ('e \text{ list}, 's) \text{ uepr} \Rightarrow ('s, 'e) \text{ action } (\Phi'(-, -, -))$  **where**  
[*upred-defs*]:  $\Phi(s, \sigma, t) = ([s]_{S<} \wedge \$tr' =_u \$tr \hat{\ }_u [t]_{S<} \wedge [\langle \sigma \rangle_a]_s)$

Predicate  $\Phi(s, \sigma, t)$  states that if the initial state satisfies  $s$ , and the trace  $t$  is performed, then afterwards the state update  $\sigma$  is executed.

**lemma** *unrest-csp-do* [*unrest*]:  
 $\llbracket x \bowtie (\$tr)_v; x \bowtie (\$tr')_v; x \bowtie (\$st)_v; x \bowtie (\$st')_v \rrbracket \Longrightarrow x \# \Phi(s, \sigma, t)$   
**by** (*simp-all add: csp-do-def alpha-in-var alpha-out-var prod-as-plus unrest lens-indep-sym*)

**lemma** *csp-do-CRR* [*closure*]:  $\Phi(s, \sigma, t)$  *is CRR*  
**by** (*rel-auto*)

**lemma** *csp-do-R4-closed* [*closure*]:  
 $\Phi(b, \sigma, \text{bop } \text{Cons } x \text{ } xs)$  *is R4*  
**by** (*rel-auto, simp add: Prefix-Order.strict-prefixI'*)

**lemma** *st-pred-conj-csp-do* [*rpred*]:  
 $([b]_{S<} \wedge \Phi(s, \sigma, t)) = \Phi(b \wedge s, \sigma, t)$   
**by** (*rel-auto*)

**lemma** *trea-subst-csp-do* [*usubst*]:  
 $(\Phi(s, \sigma, t_2)) \llbracket t_1 \rrbracket_t = \Phi(s, \sigma, t_1 \hat{\ }_u t_2)$   
**apply** (*rel-auto*)  
**apply** (*metis append.assoc less-eq-list-def prefix-concat-minus*)  
**apply** (*simp add: list-concat-minus-list-concat*)

**done**

**lemma** *st-subst-csp-do* [*usubst*]:  
 $[\sigma]_{S\sigma} \dagger \Phi(s, \rho, t) = \Phi(\sigma \dagger s, \rho \circ \sigma, \sigma \dagger t)$   
**by** (*rel-auto*)

**lemma** *csp-init-do* [*rpred*]:  $(\mathcal{I}(s1, t) \wedge \Phi(s2, \sigma, t)) = \Phi(s1 \wedge s2, \sigma, t)$   
**by** (*rel-auto*)

**lemma** *csp-do-false* [*rpred*]:  $\Phi(\text{false}, s, t) = \text{false}$   
**by** (*rel-auto*)

**lemma** *csp-do-assign* [*rpred*]:  
**assumes**  $P$  *is CRR*  
**shows**  $\Phi(s, \sigma, t) ;; P = ([s]_{S<} \wedge ([\sigma]_{S\sigma} \dagger P)) \llbracket t \rrbracket_t$

**proof** –

**have**  $\Phi(s, \sigma, t) ;; CRR(P) = ([s]_{S<} \wedge ([\sigma]_{S\sigma} \dagger CRR(P))) \llbracket t \rrbracket_t$   
**by** (*rel-blast*)

**thus** *?thesis*

**by** (*simp add: Healthy-if assms*)

**qed**

**lemma** *subst-state-csp-enable* [*usubst*]:  
 $[\sigma]_{S\sigma} \dagger \mathcal{E}(s, t_2, e) = \mathcal{E}(\sigma \dagger s, \sigma \dagger t_2, \sigma \dagger e)$

by (*rel-auto*)

**lemma** *csp-do-assign-enable* [*rpred*]:

$\Phi(s_1, \sigma, t_1) ;; \mathcal{E}(s_2, t_2, e) = \mathcal{E}(s_1 \wedge \sigma \dagger s_2, t_1 \hat{^u} (\sigma \dagger t_2), (\sigma \dagger e))$   
by (*simp add: rpred closure usubst*)

**lemma** *csp-do-assign-do* [*rpred*]:

$\Phi(s_1, \sigma, t_1) ;; \Phi(s_2, \varrho, t_2) = \Phi(s_1 \wedge (\sigma \dagger s_2), \varrho \circ \sigma, t_1 \hat{^u} (\sigma \dagger t_2))$   
by (*rel-auto*)

**lemma** *csp-do-cond* [*rpred*]:

$\Phi(s_1, \sigma, t_1) \triangleleft b \triangleright_R \Phi(s_2, \varrho, t_2) = \Phi(s_1 \triangleleft b \triangleright s_2, \sigma \triangleleft b \triangleright_s \varrho, t_1 \triangleleft b \triangleright t_2)$   
by (*rel-auto*)

**lemma** *rea-assm-csp-do* [*rpred*]:

$[b]^\top_r ;; \Phi(s, \sigma, t) = \Phi(b \wedge s, \sigma, t)$   
by (*rel-auto*)

**lemma** *csp-do-skip* [*rpred*]:

**assumes** *P is CRR*  
**shows**  $\Phi(\text{true}, \text{id}, t) ;; P = P[[t]]_t$

**proof** –

**have**  $\Phi(\text{true}, \text{id}, t) ;; \text{CRR}(P) = (\text{CRR } P)[[t]]_t$   
**by** (*rel-auto*)

**thus** *?thesis*

**by** (*simp add: Healthy-if assms*)

**qed**

**lemma** *wp-rea-csp-do-lemma*:

**fixes** *P* :: (*'σ, 'φ*) *action*  
**assumes**  $\$ok \# P \$wait \# P \$ref \# P$   
**shows**  $([\langle \sigma \rangle_a]_S \wedge \$tr' =_u \$tr \hat{^u} [t]_{S<}) ;; P = ([\sigma]_{S\sigma} \dagger P)[[tr \hat{^u} [t]_{S<}/\$tr]]$   
**using** *assms* **by** (*rel-auto, meson*)

**lemma** *wp-rea-csp-do* [*wp*]:

**fixes** *P* :: (*'σ, 'φ*) *action*  
**assumes** *P is CRR*  
**shows**  $\Phi(s, \sigma, t) \text{wp}_r P = (\mathcal{I}(s, t) \Rightarrow_r ([\sigma]_{S\sigma} \dagger P)[[t]]_t)$

**proof** –

**have**  $\Phi(s, \sigma, t) \text{wp}_r \text{CRR}(P) = (\mathcal{I}(s, t) \Rightarrow_r ([\sigma]_{S\sigma} \dagger \text{CRR}(P))[[t]]_t)$   
**by** (*rel-blast*)

**thus** *?thesis*

**by** (*simp add: assms Healthy-if*)

**qed**

**lemma** *csp-do-power-Suc* [*rpred*]:

$\Phi(\text{true}, \text{id}, t) \hat{^} (\text{Suc } i) = \Phi(\text{true}, \text{id}, \text{iter}[\text{Suc } i](t))$   
by (*induct i, (rel-auto)+*)

**lemma** *csp-power-do-comp* [*rpred*]:

**assumes** *P is CRR*  
**shows**  $\Phi(\text{true}, \text{id}, t) \hat{^} i ;; P = \Phi(\text{true}, \text{id}, \text{iter}[i](t)) ;; P$   
**apply** (*cases i*)  
**apply** (*simp-all add: rpred usubst assms closure*)  
**done**

**lemma** *wp-rea-csp-do-skip* [*wp*]:  
**fixes**  $Q :: ('\sigma, '\varphi)$  *action*  
**assumes**  $P$  *is CRR*  
**shows**  $\Phi(s, id, t) \text{ wp}_r P = (\mathcal{I}(s, t) \Rightarrow_r P \llbracket t \rrbracket_t)$   
**proof** –  
**have**  $\Phi(s, id, t) \text{ wp}_r P = \Phi(s, id, t) \text{ wp}_r P$   
**by** (*simp add: skip-r-def*)  
**thus** ?thesis **by** (*simp add: wp assms usubst alpha*)  
**qed**

**lemma** *msubst-csp-do* [*usubst*]:  
 $\Phi(s(x), \sigma, t(x)) \llbracket x \rightarrow [v]_{S \leftarrow} \rrbracket = \Phi(s(x) \llbracket x \rightarrow v \rrbracket, \sigma, t(x) \llbracket x \rightarrow v \rrbracket)$   
**by** (*rel-auto*)

**end**

## 4 Stateful-Failure Healthiness Conditions

**theory** *utp-sfrd-healths*  
**imports** *utp-sfrd-rel*  
**begin**

## 5 Definitions

We here define extra healthiness conditions for stateful-failure reactive designs.

**abbreviation**  $CSP1 :: ((' \sigma, '\varphi) \text{ st-csp} \times (' \sigma, '\varphi) \text{ st-csp}) \text{ health}$   
**where**  $CSP1(P) \equiv RD1(P)$

**abbreviation**  $CSP2 :: ((' \sigma, '\varphi) \text{ st-csp} \times (' \sigma, '\varphi) \text{ st-csp}) \text{ health}$   
**where**  $CSP2(P) \equiv RD2(P)$

**abbreviation**  $CSP :: ((' \sigma, '\varphi) \text{ st-csp} \times (' \sigma, '\varphi) \text{ st-csp}) \text{ health}$   
**where**  $CSP(P) \equiv SRD(P)$

**definition**  $STOP :: '\varphi \text{ rel-csp}$  **where**  
[*upred-defs*]:  $STOP = CSP1(\$ok' \wedge R3c(\$tr' =_u \$tr \wedge \$wait'))$

**definition**  $SKIP :: '\varphi \text{ rel-csp}$  **where**  
[*upred-defs*]:  $SKIP = \mathbf{R}_s(\exists \$ref \cdot CSP1(\$ref \cdot II))$

**definition**  $Stop :: (' \sigma, '\varphi) \text{ action}$  **where**  
[*upred-defs*]:  $Stop = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \$wait'))$

**definition**  $Skip :: (' \sigma, '\varphi) \text{ action}$  **where**  
[*upred-defs*]:  $Skip = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr \wedge \neg \$wait' \wedge \$st' =_u \$st))$

**definition**  $CSP3 :: ((' \sigma, '\varphi) \text{ st-csp} \times (' \sigma, '\varphi) \text{ st-csp}) \text{ health}$  **where**  
[*upred-defs*]:  $CSP3(P) = (Skip ;; P)$

**definition**  $CSP4 :: ((' \sigma, '\varphi) \text{ st-csp} \times (' \sigma, '\varphi) \text{ st-csp}) \text{ health}$  **where**  
[*upred-defs*]:  $CSP4(P) = (P ;; Skip)$

**definition**  $NCSP :: ((' \sigma, '\varphi) \text{ st-csp} \times (' \sigma, '\varphi) \text{ st-csp}) \text{ health}$  **where**

[upred-defs]:  $NCSP = CSP3 \circ CSP4 \circ CSP$

Productive and normal processes

**abbreviation**  $PCSP \equiv Productive \circ NCSP$

Instantaneous and normal processes

**abbreviation**  $ICSP \equiv ISRD1 \circ NCSP$

## 5.1 Healthiness condition properties

$SKIP$  is the same as  $Skip$ , and  $STOP$  is the same as  $Stop$ , when we consider stateless CSP processes. This is because any reference to the  $st$  variable degenerates when the alphabet type coerces its type to be empty. We therefore need not consider  $SKIP$  and  $STOP$  actions.

**theorem** *SKIP-is-Skip*:  $SKIP = Skip$   
by (rel-auto)

**theorem** *STOP-is-Stop*:  $STOP = Stop$   
by (rel-auto)

**theorem** *Skip-UTP-form*:  $Skip = \mathbf{R}_s(\exists \$ref \cdot CSP1(II))$   
by (rel-auto)

**lemma** *Skip-is-CSP* [closure]:  
*Skip is CSP*  
by (simp add: Skip-def RHS-design-is-SRD unrest)

**lemma** *Skip-RHS-tri-design*:  
 $Skip = \mathbf{R}_s(true \vdash (false \diamond (\$tr' =_u \$tr \wedge \$st' =_u \$st)))$   
by (rel-auto)

**lemma** *Skip-RHS-tri-design'* [rdes-def]:  
 $Skip = \mathbf{R}_s(true_r \vdash (false \diamond \Phi(true, id, \langle \rangle)))$   
by (rel-auto)

**lemma** *Stop-is-CSP* [closure]:  
*Stop is CSP*  
by (simp add: Stop-def RHS-design-is-SRD unrest)

**lemma** *Stop-RHS-tri-design*:  $Stop = \mathbf{R}_s(true \vdash (\$tr' =_u \$tr) \diamond false)$   
by (rel-auto)

**lemma** *Stop-RHS-rdes-def* [rdes-def]:  $Stop = \mathbf{R}_s(true_r \vdash \mathcal{E}(true, \langle \rangle, \{\}_u) \diamond false)$   
by (rel-auto)

**lemma** *preR-Skip* [rdes]:  $pre_R(Skip) = true_r$   
by (rel-auto)

**lemma** *periR-Skip* [rdes]:  $peri_R(Skip) = false$   
by (rel-auto)

**lemma** *postR-Skip* [rdes]:  $post_R(Skip) = \Phi(true, id, \langle \rangle)$   
by (rel-auto)

**lemma** *Productive-Stop* [closure]:

*Stop is Productive*

by (*simp add: Stop-RHS-tri-design Healthy-def Productive-RHS-design-form unrest*)

**lemma** *Skip-left-lemma:*

**assumes** *P is CSP*

**shows**  $Skip \;; P = \mathbf{R}_s ((\forall \$ref \cdot pre_R P) \vdash (\exists \$ref \cdot cmt_R P))$

**proof** –

**have**  $Skip \;; P =$

$\mathbf{R}_s ((\$tr' =_u \$tr \wedge \$st' =_u \$st) wp_r pre_R P \vdash$   
 $(\$tr' =_u \$tr \wedge \$st' =_u \$st) \;; peri_R P \diamond$   
 $(\$tr' =_u \$tr \wedge \$st' =_u \$st) \;; post_R P)$

by (*simp add: SRD-composition-wp alpha rdes closure wp assms rpred C1, rel-auto*)

**also have**  $\dots = \mathbf{R}_s ((\forall \$ref \cdot pre_R P) \vdash$

$(\$tr' =_u \$tr \wedge \neg \$wait' \wedge \$st' =_u \$st) \;; ((\exists \$st \cdot [II]_D \triangleleft \$wait \triangleright cmt_R P))$

by (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)

**also have**  $\dots = \mathbf{R}_s ((\forall \$ref \cdot pre_R P) \vdash (\exists \$ref \cdot cmt_R P))$

by (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)

**finally show** *?thesis* .

**qed**

**lemma** *Skip-left-unit-ref-unrest:*

**assumes** *P is CSP*  $\$ref \# P \llbracket false/\$wait \rrbracket$

**shows**  $Skip \;; P = P$

**using** *assms*

by (*simp add: Skip-left-lemma*)

(*metis SRD-reactive-design-alt all-unrest cmt-unrest-ref cmt-wait-false ex-unrest pre-unrest-ref pre-wait-false*)

**lemma** *CSP3-intro:*

$\llbracket P \text{ is CSP}; \$ref \# P \llbracket false/\$wait \rrbracket \rrbracket \implies P \text{ is CSP3}$

by (*simp add: CSP3-def Healthy-def' Skip-left-unit-ref-unrest*)

**lemma** *ref-unrest-RHS-design:*

**assumes**  $\$ref \# P \ \$ref \# Q_1 \ \$ref \# Q_2$

**shows**  $\$ref \# (\mathbf{R}_s(P \vdash Q_1 \diamond Q_2)) \ f$

by (*simp add: RHS-def R1-def R2c-def R2s-def R3h-def design-def unrest usubst assms*)

**lemma** *CSP3-SRD-intro:*

**assumes** *P is CSP*  $\$ref \# pre_R(P) \ \$ref \# peri_R(P) \ \$ref \# post_R(P)$

**shows** *P is CSP3*

**proof** –

**have**  $P: \mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P)) = P$

by (*simp add: SRD-reactive-design-alt assms(1) wait'-cond-peri-post-cmt[THEN sym]*)

**have**  $\mathbf{R}_s(pre_R(P) \vdash peri_R(P) \diamond post_R(P))$  *is CSP3*

by (*rule CSP3-intro, simp add: assms P, simp add: ref-unrest-RHS-design assms*)

**thus** *?thesis*

by (*simp add: P*)

**qed**

**lemma** *Skip-unrest-ref [unrest]:*  $\$ref \# Skip \llbracket false/\$wait \rrbracket$

by (*simp add: Skip-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest*)

**lemma** *Skip-unrest-ref' [unrest]:*  $\$ref' \# Skip \llbracket false/\$wait \rrbracket$

by (*simp add: Skip-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest*)

**lemma** *CSP3-iff:*

**assumes**  $P$  is CSP  
**shows**  $P$  is CSP3  $\longleftrightarrow$  ( $\$ref \# P[[false/\$wait]]$ )

**proof**

**assume** 1:  $P$  is CSP3  
**have**  $\$ref \# (Skip ;; P)[[false/\$wait]]$   
**by** (*simp add: usubst unrest*)  
**with** 1 **show**  $\$ref \# P[[false/\$wait]]$   
**by** (*metis CSP3-def Healthy-def*)

**next**

**assume** 1:  $\$ref \# P[[false/\$wait]]$   
**show**  $P$  is CSP3  
**by** (*simp add: 1 CSP3-intro assms*)

**qed**

**lemma** *CSP3-unrest-ref* [*unrest*]:

**assumes**  $P$  is CSP  $P$  is CSP3  
**shows**  $\$ref \# pre_R(P) \ \$ref \# peri_R(P) \ \$ref \# post_R(P)$

**proof** –

**have**  $a: (\$ref \# P[[false/\$wait]])$   
**using** *CSP3-iff assms* **by** *blast*  
**from**  $a$  **show**  $\$ref \# pre_R(P)$   
**by** (*rel-blast*)  
**from**  $a$  **show**  $\$ref \# peri_R(P)$   
**by** (*rel-blast*)  
**from**  $a$  **show**  $\$ref \# post_R(P)$   
**by** (*rel-blast*)

**qed**

**lemma** *CSP3-rdes*:

**assumes**  $P$  is RR  $Q$  is RR  $R$  is RR  
**shows**  $CSP3(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\forall \$ref \cdot P) \vdash (\exists \$ref \cdot Q) \diamond (\exists \$ref \cdot R))$   
**by** (*simp add: CSP3-def Skip-left-lemma closure assms rdes, rel-auto*)

**lemma** *CSP3-form*:

**assumes**  $P$  is CSP  
**shows**  $CSP3(P) = \mathbf{R}_s((\forall \$ref \cdot pre_R(P)) \vdash (\exists \$ref \cdot peri_R(P)) \diamond (\exists \$ref \cdot post_R(P)))$   
**by** (*simp add: CSP3-def Skip-left-lemma assms, rel-auto*)

**lemma** *CSP3-Skip* [*closure*]:

*Skip* is CSP3  
**by** (*rule CSP3-intro, simp add: Skip-is-CSP, simp add: Skip-def unrest*)

**lemma** *CSP3-Stop* [*closure*]:

*Stop* is CSP3  
**by** (*rule CSP3-intro, simp add: Stop-is-CSP, simp add: Stop-def unrest*)

**lemma** *CSP3-Idempotent* [*closure*]: *Idempotent CSP3*

**by** (*metis (no-types, lifting) CSP3-Skip CSP3-def Healthy-if Idempotent-def seqr-assoc*)

**lemma** *CSP3-Continuous*: *Continuous CSP3*

**by** (*simp add: Continuous-def CSP3-def seq-Sup-distl*)

**lemma** *Skip-right-lemma*:

**assumes**  $P$  is CSP  
**shows**  $P ;; Skip = \mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot cmt_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot cmt_R P)))$

**proof** –

**have**  $P ;; \text{Skip} = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash (\exists \$st' \cdot \text{peri}_R P) \diamond \text{post}_R P ;; (\$tr' =_u \$tr \wedge \$st' =_u \$st))$

**by** (*simp add: SRD-composition-wp closure assms wp rdes rpred, rel-auto*)

**also have**  $\dots = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\text{cmt}_R P ;; (\exists \$st \cdot [II]_D)) \triangleleft \$wait' \triangleright (\text{cmt}_R P ;; (\$tr' =_u \$tr \wedge \neg \$wait \wedge \$st' =_u \$st))))$

**by** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)

**also have**  $\dots = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\text{cmt}_R P ;; (\$tr' =_u \$tr \wedge \neg \$wait \wedge \$st' =_u \$st))))$

**by** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)

**also have**  $\dots = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P)))$

**by** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)

**finally show** *?thesis* .

**qed**

**lemma** *Skip-right-tri-lemma:*

**assumes**  $P \text{ is CSP}$

**shows**  $P ;; \text{Skip} = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{peri}_R P) \diamond (\exists \$ref' \cdot \text{post}_R P)))$

**proof** –

**have**  $((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P)) = ((\exists \$st' \cdot \text{peri}_R P) \diamond (\exists \$ref' \cdot \text{post}_R P))$

**by** (*rel-auto*)

**thus** *?thesis* **by** (*simp add: Skip-right-lemma[OF assms]*)

**qed**

**lemma** *CSP<sub>4</sub>-intro:*

**assumes**  $P \text{ is CSP } (\neg_r \text{pre}_R(P)) ;; R1(\text{true}) = (\neg_r \text{pre}_R(P))$

$\$st' \# (\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \$ref' \# (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket$

**shows**  $P \text{ is CSP}_4$

**proof** –

**have**  $\text{CSP}_4(P) = \mathbf{R}_s ((\neg_r \text{pre}_R P) \text{wp}_r \text{false} \vdash ((\exists \$st' \cdot \text{cmt}_R P) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P)))$

**by** (*simp add: CSP<sub>4</sub>-def Skip-right-lemma assms(1)*)

**also have**  $\dots = \mathbf{R}_s (\text{pre}_R(P) \vdash ((\exists \$st' \cdot \text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\exists \$ref' \cdot \text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket))$

**by** (*simp add: wp-rea-def assms(2) rpred closure cond-var-subst-left cond-var-subst-right*)

**also have**  $\dots = \mathbf{R}_s (\text{pre}_R(P) \vdash ((\exists \$st' \cdot (\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket) \triangleleft \$wait' \triangleright (\exists \$ref' \cdot (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket)))$

**by** (*simp add: usubst unrest*)

**also have**  $\dots = \mathbf{R}_s (\text{pre}_R P \vdash ((\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket))$

**by** (*simp add: ex-unrest assms*)

**also have**  $\dots = \mathbf{R}_s (\text{pre}_R P \vdash \text{cmt}_R P)$

**by** (*simp add: cond-var-split*)

**also have**  $\dots = P$

**by** (*simp add: SRD-reactive-design-alt assms(1)*)

**finally show** *?thesis*

**by** (*simp add: Healthy-def'*)

**qed**

**lemma** *CSP<sub>4</sub>-RC-intro:*

**assumes**  $P \text{ is CSP } \text{pre}_R(P) \text{ is RC}$

$\$st' \# (\text{cmt}_R P) \llbracket \text{true}/\$wait' \rrbracket \$ref' \# (\text{cmt}_R P) \llbracket \text{false}/\$wait' \rrbracket$

**shows**  $P \text{ is CSP}_4$

**proof** –

**have**  $(\neg_r \text{pre}_R(P)) ;; R1(\text{true}) = (\neg_r \text{pre}_R(P))$



**by** (*metis* (*no-types*, *lifting*) *R1-seqr-closure* *assms*(2) *rea-not-R1* *rea-not-false* *rea-not-not* *wp-rea-RC-false* *wp-rea-def*)  
**thus** *?thesis*  
**by** (*simp* *add*: *CSP4-intro* *assms*)  
**qed**

**lemma** *CSP4-rdes*:

**assumes** *P* is *RR* *Q* is *RR* *R* is *RR*  
**shows**  $CSP_4(\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\neg_r P) \text{ wp}_r \text{ false} \vdash ((\exists \$st' \cdot Q) \diamond (\exists \$ref' \cdot R)))$   
**by** (*simp* *add*: *CSP4-def* *Skip-right-lemma* *closure* *assms* *rdes*, *rel-auto*, *blast+*)

**lemma** *CSP4-form*:

**assumes** *P* is *CSP*  
**shows**  $CSP_4(P) = \mathbf{R}_s((\neg_r \text{pre}_R P) \text{ wp}_r \text{ false} \vdash ((\exists \$st' \cdot \text{peri}_R P) \diamond (\exists \$ref' \cdot \text{post}_R P)))$   
**by** (*simp* *add*: *CSP4-def* *Skip-right-tri-lemma* *assms*)

**lemma** *Skip-srdes-right-unit*:

*(Skip* :: (*'σ*, *'φ*) *action*) ;;  $II_R = \text{Skip}$   
**by** (*rdes-simp*)

**lemma** *Skip-srdes-left-unit*:

$II_R$  ;; (*Skip* :: (*'σ*, *'φ*) *action*) = *Skip*  
**by** (*rdes-eq*)

**lemma** *CSP4-right-subsumes-RD3*:  $RD3(CSP_4(P)) = CSP_4(P)$

**by** (*metis* (*no-types*, *hide-lams*) *CSP4-def* *RD3-def* *Skip-srdes-right-unit* *seqr-assoc*)

**lemma** *CSP4-implies-RD3*: *P* is *CSP4*  $\implies$  *P* is *RD3*

**by** (*metis* *CSP4-right-subsumes-RD3* *Healthy-def*)

**lemma** *CSP4-tri-intro*:

**assumes** *P* is *CSP*  $(\neg_r \text{pre}_R(P))$  ;;  $R1(\text{true}) = (\neg_r \text{pre}_R(P)) \$st' \# \text{peri}_R(P) \$ref' \# \text{post}_R(P)$   
**shows** *P* is *CSP4*  
**using** *assms*  
**by** (*rule-tac* *CSP4-intro*, *simp-all* *add*: *pre\_R-def* *peri\_R-def* *post\_R-def* *usubst* *cmt\_R-def*)

**lemma** *CSP4-NSRD-intro*:

**assumes** *P* is *NSRD*  $\$ref' \# \text{post}_R(P)$   
**shows** *P* is *CSP4*  
**by** (*simp* *add*: *CSP4-tri-intro* *NSRD-is-SRD* *NSRD-neg-pre-unit* *NSRD-st'-unrest-peri* *assms*)

**lemma** *CSP3-commutes-CSP4*:  $CSP3(CSP_4(P)) = CSP_4(CSP3(P))$

**by** (*simp* *add*: *CSP3-def* *CSP4-def* *seqr-assoc*)

**lemma** *NCSP-implies-CSP* [*closure*]: *P* is *NCSP*  $\implies$  *P* is *CSP*

**by** (*metis* (*no-types*, *hide-lams*) *CSP3-def* *CSP4-def* *Healthy-def* *NCSP-def* *SRD-idem* *SRD-seqr-closure* *Skip-is-CSP* *comp-apply*)

**lemma** *NCSP-elim* [*RD-elim*]:

$\llbracket X \text{ is } NCSP; P(\mathbf{R}_s(\text{pre}_R(X) \vdash \text{peri}_R(X) \diamond \text{post}_R(X))) \rrbracket \implies P(X)$   
**by** (*simp* *add*: *SRD-reactive-tri-design* *closure*)

**lemma** *NCSP-implies-CSP3* [*closure*]:

*P* is *NCSP*  $\implies$  *P* is *CSP3*

**by** (*metis* (*no-types*, *lifting*) *CSP3-def* *Healthy-def'* *NCSP-def* *Skip-is-CSP* *Skip-left-unit-ref-unrest*)

*Skip-unrest-ref comp-apply seqr-assoc*)

**lemma** *NCSP-implies-CSP4* [closure]:

*P is NCSP*  $\implies$  *P is CSP4*

**by** (*metis* (*no-types*, *hide-lams*) *CSP3-commutes-CSP4* *Healthy-def* *NCSP-def* *NCSP-implies-CSP* *NCSP-implies-CSP3* *comp-apply*)

**lemma** *NCSP-implies-RD3* [closure]: *P is NCSP*  $\implies$  *P is RD3*

**by** (*metis* *CSP3-commutes-CSP4* *CSP4-right-subsumes-RD3* *Healthy-def* *NCSP-def* *comp-apply*)

**lemma** *NCSP-implies-NSRD* [closure]: *P is NCSP*  $\implies$  *P is NSRD*

**by** (*simp* *add*: *NCSP-implies-CSP* *NCSP-implies-RD3* *SRD-RD3-implies-NSRD*)

**lemma** *NCSP-subset-implies-CSP* [closure]:

$A \subseteq \llbracket \text{NCSP} \rrbracket_H \implies A \subseteq \llbracket \text{CSP} \rrbracket_H$

**using** *NCSP-implies-CSP* **by** *blast*

**lemma** *NCSP-subset-implies-NSRD* [closure]:

$A \subseteq \llbracket \text{NCSP} \rrbracket_H \implies A \subseteq \llbracket \text{NSRD} \rrbracket_H$

**using** *NCSP-implies-NSRD* **by** *blast*

**lemma** *CSP-Healthy-subset-member*:  $\llbracket P \in A; A \subseteq \llbracket \text{CSP} \rrbracket_H \rrbracket \implies P \text{ is } \text{CSP}$

**by** (*simp* *add*: *is-Healthy-subset-member*)

**lemma** *CSP3-Healthy-subset-member*:  $\llbracket P \in A; A \subseteq \llbracket \text{CSP3} \rrbracket_H \rrbracket \implies P \text{ is } \text{CSP3}$

**by** (*simp* *add*: *is-Healthy-subset-member*)

**lemma** *CSP4-Healthy-subset-member*:  $\llbracket P \in A; A \subseteq \llbracket \text{CSP4} \rrbracket_H \rrbracket \implies P \text{ is } \text{CSP4}$

**by** (*simp* *add*: *is-Healthy-subset-member*)

**lemma** *NCSP-Healthy-subset-member*:  $\llbracket P \in A; A \subseteq \llbracket \text{NCSP} \rrbracket_H \rrbracket \implies P \text{ is } \text{NCSP}$

**by** (*simp* *add*: *is-Healthy-subset-member*)

**lemma** *NCSP-intro*:

**assumes** *P is CSP* *P is CSP3* *P is CSP4*

**shows** *P is NCSP*

**by** (*metis* *Healthy-def* *NCSP-def* *assms* *comp-eq-dest-lhs*)

**lemma** *Skip-left-unit*: *P is NCSP*  $\implies$  *Skip* ;; *P* = *P*

**by** (*metis* (*full-types*) *CSP3-def* *Healthy-if* *NCSP-implies-CSP3*)

**lemma** *Skip-right-unit*: *P is NCSP*  $\implies$  *P* ;; *Skip* = *P*

**by** (*metis* (*full-types*) *CSP4-def* *Healthy-if* *NCSP-implies-CSP4*)

**lemma** *NCSP-NSRD-intro*:

**assumes** *P is NSRD*  $\$ref \# pre_R(P) \$ref \# peri_R(P) \$ref \# post_R(P) \$ref' \# post_R(P)$

**shows** *P is NCSP*

**by** (*simp* *add*: *CSP3-SRD-intro* *CSP4-NSRD-intro* *NCSP-intro* *NSRD-is-SRD* *assms*)

**lemma** *CSP4-neg-pre-unit*:

**assumes** *P is CSP* *P is CSP4*

**shows**  $(\neg_r pre_R(P)) ;; R1(true) = (\neg_r pre_R(P))$

**by** (*simp* *add*: *CSP4-implies-RD3* *NSRD-neg-pre-unit* *SRD-RD3-implies-NSRD* *assms(1)* *assms(2)*)

**lemma** *NSRD-CSP4-intro*:

**assumes**  $P$  is CSP  $P$  is CSP<sub>4</sub>  
**shows**  $P$  is NSRD  
**by** (*simp add: CSP<sub>4</sub>-implies-RD3 SRD-RD3-implies-NSRD assms(1) assms(2)*)

**lemma** *NCSP-form:*

$NCSP\ P = \mathbf{R}_s\ ((\forall\ \$ref\ \cdot\ (\neg_r\ pre_R(P))\ wp_r\ false) \vdash ((\exists\ \$ref\ \cdot\ \exists\ \$st'\ \cdot\ peri_R(P)) \diamond (\exists\ \$ref\ \cdot\ \exists\ \$ref'\ \cdot\ post_R(P))))$

**proof** –

**have**  $NCSP\ P = CSP3\ (CSP4\ (NSRD\ P))$

**by** (*metis (no-types, hide-lams) CSP<sub>4</sub>-def NCSP-def NSRD-alt-def RA1 RD3-def Skip-srdes-left-unit o-apply*)

**also**

**have**  $\dots = \mathbf{R}_s\ ((\forall\ \$ref\ \cdot\ (\neg_r\ pre_R\ (NSRD\ P))\ wp_r\ false) \vdash (\exists\ \$ref\ \cdot\ \exists\ \$st'\ \cdot\ peri_R\ (NSRD\ P)) \diamond (\exists\ \$ref\ \cdot\ \exists\ \$ref'\ \cdot\ post_R\ (NSRD\ P)))$

**by** (*simp add: CSP<sub>3</sub>-form CSP<sub>4</sub>-form closure unrest rdes, rel-auto*)

**also have**  $\dots = \mathbf{R}_s\ ((\forall\ \$ref\ \cdot\ (\neg_r\ pre_R(P))\ wp_r\ false) \vdash ((\exists\ \$ref\ \cdot\ \exists\ \$st'\ \cdot\ peri_R(P)) \diamond (\exists\ \$ref\ \cdot\ \exists\ \$ref'\ \cdot\ post_R(P))))$

**by** (*simp add: NSRD-form rdes closure, rel-blast*)

**finally show** *?thesis* .

**qed**

**lemma** *CSP<sub>4</sub>-st'-unrest-peri [unrest]:*

**assumes**  $P$  is CSP  $P$  is CSP<sub>4</sub>

**shows**  $\$st' \# peri_R(P)$

**by** (*simp add: NSRD-CSP<sub>4</sub>-intro NSRD-st'-unrest-peri assms*)

**lemma** *CSP<sub>4</sub>-healthy-form:*

**assumes**  $P$  is CSP  $P$  is CSP<sub>4</sub>

**shows**  $P = \mathbf{R}_s((\neg_r\ pre_R\ P)\ wp_r\ false \vdash ((\exists\ \$st'\ \cdot\ peri_R(P)) \diamond (\exists\ \$ref'\ \cdot\ post_R(P))))$

**proof** –

**have**  $P = \mathbf{R}_s((\neg_r\ pre_R\ P)\ wp_r\ false \vdash ((\exists\ \$st'\ \cdot\ cmt_R\ P) \triangleleft \$wait' \triangleright (\exists\ \$ref'\ \cdot\ cmt_R\ P)))$

**by** (*metis CSP<sub>4</sub>-def Healthy-def Skip-right-lemma assms(1) assms(2)*)

**also have**  $\dots = \mathbf{R}_s((\neg_r\ pre_R\ P)\ wp_r\ false \vdash ((\exists\ \$st'\ \cdot\ cmt_R\ P) \llbracket true/\$wait' \rrbracket \triangleleft \$wait' \triangleright (\exists\ \$ref'\ \cdot\ cmt_R\ P) \llbracket false/\$wait' \rrbracket))$

**by** (*metis (no-types, hide-lams) subst-wait'-left-subst subst-wait'-right-subst wait'-cond-def*)

**also have**  $\dots = \mathbf{R}_s((\neg_r\ pre_R\ P)\ wp_r\ false \vdash ((\exists\ \$st'\ \cdot\ peri_R(P)) \diamond (\exists\ \$ref'\ \cdot\ post_R(P))))$

**by** (*simp add: wait'-cond-def usubst peri\_R-def post\_R-def cmt\_R-def unrest*)

**finally show** *?thesis* .

**qed**

**lemma** *CSP<sub>4</sub>-ref'-unrest-pre [unrest]:*

**assumes**  $P$  is CSP  $P$  is CSP<sub>4</sub>

**shows**  $\$ref' \# pre_R(P)$

**proof** –

**have**  $pre_R(P) = pre_R(\mathbf{R}_s((\neg_r\ pre_R\ P)\ wp_r\ false \vdash ((\exists\ \$st'\ \cdot\ peri_R(P)) \diamond (\exists\ \$ref'\ \cdot\ post_R(P))))$

**using** *CSP<sub>4</sub>-healthy-form assms(1) assms(2)* **by** *fastforce*

**also have**  $\dots = (\neg_r\ pre_R\ P)\ wp_r\ false$

**by** (*simp add: rea-pre-RHS-design wp-rea-def usubst unrest*)

*CSP<sub>4</sub>-neg-pre-unit R1-rea-not R2c-preR R2c-rea-not assms*)

**also have**  $\$ref' \# \dots$

**by** (*simp add: wp-rea-def unrest*)

**finally show** *?thesis* .

**qed**

**lemma** *NCSP-set-unrest-pre-wait'*:  
**assumes**  $A \subseteq \llbracket \text{NCSP} \rrbracket_H$   
**shows**  $\bigwedge P. P \in A \implies \$wait' \# pre_R(P)$   
**proof** –  
**fix**  $P$   
**assume**  $P \in A$   
**hence**  $P$  is NSRD  
**using** *NCSP-implies-NSRD assms by auto*  
**thus**  $\$wait' \# pre_R(P)$   
**using** *NSRD-wait'-unrest-pre by blast*  
**qed**

**lemma** *CSP4-set-unrest-pre-st'*:  
**assumes**  $A \subseteq \llbracket \text{CSP} \rrbracket_H$   $A \subseteq \llbracket \text{CSP}_4 \rrbracket_H$   
**shows**  $\bigwedge P. P \in A \implies \$st' \# pre_R(P)$   
**proof** –  
**fix**  $P$   
**assume**  $P \in A$   
**hence**  $P$  is NSRD  
**using** *NSRD-CSP4-intro assms(1) assms(2) by blast*  
**thus**  $\$st' \# pre_R(P)$   
**using** *NSRD-st'-unrest-pre by blast*  
**qed**

**lemma** *CSP4-ref'-unrest-post [unrest]*:  
**assumes**  $P$  is CSP  $P$  is CSP<sub>4</sub>  
**shows**  $\$ref' \# post_R(P)$   
**proof** –  
**have**  $post_R(P) = post_R(\mathbf{R}_s((\neg_r pre_R P) wp_r false \vdash ((\exists \$st' \cdot peri_R(P)) \diamond (\exists \$ref' \cdot post_R(P))))))$   
**using** *CSP4-healthy-form assms(1) assms(2) by fastforce*  
**also have**  $\dots = R1 (R2c ((\neg_r pre_R P) wp_r false \Rightarrow_r (\exists \$ref' \cdot post_R P)))$   
**by** (*simp add: rea-post-RHS-design usubst unrest wp-rea-def*)  
**also have**  $\$ref' \# \dots$   
**by** (*simp add: R1-def R2c-def wp-rea-def unrest*)  
**finally show** *?thesis* .  
**qed**

**lemma** *CSP3-Chaos [closure]: Chaos is CSP3*  
**by** (*simp add: Chaos-def, rule CSP3-intro, simp-all add: RHS-design-is-SRD unrest*)

**lemma** *CSP4-Chaos [closure]: Chaos is CSP4*  
**by** (*rule CSP4-tri-intro, simp-all add: closure rdes unrest*)

**lemma** *NCSP-Chaos [closure]: Chaos is NCSP*  
**by** (*simp add: NCSP-intro closure*)

**lemma** *CSP3-Miracle [closure]: Miracle is CSP3*  
**by** (*simp add: Miracle-def, rule CSP3-intro, simp-all add: RHS-design-is-SRD unrest*)

**lemma** *CSP4-Miracle [closure]: Miracle is CSP4*  
**by** (*rule CSP4-tri-intro, simp-all add: closure rdes unrest*)

**lemma** *NCSP-Miracle [closure]: Miracle is NCSP*  
**by** (*simp add: NCSP-intro closure*)

**lemma** *NCSP-seqr-closure* [*closure*]:  
**assumes** *P is NCSP Q is NCSP*  
**shows** *P ;; Q is NCSP*  
**by** (*metis (no-types, lifting) CSP3-def CSP4-def Healthy-def' NCSP-implies-CSP NCSP-implies-CSP3 NCSP-implies-CSP4 NCSP-intro SRD-seqr-closure assms(1) assms(2) seqr-assoc*)

**lemma** *CSP4-Skip* [*closure*]: *Skip is CSP4*  
**apply** (*rule CSP4-intro, simp-all add: Skip-is-CSP*)  
**apply** (*simp-all add: Skip-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest R2c-true*)  
**done**

**lemma** *NCSP-Skip* [*closure*]: *Skip is NCSP*  
**by** (*metis CSP3-Skip CSP4-Skip Healthy-def NCSP-def Skip-is-CSP comp-apply*)

**lemma** *CSP4-Stop* [*closure*]: *Stop is CSP4*  
**apply** (*rule CSP4-intro, simp-all add: Stop-is-CSP*)  
**apply** (*simp-all add: Stop-def rea-pre-RHS-design rea-cmt-RHS-design usubst unrest R2c-true*)  
**done**

**lemma** *NCSP-Stop* [*closure*]: *Stop is NCSP*  
**by** (*metis CSP3-Stop CSP4-Stop Healthy-def NCSP-def Stop-is-CSP comp-apply*)

**lemma** *CSP4-Idempotent*: *Idempotent CSP4*  
**by** (*metis (no-types, lifting) CSP3-Skip CSP3-def CSP4-def Healthy-if Idempotent-def seqr-assoc*)

**lemma** *CSP4-Continuous*: *Continuous CSP4*  
**by** (*simp add: Continuous-def CSP4-def seq-Sup-distr*)

**lemma** *preR-Stop* [*rdes*]: *pre<sub>R</sub>(Stop) = true<sub>r</sub>*  
**by** (*simp add: Stop-def Stop-is-CSP rea-pre-RHS-design unrest usubst R2c-true*)

**lemma** *periR-Stop* [*rdes*]: *peri<sub>R</sub>(Stop) =  $\mathcal{E}(\text{true}, \langle \rangle, \{ \}_u)$*   
**by** (*rel-auto*)

**lemma** *postR-Stop* [*rdes*]: *post<sub>R</sub>(Stop) = false*  
**by** (*rel-auto*)

**lemma** *cmtR-Stop* [*rdes*]: *cmt<sub>R</sub>(Stop) = ( $\$tr' =_u \$tr \wedge \$wait'$ )*  
**by** (*rel-auto*)

**lemma** *NCSP-Idempotent* [*closure*]: *Idempotent NCSP*  
**by** (*clarsimp simp add: NCSP-def Idempotent-def*)  
(*metis (no-types, hide-lams) CSP3-Idempotent CSP3-def CSP4-Idempotent CSP4-def Healthy-def Idempotent-def SRD-idem SRD-seqr-closure Skip-is-CSP seqr-assoc*)

**lemma** *NCSP-Continuous* [*closure*]: *Continuous NCSP*  
**by** (*simp add: CSP3-Continuous CSP4-Continuous Continuous-comp NCSP-def SRD-Continuous*)

**lemma** *preR-CRR* [*closure*]: *P is NCSP  $\implies$  pre<sub>R</sub>(P) is CRR*  
**by** (*rule CRR-intro, simp-all add: closure unrest*)

**lemma** *periR-CRR* [*closure*]: *P is NCSP  $\implies$  peri<sub>R</sub>(P) is CRR*  
**by** (*rule CRR-intro, simp-all add: closure unrest*)

**lemma** *postR-CRR* [*closure*]: *P is NCSP  $\implies$  post<sub>R</sub>(P) is CRR*

by (rule *CRR-intro*, *simp-all add: closure unrest*)

**lemma** *NCSP-rdes-intro* [closure]:

**assumes** *P* is CRC *Q* is CRR *R* is CRR

$\$st' \# Q \$ref' \# R$

**shows**  $\mathbf{R}_s(P \vdash Q \diamond R)$  is NCSP

**apply** (rule *NCSP-intro*)

**apply** (*simp-all add: closure assms*)

**apply** (rule *CSP3-SRD-intro*)

**apply** (*simp-all add: rdes closure assms unrest*)

**apply** (rule *CSP4-tri-intro*)

**apply** (*simp-all add: rdes closure assms unrest*)

**apply** (*metis (no-types, lifting) CRC-implies-RC R1-seqr-closure assms(1) rea-not-R1 rea-not-false rea-not-not wp-rea-RC-false wp-rea-def*)

**done**

**lemma** *NCSP-preR-CRC* [closure]:

**assumes** *P* is NCSP

**shows**  $pre_R(P)$  is CRC

**by** (rule *CRC-intro*, *simp-all add: closure assms unrest*)

**lemma** *CSP3-Sup-closure* [closure]:

$A \subseteq \llbracket CSP3 \rrbracket_H \implies (\bigcap A)$  is CSP3

**apply** (*auto simp add: CSP3-def Healthy-def seq-Sup-distl*)

**apply** (rule *cong[of Sup]*)

**apply** (*simp*)

**using** *image-iff* **apply** *force*

**done**

**lemma** *CSP4-Sup-closure* [closure]:

$A \subseteq \llbracket CSP4 \rrbracket_H \implies (\bigcap A)$  is CSP4

**apply** (*auto simp add: CSP4-def Healthy-def seq-Sup-distr*)

**apply** (rule *cong[of Sup]*)

**apply** (*simp*)

**using** *image-iff* **apply** *force*

**done**

**lemma** *NCSP-Sup-closure* [closure]:  $\llbracket A \subseteq \llbracket NCSP \rrbracket_H; A \neq \{\} \rrbracket \implies (\bigcap A)$  is NCSP

**apply** (rule *NCSP-intro*, *simp-all add: closure*)

**apply** (*metis (no-types, lifting) Ball-Collect CSP3-Sup-closure NCSP-implies-CSP3*)

**apply** (*metis (no-types, lifting) Ball-Collect CSP4-Sup-closure NCSP-implies-CSP4*)

**done**

**lemma** *NCSP-SUP-closure* [closure]:  $\llbracket \bigwedge i. P(i) \text{ is NCSP}; A \neq \{\} \rrbracket \implies (\bigcap_{i \in A} P(i))$  is NCSP

**by** (*metis (mono-tags, lifting) Ball-Collect NCSP-Sup-closure image-iff image-is-empty*)

**lemma** *PCSP-implies-NCSP* [closure]:

**assumes** *P* is PCSP

**shows** *P* is NCSP

**proof** –

**have**  $P = \text{Productive}(\text{NCSP}(\text{NCSP } P))$

**by** (*metis (no-types, hide-lams) Healthy-def' Idempotent-def NCSP-Idempotent assms comp-apply*)

**also have**  $\dots = \mathbf{R}_s((\forall \$ref \cdot (\neg_r pre_R(\text{NCSP } P)) wp_r false) \vdash$

$(\exists \$ref \cdot \exists \$st' \cdot peri_R(\text{NCSP } P)) \diamond$

$((\exists \$ref \cdot \exists \$ref' \cdot post_R (NCSP P)) \wedge \$tr <_u \$tr')$   
**by** (*simp add: NCSP-form Productive-RHS-design-form unrest closure*)  
**also have ... is NCSP**  
**apply** (*rule NCSP-rdes-intro*)  
**apply** (*rule CRC-intro*)  
**apply** (*simp-all add: unrest ex-unrest all-unrest closure*)  
**done**  
**finally show ?thesis .**  
**qed**

**lemma PCSP-elim [RD-elim]:**  
**assumes**  $X$  *is PCSP*  $P$  ( $\mathbf{R}_s ((pre_R X) \vdash peri_R X \diamond (R_4(post_R X))))$   
**shows**  $P X$   
**by** (*metis R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP Productive-form assms comp-apply*)

**lemma ICSP-implies-NCSP [closure]:**  
**assumes**  $P$  *is ICSP*  
**shows**  $P$  *is NCSP*

**proof** –

**have**  $P = ISRD1(NCSP(NCSP P))$   
**by** (*metis (no-types, hide-lams) Healthy-def' Idempotent-def NCSP-Idempotent assms comp-apply*)  
**also have ... = ISRD1** ( $\mathbf{R}_s ((\forall \$ref \cdot (\neg_r pre_R (NCSP P)) wp_r false) \vdash$   
 $(\exists \$ref \cdot \exists \$st' \cdot peri_R (NCSP P)) \diamond$   
 $(\exists \$ref \cdot \exists \$ref' \cdot post_R (NCSP P))))$ )  
**by** (*simp add: NCSP-form*)  
**also have ... =  $\mathbf{R}_s$**  ( $(\forall \$ref \cdot (\neg_r pre_R (NCSP P)) wp_r false) \vdash$   
 $false \diamond$   
 $((\exists \$ref \cdot \exists \$ref' \cdot post_R (NCSP P)) \wedge \$tr' =_u \$tr)$ )  
**by** (*simp-all add: ISRD1-RHS-design-form closure rdes unrest*)  
**also have ... is NCSP**  
**apply** (*rule NCSP-rdes-intro*)  
**apply** (*rule CRC-intro*)  
**apply** (*simp-all add: unrest ex-unrest all-unrest closure*)  
**done**  
**finally show ?thesis .**  
**qed**

**lemma ICSP-implies-ISR1 [closure]:**

**assumes**  $P$  *is ICSP*  
**shows**  $P$  *is ISR1*  
**by** (*metis (no-types, hide-lams) Healthy-def ICSP-implies-NCSP ISR1-def NCSP-implies-ISR1 assms comp-apply*)

**lemma ICSP-elim [RD-elim]:**

**assumes**  $X$  *is ICSP*  $P$  ( $\mathbf{R}_s ((pre_R X) \vdash false \diamond (post_R X \wedge \$tr' =_u \$tr))$ )  
**shows**  $P X$   
**by** (*metis Healthy-if NCSP-implies-CSP ICSP-implies-NCSP ISR1-form assms comp-apply*)

**lemma ICSP-Stop-right-zero-lemma:**

$(P \wedge (\$tr' =_u \$tr)) ;; true_r = true_r \implies (P \wedge (\$tr' =_u \$tr)) ;; (\$tr' =_u \$tr) = (\$tr' =_u \$tr)$   
**by** (*rel-blast*)

**lemma ICSP-Stop-right-zero:**

**assumes**  $P$  *is ICSP*  $pre_R(P) = true_r$   $post_R(P) ;; true_r = true_r$   
**shows**  $P ;; Stop = Stop$

**proof** –

**from**  $assms(3)$  **have**  $1:(post_R P \wedge \$tr' =_u \$tr) ;; true_r = true_r$

**by** ( $rel-auto$ ,  $metis$  ( $full-types$ ,  $hide-lams$ )  $dual-order.antisym$   $order-refl$ )

**show**  $?thesis$

**by** ( $rdes-simp$   $cls: assms(1)$ ,  $simp$   $add: csp-enable-nothing$   $assms(2)$   $ICSP-Stop-right-zero-lemma[OF$   
 $1]$ )

**qed**

**lemma**  $ICSP-intro: \llbracket P \text{ is NCSP}; P \text{ is ISR D1} \rrbracket \implies P \text{ is ICSP}$

**using**  $Healthy-comp$  **by**  $blast$

**lemma**  $seq-ICSP-closed$  [ $closure$ ]:

**assumes**  $P \text{ is ICSP}$   $Q \text{ is ICSP}$

**shows**  $P ;; Q \text{ is ICSP}$

**by** ( $meson$   $ICSP-implies-ISR D$   $ICSP-implies-NCSP$   $ICSP-intro$   $ISR D-implies-ISR D1$   $NCSP-seqr-closure$   
 $assms$   $seq-ISR D-closed$ )

**lemma**  $Miracle-ICSP$  [ $closure$ ]:  $Miracle \text{ is ICSP}$

**by** ( $rule$   $ICSP-intro$ ,  $simp$   $add: closure$ ,  $simp$   $add: ISR D1-rdes-intro$   $rdes-def$   $closure$ )

## 5.2 CSP theories

**typedecl**  $TCSP$

**abbreviation**  $TCSP \equiv UTHY(TCSP, ('\sigma, '\varphi) \text{ st-csp})$

**overloading**

$tjsp-hcond == utp-hcond :: (TCSP, ('\sigma, '\varphi) \text{ st-csp}) \text{ uthy} \Rightarrow (('\sigma, '\varphi) \text{ st-csp} \times ('\sigma, '\varphi) \text{ st-csp}) \text{ health}$

$tjsp-unit == utp-unit :: (TCSP, ('\sigma, '\varphi) \text{ st-csp}) \text{ uthy} \Rightarrow ('\sigma, '\varphi) \text{ action}$

**begin**

**definition**  $tjsp-hcond :: (TCSP, ('\sigma, '\varphi) \text{ st-csp}) \text{ uthy} \Rightarrow (('\sigma, '\varphi) \text{ st-csp} \times ('\sigma, '\varphi) \text{ st-csp}) \text{ health}$  **where**

[ $upred-defs$ ]:  $tjsp-hcond T = NCSP$

**definition**  $tjsp-unit :: (TCSP, ('\sigma, '\varphi) \text{ st-csp}) \text{ uthy} \Rightarrow ('\sigma, '\varphi) \text{ action}$  **where**

[ $upred-defs$ ]:  $tjsp-unit T = Skip$

**end**

**interpretation**  $csp-theory: utp-theory-kleene$   $UTHY(TCSP, ('\sigma, '\varphi) \text{ st-csp})$

**rewrites**  $\bigwedge P. P \in \text{carrier}(\text{uthy-order } TCSP) \longleftrightarrow P \text{ is NCSP}$

**and**  $P \text{ is } \mathcal{H}_{TCSP} \longleftrightarrow P \text{ is NCSP}$

**and**  $\mathcal{I}L_{TCSP} = Skip$

**and**  $\top_{TCSP} = Miracle$

**and**  $\text{carrier}(\text{uthy-order } TCSP) \rightarrow \text{carrier}(\text{uthy-order } TCSP) \equiv \llbracket NCSP \rrbracket_H \rightarrow \llbracket NCSP \rrbracket_H$

**and**  $A \subseteq \text{carrier}(\text{uthy-order } TCSP) \longleftrightarrow A \subseteq \llbracket NCSP \rrbracket_H$

**and**  $le(\text{uthy-order } TCSP) = op \sqsubseteq$

**proof** –

**interpret**  $lat: utp-theory-continuous$   $UTHY(TCSP, ('\sigma, '\varphi) \text{ st-csp})$

**by** ( $unfold-locale$ ,  $simp-all$   $add: tjsp-hcond-def$   $closure$   $Healthy-if$ )

**show**  $1: \top_{TCSP} = (Miracle :: ('\sigma, '\varphi) \text{ action})$

**by** ( $metis$   $NCSP-Miracle$   $NCSP-implies-CSP$   $lat.top-healthy$   $lat.utp-theory-continuous-axioms$   $srdes-theory-continuous.$   
 $tjsp-hcond-def$   $upred-semiring.add-commute$   $utp-theory-continuous.meet-top$ )

**thus**  $utp-theory-kleene$   $UTHY(TCSP, ('\sigma, '\varphi) \text{ st-csp})$

**by** ( $unfold-locale$ ,  $simp-all$   $add: tjsp-hcond-def$   $tjsp-unit-def$   $Skip-left-unit$   $Skip-right-unit$   $closure$   
 $Healthy-if$   $Miracle-left-zero$  )

**qed** ( $simp-all$   $add: tjsp-hcond-def$   $tjsp-unit-def$   $closure$   $Healthy-if$ )



**declare** *csp-theory.top-healthy* [*simp del*]  
**declare** *csp-theory.bottom-healthy* [*simp del*]

**abbreviation** *TestC* (*test<sub>C</sub>*) **where**  
*test<sub>C</sub> P*  $\equiv$  *utest TCSP P*

**abbreviation** *StarC* :: (*'σ, 'φ*) *action*  $\Rightarrow$  (*'σ, 'φ*) *action* (*-<sup>\*C</sup>* [999] 999) **where**  
*StarC P*  $\equiv$  *P<sup>★</sup>TCSP*

**lemma** *csp-bottom-Chaos*:  $\perp_{TCSP} = \text{Chaos}$   
**using** *NCSP-Chaos NCSP-implies-CSP* **by** *auto*

**lemma** *csp-top-Miracle*:  $\top_{TCSP} = \text{Miracle}$   
**by** (*simp add: csp-theory.healthy-top csp-theory.utp-theory-mono-axioms utp-theory-mono.healthy-top*)

### 5.3 Algebraic laws

**lemma** *Stop-left-zero*:  
**assumes** *P is CSP*  
**shows** *Stop ; P = Stop*  
**by** (*simp add: NSRD-seq-post-false assms NCSP-implies-NSRD NCSP-Stop postR-Stop*)

**end**

## 6 Stateful-Failure Reactive Contracts

**theory** *utp-sfrd-contracts*  
**imports** *utp-sfrd-healths*  
**begin**

**definition** *mk-CRD* :: *'s upred*  $\Rightarrow$  (*'e list*  $\Rightarrow$  *'e set*  $\Rightarrow$  *'s upred*)  $\Rightarrow$  (*'e list*  $\Rightarrow$  *'s hrel*)  $\Rightarrow$  (*'s, 'e*) *action*  
**where**

[*rdes-def*]: *mk-CRD P Q R* =  $\mathbf{R}_s([P]_{S<} \vdash [Q \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref \ ' \rrbracket \diamond [R(x)]_S \llbracket x \rightarrow \&tt \rrbracket \rrbracket$

**syntax**

*-ref-var* :: *logic*  
*-mk-CRD* :: *logic*  $\Rightarrow$  *logic*  $\Rightarrow$  *logic*  $\Rightarrow$  *logic* (*[-/*  $\vdash$  *-/*  $|$  *-]*<sub>*C*</sub>)

**parse-translation**  $\ll$

*let*

*fun ref-var-tr []* = *Syntax.free refs*  
 $|$  *ref-var-tr -* = *raise Match*;

*in*

$\llbracket (@\{syntax-const \ -ref-var\}, K \ ref-var-tr) \rrbracket$

*end*

$\gg$

**translations**

$[P \vdash Q \mid R]_C \Rightarrow \text{CONST } mk-CRD \ P \ (\lambda \ -trace-var \ -ref-var. \ Q) \ (\lambda \ -trace-var. \ R)$   
 $[P \vdash Q \mid R]_C \Leftarrow \text{CONST } mk-CRD \ P \ (\lambda \ x \ r. \ Q) \ (\lambda \ y. \ R)$

**lemma** *CSP-mk-CRD [closure]*:  $[P \vdash Q \ trace \ refs \mid R(trace)]_C$  *is CSP*  
**by** (*simp add: mk-CRD-def closure unrest*)

**lemma** *preR-mk-CRD [rdes]*:  $pre_R([P \vdash Q \ trace \ refs \mid R(trace)]_C) = [P]_{S<}$

by (simp add: mk-CRD-def rea-pre-RHS-design usubst unrest R2c-not R2c-lift-state-pre rea-st-cond-def, rel-auto)

**lemma** *periR-mk-CRD* [rdes]:  $peri_R([P \vdash Q \text{ trace refs} \mid R(\text{trace})]_C) = ([P]_{S<} \Rightarrow_r ([Q \text{ trace refs}]_{S<})[(\text{trace}, \text{refs}) \rightarrow (\&tt, \$r)])$   
 by (simp add: mk-CRD-def rea-peri-RHS-design usubst unrest R2c-not R2c-lift-state-pre impl-alt-def R2c-disj R2c-msubst-tt R1-disj, rel-auto)

**lemma** *postR-mk-CRD* [rdes]:  $post_R([P \vdash Q \text{ trace refs} \mid R(\text{trace})]_C) = ([P]_{S<} \Rightarrow_r ([R(\text{trace})]_S)[\text{trace} \rightarrow \&tt])$   
 by (simp add: mk-CRD-def rea-post-RHS-design usubst unrest R2c-not R2c-lift-state-pre impl-alt-def R2c-disj R2c-msubst-tt R1-disj, rel-auto)

Refinement introduction law for contracts

**lemma** *CRD-contract-refine*:

**assumes**

$Q$  is CSP ‘ $[P_1]_{S<} \Rightarrow pre_R Q$ ’  
 ‘ $[P_1]_{S<} \wedge peri_R Q \Rightarrow [P_2 \ t \ r]_{S<}[\![t \rightarrow \&tt]\!] [r \rightarrow \$ref \ ]$ ’  
 ‘ $[P_1]_{S<} \wedge post_R Q \Rightarrow [P_3 \ x]_S[\![x \rightarrow \&tt]\!]$ ’

**shows**  $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq Q$

**proof** –

**have**  $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq \mathbf{R}_s(pre_R(Q) \vdash peri_R(Q) \diamond post_R(Q))$

**using** *assms* by (simp add: mk-CRD-def, rule-tac sdes-tri-refine-intro, rel-auto+)

**thus** *?thesis*

by (simp add: SRD-reactive-tri-design *assms*(1))

qed

**lemma** *CRD-contract-refine'*:

**assumes**

$Q$  is CSP ‘ $[P_1]_{S<} \Rightarrow pre_R Q$ ’  
 $[P_2 \ t \ r]_{S<}[\![t \rightarrow \&tt]\!] [r \rightarrow \$ref \ ] \sqsubseteq ([P_1]_{S<} \wedge peri_R Q)$   
 $[P_3 \ x]_S[\![x \rightarrow \&tt]\!] \sqsubseteq ([P_1]_{S<} \wedge post_R Q)$

**shows**  $[P_1 \vdash P_2 \text{ trace refs} \mid P_3(\text{trace})]_C \sqsubseteq Q$

**using** *assms* by (rule-tac *CRD-contract-refine*, simp-all add: *refBy-order*)

**lemma** *CRD-refine-CRD*:

**assumes**

$[P_1]_{S<} \Rightarrow ([Q_1]_{S<} :: ('e, 's) \text{ action})$   
 $([P_2 \ x \ r]_{S<}[\![x \rightarrow \&tt]\!] [r \rightarrow \$ref \ ]) \sqsubseteq ([P_1]_{S<} \wedge [Q_2 \ x \ r]_{S<}[\![x \rightarrow \&tt]\!] [r \rightarrow \$ref \ ]) :: ('e, 's) \text{ action}$   
 $[P_3 \ x]_S[\![x \rightarrow \&tt]\!] \sqsubseteq ([P_1]_{S<} \wedge [Q_3 \ x]_S[\![x \rightarrow \&tt]\!] :: ('e, 's) \text{ action})$

**shows**  $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq [Q_1 \vdash Q_2 \text{ trace refs} \mid Q_3 \text{ trace}]_C$

**using** *assms*

by (simp add: mk-CRD-def, rule-tac sdes-tri-refine-intro, rel-auto+)

**lemma** *CRD-refine-rdes*:

**assumes**

$[P_1]_{S<} \Rightarrow Q_1$   
 $([P_2 \ x \ r]_{S<}[\![x \rightarrow \&tt]\!] [r \rightarrow \$ref \ ]) \sqsubseteq ([P_1]_{S<} \wedge Q_2)$   
 $[P_3 \ x]_S[\![x \rightarrow \&tt]\!] \sqsubseteq ([P_1]_{S<} \wedge Q_3)$

**shows**  $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq$

$\mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$

**using** *assms*

by (simp add: mk-CRD-def, rule-tac sdes-tri-refine-intro, rel-auto+)

**lemma** *CRD-refine-rdes'*:

**assumes**

$Q_2$  is RR

$Q_3$  is *RR*  
 $\text{'}[P_1]_{S<} \Rightarrow Q_1 \text{'}$   
 $\bigwedge t. ([P_2 \ t \ r]_{S<} \llbracket r \rightarrow \$ref' \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge Q_2 \llbracket \langle, \ll t \gg / \$tr, \$tr' \rrbracket)$   
 $\bigwedge t. [P_3 \ t]_{S'} \sqsubseteq ([P_1]_{S<} \wedge Q_3 \llbracket \langle, \ll t \gg / \$tr, \$tr' \rrbracket)$   
**shows**  $([P_1 \vdash P_2 \text{ trace refs} \mid P_3 \text{ trace}]_C :: ('e, 's) \text{ action}) \sqsubseteq$   
 $\mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3)$   
**proof** (*simp add: mk-CRD-def, rule srdes-tri-refine-intro*)  
**show**  $\text{'}[P_1]_{S<} \Rightarrow Q_1 \text{'}$  **by** (*fact assms(3)*)  
  
**have**  $\bigwedge t. ([P_2 \ t \ r]_{S<} \llbracket r \rightarrow \$ref' \rrbracket) \sqsubseteq ([P_1]_{S<} \wedge (RR \ Q_2) \llbracket \langle, \ll t \gg / \$tr, \$tr' \rrbracket)$   
**by** (*simp add: assms Healthy-if*)  
**hence**  $\text{'}[P_1]_{S<} \wedge RR(Q_2) \Rightarrow [P_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref' \rrbracket \text{'}$   
**by** (*rel-simp; meson*)  
**thus**  $\text{'}[P_1]_{S<} \wedge Q_2 \Rightarrow [P_2 \ x \ r]_{S<} \llbracket x \rightarrow \&tt \rrbracket \llbracket r \rightarrow \$ref' \rrbracket \text{'}$   
**by** (*simp add: Healthy-if assms*)  
  
**have**  $\bigwedge t. [P_3 \ t]_{S'} \sqsubseteq ([P_1]_{S<} \wedge (RR \ Q_3) \llbracket \langle, \ll t \gg / \$tr, \$tr' \rrbracket)$   
**by** (*simp add: assms Healthy-if*)  
**hence**  $\text{'}[P_1]_{S<} \wedge (RR \ Q_3) \Rightarrow [P_3 \ x]_{S'} \llbracket x \rightarrow \&tt \rrbracket \text{'}$   
**by** (*rel-simp; meson*)  
**thus**  $\text{'}[P_1]_{S<} \wedge Q_3 \Rightarrow [P_3 \ x]_{S'} \llbracket x \rightarrow \&tt \rrbracket \text{'}$   
**by** (*simp add: Healthy-if assms*)  
**qed**  
  
**end**

## 7 External Choice

**theory** *utp-sfrd-extchoice*  
**imports**  
*utp-sfrd-healths*  
*utp-sfrd-rel*  
**begin**

### 7.1 Definitions and syntax

**definition** *ExtChoice* ::

$('\sigma, '\varphi) \text{ action set} \Rightarrow (''\sigma, ''\varphi) \text{ action where}$   
 $[upred-defs]: \text{ExtChoice } A = \mathbf{R}_s((\bigsqcup P \in A \cdot pre_R(P)) \vdash ((\bigsqcup P \in A \cdot cmt_R(P)) \triangleleft \$tr' =_u \$tr \wedge \$wait'$   
 $\triangleright (\bigsqcap P \in A \cdot cmt_R(P))))$

**syntax**

$-ExtChoice :: pptrn \Rightarrow 'a \text{ set} \Rightarrow 'b \Rightarrow 'b \ ((\exists \square - \in - \cdot / -) [0, 0, 10] 10)$   
 $-ExtChoice-simp :: pptrn \Rightarrow 'b \Rightarrow 'b \ ((\exists \square - \cdot / -) [0, 10] 10)$

**translations**

$\square P \in A \cdot B \quad \equiv \text{CONST } ExtChoice \ ((\lambda P. B) \text{' } A)$   
 $\square P \cdot B \quad \equiv \text{CONST } ExtChoice \ (\text{CONST } range \ (\lambda P. B))$

**definition** *extChoice* ::

$('\sigma, '\varphi) \text{ action} \Rightarrow (''\sigma, ''\varphi) \text{ action} \Rightarrow (''\sigma, ''\varphi) \text{ action (infixl } \square 65) \text{ where}$   
 $[upred-defs]: P \square Q \equiv ExtChoice \ \{P, Q\}$

Small external choice as an indexed big external choice.

**lemma** *extChoice-alt-def*:

$P \sqcap Q = (\Box i :: \text{nat} \in \{0,1\} \cdot P \triangleleft \ll i = 0 \gg \triangleright Q)$   
 by (simp add: extChoice-def ExtChoice-def, unliteralise, simp)

## 7.2 Basic laws

## 7.3 Algebraic laws

**lemma** *ExtChoice-empty*:  $\text{ExtChoice } \{\} = \text{Stop}$   
 by (simp add: ExtChoice-def cond-def Stop-def)

**lemma** *ExtChoice-single*:  
 $P \text{ is CSP} \implies \text{ExtChoice } \{P\} = P$   
 by (simp add: ExtChoice-def usup-and uinf-or SRD-reactive-design-alt)

## 7.4 Reactive design calculations

**lemma** *ExtChoice-rdes*:  
 assumes  $\bigwedge i. \$ok' \nmid P(i) \ A \neq \{\}$   
 shows  $(\Box i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) = \mathbf{R}_s((\bigsqcup i \in A \cdot P(i)) \vdash ((\bigsqcup i \in A \cdot Q(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\bigsqcup i \in A \cdot Q(i))))$

**proof** –

have  $(\Box i \in A \cdot \mathbf{R}_s(P(i) \vdash Q(i))) =$   
 $\mathbf{R}_s((\bigsqcup i \in A \cdot \text{pre}_R(\mathbf{R}_s(P \ i \vdash Q \ i))) \vdash$   
 $((\bigsqcup i \in A \cdot \text{cmt}_R(\mathbf{R}_s(P \ i \vdash Q \ i)))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\bigsqcup i \in A \cdot \text{cmt}_R(\mathbf{R}_s(P \ i \vdash Q \ i))))$

by (simp add: ExtChoice-def)

also have ... =

$\mathbf{R}_s((\bigsqcup i \in A \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$   
 $((\bigsqcup i \in A \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\bigsqcup i \in A \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))))$

by (simp add: rea-pre-RHS-design rea-cmt-RHS-design)

also have ... =

$\mathbf{R}_s((\bigsqcup i \in A \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$   
 $R1(R2c$   
 $((\bigsqcup i \in A \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\bigsqcup i \in A \cdot R1(R2c(\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))))$

by (metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c)

also have ... =

$\mathbf{R}_s((\bigsqcup i \in A \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$   
 $R1(R2c$   
 $((\bigsqcup i \in A \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\bigsqcup i \in A \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$

by (simp add: R2c-UINF R2c-cond R1-cond R1-idem R1-R2c-commute R2c-idem R1-UINF assms R1-USUP R2c-USUP)

also have ... =

$\mathbf{R}_s((\bigsqcup i \in A \cdot R1(R2c(\text{pre}_s \dagger P(i)))) \vdash$   
 $\text{cmt}_s \dagger$   
 $((\bigsqcup i \in A \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\bigsqcup i \in A \cdot (\text{cmt}_s \dagger (P(i) \Rightarrow Q(i))))$

by (metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c rdes-export-cmt)

also have ... =

$\mathbf{R}_s ((\bigsqcup i \in A \cdot R1 (R2c (pre_s \dagger P(i)))) \vdash$   
 $cmt_s \dagger$   
 $((\bigsqcup i \in A \cdot (P(i) \Rightarrow Q(i)))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(\prod i \in A \cdot (P(i) \Rightarrow Q(i))))$   
**by** (*simp add: usubst*)  
**also have ... =**  
 $\mathbf{R}_s ((\bigsqcup i \in A \cdot R1 (R2c (pre_s \dagger P(i)))) \vdash$   
 $((\bigsqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod i \in A \cdot (P(i) \Rightarrow Q(i))))$   
**by** (*simp add: rdes-export-cmt*)  
**also have ... =**  
 $\mathbf{R}_s ((R1(R2c(\bigsqcup i \in A \cdot (pre_s \dagger P(i)))) \vdash$   
 $((\bigsqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod i \in A \cdot (P(i) \Rightarrow Q(i))))$   
**by** (*simp add: not-UINF R1-UINF R2c-UINF assms*)  
**also have ... =**  
 $\mathbf{R}_s ((R2c(\bigsqcup i \in A \cdot (pre_s \dagger P(i)))) \vdash$   
 $((\bigsqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod i \in A \cdot (P(i) \Rightarrow Q(i))))$   
**by** (*simp add: R1-design-R1-pre*)  
**also have ... =**  
 $\mathbf{R}_s (((\bigsqcup i \in A \cdot (pre_s \dagger P(i)))) \vdash$   
 $((\bigsqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod i \in A \cdot (P(i) \Rightarrow Q(i))))$   
**by** (*metis (no-types, lifting) RHS-design-R2c-pre*)  
**also have ... =**  
 $\mathbf{R}_s (([\$ok \mapsto_s true, \$wait \mapsto_s false] \dagger (\bigsqcup i \in A \cdot P(i))) \vdash$   
 $((\bigsqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod i \in A \cdot (P(i) \Rightarrow Q(i))))$   
**proof** –  
**from** *assms* **have**  $\bigwedge i. pre_s \dagger P(i) = [\$ok \mapsto_s true, \$wait \mapsto_s false] \dagger P(i)$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*simp add: usubst*)  
**qed**  
**also have ... =**  
 $\mathbf{R}_s ((\bigsqcup i \in A \cdot P(i)) \vdash ((\bigsqcup i \in A \cdot (P(i) \Rightarrow Q(i))) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod i \in A \cdot (P(i) \Rightarrow$   
 $Q(i))))$   
**by** (*simp add: rdes-export-pre not-UINF*)  
**also have ... =**  $\mathbf{R}_s ((\bigsqcup i \in A \cdot P(i)) \vdash ((\bigsqcup i \in A \cdot Q(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod i \in A \cdot Q(i))))$   
**by** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto, blast+*)

**finally show** *?thesis* .  
**qed**

**lemma** *ExtChoice-tri-rdes*:

**assumes**  $\bigwedge i. \$ok' \# P_1(i) \ A \neq \{\}$   
**shows**  $(\square i \in A \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$   
 $\mathbf{R}_s ((\bigsqcup i \in A \cdot P_1(i)) \vdash (((\bigsqcup i \in A \cdot P_2(i)) \triangleleft \$tr' =_u \$tr \triangleright (\prod i \in A \cdot P_2(i))) \diamond (\prod i \in A \cdot$   
 $P_3(i))))$   
**proof** –  
**have**  $(\square i \in A \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$   
 $\mathbf{R}_s ((\bigsqcup i \in A \cdot P_1(i)) \vdash (((\bigsqcup i \in A \cdot P_2(i) \diamond P_3(i)) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod i \in A \cdot P_2(i) \diamond$   
 $P_3(i))))$   
**by** (*simp add: ExtChoice-rdes assms*)  
**also**  
**have ... =**  
 $\mathbf{R}_s ((\bigsqcup i \in A \cdot P_1(i)) \vdash (((\bigsqcup i \in A \cdot P_2(i) \diamond P_3(i)) \triangleleft \$wait' \wedge \$tr' =_u \$tr \triangleright (\prod i \in A \cdot P_2(i) \diamond$   
 $P_3(i))))$

by (*simp add: conj-comm*)  
 also  
 have ... =  
 $\mathbf{R}_s ((\sqcup i \in A \cdot P_1(i)) \vdash (((\sqcup i \in A \cdot P_2(i) \diamond P_3(i)) \triangleleft \$tr' =_u \$tr \triangleright (\prod i \in A \cdot P_2(i) \diamond P_3(i))) \diamond (\prod i \in A \cdot P_2(i) \diamond P_3(i))))$   
 by (*simp add: cond-conj wait'-cond-def*)  
 also  
 have ... =  $\mathbf{R}_s ((\sqcup i \in A \cdot P_1(i)) \vdash (((\sqcup i \in A \cdot P_2(i)) \triangleleft \$tr' =_u \$tr \triangleright (\prod i \in A \cdot P_2(i))) \diamond (\prod i \in A \cdot P_3(i))))$   
 by (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
 finally show ?thesis .  
 qed

**lemma** *ExtChoice-tri-rdes'* [*rdes-def*]:  
 assumes  $\bigwedge i . \$ok' \# P_1(i) \ A \neq \{\}$   
 shows  $(\prod i \in A \cdot \mathbf{R}_s(P_1(i) \vdash P_2(i) \diamond P_3(i))) =$   
 $\mathbf{R}_s ((\sqcup i \in A \cdot P_1(i)) \vdash (((\sqcup i \in A \cdot R5(P_2(i))) \vee (\prod i \in A \cdot R4(P_2(i)))) \diamond (\prod i \in A \cdot P_3(i))))$   
 by (*simp add: ExtChoice-tri-rdes assms, rel-auto, simp-all add: less-le assms*)

**lemma** *ExtChoice-tri-rdes-def* [*rdes-def*]:  
 assumes  $A \subseteq \llbracket CSP \rrbracket_H$   
 shows  $ExtChoice \ A = \mathbf{R}_s ((\prod P \in A \cdot pre_R \ P) \vdash (((\prod P \in A \cdot peri_R \ P) \triangleleft \$tr' =_u \$tr \triangleright (\prod P \in A \cdot post_R \ P)) \diamond (\prod P \in A \cdot post_R \ P)))$   
**proof** –  
 have  $((\prod P \in A \cdot cmt_R \ P) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\prod P \in A \cdot cmt_R \ P)) =$   
 $((\prod P \in A \cdot cmt_R \ P) \triangleleft \$tr' =_u \$tr \triangleright (\prod P \in A \cdot cmt_R \ P)) \diamond (\prod P \in A \cdot cmt_R \ P)$   
 by (*rel-auto*)  
 also have ... =  $((\prod P \in A \cdot peri_R \ P) \triangleleft \$tr' =_u \$tr \triangleright (\prod P \in A \cdot peri_R \ P)) \diamond (\prod P \in A \cdot post_R \ P)$   
 by (*rel-auto*)  
 finally show ?thesis  
 by (*simp add: ExtChoice-def*)  
 qed

**lemma** *extChoice-rdes*:  
 assumes  $\$ok' \# P_1 \ \$ok' \# Q_1$   
 shows  $\mathbf{R}_s(P_1 \vdash P_2) \square \mathbf{R}_s(Q_1 \vdash Q_2) = \mathbf{R}_s ((P_1 \wedge Q_1) \vdash ((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \vee Q_2)))$   
**proof** –  
 have  $(\square i :: nat \in \{0, 1\} \cdot \mathbf{R}_s (P_1 \vdash P_2) \triangleleft \ll i = 0 \gg \triangleright \mathbf{R}_s (Q_1 \vdash Q_2)) = (\square i :: nat \in \{0, 1\} \cdot \mathbf{R}_s ((P_1 \vdash P_2) \triangleleft \ll i = 0 \gg \triangleright (Q_1 \vdash Q_2)))$   
 by (*simp only: RHS-cond R2c-lit*)  
 also have ... =  $(\square i :: nat \in \{0, 1\} \cdot \mathbf{R}_s ((P_1 \triangleleft \ll i = 0 \gg \triangleright Q_1) \vdash (P_2 \triangleleft \ll i = 0 \gg \triangleright Q_2)))$   
 by (*simp add: design-condr*)  
 also have ... =  $\mathbf{R}_s ((P_1 \wedge Q_1) \vdash ((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \vee Q_2)))$   
 apply (*subst ExtChoice-rdes, simp-all add: assms unrest*)  
 apply *unliteralise*  
 apply (*simp add: uinf-or usup-and*)  
 done  
 finally show ?thesis by (*simp add: extChoice-alt-def*)  
 qed

**lemma** *extChoice-tri-rdes*:  
 assumes  $\$ok' \# P_1 \ \$ok' \# Q_1$   
 shows  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \square \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$   
 $\mathbf{R}_s ((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$

**proof** –

**have**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \square \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$   
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash ((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$   
**by** (*simp add: extChoice-rdes assms*)  
**also**  
**have**  $\dots = \mathbf{R}_s((P_1 \wedge Q_1) \vdash ((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$wait' \wedge \$tr' =_u \$tr \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)))$   
**by** (*simp add: conj-comm*)  
**also**  
**have**  $\dots = \mathbf{R}_s((P_1 \wedge Q_1) \vdash$   
 $((P_2 \diamond P_3 \wedge Q_2 \diamond Q_3) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \diamond P_3 \vee Q_2 \diamond Q_3)) \diamond (P_2 \diamond P_3 \vee Q_2 \diamond Q_3))$   
**by** (*simp add: cond-conj wait'-cond-def*)  
**also**  
**have**  $\dots = \mathbf{R}_s((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$   
**by** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
**finally show** *?thesis* .  
**qed**

**lemma** *extChoice-rdes-def*:

**assumes**  $P_1$  is RR  $Q_1$  is RR  
**shows**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \square \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$   
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash (((P_2 \wedge Q_2) \triangleleft \$tr' =_u \$tr \triangleright (P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$   
**by** (*subst extChoice-tri-rdes, simp-all add: assms unrest*)

**lemma** *extChoice-rdes-def' [rdes-def]*:

**assumes**  $P_1$  is RR  $Q_1$  is RR  
**shows**  $\mathbf{R}_s(P_1 \vdash P_2 \diamond P_3) \square \mathbf{R}_s(Q_1 \vdash Q_2 \diamond Q_3) =$   
 $\mathbf{R}_s((P_1 \wedge Q_1) \vdash ((R5(P_2 \wedge Q_2) \vee R4(P_2 \vee Q_2)) \diamond (P_3 \vee Q_3)))$   
**by** (*simp add: extChoice-rdes-def assms, rel-auto, simp-all add: less-le*)

**lemma** *CSP-ExtChoice [closure]*:

$ExtChoice A$  is CSP  
**by** (*simp add: ExtChoice-def RHS-design-is-SRD unrest*)

**lemma** *CSP-extChoice [closure]*:

$P \square Q$  is CSP  
**by** (*simp add: CSP-ExtChoice extChoice-def*)

**lemma** *preR-ExtChoice [rdes]*:

**assumes**  $A \neq \{\}$   $A \subseteq \llbracket CSP \rrbracket_H$   
**shows**  $pre_R(ExtChoice A) = (\bigsqcup P \in A \cdot pre_R(P))$

**proof** –

**have**  $pre_R(ExtChoice A) = (R1 (R2c ((\bigsqcup P \in A \cdot pre_R P))))$   
**by** (*simp add: ExtChoice-def rea-pre-RHS-design usubst unrest*)  
**also from** *assms* **have**  $\dots = (R1 (R2c (\bigsqcup P \in A \cdot (pre_R(CSP(P)))))$   
**by** (*metis USUP-healthy*)  
**also from** *assms* **have**  $\dots = (\bigsqcup P \in A \cdot (pre_R(CSP(P))))$   
**by** (*rel-auto*)  
**also from** *assms* **have**  $\dots = (\bigsqcup P \in A \cdot (pre_R(P)))$   
**by** (*metis USUP-healthy*)  
**finally show** *?thesis* .  
**qed**

**lemma** *preR-ExtChoice-ind [rdes]*:

**assumes**  $A \neq \{\} \wedge P. P \in A \implies F(P)$  is CSP  
**shows**  $pre_R(\bigsqcup P \in A \cdot F(P)) = (\bigsqcup P \in A \cdot pre_R(F(P)))$

using *assms* by (*subst preR-ExtChoice*, *auto*)

lemma *periR-ExtChoice* [*rdes*]:

assumes  $A \subseteq \llbracket \text{NCSP} \rrbracket_H A \neq \{\}$

shows  $\text{peri}_R(\text{ExtChoice } A) = ((\bigsqcup P \in A \cdot \text{pre}_R(P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcap P \in A \cdot \text{peri}_R P)$

proof –

have  $\text{peri}_R(\text{ExtChoice } A) = \text{peri}_R(\mathbf{R}_s((\bigsqcup P \in A \cdot \text{pre}_R P)) \vdash ((\bigsqcup P \in A \cdot \text{peri}_R P) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcap P \in A \cdot \text{peri}_R P)) \diamond (\bigsqcap P \in A \cdot \text{post}_R P))$

by (*simp add: ExtChoice-tri-rdes-def assms closure*)

also have  $\dots = \text{peri}_R(\mathbf{R}_s((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \vdash ((\bigsqcup P \in A \cdot \text{peri}_R(\text{NCSP } P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcap P \in A \cdot \text{peri}_R(\text{NCSP } P)) \diamond (\bigsqcap P \in A \cdot \text{post}_R P)))$

by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

also have  $\dots = R1(R2c((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R(\text{NCSP } P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcap P \in A \cdot \text{peri}_R(\text{NCSP } P))))$

proof –

have  $(\bigsqcup P \in A \cdot [\$ok \mapsto_s \text{true}, \$ok' \mapsto_s \text{true}, \$wait \mapsto_s \text{false}, \$wait' \mapsto_s \text{true}] \dagger \text{pre}_R(\text{NCSP } P)) = (\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P))$

by (*rule USUP-cong, simp add: closure usubst unrest assms*)

thus *?thesis*

by (*simp add: rea-peri-RHS-design Healthy-Idempotent SRD-Idempotent usubst unrest assms*)

qed

also have  $\dots = R1((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R(\text{NCSP } P)) \triangleleft \$tr' =_u \$tr \triangleright (\bigsqcap P \in A \cdot \text{peri}_R(\text{NCSP } P)))$

by (*simp add: R2c-rea-impl R2c-condr R2c-UINF R2c-preR R2c-periR R2c-tr'-minus-tr R2c-USUP closure*)

also have  $\dots = (((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R(\text{NCSP } P))) \triangleleft \$tr' =_u \$tr \triangleright ((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\bigsqcap P \in A \cdot \text{peri}_R(\text{NCSP } P))))$

by (*simp add: R1-rea-impl R1-cond R1-USUP R1-UINF assms Healthy-if closure, rel-auto*)

also have  $\dots = (((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R(\text{NCSP } P))) \triangleleft \$tr' =_u \$tr \triangleright ((\bigsqcap P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r \text{peri}_R(\text{NCSP } P)))$

by (*simp add: UINF-rea-impl[THEN sym]*)

also have  $\dots = (((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R(\text{NCSP } P))) \triangleleft \$tr' =_u \$tr \triangleright ((\bigsqcap P \in A \cdot \text{peri}_R(\text{NCSP } P))))$

by (*simp add: SRD-peri-under-pre closure assms unrest*)

also have  $\dots = (((\bigsqcup P \in A \cdot \text{pre}_R P) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R P)) \triangleleft \$tr' =_u \$tr \triangleright ((\bigsqcap P \in A \cdot \text{peri}_R P)))$

by (*simp add: UINF-healthy[OF assms(1), THEN sym] USUP-healthy[OF assms(1), THEN sym]*)

finally show *?thesis* .

qed

lemma *periR-ExtChoice'*:

assumes  $A \subseteq \llbracket \text{NCSP} \rrbracket_H A \neq \{\}$



**shows**  $\text{peri}_R(\text{ExtChoice } A) = (R5((\bigsqcup P \in A \cdot \text{pre}_R(P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R P)) \vee (\prod P \in A \cdot R4(\text{peri}_R P)))$   
**using**  $\text{assms}(2)$   
**by** ( $\text{simp add: periR-ExtChoice assms}(1), \text{rel-auto}$ )

**lemma**  $\text{periR-ExtChoice-ind}$  [ $\text{rdes}$ ]:

**assumes**  $\bigwedge P. P \in A \implies F(P)$  is NCSP  $A \neq \{\}$   
**shows**  $\text{peri}_R(\bigsqcup P \in A \cdot F(P)) = ((\bigsqcup P \in A \cdot \text{pre}_R(F P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R (F P))) \triangleleft \$tr' =_u \$tr \triangleright (\prod P \in A \cdot \text{peri}_R (F P))$   
**using**  $\text{assms}$  **by** ( $\text{subst periR-ExtChoice}, \text{auto simp add: closure unrest}$ )

**lemma**  $\text{periR-ExtChoice-ind}'$ :

**assumes**  $\bigwedge P. P \in A \implies F(P)$  is NCSP  $A \neq \{\}$   
**shows**  $\text{peri}_R(\bigsqcup P \in A \cdot F(P)) = (R5((\bigsqcup P \in A \cdot \text{pre}_R(F P)) \Rightarrow_r (\bigsqcup P \in A \cdot \text{peri}_R (F P))) \vee (\prod P \in A \cdot R4(\text{peri}_R (F P))))$   
**using**  $\text{assms}$  **by** ( $\text{subst periR-ExtChoice}', \text{auto simp add: closure unrest}$ )

**lemma**  $\text{postR-ExtChoice}$  [ $\text{rdes}$ ]:

**assumes**  $A \subseteq \llbracket \text{NCSP} \rrbracket_H A \neq \{\}$   
**shows**  $\text{post}_R(\text{ExtChoice } A) = (\prod P \in A \cdot \text{post}_R P)$

**proof** –

**have**  $\text{post}_R(\text{ExtChoice } A) = \text{post}_R(\mathbf{R}_s((\bigsqcup P \in A \cdot \text{pre}_R P) \vdash ((\bigsqcup P \in A \cdot \text{peri}_R P) \triangleleft \$tr' =_u \$tr \triangleright (\prod P \in A \cdot \text{peri}_R P)) \diamond (\prod P \in A \cdot \text{post}_R P)))$   
**by** ( $\text{simp add: ExtChoice-tri-rdes-def closure assms}$ )

**also have**  $\dots = \text{post}_R(\mathbf{R}_s((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \vdash ((\bigsqcup P \in A \cdot \text{peri}_R P) \triangleleft \$tr' =_u \$tr \triangleright (\prod P \in A \cdot \text{peri}_R P)) \diamond (\prod P \in A \cdot \text{post}_R(\text{NCSP } P))))$   
**by** ( $\text{simp add: UINF-healthy[OF assms}(1), \text{THEN sym}] \text{USUP-healthy[OF assms}(1), \text{THEN sym}]$ )

**also have**  $\dots = R1(R2c((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\prod P \in A \cdot \text{post}_R(\text{NCSP } P))))$

**proof** –

**have**  $(\bigsqcup P \in A \cdot [\text{\$ok} \mapsto_s \text{true}, \text{\$ok}' \mapsto_s \text{true}, \text{\$wait} \mapsto_s \text{false}, \text{\$wait}' \mapsto_s \text{false}] \dagger \text{pre}_R(\text{NCSP } P)) = (\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P))$   
**by** ( $\text{rule USUP-cong}, \text{simp add: usubst closure unrest assms}$ )  
**thus**  $?thesis$   
**by** ( $\text{simp add: rea-post-RHS-design Healthy-Idempotent SRD-Idempotent usubst unrest assms}$ )

**qed**

**also have**  $\dots = R1((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\prod P \in A \cdot \text{post}_R(\text{NCSP } P)))$

**by** ( $\text{simp add: R2c-rea-impl R2c-condr R2c-UINF R2c-preR R2c-postR R2c-tr'-minus-tr R2c-USUP closure}$ )

**also from**  $\text{assms}(2)$  **have**  $\dots = ((\bigsqcup P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r (\prod P \in A \cdot \text{post}_R(\text{NCSP } P)))$

**by** ( $\text{simp add: R1-rea-impl R1-cond R1-USUP R1-UINF assms Healthy-if closure}$ )

**also have**  $\dots = (\prod P \in A \cdot \text{pre}_R(\text{NCSP } P)) \Rightarrow_r \text{post}_R(\text{NCSP } P)$

**by** ( $\text{simp add: UINF-rea-impl}$ )

**also have**  $\dots = (\prod P \in A \cdot \text{post}_R(\text{NCSP } P))$

**by** ( $\text{simp add: SRD-post-under-pre closure assms unrest}$ )

**finally show**  $?thesis$

**by** ( $\text{simp add: UINF-healthy[OF assms}(1), \text{THEN sym}] \text{USUP-healthy[OF assms}(1), \text{THEN sym}]$ )

**qed**

**lemma**  $\text{postR-ExtChoice-ind}$  [ $\text{rdes}$ ]:

**assumes**  $\bigwedge P. P \in A \implies F(P)$  is NCSP  $A \neq \{\}$   
**shows**  $\text{post}_R(\bigsqcup P \in A \cdot F(P)) = (\prod P \in A \cdot \text{post}_R(F(P)))$

**using** *assms* **by** (*subst postR-ExtChoice*, *auto simp add: closure unrest*)

**lemma** *preR-extChoice*:

**assumes** *P is CSP Q is CSP*  $\$wait' \# pre_R(P) \$wait' \# pre_R(Q)$   
**shows**  $pre_R(P \sqcap Q) = (pre_R(P) \wedge pre_R(Q))$   
**by** (*simp add: extChoice-def preR-ExtChoice assms usup-and*)

**lemma** *preR-extChoice' [rdes]*:

**assumes** *P is NCSP Q is NCSP*  
**shows**  $pre_R(P \sqcap Q) = (pre_R(P) \wedge pre_R(Q))$   
**by** (*simp add: preR-extChoice closure assms unrest*)

**lemma** *periR-extChoice [rdes]*:

**assumes** *P is NCSP Q is NCSP*  
**shows**  $peri_R(P \sqcap Q) = ((pre_R(P) \wedge pre_R(Q)) \Rightarrow_r peri_R(P) \wedge peri_R(Q)) \triangleleft \$tr' =_u \$tr \triangleright (peri_R(P) \vee peri_R(Q))$   
**using** *assms*  
**by** (*simp add: extChoice-def, subst periR-ExtChoice, auto simp add: usup-and uinf-or*)

**lemma** *postR-extChoice [rdes]*:

**assumes** *P is NCSP Q is NCSP*  
**shows**  $post_R(P \sqcap Q) = (post_R(P) \vee post_R(Q))$   
**using** *assms*  
**by** (*simp add: extChoice-def, subst postR-ExtChoice, auto simp add: usup-and uinf-or*)

**lemma** *ExtChoice-cong*:

**assumes**  $\bigwedge P. P \in A \implies F(P) = G(P)$   
**shows**  $(\sqcap P \in A \cdot F(P)) = (\sqcap P \in A \cdot G(P))$   
**using** *assms image-cong* **by** *force*

**lemma** *ref-unrest-ExtChoice*:

**assumes**  
 $\bigwedge P. P \in A \implies \$ref \# pre_R(P)$   
 $\bigwedge P. P \in A \implies \$ref \# cmt_R(P)$   
**shows**  $\$ref \# (ExtChoice A) \llbracket false / \$wait \rrbracket$

**proof** –

**have**  $\bigwedge P. P \in A \implies \$ref \# pre_R(P \llbracket 0 / \$tr \rrbracket)$   
**using** *assms* **by** (*rel-blast*)

**with** *assms* **show** *?thesis*

**by** (*simp add: ExtChoice-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest*)

**qed**

**lemma** *CSP4-ExtChoice*:

**assumes**  $A \subseteq \llbracket NCSP \rrbracket_H$   
**shows** *ExtChoice A is CSP4*

**proof** (*cases A = {}*)

**case** *True* **thus** *?thesis*

**by** (*simp add: ExtChoice-empty Healthy-def CSP4-def, simp add: Skip-is-CSP Stop-left-zero*)

**next**

**case** *False*

**have**  $1: (\neg_r (\neg_r pre_R (ExtChoice A)) ;;_h R1 true) = pre_R (ExtChoice A)$

**proof** –

**have**  $\bigwedge P. P \in A \implies (\neg_r pre_R(P)) ;; R1 true = (\neg_r pre_R(P))$

**by** (*simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-neg-pre-unit assms*)

**thus** *?thesis*

```

apply (simp add: False preR-ExtChoice closure NCSP-set-unrest-pre-wait' assms not-UINF seq-UINF-distr
not-USUP)
  apply (rule USUP-cong)
  apply (simp add: rpred assms closure)
  done
qed
have 2: $st' \# peri_R (ExtChoice A)
proof -
  have a:  $\bigwedge P. P \in A \implies \$st' \# pre_R(P)$ 
    by (simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-st'-unrest-pre assms)
  have b:  $\bigwedge P. P \in A \implies \$st' \# peri_R(P)$ 
    by (simp add: NCSP-Healthy-subset-member NCSP-implies-NSRD NSRD-st'-unrest-peri assms)
  from a b show ?thesis
  apply (subst periR-ExtChoice)
  apply (simp-all add: assms closure unrest CSP4-set-unrest-pre-st' NCSP-set-unrest-pre-wait'
False)
  done
qed
have 3: $ref' \# post_R (ExtChoice A)
proof -
  have a:  $\bigwedge P. P \in A \implies \$ref' \# pre_R(P)$ 
    by (simp add: CSP4-ref'-unrest-pre CSP-Healthy-subset-member NCSP-Healthy-subset-member
NCSP-implies-CSP4 NCSP-subset-implies-CSP assms)
  have b:  $\bigwedge P. P \in A \implies \$ref' \# post_R(P)$ 
    by (simp add: CSP4-ref'-unrest-post CSP-Healthy-subset-member NCSP-Healthy-subset-member
NCSP-implies-CSP4 NCSP-subset-implies-CSP assms)
  from a b show ?thesis
  by (subst postR-ExtChoice, simp-all add: assms CSP4-set-unrest-pre-st' NCSP-set-unrest-pre-wait'
unrest False)
qed
show ?thesis
  by (rule CSP4-tri-intro, simp-all add: 1 2 3 assms closure)
  (metis 1 R1-seqr-closure rea-not-R1 rea-not-not rea-true-R1)
qed

```

```

lemma CSP4-extChoice [closure]:
  assumes P is NCSP Q is NCSP
  shows P  $\square$  Q is CSP4
  by (simp add: extChoice-def, rule CSP4-ExtChoice, simp-all add: assms)

```

```

lemma NCSP-ExtChoice [closure]:
  assumes A  $\subseteq \llbracket NCSP \rrbracket_H$ 
  shows ExtChoice A is NCSP
proof (cases A = {})
  case True
  then show ?thesis by (simp add: ExtChoice-empty closure)
next
  case False
  show ?thesis
proof (rule NCSP-intro)
  from assms have cls: A  $\subseteq \llbracket CSP \rrbracket_H$  A  $\subseteq \llbracket CSP3 \rrbracket_H$  A  $\subseteq \llbracket CSP4 \rrbracket_H$ 
  using NCSP-implies-CSP NCSP-implies-CSP3 NCSP-implies-CSP4 by blast+
  have wu:  $\bigwedge P. P \in A \implies \$wait' \# pre_R(P)$ 
  using NCSP-implies-NSRD NSRD-wait'-unrest-pre assms by force
  show 1:ExtChoice A is CSP

```

by (metis (mono-tags) Ball-Collect CSP-ExtChoice NCSP-implies-CSP assms)  
 from cls show ExtChoice A is CSP3  
 by (rule-tac CSP3-SRD-intro, simp-all add: CSP-Healthy-subset-member CSP3-Healthy-subset-member  
 closure rdes unrest wu assms 1 False)  
 from cls show ExtChoice A is CSP4  
 by (simp add: CSP4-ExtChoice assms)  
 qed  
 qed

**lemma** ExtChoice-NCSP-closed [closure]:  
 assumes  $\bigwedge i. i \in I \implies P(i)$  is NCSP  
 shows  $(\bigcap_{i \in I} P(i))$  is NCSP  
 by (simp add: NCSP-ExtChoice assms image-subset-iff)

**lemma** NCSP-extChoice [closure]:  
 assumes P is NCSP Q is NCSP  
 shows  $P \sqcap Q$  is NCSP  
 by (simp add: NCSP-ExtChoice assms extChoice-def)

## 7.5 Productivity and Guardedness

**lemma** Productive-ExtChoice [closure]:  
 assumes  $A \neq \{\}$   $A \subseteq \llbracket \text{NCSP} \rrbracket_H$   $A \subseteq \llbracket \text{Productive} \rrbracket_H$   
 shows ExtChoice A is Productive

**proof** –

have 1:  $\bigwedge P. P \in A \implies \$wait' \# pre_R(P)$   
 using NCSP-implies-NSRD NSRD-wait'-unrest-pre assms(2) by blast  
 show ?thesis

**proof** (rule Productive-intro, simp-all add: assms closure rdes 1 unrest)

have  $((\bigcap_{P \in A} pre_R P) \wedge (\bigcap_{P \in A} post_R P)) =$   
 $((\bigcap_{P \in A} pre_R P) \wedge (\bigcap_{P \in A} (pre_R P \wedge post_R P)))$   
 by (rel-auto)

**moreover** have  $(\bigcap_{P \in A} (pre_R P \wedge post_R P)) = (\bigcap_{P \in A} (pre_R P \wedge post_R P) \wedge \$tr <_u$   
 $\$tr')$

by (rule UINF-cong, metis (no-types, lifting) 1 Ball-Collect NCSP-implies-CSP Productive-post-refines-tr-increase  
 assms utp-pred-laws.inf.absorb1)

**ultimately** show  $(\$tr' >_u \$tr) \sqsubseteq ((\bigcap_{P \in A} pre_R P) \wedge ((\bigcap_{P \in A} post_R P)))$   
 by (rel-auto)

qed  
 qed

**lemma** Productive-extChoice [closure]:  
 assumes P is NCSP Q is NCSP P is Productive Q is Productive  
 shows  $P \sqcap Q$  is Productive  
 by (simp add: extChoice-def Productive-ExtChoice assms)

**lemma** ExtChoice-Guarded [closure]:  
 assumes  $\bigwedge P. P \in A \implies \text{Guarded } P$   
 shows Guarded  $(\lambda X. \bigcap_{P \in A} P(X))$

**proof** (rule GuardedI)

fix X n

have  $\bigwedge Y. ((\bigcap_{P \in A} P Y) \wedge gvert(n+1)) = ((\bigcap_{P \in A} (P Y \wedge gvert(n+1))) \wedge gvert(n+1))$

**proof** –

fix Y

let ?lhs =  $((\bigcap_{P \in A} P Y) \wedge gvert(n+1))$  and ?rhs =  $((\bigcap_{P \in A} (P Y \wedge gvert(n+1))) \wedge gvert(n+1))$

**have**  $a: ?lhs \llbracket false/\$ok \rrbracket = ?rhs \llbracket false/\$ok \rrbracket$   
**by** (*rel-auto*)  
**have**  $b: ?lhs \llbracket true/\$ok \rrbracket \llbracket true/\$wait \rrbracket = ?rhs \llbracket true/\$ok \rrbracket \llbracket true/\$wait \rrbracket$   
**by** (*rel-auto*)  
**have**  $c: ?lhs \llbracket true/\$ok \rrbracket \llbracket false/\$wait \rrbracket = ?rhs \llbracket true/\$ok \rrbracket \llbracket false/\$wait \rrbracket$   
**by** (*simp add: ExtChoice-def RHS-def R1-def R2c-def R2s-def R3h-def design-def usubst unrest, rel-blast*)  
**show**  $?lhs = ?rhs$   
**using**  $a\ b\ c$   
**by** (*rule-tac bool-eq-splitI[of in-var ok], simp, rule-tac bool-eq-splitI[of in-var wait], simp-all*)  
**qed**  
**moreover have**  $((\Box P \in A \cdot (P\ X \wedge gvirt(n+1))) \wedge gvirt(n+1)) = ((\Box P \in A \cdot (P\ (X \wedge gvirt(n)) \wedge gvirt(n+1))) \wedge gvirt(n+1))$   
**proof** –  
**have**  $(\Box P \in A \cdot (P\ X \wedge gvirt(n+1))) = (\Box P \in A \cdot (P\ (X \wedge gvirt(n)) \wedge gvirt(n+1)))$   
**proof** (*rule ExtChoice-cong*)  
**fix**  $P$  **assume**  $P \in A$   
**thus**  $(P\ X \wedge gvirt(n+1)) = (P\ (X \wedge gvirt(n)) \wedge gvirt(n+1))$   
**using** *Guarded-def assms by blast*  
**qed**  
**thus**  $?thesis$  **by** *simp*  
**qed**  
**ultimately show**  $((\Box P \in A \cdot P\ X) \wedge gvirt(n+1)) = ((\Box P \in A \cdot (P\ (X \wedge gvirt(n)))) \wedge gvirt(n+1))$   
**by** *simp*  
**qed**

**lemma** *extChoice-Guarded* [*closure*]:  
**assumes** *Guarded P Guarded Q*  
**shows** *Guarded*  $(\lambda X. P(X) \Box Q(X))$   
**proof** –  
**have** *Guarded*  $(\lambda X. \Box F \in \{P, Q\} \cdot F(X))$   
**by** (*rule ExtChoice-Guarded, auto simp add: assms*)  
**thus**  $?thesis$   
**by** (*simp add: extChoice-def*)  
**qed**

## 7.6 Algebraic laws

**lemma** *extChoice-comm*:  
 $P \Box Q = Q \Box P$   
**by** (*unfold extChoice-def, simp add: insert-commute*)

**lemma** *extChoice-idem*:  
 $P$  *is CSP*  $\implies P \Box P = P$   
**by** (*unfold extChoice-def, simp add: ExtChoice-single*)

**lemma** *extChoice-assoc*:  
**assumes**  $P$  *is CSP*  $Q$  *is CSP*  $R$  *is CSP*  
**shows**  $P \Box Q \Box R = P \Box (Q \Box R)$   
**proof** –  
**have**  $P \Box Q \Box R = \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) \Box \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{cmt}_R(Q)) \Box \mathbf{R}_s(\text{pre}_R(R) \vdash \text{cmt}_R(R))$   
**by** (*simp add: SRD-reactive-design-alt assms(1) assms(2) assms(3)*)  
**also have**  $\dots =$   
 $\mathbf{R}_s(((\text{pre}_R P \wedge \text{pre}_R Q) \wedge \text{pre}_R R) \vdash$   
 $((\text{cmt}_R P \wedge \text{cmt}_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\text{cmt}_R P \vee \text{cmt}_R Q) \wedge \text{cmt}_R R)$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$

$((cmt_R P \wedge cmt_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R P \vee cmt_R Q) \vee cmt_R R))$   
 by (*simp add: extChoice-rdes unrest*)  
 also have ... =  
 $\mathbf{R}_s (((pre_R P \wedge pre_R Q) \wedge pre_R R) \vdash$   
 $((cmt_R P \wedge cmt_R Q) \wedge cmt_R R)$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $((cmt_R P \vee cmt_R Q) \vee cmt_R R)))$   
 by (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
 also have ... =  
 $\mathbf{R}_s ((pre_R P \wedge pre_R Q \wedge pre_R R) \vdash$   
 $((cmt_R P \wedge (cmt_R Q \wedge cmt_R R))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(cmt_R P \vee (cmt_R Q \vee cmt_R R))))$   
 by (*simp add: conj-assoc disj-assoc*)  
 also have ... =  
 $\mathbf{R}_s ((pre_R P \wedge pre_R Q \wedge pre_R R) \vdash$   
 $((cmt_R P \wedge (cmt_R Q \wedge cmt_R R) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R Q \vee cmt_R R))$   
 $\triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright$   
 $(cmt_R P \vee (cmt_R Q \wedge cmt_R R) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (cmt_R Q \vee cmt_R R))))$   
 by (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
 also have ... =  $\mathbf{R}_s(pre_R(P) \vdash cmt_R(P)) \square (\mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q)) \square \mathbf{R}_s(pre_R(R) \vdash cmt_R(R)))$   
 by (*simp add: extChoice-rdes unrest*)  
 also have ... =  $P \square (Q \square R)$   
 by (*simp add: SRD-reactive-design-alt assms(1) assms(2) assms(3)*)  
 finally show ?thesis .  
 qed

**lemma** *extChoice-Stop:*

assumes *Q is CSP*

shows  $Stop \square Q = Q$

using *assms*

**proof** –

have  $Stop \square Q = \mathbf{R}_s (true \vdash (\$tr' =_u \$tr \wedge \$wait')) \square \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q))$

by (*simp add: Stop-def SRD-reactive-design-alt assms*)

also have ... =  $\mathbf{R}_s (pre_R Q \vdash (((\$tr' =_u \$tr \wedge \$wait') \wedge cmt_R Q) \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright (\$tr' =_u \$tr \wedge \$wait' \vee cmt_R Q)))$

by (*simp add: extChoice-rdes unrest*)

also have ... =  $\mathbf{R}_s (pre_R Q \vdash (cmt_R Q \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright cmt_R Q))$

by (*metis (no-types, lifting) cond-def eq-upred-sym neg-conj-cancell1 utp-pred-laws.inf.left-idem*)

also have ... =  $\mathbf{R}_s (pre_R Q \vdash cmt_R Q)$

by (*simp add: cond-idem*)

also have ... =  $Q$

by (*simp add: SRD-reactive-design-alt assms*)

finally show ?thesis .

qed

**lemma** *extChoice-Chaos:*

assumes *Q is CSP*

shows  $Chaos \square Q = Chaos$

**proof** –

have  $Chaos \square Q = \mathbf{R}_s (false \vdash true) \square \mathbf{R}_s(pre_R(Q) \vdash cmt_R(Q))$

by (*simp add: Chaos-def SRD-reactive-design-alt assms*)

also have ... =  $\mathbf{R}_s (false \vdash (cmt_R Q \triangleleft \$tr' =_u \$tr \wedge \$wait' \triangleright true))$

by (*simp add: extChoice-rdes unrest*)

also have ... =  $\mathbf{R}_s (false \vdash true)$

by (rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto)  
 also have ... = Chaos  
 by (simp add: Chaos-def)  
 finally show ?thesis .  
 qed

lemma extChoice-Dist:

assumes  $P$  is CSP  $S \subseteq \llbracket \text{CSP} \rrbracket_H S \neq \{\}$   
 shows  $P \sqcap (\prod S) = (\prod_{Q \in S} P \sqcap Q)$

proof –

let  $?S1 = \text{pre}_R \text{ ` } S$  and  $?S2 = \text{cmt}_R \text{ ` } S$

have  $P \sqcap (\prod S) = P \sqcap (\prod_{Q \in S} \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{cmt}_R(Q)))$

by (simp add: SRD-as-reactive-design[THEN sym] Healthy-SUPREMUM UINF-as-Sup-collect assms)

also have ... =  $\mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) \sqcap \mathbf{R}_s(\prod_{Q \in S} \text{pre}_R(Q) \vdash \prod_{Q \in S} \text{cmt}_R(Q))$

by (simp add: RHS-design-USUP SRD-reactive-design-alt assms)

also have ... =  $\mathbf{R}_s((\text{pre}_R(P) \wedge (\prod_{Q \in S} \text{pre}_R(Q))) \vdash$   
 $((\text{cmt}_R(P) \wedge (\prod_{Q \in S} \text{cmt}_R(Q)))$   
 $\triangleleft \text{\$tr}' =_u \text{\$tr} \wedge \text{\$wait}' \triangleright$   
 $(\text{cmt}_R(P) \vee (\prod_{Q \in S} \text{cmt}_R(Q))))$

by (simp add: extChoice-rdes unrest)

also have ... =  $\mathbf{R}_s((\prod_{Q \in S} \text{pre}_R P \wedge \text{pre}_R Q) \vdash$   
 $(\prod_{Q \in S} \text{cmt}_R P \wedge \text{cmt}_R Q) \triangleleft \text{\$tr}' =_u \text{\$tr} \wedge \text{\$wait}' \triangleright (\text{cmt}_R P \vee \text{cmt}_R Q))$

by (simp add: conj-USUP-dist conj-UINF-dist disj-UINF-dist cond-UINF-dist assms)

also have ... =  $(\prod_{Q \in S} \mathbf{R}_s((\text{pre}_R P \wedge \text{pre}_R Q) \vdash$   
 $((\text{cmt}_R P \wedge \text{cmt}_R Q) \triangleleft \text{\$tr}' =_u \text{\$tr} \wedge \text{\$wait}' \triangleright (\text{cmt}_R P \vee \text{cmt}_R Q))))$

by (simp add: assms RHS-design-USUP)

also have ... =  $(\prod_{Q \in S} \mathbf{R}_s(\text{pre}_R(P) \vdash \text{cmt}_R(P)) \sqcap \mathbf{R}_s(\text{pre}_R(Q) \vdash \text{cmt}_R(Q)))$

by (simp add: extChoice-rdes unrest)

also have ... =  $(\prod_{Q \in S} P \sqcap \text{CSP}(Q))$

by (simp add: UINF-as-Sup-collect, metis (no-types, lifting) Healthy-if SRD-as-reactive-design assms(1))

also have ... =  $(\prod_{Q \in S} P \sqcap Q)$

by (rule SUP-cong, simp-all add: Healthy-subset-member[OF assms(2)])

finally show ?thesis .

qed

lemma extChoice-dist:

assumes  $P$  is CSP  $Q$  is CSP  $R$  is CSP

shows  $P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap (P \sqcap R)$

using assms extChoice-Dist[of  $P \{Q, R\}$ ] by simp

lemma ExtChoice-seq-distr:

assumes  $\bigwedge i. i \in A \implies P i$  is PCSP  $Q$  is NCSP

shows  $(\prod_{i \in A} P i) ;; Q = (\prod_{i \in A} P i ;; Q)$

proof (cases  $A = \{\}$ )

case True

then show ?thesis

by (simp add: ExtChoice-empty NCSP-implies-CSP Stop-left-zero assms(2))

next

case False

show ?thesis

proof –

have  $1: (\prod_{i \in A} P i) = (\prod_{i \in A} (\mathbf{R}_s((\text{pre}_R(P i)) \vdash \text{peri}_R(P i) \diamond (R4(\text{post}_R(P i))))))$   
 (is  $?X = ?Y$ )

by (rule ExtChoice-cong, metis (no-types, hide-lams) R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP)

*Productive-form assms(1) comp-apply*  
**have**  $2: (\Box i \in A \cdot P i ;; Q) = (\Box i \in A \cdot (\mathbf{R}_s ((pre_R (P i)) \vdash peri_R (P i) \diamond (R4(post_R (P i)))))) ;; Q)$   
**(is**  $?X = ?Y$   
**by** (rule *ExtChoice-cong*, *metis* (no-types, hide-lams) *R4-def Healthy-if NCSP-implies-CSP PCSP-implies-NCSP*  
*Productive-form assms(1) comp-apply*)  
**show** *?thesis*  
**by** (simp add: 1 2, rdes-eq cls: *assms False cong: ExtChoice-cong USUP-cong*)  
**qed**  
**qed**

**lemma** *extChoice-seq-distr*:  
**assumes** *P is PCSP Q is PCSP R is NCSP*  
**shows**  $(P \Box Q) ;; R = (P ;; R \Box Q ;; R)$   
**by** (rdes-eq cls: *assms*)

**lemma** *extChoice-seq-distl*:  
**assumes** *P is ICSP Q is ICSP R is NCSP*  
**shows**  $P ;; (Q \Box R) = (P ;; Q \Box P ;; R)$   
**by** (rdes-eq cls: *assms*)

**end**

## 8 Stateful-Failure Programs

**theory** *utp-sfrd-prog*  
**imports**  
*utp-sfrd-extchoice*  
**begin**

### 8.1 Conditionals

**lemma** *NCSP-cond-srea* [*closure*]:  
**assumes** *P is NCSP Q is NCSP*  
**shows**  $P \triangleleft b \triangleright_R Q$  *is NCSP*  
**by** (rule *NCSP-NSRD-intro*, simp-all add: *closure rdes assms unrest*)

### 8.2 Guarded commands

**lemma** *GuardedCommR-NCSP-closed* [*closure*]:  
**assumes** *P is NCSP*  
**shows**  $g \rightarrow_R P$  *is NCSP*  
**by** (simp add: *gcmd-def closure assms*)

### 8.3 Alternation

**lemma** *AlternateR-NCSP-closed* [*closure*]:  
**assumes**  $\bigwedge i. i \in A \implies P(i)$  *is NCSP* *Q is NCSP*  
**shows**  $(if_R i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q fi)$  *is NCSP*  
**proof** (cases  $A = \{\}$ )  
**case** *True*  
**then show** *?thesis*  
**by** (simp add: *assms*)  
**next**  
**case** *False*  
**then show** *?thesis*



by (*simp add: AlternateR-def closure assms*)  
qed

## 8.4 While Loops

**lemma** *NSRD-coerce-NCSP*:

*P is NSRD  $\implies$  Skip ;; P ;; Skip is NCSP*

by (*metis (no-types, hide-lams) CSP3-Skip CSP3-def CSP4-def Healthy-def NCSP-Skip NCSP-implies-CSP NCSP-intro NSRD-is-SRD RA1 SRD-seqr-closure*)

**definition** *WhileC* :: *'s upred  $\implies$  ('s, 'e) action  $\implies$  ('s, 'e) action (while<sub>C</sub> - do - od) where*  
*while<sub>C</sub> b do P od = Skip ;; while<sub>R</sub> b do P od ;; Skip*

**lemma** *WhileC-NCSP-closed* [closure]:

**assumes** *P is NCSP P is Productive*

**shows** *while<sub>C</sub> b do P od is NCSP*

by (*simp add: WhileC-def NSRD-coerce-NCSP assms closure*)

## 8.5 Assignment

**definition** *AssignsCSP* :: *'σ usubst  $\implies$  ('σ, 'φ) action ( $\langle\cdot\rangle_C$ ) where*  
*[upred-defs]: AssignsCSP σ =  $\mathbf{R}_s(\text{true} \vdash \text{false} \diamond (\$tr' =_u \$tr \wedge [\langle\sigma\rangle_a]_s))$*

**syntax**

*-assigns-csp* :: *svids  $\implies$  uexprs  $\implies$  logic ('(-) :=<sub>C</sub> '(-))*

*-assigns-csp* :: *svids  $\implies$  uexprs  $\implies$  logic (**infixr** :=<sub>C</sub> 90)*

**translations**

*-assigns-csp xs vs  $\implies$  CONST AssignsCSP (-mk-usubst (CONST id) xs vs)*

*-assigns-csp x v  $\leq$  CONST AssignsCSP (CONST subst-upd (CONST id) x v)*

*-assigns-csp x v  $\leq$  -assigns-csp (-spvar x) v*

*x, y :=<sub>C</sub> u, v  $\leq$  CONST AssignsCSP (CONST subst-upd (CONST subst-upd (CONST id) (CONST svar x) u) (CONST svar y) v)*

**lemma** *preR-AssignsCSP* [rdes]: *pre<sub>R</sub>( $\langle\sigma\rangle_C$ ) = true<sub>r</sub>*

by (*rel-auto*)

**lemma** *periR-AssignsCSP* [rdes]: *peri<sub>R</sub>( $\langle\sigma\rangle_C$ ) = false*

by (*rel-auto*)

**lemma** *postR-AssignsCSP* [rdes]: *post<sub>R</sub>( $\langle\sigma\rangle_C$ ) =  $\Phi(\text{true}, \sigma, \langle\rangle)$*

by (*rel-auto*)

**lemma** *AssignsCSP-rdes-def* [rdes-def] :  *$\langle\sigma\rangle_C = \mathbf{R}_s(\text{true}_r \vdash \text{false} \diamond \Phi(\text{true}, \sigma, \langle\rangle))$*

by (*rel-auto*)

**lemma** *AssignsCSP-CSP* [closure]:  *$\langle\sigma\rangle_C$  is CSP*

by (*simp add: AssignsCSP-def RHS-tri-design-is-SRD unrest*)

**lemma** *AssignsCSP-CSP3* [closure]:  *$\langle\sigma\rangle_C$  is CSP3*

by (*rule CSP3-intro, simp add: closure, rel-auto*)

**lemma** *AssignsCSP-CSP4* [closure]:  *$\langle\sigma\rangle_C$  is CSP4*

by (*rule CSP4-intro, simp add: closure, rel-auto+*)

**lemma** *AssignsCSP-NCSP* [closure]:  *$\langle\sigma\rangle_C$  is NCSP*

by (simp add: AssignsCSP-CSP AssignsCSP-CSP3 AssignsCSP-CSP4 NCSP-intro)

**lemma** *AssignsCSP-ICSP* [closure]:  $\langle \sigma \rangle_C$  is ICSP  
**apply** (rule ICSP-intro, simp add: closure, simp add: rdes-def)  
**apply** (rule ISRD1-rdes-intro)  
**apply** (simp-all add: closure)  
**apply** (rel-auto)  
**done**

## 8.6 Assignment with update

There are different collections that we would like to assign to, but they all have different types and perhaps more importantly different conditions on the update being well defined. For example, for a list well-definedness equates to the index being less than the length etc. Thus we here set up a polymorphic constant for CSP assignment updates that can be specialised to different types.

**definition** *AssignCSP-update* ::  
 $(f \Rightarrow k \text{ set}) \Rightarrow (f \Rightarrow k \Rightarrow v \Rightarrow f) \Rightarrow (f \Longrightarrow \sigma) \Rightarrow$   
 $(k, \sigma) \text{ uexpr} \Rightarrow (v, \sigma) \text{ uexpr} \Rightarrow (\sigma, \varphi) \text{ action}$  **where**  
[upred-defs, rdes-def]: *AssignCSP-update* domf updatef x k v =  
 $\mathbf{R}_s([k \in_u \text{uop domf } (\&x)]_{S<} \vdash \text{false} \diamond \Phi(\text{true}, [x \mapsto_s \text{trop updatef } (\&x) k v], \langle \rangle))$

All different assignment updates have the same syntax; the type resolves which implementation to use.

**syntax**

-csp-assign-upd :: *svid*  $\Rightarrow$  *logic*  $\Rightarrow$  *logic*  $\Rightarrow$  *logic* (-[-] :=<sub>C</sub> - [0,0,72] 72)

**translations**

$x[k] :=_C v == \text{CONST } \text{AssignCSP-update } (\text{CONST } \text{uom}) (\text{CONST } \text{uupd}) x k v$

**lemma** *AssignCSP-update-CSP* [closure]:  
*AssignCSP-update* domf updatef x k v is CSP  
**by** (simp add: AssignCSP-update-def RHS-tri-design-is-SRD unrest)

**lemma** *preR-AssignCSP-update* [rdes]:  
 $\text{pre}_R(\text{AssignCSP-update domf updatef } x k v) = [k \in_u \text{uop domf } (\&x)]_{S<}$   
**by** (rel-auto)

**lemma** *periR-AssignCSP-update* [rdes]:  
 $\text{peri}_R(\text{AssignCSP-update domf updatef } x k v) = [k \notin_u \text{uop domf } (\&x)]_{S<}$   
**by** (rel-simp)

**lemma** *post-AssignCSP-update* [rdes]:  
 $\text{post}_R(\text{AssignCSP-update domf updatef } x k v) =$   
 $(\Phi(\text{true}, [x \mapsto_s \text{trop updatef } (\&x) k v], \langle \rangle) \triangleleft k \in_u \text{uop domf } (\&x) \triangleright_R R1(\text{true}))$   
**by** (rel-auto)

**lemma** *AssignCSP-update-NCSP* [closure]:  
*AssignCSP-update* domf updatef x k v is NCSP

**proof** (rule NCSP-intro)

**show** (*AssignCSP-update* domf updatef x k v) is CSP

**by** (simp add: closure)

**show** (*AssignCSP-update* domf updatef x k v) is CSP3

**by** (rule CSP3-SRD-intro, simp-all add: csp-do-def closure rdes unrest)

**show** (*AssignCSP-update domf updatef x k v*) is *CSP4*  
 by (*rule CSP4-tri-intro, simp-all add: csp-do-def closure rdes unrest, rel-auto*)  
**qed**

## 8.7 State abstraction

**lemma** *ref-unrest-abs-st* [*unrest*]:  
 $\$ref \# P \Longrightarrow \$ref \# \langle P \rangle_S$   
 $\$ref' \# P \Longrightarrow \$ref' \# \langle P \rangle_S$   
 by (*rel-simp*)<sup>+</sup>

**lemma** *NCSP-state-srea* [*closure*]: *P* is *NCSP*  $\Longrightarrow$  *state 'a · P* is *NCSP*  
**apply** (*rule NCSP-NSRD-intro*)  
**apply** (*simp-all add: closure rdes*)  
**apply** (*simp-all add: state-srea-def unrest closure*)  
**done**

## 8.8 Assumptions

**definition** *AssumeCircus* ( $\{-\}_C$ ) **where**  
*[rdes-def]*:  $\{b\}_C = \mathbf{R}_s(\mathcal{I}(b, \langle \rangle) \vdash (false \diamond \Phi(true, id, \langle \rangle)))$

## 8.9 Guards

**definition** *GuardCSP* ::

$'\sigma \text{ cond} \Rightarrow$   
 $('\sigma, '\varphi) \text{ action} \Rightarrow$   
 $('\sigma, '\varphi) \text{ action (infixr } \&_u \ 70) \text{ where}$   
*[upred-defs]*:  $g \ \&_u \ A = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r \text{pre}_R(A)) \vdash ((\lceil g \rceil_{S<} \wedge \text{cmt}_R(A)) \vee (\lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$

**lemma** *Guard-tri-design*:

$g \ \&_u \ P = \mathbf{R}_s((\lceil g \rceil_{S<} \Rightarrow_r \text{pre}_R(P)) \vdash (\text{peri}_R(P) \triangleleft \lceil g \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (\lceil g \rceil_{S<} \wedge \text{post}_R(P)))$   
**proof** –  
 have  $(\lceil g \rceil_{S<} \wedge \text{cmt}_R(P) \vee \lceil \neg g \rceil_{S<} \wedge \$tr' =_u \$tr \wedge \$wait') = (\text{peri}_R(P) \triangleleft \lceil g \rceil_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (\lceil g \rceil_{S<} \wedge \text{post}_R(P))$   
 by (*rel-auto*)  
 thus *?thesis* by (*simp add: GuardCSP-def*)  
**qed**

**lemma** *csp-do-cond-conj*:

**assumes** *P* is *CRR*  
**shows**  $(\lceil b \rceil_{S<} \wedge P) = \Phi(b, id, \langle \rangle) ;; P$   
**proof** –  
 have  $(\lceil b \rceil_{S<} \wedge \text{CRR}(P)) = \Phi(b, id, \langle \rangle) ;; \text{CRR}(P)$   
 by (*rel-auto*)  
 thus *?thesis*  
 by (*simp add: Healthy-if assms*)  
**qed**

**lemma** *Guard-rdes-def* [*rdes-def*]:

**assumes** *P* is *RR* *Q* is *CRR* *R* is *CRR*  
**shows**  $g \ \&_u \ \mathbf{R}_s(P \vdash Q \diamond R) = \mathbf{R}_s((\mathcal{I}(g, \langle \rangle) \Rightarrow_r P) \vdash ((\Phi(g, id, \langle \rangle) ;; Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{u\})) \diamond (\Phi(g, id, \langle \rangle) ;; R))$   
 (*is ?lhs = ?rhs*)  
**proof** –

**have**  $?lhs = \mathbf{R}_s (([g]_{S<} \Rightarrow_r P) \vdash ((P \Rightarrow_r Q) \triangleleft [g]_{S<} \triangleright (\$tr' =_u \$tr)) \diamond ([g]_{S<} \wedge (P \Rightarrow_r R)))$   
**by** (*simp add: Guard-tri-design rdes assms closure*)  
**also have**  $\dots = \mathbf{R}_s ((\mathcal{I}(g, \langle \rangle) \Rightarrow_r P) \vdash (([g]_{S<} \wedge Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{u\})) \diamond ([g]_{S<} \wedge R))$   
**by** (*rel-auto*)  
**also have**  $\dots = \mathbf{R}_s ((\mathcal{I}(g, \langle \rangle) \Rightarrow_r P) \vdash ((\Phi(g, id, \langle \rangle) ;; Q) \vee \mathcal{E}(\neg g, \langle \rangle, \{u\})) \diamond (\Phi(g, id, \langle \rangle) ;; R))$   
**by** (*simp add: assms(2) assms(3) csp-do-cond-conj*)  
**finally show** *?thesis* .  
**qed**

**lemma** *Guard-rdes-def'*:

**assumes**  $\$ok' \# P$   
**shows**  $g \&_u (\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(([g]_{S<} \Rightarrow_r P) \vdash ([g]_{S<} \wedge Q \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**proof** –  
**have**  $g \&_u (\mathbf{R}_s(P \vdash Q)) = \mathbf{R}_s(([g]_{S<} \Rightarrow_r pre_R (\mathbf{R}_s(P \vdash Q))) \vdash ([g]_{S<} \wedge cmt_R (\mathbf{R}_s(P \vdash Q)) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*simp add: GuardCSP-def*)  
**also have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash ([g]_{S<} \wedge R1(R2c(cmt_s \dagger (P \Rightarrow Q))) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*simp add: rea-pre-RHS-design rea-cmt-RHS-design*)  
**also have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash R1(R2c([g]_{S<} \wedge R1(R2c(cmt_s \dagger (P \Rightarrow Q)))) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c*)  
**also have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash R1(R2c([g]_{S<} \wedge (cmt_s \dagger (P \Rightarrow Q)) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*simp add: R1-R2c-commute R1-disj R1-extend-conj' R1-idem R2c-and R2c-disj R2c-idem*)  
**also have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash ([g]_{S<} \wedge (cmt_s \dagger (P \Rightarrow Q)) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*metis (no-types, lifting) RHS-design-export-R1 RHS-design-export-R2c*)  
**also have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash cmt_s \dagger ([g]_{S<} \wedge (cmt_s \dagger (P \Rightarrow Q)) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*simp add: rdes-export-cmt*)  
**also have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash cmt_s \dagger ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*simp add: usubst*)  
**also have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r R1(R2c(pre_s \dagger P))) \vdash ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*simp add: rdes-export-cmt*)  
**also from** *assms* **have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r (pre_s \dagger P)) \vdash ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*rel-auto*)  
**also have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r pre_s \dagger P) \llbracket true, false / \$ok, \$wait \rrbracket \vdash ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*simp add: rdes-export-pre*)  
**also from** *assms* **have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r P) \llbracket true, false / \$ok, \$wait \rrbracket \vdash ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*rel-auto*)  
**also from** *assms* **have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r P) \vdash ([g]_{S<} \wedge (P \Rightarrow Q) \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*simp add: rdes-export-pre*)  
**also have**  $\dots = \mathbf{R}_s(([g]_{S<} \Rightarrow_r P) \vdash ([g]_{S<} \wedge Q \vee [\neg g]_{S<} \wedge \$tr' =_u \$tr \wedge \$wait'))$   
**by** (*rule cong[of  $\mathbf{R}_s \mathbf{R}_s$ ], simp, rel-auto*)  
**finally show** *?thesis* .  
**qed**

**lemma** *CSP-Guard [closure]*:  $b \&_u P$  is CSP

by (simp add: GuardCSP-def, rule RHS-design-is-SRD, simp-all add: unrest)

**lemma** *preR-Guard* [rdes]:  $P$  is CSP  $\implies \text{pre}_R(b \ \&_u \ P) = ([b]_{S<} \Rightarrow_r \text{pre}_R \ P)$

by (simp add: Guard-tri-design rea-pre-RHS-design usubst unrest R2c-preR R2c-lift-state-pre R2c-rea-impl R1-rea-impl R1-preR Healthy-if, rel-auto)

**lemma** *periR-Guard* [rdes]:

assumes  $P$  is NCSP

shows  $\text{peri}_R(b \ \&_u \ P) = (\text{peri}_R \ P \triangleleft b \triangleright_R \ \mathcal{E}(\text{true}, \langle \rangle, \{ \}_u))$

**proof** –

have  $\text{peri}_R(b \ \&_u \ P) = (([b]_{S<} \Rightarrow_r \text{pre}_R \ P) \Rightarrow_r (\text{peri}_R \ P \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr)))$

by (simp add: assms Guard-tri-design rea-peri-RHS-design usubst unrest R1-rea-impl R2c-rea-not R2c-rea-impl R2c-preR R2c-periR R2c-tr'-minus-tr R2c-lift-state-pre R2c-condr closure Healthy-if R1-cond R1-tr'-eq-tr)

also have  $\dots = ((\text{pre}_R \ P \Rightarrow_r \text{peri}_R \ P) \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr))$

by (rel-auto)

also have  $\dots = (\text{peri}_R \ P \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr))$

by (simp add: SRD-peri-under-pre add: unrest closure assms)

finally show *?thesis*

by rel-auto

qed

**lemma** *postR-Guard* [rdes]:

assumes  $P$  is NCSP

shows  $\text{post}_R(b \ \&_u \ P) = ([b]_{S<} \wedge \text{post}_R \ P)$

**proof** –

have  $\text{post}_R(b \ \&_u \ P) = (([b]_{S<} \Rightarrow_r \text{pre}_R \ P) \Rightarrow_r ([b]_{S<} \wedge \text{post}_R \ P))$

by (simp add: Guard-tri-design rea-post-RHS-design usubst unrest R2c-rea-not R2c-and R2c-rea-impl R2c-preR R2c-postR R2c-tr'-minus-tr R2c-lift-state-pre R2c-condr R1-rea-impl R1-extend-conj' R1-post-SRD closure assms)

also have  $\dots = ([b]_{S<} \wedge (\text{pre}_R \ P \Rightarrow_r \text{post}_R \ P))$

by (rel-auto)

also have  $\dots = ([b]_{S<} \wedge \text{post}_R \ P)$

by (simp add: SRD-post-under-pre add: unrest closure assms)

also have  $\dots = ([b]_{S<} \wedge \text{post}_R \ P)$

by (metis CSP-Guard R1-extend-conj R1-post-SRD calculation rea-st-cond-def)

finally show *?thesis* .

qed

**lemma** *CSP3-Guard* [closure]:

assumes  $P$  is CSP  $P$  is CSP3

shows  $b \ \&_u \ P$  is CSP3

**proof** –

from assms have  $1:\$ref \ \# \ P \llbracket \text{false}/\$wait \rrbracket$

by (simp add: CSP-Guard CSP3-iff)

hence  $\$ref \ \# \ \text{pre}_R (P \llbracket 0/\$tr \rrbracket) \ \$ref \ \# \ \text{pre}_R \ P \ \$ref \ \# \ \text{cmt}_R \ P$

by (pred-blast)+

hence  $\$ref \ \# \ (b \ \&_u \ P) \llbracket \text{false}/\$wait \rrbracket$

by (simp add: CSP3-iff GuardCSP-def RHS-def R1-def R2c-def R2s-def R3h-def design-def unrest usubst)

thus *?thesis*

by (metis CSP3-intro CSP-Guard)

qed

**lemma** *CSP4-Guard* [closure]:

**assumes**  $P$  is NCSP  
**shows**  $b \&_u P$  is CSP<sub>4</sub>  
**proof** (rule CSP<sub>4</sub>-tri-intro[OF CSP-Guard])  
**show**  $(\neg_r \text{pre}_R (b \&_u P)) \;; R1 \text{ true} = (\neg_r \text{pre}_R (b \&_u P))$   
**proof** –  
**have**  $a: (\neg_r \text{pre}_R P) \;; R1 \text{ true} = (\neg_r \text{pre}_R P)$   
**by** (simp add: CSP<sub>4</sub>-neg-pre-unit assms closure)  
**have**  $(\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) \;; R1 \text{ true} = (\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P))$   
**proof** –  
**have**  $1: (\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) = ([b]_{S<} \wedge (\neg_r \text{pre}_R P))$   
**by** (rel-auto)  
**also have**  $2: \dots = ([b]_{S<} \wedge ((\neg_r \text{pre}_R P) \;; R1 \text{ true}))$   
**by** (simp add: a)  
**also have**  $3: \dots = (\neg_r ([b]_{S<} \Rightarrow_r \text{pre}_R P)) \;; R1 \text{ true}$   
**by** (rel-auto)  
**finally show** ?thesis ..  
**qed**  
**thus** ?thesis  
**by** (simp add: preR-Guard periR-Guard NSRD-CSP<sub>4</sub>-intro closure assms unrest)  
**qed**  
**show**  $\$st' \# \text{peri}_R (b \&_u P)$   
**by** (simp add: preR-Guard periR-Guard NSRD-CSP<sub>4</sub>-intro closure assms unrest)  
**show**  $\$ref' \# \text{post}_R (b \&_u P)$   
**by** (simp add: preR-Guard postR-Guard NSRD-CSP<sub>4</sub>-intro closure assms unrest)  
**qed**

**lemma** NCSP-Guard [closure]:

**assumes**  $P$  is NCSP  
**shows**  $b \&_u P$  is NCSP  
**proof** –  
**have**  $P$  is CSP  
**using** NCSP-implies-CSP assms **by** blast  
**then show** ?thesis  
**by** (metis (no-types) CSP<sub>3</sub>-Guard CSP<sub>3</sub>-commutes-CSP<sub>4</sub> CSP<sub>4</sub>-Guard CSP<sub>4</sub>-Idempotent CSP-Guard Healthy-Idempotent Healthy-def NCSP-def assms comp-apply)  
**qed**

**lemma** Productive-Guard [closure]:

**assumes**  $P$  is CSP  $P$  is Productive  $\$wait' \# \text{pre}_R(P)$   
**shows**  $b \&_u P$  is Productive  
**proof** –  
**have**  $b \&_u P = b \&_u \mathbf{R}_s(\text{pre}_R(P) \vdash \text{peri}_R(P) \diamond (\text{post}_R(P) \wedge \$tr <_u \$tr'))$   
**by** (metis Healthy-def Productive-form assms(1) assms(2))  
**also have**  $\dots =$   
 $\mathbf{R}_s (([b]_{S<} \Rightarrow_r \text{pre}_R P) \vdash$   
 $((\text{pre}_R P \Rightarrow_r \text{peri}_R P) \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr)) \diamond ([b]_{S<} \wedge (\text{pre}_R P \Rightarrow_r \text{post}_R P \wedge \$tr' >_u$   
 $\$tr)))$   
**by** (simp add: Guard-tri-design rea-pre-RHS-design rea-peri-RHS-design rea-post-RHS-design unrest assms  
 $usubst R1\text{-preR Healthy-if R1\text{-rea-impl R1\text{-peri-SRD R1\text{-extend-conj}' R2c\text{-preR R2c\text{-not R2c\text{-rea-impl}}$   
 $R2c\text{-periR R2c\text{-postR R2c\text{-and R2c\text{-tr-less-tr}' R1\text{-tr-less-tr}'$ )  
**also have**  $\dots = \mathbf{R}_s (([b]_{S<} \Rightarrow_r \text{pre}_R P) \vdash (\text{peri}_R P \triangleleft [b]_{S<} \triangleright (\$tr' =_u \$tr)) \diamond (([b]_{S<} \wedge \text{post}_R P)$   
 $\wedge \$tr' >_u \$tr))$   
**by** (rel-auto)

**also have** ... = *Productive*( $b \&_u P$ )  
**by** (*simp add: Productive-def Guard-tri-design RHS-tri-design-par unrest*)  
**finally show** ?thesis  
**by** (*simp add: Healthy-def'*)  
**qed**

## 8.10 Basic events

**definition**  $do_u$  ::

$(\prime\varphi, \prime\sigma) uexpr \Rightarrow (\prime\sigma, \prime\varphi)$  **action where**  
*[upred-defs]*:  $do_u e = ((\$tr' =_u \$tr \wedge \lceil e \rceil_{S<} \notin_u \$ref') \triangleleft \$wait' \triangleright (\$tr' =_u \$tr \hat{^}_u \langle \lceil e \rceil_{S<} \rangle \wedge \$st' =_u \$st))$

**definition**  $DoCSP$  ::  $(\prime\varphi, \prime\sigma) uexpr \Rightarrow (\prime\sigma, \prime\varphi)$  **action** ( $do_C$ ) **where**

*[upred-defs]*:  $DoCSP a = \mathbf{R}_s(true \vdash do_u a)$

**lemma**  $R1-DoAct$ :  $R1(do_u(a)) = do_u(a)$

**by** (*rel-auto*)

**lemma**  $R2c-DoAct$ :  $R2c(do_u(a)) = do_u(a)$

**by** (*rel-auto*)

**lemma**  $DoCSP-alt-def$ :  $do_C(a) = R3h(CSP1(\$ok' \wedge do_u(a)))$

**apply** (*simp add: DoCSP-def RHS-def design-def impl-alt-def R1-R3h-commute R2c-R3h-commute R2c-disj*)

*R2c-not R2c-ok R2c-ok' R2c-and R2c-DoAct R1-disj R1-extend-conj' R1-DoAct*)

**apply** (*rel-auto*)

**done**

**lemma**  $DoAct-unrests$  [*unrest*]:

$\$ok \# do_u(a) \ \$wait \# do_u(a)$

**by** (*pred-auto*)<sup>+</sup>

**lemma**  $DoCSP-RHS-tri$  [*rdes-def*]:

$do_C(a) = \mathbf{R}_s(true_r \vdash (\mathcal{E}(true, \langle \rangle, \{a\}_u) \diamond \Phi(true, id, \langle a \rangle)))$

**by** (*simp add: DoCSP-def do\_u-def wait'-cond-def, rel-auto*)

**lemma**  $CSP-DoCSP$  [*closure*]:  $do_C(a)$  is  $CSP$

**by** (*simp add: DoCSP-def do\_u-def RHS-design-is-SRD unrest*)

**lemma**  $preR-DoCSP$  [*rdes*]:  $pre_R(do_C(a)) = true_r$

**by** (*simp add: DoCSP-def rea-pre-RHS-design unrest usubst R2c-true*)

**lemma**  $periR-DoCSP$  [*rdes*]:  $peri_R(do_C(a)) = \mathcal{E}(true, \langle \rangle, \{a\}_u)$

**by** (*rel-auto*)

**lemma**  $postR-DoCSP$  [*rdes*]:  $post_R(do_C(a)) = \Phi(true, id, \langle a \rangle)$

**by** (*rel-auto*)

**lemma**  $CSP3-DoCSP$  [*closure*]:  $do_C(a)$  is  $CSP3$

**by** (*rule CSP3-intro[OF CSP-DoCSP]*)

(*simp add: DoCSP-def do\_u-def RHS-def design-def R1-def R2c-def R2s-def R3h-def unrest usubst*)

**lemma**  $CSP4-DoCSP$  [*closure*]:  $do_C(a)$  is  $CSP4$

**by** (*rule CSP4-tri-intro[OF CSP-DoCSP], simp-all add: preR-DoCSP periR-DoCSP postR-DoCSP unrest*)

**lemma** *NCSP-DoCSP* [closure]:  $do_C(a)$  is NCSP  
 by (metis *CSP3-DoCSP CSP4-DoCSP CSP-DoCSP Healthy-def NCSP-def comp-apply*)

**lemma** *Productive-DoCSP* [closure]:  
 $(do_C a :: ('\sigma, '\psi)$  action) is Productive

**proof** –

have  $((\Phi(true, id, \langle a \rangle) \wedge \$tr' >_u \$tr) :: ('\sigma, '\psi)$  action)  
 =  $(\Phi(true, id, \langle a \rangle))$

by (rel-auto, simp add: *Prefix-Order.strict-prefixI'*)

hence  $Productive(do_C a) = do_C a$

by (simp add: *Productive-RHS-design-form DoCSP-RHS-tri unrest*)

thus ?thesis

by (simp add: *Healthy-def*)

qed

**lemma** *PCSP-DoCSP* [closure]:

$(do_C a :: ('\sigma, '\psi)$  action) is PCSP

by (simp add: *Healthy-comp NCSP-DoCSP Productive-DoCSP*)

**lemma** *wp-rea-DoCSP-lemma*:

fixes  $P :: ('\sigma, '\varphi)$  action

assumes  $\$ok \# P$   $\$wait \# P$

shows  $(\$tr' =_u \$tr \hat{=} \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) ;; P = (\exists \$ref \cdot P[\$tr \hat{=} \langle [a]_{S<} \rangle / \$tr])$

using *assms*

by (rel-auto, meson)

**lemma** *wp-rea-DoCSP*:

assumes  $P$  is NCSP

shows  $(\$tr' =_u \$tr \hat{=} \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) wp_r pre_R P =$

$(\neg_r (\neg_r pre_R P)[\$tr \hat{=} \langle [a]_{S<} \rangle / \$tr])$

by (simp add: *wp-rea-def wp-rea-DoCSP-lemma unrest usubst ex-unrest assms closure*)

**lemma** *wp-rea-DoCSP-alt*:

assumes  $P$  is NCSP

shows  $(\$tr' =_u \$tr \hat{=} \langle [a]_{S<} \rangle \wedge \$st' =_u \$st) wp_r pre_R P =$

$(\$tr' \geq_u \$tr \hat{=} \langle [a]_{S<} \rangle \Rightarrow_r (pre_R P)[\$tr \hat{=} \langle [a]_{S<} \rangle / \$tr])$

by (simp add: *wp-rea-DoCSP assms rea-not-def R1-def usubst unrest, rel-auto*)

## 8.11 Event prefix

**definition** *PrefixCSP* ::

$('\varphi, '\sigma)$  uexpr  $\Rightarrow$

$('\sigma, '\varphi)$  action  $\Rightarrow$

$('\sigma, '\varphi)$  action  $(- \rightarrow_C - [81, 80] 80)$  **where**

[upred-defs]:  $PrefixCSP a P = (do_C(a) ;; CSP(P))$

**abbreviation** *OutputCSP*  $c v P \equiv PrefixCSP (c \cdot v)_u P$

**lemma** *CSP-PrefixCSP* [closure]:  $PrefixCSP a P$  is CSP

by (simp add: *PrefixCSP-def closure*)

**lemma** *CSP3-PrefixCSP* [closure]:

$PrefixCSP a P$  is CSP3

by (metis (no-types, hide-lams) *CSP3-DoCSP CSP3-def Healthy-def PrefixCSP-def seqr-assoc*)



**lemma** *CSP4-PrefixCSP* [closure]:

**assumes** *P is CSP P is CSP4*

**shows** *PrefixCSP a P is CSP4*

**by** (*metis (no-types, hide-lams) CSP4-def Healthy-def PrefixCSP-def assms(1) assms(2) seqr-assoc*)

**lemma** *NCSP-PrefixCSP* [closure]:

**assumes** *P is NCSP*

**shows** *PrefixCSP a P is NCSP*

**by** (*metis (no-types, hide-lams) CSP3-PrefixCSP CSP3-commutes-CSP4 CSP4-Idempotent CSP4-PrefixCSP CSP-PrefixCSP Healthy-Idempotent Healthy-def NCSP-def NCSP-implies-CSP assms comp-apply*)

**lemma** *Productive-PrefixCSP* [closure]: *P is NCSP  $\implies$  PrefixCSP a P is Productive*

**by** (*simp add: Healthy-if NCSP-DoCSP NCSP-implies-NSRD NSRD-is-SRD PrefixCSP-def Productive-DoCSP Productive-seq-1*)

**lemma** *PCSP-PrefixCSP* [closure]: *P is NCSP  $\implies$  PrefixCSP a P is PCSP*

**by** (*simp add: Healthy-comp NCSP-PrefixCSP Productive-PrefixCSP*)

**lemma** *PrefixCSP-Guarded* [closure]: *Guarded (PrefixCSP a)*

**proof** –

**have** *PrefixCSP a = ( $\lambda X. do_C(a) ;; CSP(X)$ )*

**by** (*simp add: fun-eq-iff PrefixCSP-def*)

**thus** *?thesis*

**using** *Guarded-if-Productive NCSP-DoCSP NCSP-implies-NSRD Productive-DoCSP* **by** *auto*

**qed**

**lemma** *PrefixCSP-type* [closure]: *PrefixCSP a  $\in$   $\llbracket H \rrbracket_H \rightarrow \llbracket CSP \rrbracket_H$*

**using** *CSP-PrefixCSP* **by** *blast*

**lemma** *PrefixCSP-Continuous* [closure]: *Continuous (PrefixCSP a)*

**by** (*simp add: Continuous-def PrefixCSP-def ContinuousD[OF SRD-Continuous] seq-Sup-distl*)

**lemma** *PrefixCSP-RHS-tri-lemma1*:

*R1 (R2s ( $\$tr' =_u \$tr \hat{=} \langle [a]_{S<} \rangle \wedge [II]_R$ )) = ( $\$tr' =_u \$tr \hat{=} \langle [a]_{S<} \rangle \wedge [II]_R$ )*

**by** (*rel-auto*)

**lemma** *PrefixCSP-RHS-tri-lemma2*:

**fixes** *P :: (' $\sigma$ , ' $\varphi$ ) action*

**assumes**  *$\$ok \# P \$wait \# P$*

**shows** (*( $\$tr' =_u \$tr \hat{=} \langle [a]_{S<} \rangle \wedge \$st' =_u \$st$ )  $\wedge$   $\neg \$wait'$ ) ;; *P = ( $\exists \$ref \cdot P[\$tr \hat{=} \langle [a]_{S<} \rangle / \$tr]$ )**

**using** *assms*

**by** (*rel-auto, meson, fastforce*)

**lemma** *tr-extend-seqr*:

**fixes** *P :: (' $\sigma$ , ' $\varphi$ ) action*

**assumes**  *$\$ok \# P \$wait \# P \$ref \# P$*

**shows** ( *$\$tr' =_u \$tr \hat{=} \langle [a]_{S<} \rangle \wedge \$st' =_u \$st$ ) ;; *P = P[\\$tr \hat{=} \langle [a]\_{S<} \rangle / \\$tr]**

**using** *assms* **by** (*simp add: wp-rea-DoCSP-lemma assms unrest ex-unrest*)

**lemma** *trace-ext-R1-closed* [closure]: *P is R1  $\implies$  P[\\$tr \hat{=} e / \\$tr] is R1*

**by** (*rel-blast*)

**lemma** *preR-PrefixCSP-NCSP* [rdes]:

**assumes** *P is NCSP*

**shows** *pre<sub>R</sub>(PrefixCSP a P) = ( $\mathcal{I}(true, \langle a \rangle) \Rightarrow_r (pre_R P)[\langle a \rangle]_t$ )*

by (*simp add: PrefixCSP-def assms closure rdes rpred Healthy-if wp usubst unrest*)

**lemma** *periR-PrefixCSP* [*rdes*]:

**assumes** *P is NCSP*

**shows**  $\text{peri}_R(\text{PrefixCSP } a \ P) = (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee (\text{peri}_R \ P) \llbracket \langle a \rangle \rrbracket_t)$

**proof** –

**have**  $\text{peri}_R(\text{PrefixCSP } a \ P) = \text{peri}_R(\text{do}_C \ a \ ;; \ P)$

**by** (*simp add: PrefixCSP-def closure assms Healthy-if*)

**also have**  $\dots = ((\mathcal{I}(\text{true}, \langle a \rangle) \Rightarrow_r \text{pre}_R \ P \llbracket \langle a \rangle \rrbracket_t) \Rightarrow_r \ \$tr' =_u \ \$tr \wedge [a]_{S <} \notin_u \ \$ref' \vee \text{peri}_R \ P \llbracket \langle a \rangle \rrbracket_t)$

**by** (*simp add: assms NSRD-CSP<sub>4</sub>-intro csp-enable-tr-empty closure rdes unrest ex-unrest usubst rpred wp*)

**also have**  $\dots = (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee ((\mathcal{I}(\text{true}, \langle a \rangle) \Rightarrow_r \text{pre}_R \ P \llbracket \langle a \rangle \rrbracket_t) \Rightarrow_r \text{peri}_R \ P \llbracket \langle a \rangle \rrbracket_t))$

**by** (*rel-auto*)

**also have**  $\dots = (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee ((\text{pre}_R(P) \Rightarrow_r \text{peri}_R \ P) \llbracket \langle a \rangle \rrbracket_t))$

**by** (*rel-auto*)

**also have**  $\dots = (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee (\text{peri}_R \ P) \llbracket \langle a \rangle \rrbracket_t)$

**by** (*simp add: SRD-peri-under-pre assms closure unrest*)

**finally show** *?thesis* .

**qed**

**lemma** *postR-PrefixCSP* [*rdes*]:

**assumes** *P is NCSP*

**shows**  $\text{post}_R(\text{PrefixCSP } a \ P) = (\text{post}_R \ P) \llbracket \langle a \rangle \rrbracket_t$

**proof** –

**have**  $\text{post}_R(\text{PrefixCSP } a \ P) = ((\mathcal{I}(\text{true}, \langle a \rangle) \Rightarrow_r (\text{pre}_R \ P) \llbracket \langle a \rangle \rrbracket_t) \Rightarrow_r (\text{post}_R \ P) \llbracket \langle a \rangle \rrbracket_t)$

**by** (*simp add: PrefixCSP-def assms Healthy-if*)

(*simp add: assms Healthy-if wp closure rdes rpred wp-rea-DoCSP-lemma unrest ex-unrest usubst*)

**also have**  $\dots = (\mathcal{I}(\text{true}, \langle a \rangle) \wedge (\text{pre}_R \ P \Rightarrow_r \text{post}_R \ P) \llbracket \langle a \rangle \rrbracket_t)$

**by** (*rel-auto*)

**also have**  $\dots = (\mathcal{I}(\text{true}, \langle a \rangle) \wedge (\text{post}_R \ P) \llbracket \langle a \rangle \rrbracket_t)$

**by** (*simp add: SRD-post-under-pre assms closure unrest*)

**also have**  $\dots = (\text{post}_R \ P) \llbracket \langle a \rangle \rrbracket_t$

**by** (*rel-auto*)

**finally show** *?thesis* .

**qed**

**lemma** *PrefixCSP-RHS-tri*:

**assumes** *P is NCSP*

**shows**  $\text{PrefixCSP } a \ P = \mathbf{R}_s((\mathcal{I}(\text{true}, \langle a \rangle) \Rightarrow_r \text{pre}_R \ P \llbracket \langle a \rangle \rrbracket_t) \vdash (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee \text{peri}_R \ P \llbracket \langle a \rangle \rrbracket_t) \diamond \text{post}_R \ P \llbracket \langle a \rangle \rrbracket_t)$

**by** (*simp add: PrefixCSP-def Healthy-if unrest assms closure NSRD-composition-wp rdes rpred usubst wp*)

For prefix, we can chose whether to propagate the assumptions or not, hence there are two laws.

**lemma** *PrefixCSP-rdes-def-1* [*rdes-def*]:

**assumes** *P is CRC Q is CRR R is CRR*

$\$st' \# \ Q \ \$ref' \# \ R$

**shows**  $\text{PrefixCSP } a \ (\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\mathcal{I}(\text{true}, \langle a \rangle) \Rightarrow_r \ P \llbracket \langle a \rangle \rrbracket_t) \vdash (\mathcal{E}(\text{true}, \langle \rangle, \{a\}_u) \vee Q \llbracket \langle a \rangle \rrbracket_t) \diamond R \llbracket \langle a \rangle \rrbracket_t)$

**apply** (*subst PrefixCSP-RHS-tri*)

**apply** (*rule NCSP-rdes-intro*)

**apply** (*simp-all add: assms rdes closure*)

**apply** (*rel-auto*)

**done**

**lemma** *PrefixCSP-rdes-def-2*:

**assumes**  $P$  is CRC  $Q$  is CRR  $R$  is CRR  
 $\$st' \# Q \$ref' \# R$   
**shows**  $PrefixCSP\ a\ (\mathbf{R}_s(P \vdash Q \diamond R)) = \mathbf{R}_s((\mathcal{I}(true, \langle a \rangle) \Rightarrow_r P \llbracket \langle a \rangle \rrbracket_t) \vdash (\mathcal{E}(true, \langle \rangle, \{a\}_u) \vee (P \wedge Q) \llbracket \langle a \rangle \rrbracket_t))$   
 $\diamond (P \wedge R) \llbracket \langle a \rangle \rrbracket_t$   
**apply** (*subst PrefixCSP-RHS-tri*)  
**apply** (*rule NCSP-rdes-intro*)  
**apply** (*simp-all add: assms rdes closure*)  
**apply** (*rel-auto*)  
**done**

## 8.12 Guarded external choice

**abbreviation** *GuardedChoiceCSP* ::  $'\vartheta$  set  $\Rightarrow (' \vartheta \Rightarrow (' \sigma, ' \vartheta)$  action)  $\Rightarrow (' \sigma, ' \vartheta)$  action **where**  
*GuardedChoiceCSP*  $A\ P \equiv (\square x \in A \cdot PrefixCSP \llangle x \rrangle (P(x)))$

**syntax**

-*GuardedChoiceCSP* :: logic  $\Rightarrow$  logic  $\Rightarrow$  logic  $\Rightarrow$  logic ( $\square - \in - \rightarrow - [0,0,85]$  86)

**translations**

$\square x \in A \rightarrow P == CONST\ GuardedChoiceCSP\ A\ (\lambda x. P)$

**lemma** *GuardedChoiceCSP [rdes-def]*:

**assumes**  $\bigwedge x. P(x)$  is NCSP  $A \neq \{\}$

**shows**  $(\square x \in A \rightarrow P(x)) =$

$\mathbf{R}_s((\bigcup x \in A \cdot \mathcal{I}(true, \langle x \rangle)) \Rightarrow_r pre_R (P\ x) \llbracket \langle x \rangle \rrbracket_t) \vdash$   
 $((\bigcup x \in A \cdot \mathcal{E}(true, \langle \rangle, \{x\}_u)) \triangleleft \$tr' =_u \$tr \triangleright (\bigcap x \in A \cdot peri_R (P\ x) \llbracket \langle x \rangle \rrbracket_t)) \diamond$   
 $(\bigcap x \in A \cdot post_R (P\ x) \llbracket \langle x \rangle \rrbracket_t)$

**by** (*simp add: PrefixCSP-RHS-tri assms ExtChoice-tri-rdes closure unrest, rel-auto*)

## 8.13 Input prefix

**definition** *InputCSP* ::

$('a, ' \vartheta)$  chan  $\Rightarrow ('a \Rightarrow ' \sigma$  upred)  $\Rightarrow ('a \Rightarrow (' \sigma, ' \vartheta)$  action)  $\Rightarrow (' \sigma, ' \vartheta)$  action **where**  
*[upred-defs]*: *InputCSP*  $c\ A\ P = (\square x \in UNIV \cdot A(x) \ \&_u\ PrefixCSP\ (c \cdot \langle x \rangle)_u\ (P\ x))$

**definition** *InputVarCSP* ::  $('a, ' \vartheta)$  chan  $\Rightarrow ('a \Longrightarrow ' \sigma) \Rightarrow ('a \Rightarrow ' \sigma$  upred)  $\Rightarrow (' \sigma, ' \vartheta)$  action  $\Rightarrow (' \sigma,$

$' \vartheta)$  action **where**  
*[upred-defs, rdes-def]*: *InputVarCSP*  $c\ x\ A\ P = InputCSP\ c\ A\ (\lambda v. \langle [x \mapsto_s \langle v \rangle] \rangle_C) ;; P$

**definition** *do<sub>I</sub>* ::

$('a, ' \vartheta)$  chan  $\Rightarrow$   
 $('a \Longrightarrow (' \sigma, ' \vartheta)$  st-csp)  $\Rightarrow$   
 $('a \Rightarrow (' \sigma, ' \vartheta)$  action)  $\Rightarrow$   
 $(' \sigma, ' \vartheta)$  action **where**  
 $do_I\ c\ x\ P =$   
 $(\$tr' =_u \$tr \wedge \{e : \langle \delta_u(c) \rangle \mid P(e) \cdot (c \cdot \langle e \rangle)_u\} \cap_u \$ref' =_u \{\}_u)$   
 $\triangleleft \$wait' \triangleright$   
 $((\$tr' - \$tr) \in_u \{e : \langle \delta_u(c) \rangle \mid P(e) \cdot \langle (c \cdot \langle e \rangle)_u \rangle \wedge (c \cdot \$x')_u =_u last_u(\$tr')\})$

**lemma** *InputCSP-CSP [closure]*: *InputCSP*  $c\ A\ P$  is CSP

**by** (*simp add: CSP-ExtChoice InputCSP-def*)

**lemma** *InputCSP-NCSP [closure]*:  $\llbracket \bigwedge v. P(v)$  is NCSP  $\rrbracket \Longrightarrow InputCSP\ c\ A\ P$  is NCSP

**apply** (*simp add: InputCSP-def*)

**apply** (*rule NCSP-ExtChoice*)

**apply** (*simp add: NCSP-Guard NCSP-PrefixCSP image-Collect-subsetI top-set-def*)  
**done**

**lemma** *Productive-InputCSP* [*closure*]:

$\llbracket \bigwedge v. P(v) \text{ is NCSP} \rrbracket \implies \text{InputCSP } x \ A \ P \text{ is Productive}$

**by** (*auto simp add: InputCSP-def unrest closure intro: Productive-ExtChoice*)

**lemma** *preR-InputCSP* [*rdes*]:

**assumes**  $\bigwedge v. P(v) \text{ is NCSP}$

**shows**  $\text{pre}_R(\text{InputCSP } a \ A \ P) = (\bigsqcup v \cdot [A(v)]_{S<} \Rightarrow_r \mathcal{I}(\text{true}, \langle (a \cdot \ll v \gg)_u \rangle) \Rightarrow_r (\text{pre}_R(P(v))) \llbracket \langle (a \cdot \ll v \gg)_u \rangle \rrbracket_t)$

**by** (*simp add: InputCSP-def rdes closure assms alpha usubst unrest*)

**lemma** *periR-InputCSP* [*rdes*]:

**assumes**  $\bigwedge v. P(v) \text{ is NCSP}$

**shows**  $\text{peri}_R(\text{InputCSP } a \ A \ P) =$

$(\bigsqcup x \cdot [A(x)]_{S<} \Rightarrow_r \mathcal{E}(\text{true}, \langle \rangle, \{(a \cdot \ll x \gg)_u\}_u))$   
 $\triangleleft \$tr' =_u \$tr \triangleright$

$(\prod x \cdot [A(x)]_{S<} \wedge (\text{peri}_R(P \ x)) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t)$

**by** (*simp add: InputCSP-def rdes closure assms, rel-auto*)

**lemma** *postR-InputCSP* [*rdes*]:

**assumes**  $\bigwedge v. P(v) \text{ is NCSP}$

**shows**  $\text{post}_R(\text{InputCSP } a \ A \ P) =$

$(\prod x \cdot [A \ x]_{S<} \wedge \text{post}_R(P \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t)$

**using** *assms* **by** (*simp add: InputCSP-def rdes closure assms usubst unrest*)

**lemma** *InputCSP-rdes-def* [*rdes-def*]:

**assumes**  $\bigwedge v. P(v) \text{ is CRC} \ \bigwedge v. Q(v) \text{ is CRR} \ \bigwedge v. R(v) \text{ is CRR}$

$\bigwedge v. \$st' \ \#\ Q(v) \ \bigwedge v. \$ref' \ \#\ R(v)$

**shows**  $\text{InputCSP } a \ A \ (\lambda v. \mathbf{R}_s(P(v) \vdash Q(v) \diamond R(v))) =$

$\mathbf{R}_s(\bigsqcup v \cdot ([A(v)]_{S<} \Rightarrow_r \mathcal{I}(\text{true}, \langle (a \cdot \ll v \gg)_u \rangle) \Rightarrow_r (P \ v)) \llbracket \langle (a \cdot \ll v \gg)_u \rangle \rrbracket_t)$   
 $\vdash (\bigsqcup x \cdot \mathbf{R}_5([A(x)]_{S<} \Rightarrow_r \mathcal{E}(\text{true}, \langle \rangle, \{(a \cdot \ll x \gg)_u\}_u)))$

$\vee$

$(\prod x \cdot [A(x)]_{S<} \wedge (P \ x \wedge Q \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t)$

$\diamond (\prod x \cdot [A \ x]_{S<} \wedge (P \ x \wedge R \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t)$  (**is** *?lhs = ?rhs*)

**proof** –

**have**  $1: \text{pre}_R(?lhs) = (\bigsqcup v \cdot [A \ v]_{S<} \Rightarrow_r \mathcal{I}(\text{true}, \langle (a \cdot \ll v \gg)_u \rangle) \Rightarrow_r P \ v) \llbracket \langle (a \cdot \ll v \gg)_u \rangle \rrbracket_t$  (**is**  $- = ?A$ )

**by** (*simp add: rdes NCSP-rdes-intro assms conj-comm closure*)

**have**  $2: \text{peri}_R(?lhs) = (\bigsqcup x \cdot [A \ x]_{S<} \Rightarrow_r \mathcal{E}(\text{true}, \langle \rangle, \{(a \cdot \ll x \gg)_u\}_u)) \triangleleft \$tr' =_u \$tr \triangleright (\prod x \cdot [A \ x]_{S<} \wedge (P \ x \Rightarrow_r Q \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t)$

$\wedge (P \ x \Rightarrow_r Q \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t$

(**is**  $- = ?B$ )

**by** (*simp add: rdes NCSP-rdes-intro assms closure*)

**have**  $3: \text{post}_R(?lhs) = (\prod x \cdot [A \ x]_{S<} \wedge (P \ x \Rightarrow_r R \ x) \llbracket \langle (a \cdot \ll x \gg)_u \rangle \rrbracket_t)$

(**is**  $- = ?C$ )

**by** (*simp add: rdes NCSP-rdes-intro assms closure*)

**have**  $?lhs = \mathbf{R}_s(?A \vdash ?B \diamond ?C)$

**by** (*subst SRD-reactive-tri-design[THEN sym], simp-all add: closure 1 2 3*)

**also have**  $\dots = ?rhs$

**by** (*rel-auto*)

**finally show** *?thesis* .

**qed**

## 8.14 Algebraic laws

**lemma** *AssignCSP-conditional*:

**assumes** *vwb-lens* *x*

**shows**  $x :=_C e \triangleleft b \triangleright_R x :=_C f = x :=_C (e \triangleleft b \triangleright f)$   
**by** (*rdes-eq cls: assms*)

**lemma** *AssignsCSP-id*:  $\langle id \rangle_C = Skip$   
**by** (*rel-auto*)

**lemma** *Guard-comp*:  
 $g \&_u h \&_u P = (g \wedge h) \&_u P$   
**by** (*rule antisym, rel-blast, rel-blast*)

**lemma** *Guard-false* [*simp*]:  $false \&_u P = Stop$   
**by** (*simp add: GuardCSP-def Stop-def rpred closure alpha R1-design-R1-pre*)

**lemma** *Guard-true* [*simp*]:  
 $P \text{ is CSP} \implies true \&_u P = P$   
**by** (*simp add: GuardCSP-def alpha SRD-reactive-design-alt closure rpred*)

**lemma** *Guard-conditional*:  
**assumes**  $P \text{ is NCSP}$   
**shows**  $b \&_u P = P \triangleleft b \triangleright_R Stop$   
**by** (*rdes-eq cls: assms*)

**lemma** *Guard-expansion*:  
 $(g_1 \vee g_2) \&_u P = (g_1 \&_u P) \sqcap (g_2 \&_u P)$   
**by** (*rel-auto*)

**lemma** *Conditional-as-Guard*:  
**assumes**  $P \text{ is NCSP } Q \text{ is NCSP}$   
**shows**  $P \triangleleft b \triangleright_R Q = b \&_u P \sqcap (\neg b) \&_u Q$   
**by** (*rdes-eq cls: assms; simp add: le-less*)

**lemma** *PrefixCSP-dist*:  
 $PrefixCSP a (P \sqcap Q) = (PrefixCSP a P) \sqcap (PrefixCSP a Q)$   
**using** *Continuous-Disjunctous Disjunctuous-def PrefixCSP-Continuous* **by** *auto*

**lemma** *DoCSP-is-Prefix*:  
 $do_C(a) = PrefixCSP a Skip$   
**by** (*simp add: PrefixCSP-def Healthy-if closure, metis CSP4-DoCSP CSP4-def Healthy-def*)

**lemma** *PrefixCSP-seq*:  
**assumes**  $P \text{ is CSP } Q \text{ is CSP}$   
**shows**  $(PrefixCSP a P) ;; Q = (PrefixCSP a (P ;; Q))$   
**by** (*simp add: PrefixCSP-def seqr-assoc Healthy-if assms closure*)

**lemma** *PrefixCSP-extChoice-dist*:  
**assumes**  $P \text{ is NCSP } Q \text{ is NCSP } R \text{ is NCSP}$   
**shows**  $((a \rightarrow_C P) \sqcap (b \rightarrow_C Q)) ;; R = (a \rightarrow_C P ;; R) \sqcap (b \rightarrow_C Q ;; R)$   
**by** (*simp add: PCSP-PrefixCSP assms(1) assms(2) assms(3) extChoice-seq-distr*)

**lemma** *GuardedChoiceCSP-dist*:  
**assumes**  $\bigwedge i. i \in A \implies P(i) \text{ is NCSP } Q \text{ is NCSP}$   
**shows**  $\sqcap x \in A \rightarrow P(x) ;; Q = \sqcap x \in A \rightarrow (P(x) ;; Q)$   
**by** (*simp add: ExtChoice-seq-distr PrefixCSP-seq closure assms cong: ExtChoice-cong*)

Alternation can be re-expressed as an external choice when the guards are disjoint

**declare** *ExtChoice-tri-rdes* [*rdes-def*]  
**declare** *ExtChoice-tri-rdes'* [*rdes-def del*]

**declare** *extChoice-rdes-def* [*rdes-def*]  
**declare** *extChoice-rdes-def'* [*rdes-def del*]

**lemma** *AlternateR-as-ExtChoice*:

**assumes**  
 $\bigwedge i. i \in A \implies P(i) \text{ is NCSP } Q \text{ is NCSP}$   
 $\bigwedge i j. \llbracket i \in A; j \in A; i \neq j \rrbracket \implies (g i \wedge g j) = \text{false}$   
**shows**  $(\text{if}_R i \in A \cdot g(i) \rightarrow P(i) \text{ else } Q \text{ fi}) =$   
 $(\square i \in A \cdot g(i) \ \&_u \ P(i)) \square (\bigwedge i \in A \cdot \neg g(i)) \ \&_u \ Q$   
**proof** (*cases*  $A = \{\}$ )  
**case** *True*  
**then show** *?thesis* **by** (*simp add: ExtChoice-empty extChoice-Stop closure assms*)  
**next**  
**case** *False*  
**show** *?thesis*  
  
**proof** –  
**have**  $1: (\bigwedge i \in A \cdot g i \rightarrow_R P i) = (\bigwedge i \in A \cdot g i \rightarrow_R \mathbf{R}_s(\text{pre}_R(P i) \vdash \text{peri}_R(P i) \diamond \text{post}_R(P i)))$   
**by** (*rule UINF-cong, simp add: NCSP-implies-CSP SRD-reactive-tri-design assms(1)*)  
**have**  $2: (\square i \in A \cdot g(i) \ \&_u \ P(i)) = (\square i \in A \cdot g(i) \ \&_u \ \mathbf{R}_s(\text{pre}_R(P i) \vdash \text{peri}_R(P i) \diamond \text{post}_R(P i)))$   
**by** (*rule ExtChoice-cong, simp add: NCSP-implies-NSRD NSRD-is-SRD SRD-reactive-tri-design assms(1)*)  
**from** *assms(3)* **show** *?thesis*  
**by** (*simp add: AlternateR-def 1 2*)  
*(rdes-eq cls: assms(1–2) simps: False cong: UINF-cong ExtChoice-cong)*  
**qed**  
**qed**

**declare** *ExtChoice-tri-rdes* [*rdes-def del*]  
**declare** *ExtChoice-tri-rdes'* [*rdes-def*]

**declare** *extChoice-rdes-def* [*rdes-def del*]  
**declare** *extChoice-rdes-def'* [*rdes-def*]

**end**

## 9 Recursion in Stateful-Failures

**theory** *utp-sfrd-recursion*  
**imports** *utp-sfrd-contracts utp-sfrd-prog*  
**begin**

### 9.1 Fixed-points

The CSP weakest fixed-point is obtained simply by precomposing the body with the CSP healthiness condition.

**abbreviation** *mu-CSP* ::  $((\sigma, \varphi) \text{ action} \Rightarrow (\sigma, \varphi) \text{ action}) \Rightarrow (\sigma, \varphi) \text{ action} \ (\mu_C)$  **where**  
 $\mu_C F \equiv \mu (F \circ \text{CSP})$

**syntax**

-mu-CSP :: pptrn  $\Rightarrow$  logic  $\Rightarrow$  logic ( $\mu_C \cdot \cdot \cdot [0, 10] 10$ )

**translations**

$\mu_C X \cdot P == \text{CONST } \mu\text{-CSP } (\lambda X. P)$

**lemma** *mu-CSP-equiv*:

**assumes** *Monotonic F*  $F \in \llbracket \text{CSP} \rrbracket_H \rightarrow \llbracket \text{CSP} \rrbracket_H$

**shows**  $(\mu_R F) = (\mu_C F)$

**by** (*simp add: srd-mu-equiv assms comp-def*)

**lemma** *mu-CSP-unfold*:

*P is CSP*  $\implies (\mu_C X \cdot P ;; X) = P ;; (\mu_C X \cdot P ;; X)$

**apply** (*subst gfp-unfold*)

**apply** (*simp-all add: closure Healthy-if*)

**done**

**lemma** *mu-csp-expand* [*rdes*]:  $(\mu_C (op ;; Q)) = (\mu X \cdot Q ;; \text{CSP } X)$

**by** (*simp add: comp-def*)

**lemma** *mu-csp-basic-refine*:

**assumes**

*P is CSP Q is NCSP Q is Productive*  $\text{pre}_R(P) = \text{true}_r$   $\text{pre}_R(Q) = \text{true}_r$

$\text{peri}_R P \sqsubseteq \text{peri}_R Q$

$\text{peri}_R P \sqsubseteq \text{post}_R Q ;; \text{peri}_R P$

**shows**  $P \sqsubseteq (\mu_C X \cdot Q ;; X)$

**proof** (*rule SRD-refine-intro'*, *simp-all add: closure usubst alpha rpred rdes unrest wp seq-UINF-distr assms*)

**show**  $\text{peri}_R P \sqsubseteq (\sqcap i \cdot \text{post}_R Q \hat{=} i ;; \text{peri}_R Q)$

**proof** (*rule UINF-refines'*)

**fix** *i*

**show**  $\text{peri}_R P \sqsubseteq \text{post}_R Q \hat{=} i ;; \text{peri}_R Q$

**proof** (*induct i*)

**case** 0

**then show** ?case **by** (*simp add: assms*)

**next**

**case** (*Suc i*)

**then show** ?case

**by** (*meson assms(6) assms(7) semilattice-sup-class.le-sup-iff upower-inductl*)

**qed**

**qed**

**qed**

**lemma** *CRD-mu-basic-refine*:

**fixes** *P* :: 'e list  $\Rightarrow$  'e set  $\Rightarrow$  's upred

**assumes**

*Q is NCSP Q is Productive*  $\text{pre}_R(Q) = \text{true}_r$

$[P \ t \ r]_{S <} \llbracket (t, r) \rightarrow (\&tt, \$ref')_u \rrbracket \sqsubseteq \text{peri}_R Q$

$[P \ t \ r]_{S <} \llbracket (t, r) \rightarrow (\&tt, \$ref')_u \rrbracket \sqsubseteq \text{post}_R Q ;;_h [P \ t \ r]_{S <} \llbracket (t, r) \rightarrow (\&tt, \$ref')_u \rrbracket$

**shows**  $[\text{true} \vdash P \ \text{trace} \ \text{refs} \mid R]_C \sqsubseteq (\mu_C X \cdot Q ;; X)$

**proof** (*rule mu-csp-basic-refine*, *simp-all add: msubst-pair assms closure alpha rdes rpred Healthy-if R1-false*)

**show**  $[P \ \text{trace} \ \text{refs}]_{S <} \llbracket \text{trace} \rightarrow \&tt \rrbracket \llbracket \text{refs} \rightarrow \$ref' \rrbracket \sqsubseteq \text{peri}_R Q$

**using** *assms* **by** (*simp add: msubst-pair*)

**show**  $[P \ \text{trace} \ \text{refs}]_{S <} \llbracket \text{trace} \rightarrow \&tt \rrbracket \llbracket \text{refs} \rightarrow \$ref' \rrbracket \sqsubseteq \text{post}_R Q ;; [P \ \text{trace} \ \text{refs}]_{S <} \llbracket \text{trace} \rightarrow \&tt \rrbracket \llbracket \text{refs} \rightarrow \$ref' \rrbracket$

**using** *assms* **by** (*simp add: msubst-pair*)

qed

## 9.2 Example action expansion

**lemma** *mu-example1*:  $(\mu X \cdot \ll a \gg \rightarrow_C X) = (\prod i \cdot do_C(\ll a \gg) \wedge (i+1))$  ;; *Miracle*  
 by (*simp add: PrefixCSP-def mu-csp-form-1 closure*)

**lemma** *preR-mu-example1* [*rdes*]:  $pre_R(\mu X \cdot \ll a \gg \rightarrow_C X) = true_r$   
 by (*simp add: mu-example1 rdes closure unrest wp*)

**lemma** *periR-mu-example1* [*rdes*]:  
 $peri_R(\mu X \cdot \ll a \gg \rightarrow_C X) = (\prod i \cdot \mathcal{E}(true, iter[i](\ll a \gg)), \{\ll a \gg\}_u)$   
 by (*simp add: mu-example1 rdes rpred closure unrest wp seq-UINF-distr alpha usubst*)

**lemma** *postR-mu-example1* [*rdes*]:  
 $post_R(\mu X \cdot \ll a \gg \rightarrow_C X) = false$   
 by (*simp add: mu-example1 rdes closure unrest wp*)

end

## 10 Linking to the Failures-Divergences Model

**theory** *utp-sfrd-fdsem*  
 imports *utp-sfrd-recursion*  
 begin

### 10.1 Failures-Divergences Semantics

The following functions play a similar role to those in Roscoe's CSP semantics, and are calculated from the Circus reactive design semantics. A major difference is that these three functions account for state. Each divergence, trace, and failure is subject to an initial state. Moreover, the traces are terminating traces, and therefore also provide a final state following the given interaction. A more subtle difference from the Roscoe semantics is that the set of traces do not include the divergences. The same semantic information is present, but we construct a direct analogy with the pre-, peri- and postconditions of our reactive designs.

**definition** *divergences* ::  $('σ, 'φ) \text{ action} \Rightarrow 'σ \Rightarrow 'φ \text{ list set } (dv[-] - [0,100] 100)$  **where**  
 [*upred-defs*]:  $divergences P s = \{t \mid t. '(\neg_r pre_R(P)) \ll s \gg, \langle \rangle, \ll t \gg / \$st, \$tr, \$tr '\}$

**definition** *traces* ::  $('σ, 'φ) \text{ action} \Rightarrow 'σ \Rightarrow ('φ \text{ list} \times 'σ) \text{ set } (tr[-] - [0,100] 100)$  **where**  
 [*upred-defs*]:  $traces P s = \{(t, s') \mid t s'. '(pre_R(P) \wedge post_R(P)) \ll s \gg, \ll s' \gg, \langle \rangle, \ll t \gg / \$st, \$st', \$tr, \$tr '\}$

**definition** *failures* ::  $('σ, 'φ) \text{ action} \Rightarrow 'σ \Rightarrow ('φ \text{ list} \times 'σ) \text{ set } (fl[-] - [0,100] 100)$  **where**  
 [*upred-defs*]:  $failures P s = \{(t, r) \mid t r. '(pre_R(P) \wedge peri_R(P)) \ll r \gg, \ll s \gg, \langle \rangle, \ll t \gg / \$ref', \$st, \$tr, \$tr '\}$

**lemma** *trace-divergence-disj*:  
 assumes *P is NCSP*  $(t, s') \in tr[P]s$   $t \in dv[P]s$   
 shows *False*  
 using *assms(2,3)*  
 by (*simp add: traces-def divergences-def, rdes-simp cls:assms, rel-auto*)

**lemma** *preR-refine-divergences*:  
 assumes *P is NCSP* *Q is NCSP*  $\wedge s. dv[P]s \subseteq dv[Q]s$   
 shows  $pre_R(P) \sqsubseteq pre_R(Q)$   
**proof** (*rule CRR-refine-impl-prop, simp-all add: assms closure usubst unrest*)



**fix**  $t\ s$   
**assume**  $a$ :  $\ulcorner [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R Q \urcorner$   
**with a show**  $\ulcorner [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R P \urcorner$   
**proof** (*rule-tac ccontr*)  
**from**  $assms(\mathcal{B})[of\ s]$  **have**  $b$ :  $t \in dv[P]s \implies t \in dv[Q]s$   
**by** (*auto*)  
**assume**  $\neg \ulcorner [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R P \urcorner$   
**hence**  $\neg \ulcorner [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger CRC(pre_R P) \urcorner$   
**by** (*simp add: assms closure Healthy-if*)  
**hence**  $\ulcorner [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (\neg_r CRC(pre_R P)) \urcorner$   
**by** (*rel-auto*)  
**hence**  $\ulcorner [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (\neg_r pre_R P) \urcorner$   
**by** (*simp add: assms closure Healthy-if*)  
**with a b show** *False*  
**by** (*rel-auto*)  
**qed**  
**qed**

**lemma** *preR-eq-divergences*:  
**assumes**  $P$  is NCSP  $Q$  is NCSP  $\wedge s$ .  $dv[P]s = dv[Q]s$   
**shows**  $pre_R(P) = pre_R(Q)$   
**by** (*metis assms dual-order.antisym order-refl preR-refine-divergences*)

**lemma** *periR-refine-failures*:  
**assumes**  $P$  is NCSP  $Q$  is NCSP  $\wedge s$ .  $fl[Q]s \subseteq fl[P]s$   
**shows**  $(pre_R(P) \wedge peri_R(P)) \sqsubseteq (pre_R(Q) \wedge peri_R(Q))$   
**proof** (*rule CRR-refine-impl-prop, simp-all add: assms closure unrest subst-unrest-3*)  
**fix**  $t\ s\ r'$   
**assume**  $a$ :  $\ulcorner [\$ref' \mapsto_s \ll r' \gg, \$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R Q \wedge peri_R Q) \urcorner$   
**from**  $assms(\mathcal{B})[of\ s]$  **have**  $b$ :  $(t, r') \in fl[Q]s \implies (t, r') \in fl[P]s$   
**by** (*auto*)  
**with a show**  $\ulcorner [\$ref' \mapsto_s \ll r' \gg, \$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R P \wedge peri_R P) \urcorner$   
**by** (*simp add: failures-def*)  
**qed**

**lemma** *periR-eq-failures*:  
**assumes**  $P$  is NCSP  $Q$  is NCSP  $\wedge s$ .  $fl[P]s = fl[Q]s$   
**shows**  $(pre_R(P) \wedge peri_R(P)) = (pre_R(Q) \wedge peri_R(Q))$   
**by** (*metis (full-types) assms dual-order.antisym order-refl periR-refine-failures*)

**lemma** *postR-refine-traces*:  
**assumes**  $P$  is NCSP  $Q$  is NCSP  $\wedge s$ .  $tr[Q]s \subseteq tr[P]s$   
**shows**  $(pre_R(P) \wedge post_R(P)) \sqsubseteq (pre_R(Q) \wedge post_R(Q))$   
**proof** (*rule CRR-refine-impl-prop, simp-all add: assms closure unrest subst-unrest-5*)  
**fix**  $t\ s\ s'$   
**assume**  $a$ :  $\ulcorner [\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R Q \wedge post_R Q) \urcorner$   
**from**  $assms(\mathcal{B})[of\ s]$  **have**  $b$ :  $(t, s') \in tr[Q]s \implies (t, s') \in tr[P]s$   
**by** (*auto*)  
**with a show**  $\ulcorner [\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger (pre_R P \wedge post_R P) \urcorner$   
**by** (*simp add: traces-def*)  
**qed**

**lemma** *postR-eq-traces*:  
**assumes**  $P$  is NCSP  $Q$  is NCSP  $\wedge s$ .  $tr[P]s = tr[Q]s$   
**shows**  $(pre_R(P) \wedge post_R(P)) = (pre_R(Q) \wedge post_R(Q))$

by (*metis* *assms* *dual-order*.*antisym* *order-refl* *postR-refine-traces*)

**lemma** *circus-fd-refine-intro*:

**assumes** *P* is NCSP *Q* is NCSP  $\wedge s. dv[[Q]]s \subseteq dv[[P]]s \wedge s. fl[[Q]]s \subseteq fl[[P]]s \wedge s. tr[[Q]]s \subseteq tr[[P]]s$   
**shows**  $P \sqsubseteq Q$

**proof** (*rule* *SRD-refine-intro'*, *simp-all* *add*: *closure* *assms*)

**show** *a*: '*pre<sub>R</sub>* *P*  $\Rightarrow$  *pre<sub>R</sub>* *Q*'

**using** *assms*(1) *assms*(2) *assms*(3) *preR-refine-divergences* *refBy-order* **by** *blast*

**show** *peri<sub>R</sub>* *P*  $\sqsubseteq$  (*pre<sub>R</sub>* *P*  $\wedge$  *peri<sub>R</sub>* *Q*)

**proof** –

**have** *peri<sub>R</sub>* *P*  $\sqsubseteq$  (*pre<sub>R</sub>* *Q*  $\wedge$  *peri<sub>R</sub>* *Q*)

**by** (*metis* (*no-types*) *assms*(1) *assms*(2) *assms*(4) *periR-refine-failures* *utp-pred-laws.le-inf-iff*)

**then show** *?thesis*

**by** (*metis* *a* *refBy-order* *utp-pred-laws.inf.order-iff* *utp-pred-laws.inf-assoc*)

**qed**

**show** *post<sub>R</sub>* *P*  $\sqsubseteq$  (*pre<sub>R</sub>* *P*  $\wedge$  *post<sub>R</sub>* *Q*)

**proof** –

**have** *post<sub>R</sub>* *P*  $\sqsubseteq$  (*pre<sub>R</sub>* *Q*  $\wedge$  *post<sub>R</sub>* *Q*)

**by** (*meson* *assms*(1) *assms*(2) *assms*(5) *postR-refine-traces* *utp-pred-laws.le-inf-iff*)

**then show** *?thesis*

**by** (*metis* *a* *refBy-order* *utp-pred-laws.inf.absorb-iff1* *utp-pred-laws.inf-assoc*)

**qed**

**qed**

## 10.2 Circus Operators

**lemma** *traces-Skip*:

$tr[[Skip]]s = \{([], s)\}$

**by** (*simp* *add*: *traces-def* *rdes* *alpha* *closure*, *rel-simp*)

**lemma** *failures-Skip*:

$fl[[Skip]]s = \{\}$

**by** (*simp* *add*: *failures-def*, *rdes-calc*)

**lemma** *divergences-Skip*:

$dv[[Skip]]s = \{\}$

**by** (*simp* *add*: *divergences-def*, *rdes-calc*)

**lemma** *traces-Stop*:

$tr[[Stop]]s = \{\}$

**by** (*simp* *add*: *traces-def*, *rdes-calc*)

**lemma** *failures-Stop*:

$fl[[Stop]]s = \{([], E) \mid E. True\}$

**by** (*simp* *add*: *failures-def*, *rdes-calc*, *rel-auto*)

**lemma** *divergences-Stop*:

$dv[[Stop]]s = \{\}$

**by** (*simp* *add*: *divergences-def*, *rdes-calc*)

**lemma** *traces-AssignsCSP*:

$tr[[\langle\sigma\rangle_C]]s = \{([], \sigma(s))\}$

**by** (*simp* *add*: *traces-def* *rdes* *closure* *usubst* *alpha*, *rel-auto*)

**lemma** *failures-AssignsCSP*:

$fl[[\langle\sigma\rangle_C]]s = \{\}$

by (simp add: failures-def, rdes-calc)

**lemma** *divergences-AssignsCSP*:

$dv\llbracket\langle\sigma\rangle_C\rrbracket s = \{\}$

by (simp add: divergences-def, rdes-calc)

**lemma** *failures-Miracle*:  $fl\llbracket\text{Miracle}\rrbracket s = \{\}$

by (simp add: failures-def rdes closure usubst)

**lemma** *divergences-Miracle*:  $dv\llbracket\text{Miracle}\rrbracket s = \{\}$

by (simp add: divergences-def rdes closure usubst)

**lemma** *failures-Chaos*:  $fl\llbracket\text{Chaos}\rrbracket s = \{\}$

by (simp add: failures-def rdes, rel-auto)

**lemma** *divergences-Chaos*:  $dv\llbracket\text{Chaos}\rrbracket s = UNIV$

by (simp add: divergences-def rdes, rel-auto)

**lemma** *traces-Chaos*:  $tr\llbracket\text{Chaos}\rrbracket s = \{\}$

by (simp add: traces-def rdes closure usubst)

**lemma** *divergences-cond*:

assumes  $P$  is NCSP  $Q$  is NCSP

shows  $dv\llbracket P \triangleleft b \triangleright_R Q \rrbracket s = (if (\llbracket b \rrbracket_e s) then dv\llbracket P \rrbracket s else dv\llbracket Q \rrbracket s)$

by (rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto)

**lemma** *traces-cond*:

assumes  $P$  is NCSP  $Q$  is NCSP

shows  $tr\llbracket P \triangleleft b \triangleright_R Q \rrbracket s = (if (\llbracket b \rrbracket_e s) then tr\llbracket P \rrbracket s else tr\llbracket Q \rrbracket s)$

by (rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto)

**lemma** *failures-cond*:

assumes  $P$  is NCSP  $Q$  is NCSP

shows  $fl\llbracket P \triangleleft b \triangleright_R Q \rrbracket s = (if (\llbracket b \rrbracket_e s) then fl\llbracket P \rrbracket s else fl\llbracket Q \rrbracket s)$

by (rdes-simp cls: assms, simp add: divergences-def failures-def rdes closure rpred assms, rel-auto)

**lemma** *divergences-guard*:

assumes  $P$  is NCSP

shows  $dv\llbracket g \&_u P \rrbracket s = (if (\llbracket g \rrbracket_e s) then dv\llbracket g \&_u P \rrbracket s else \{\})$

by (rdes-simp cls: assms, simp add: divergences-def traces-def rdes closure rpred assms, rel-auto)

**lemma** *traces-do*:  $tr\llbracket do_C(e) \rrbracket s = \{(\llbracket e \rrbracket_e s, s)\}$

by (rdes-simp, simp add: traces-def rdes closure rpred, rel-auto)

**lemma** *failures-do*:  $fl\llbracket do_C(e) \rrbracket s = \{(\llbracket \cdot \rrbracket, E) \mid E. \llbracket e \rrbracket_e s \notin E\}$

by (rdes-simp, simp add: failures-def rdes closure rpred usubst, rel-auto)

**lemma** *divergences-do*:  $dv\llbracket do_C(e) \rrbracket s = \{\}$

by (rel-auto)

**lemma** *divergences-seq*:

fixes  $P :: ('s, 'e)$  action

assumes  $P$  is NCSP  $Q$  is NCSP

shows  $dv\llbracket P ;; Q \rrbracket s = dv\llbracket P \rrbracket s \cup \{t_1 @ t_2 \mid t_1 t_2 s_0. (t_1, s_0) \in tr\llbracket P \rrbracket s \wedge t_2 \in dv\llbracket Q \rrbracket s_0\}$

(is ?lhs = ?rhs)

oops

lemma *traces-seq*:

fixes  $P :: ('s, 'e)$  action

assumes  $P$  is NCSP  $Q$  is NCSP

shows  $tr[P ;; Q]_s =$

$$\begin{aligned} & \{(t_1 @ t_2, s') \mid t_1 \ t_2 \ s_0 \ s'. (t_1, s_0) \in tr[P]_s \wedge (t_2, s') \in tr[Q]_{s_0} \\ & \quad \wedge (t_1 @ t_2) \notin dv[P]_s \\ & \quad \wedge (\forall (t, s_1) \in tr[P]_s. t \leq t_1 @ t_2 \longrightarrow (t_1 @ t_2) - t \notin dv[Q]_{s_1}) \} \end{aligned}$$

(is ?lhs = ?rhs)

proof

show ?lhs  $\subseteq$  ?rhs

proof (rdes-expand cls: assms, simp add: traces-def divergences-def rdes closure assms rdes-def unrest rpred usubst, auto)

fix  $t :: 'e$  list and  $s' :: 's$

let  $?\sigma = [\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle]$

assume

$a1: ?\sigma \dagger (post_R P ;; post_R Q)'$  and

$a2: [\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger pre_R P'$  and

$a3: [\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger (post_R P \text{ wr } pre_R Q)'$

from  $a1$  have  $?\sigma \dagger (\exists tr_0. ((post_R P)[\langle tr_0 \rangle / \$tr'] ;; (post_R Q)[\langle tr_0 \rangle / \$tr]) \wedge \langle tr_0 \rangle \leq_u \$tr')$

by (simp add: R2-tr-middle assms closure)

then obtain  $tr_0$  where  $p1: ?\sigma \dagger ((post_R P)[\langle tr_0 \rangle / \$tr'] ;; (post_R Q)[\langle tr_0 \rangle / \$tr])'$  and  $tr_0: tr_0$

$\leq t$

apply (simp add: usubst)

apply (erule taut-shEx-elim)

apply (simp add: unrest-all-circus-vars-st-st' closure unrest assms)

apply (rel-auto)

done

from  $p1$  have  $?\sigma \dagger (\exists st_0. (post_R P)[\langle tr_0 \rangle / \$tr'][\langle st_0 \rangle / \$st'] ;; (post_R Q)[\langle tr_0 \rangle / \$tr][\langle st_0 \rangle / \$st])'$

by (simp add: seqr-middle[of st, THEN sym])

then obtain  $s_0$  where  $?\sigma \dagger ((post_R P)[\langle s_0 \rangle, \langle tr_0 \rangle / \$st', \$tr'] ;; (post_R Q)[\langle s_0 \rangle, \langle tr_0 \rangle / \$st, \$tr])'$

apply (simp add: usubst)

apply (erule taut-shEx-elim)

apply (simp add: unrest-all-circus-vars-st-st' closure unrest assms)

apply (rel-auto)

done

hence  $(([\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tr_0 \rangle] \dagger post_R P) ;;$

$([\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger post_R Q))'$

by (rel-auto)

hence  $(([\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tr_0 \rangle] \dagger post_R P) \wedge$

$([\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger post_R Q))'$

by (simp add: seqr-to-conj unrest-any-circus-var assms closure unrest)

hence  $postP: ([\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tr_0 \rangle] \dagger post_R P)'$  and

$postQ': ([\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger post_R Q)'$

by (rel-auto)+

from  $postQ'$  have  $[\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle] \dagger [\$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle tr_0 \rangle + (\langle t \rangle - \langle tr_0 \rangle)] \dagger post_R Q'$

using  $tr_0$  by (rel-auto)

hence  $[\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle] \dagger [\$tr \mapsto_s 0, \$tr' \mapsto_s \langle t \rangle - \langle tr_0 \rangle] \dagger post_R Q'$

by (simp add: R2-subst-tr closure assms)

hence  $postQ: [\$st \mapsto_s \langle s_0 \rangle, \$st' \mapsto_s \langle s' \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t - tr_0 \rangle] \dagger post_R Q'$

by (rel-auto)

have  $preP: [\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tr_0 \rangle] \dagger pre_R P'$

proof -

**have**  $(pre_R P)[0, \langle tr_0 \rangle / \$tr, \$tr'] \sqsubseteq (pre_R P)[0, \langle t \rangle / \$tr, \$tr']$   
**by** (*simp add: RC-prefix-refine closure assms tr0*)  
**hence**  $[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle tr_0 \rangle] \dagger pre_R P \sqsubseteq [\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger pre_R P$   
**by** (*rel-auto*)  
**thus** *?thesis*  
**by** (*simp add: taut-refine-impl a2*)  
**qed**

**have**  $preQ: '[\$st \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t - tr_0 \rangle] \dagger pre_R Q'$

**proof** –

**from** *postP a3* **have**  $'[\$st \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger pre_R Q'$   
**apply** (*simp add: wp-rea-def*)  
**apply** (*rel-auto*)  
**using** *tr0* **apply** *blast+*  
**done**

**hence**  $'[\$st \mapsto_s \langle s_0 \rangle] \dagger [\$tr \mapsto_s \langle tr_0 \rangle, \$tr' \mapsto_s \langle tr_0 \rangle + (\langle t \rangle - \langle tr_0 \rangle)] \dagger pre_R Q'$   
**by** (*rel-auto*)

**hence**  $'[\$st \mapsto_s \langle s_0 \rangle] \dagger [\$tr \mapsto_s 0, \$tr' \mapsto_s \langle t \rangle - \langle tr_0 \rangle] \dagger pre_R Q'$

**by** (*simp add: R2-subst-tr closure assms*)

**thus** *?thesis*

**by** (*rel-auto*)

**qed**

**from** *a2* **have**  $ndiv: \neg '[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle] \dagger (\neg_r pre_R P)'$   
**by** (*rel-auto*)

**have**  $t - tr_0: tr_0 @ (t - tr_0) = t$

**using** *append-minus tr0* **by** *blast*

**from** *a3*

**have**  $wpr: \bigwedge t_0 s_1.$

$'[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger pre_R P' \implies$   
 $'[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger post_R P' \implies$   
 $t_0 \leq t \implies '[\$st \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t - t_0 \rangle] \dagger (\neg_r pre_R Q)' \implies False$

**proof** –

**fix**  $t_0 s_1$

**assume**  $b:$

$'[\$st \mapsto_s \langle s \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger pre_R P'$   
 $'[\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger post_R P'$   
 $t_0 \leq t$   
 $'[\$st \mapsto_s \langle s_1 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t - t_0 \rangle] \dagger (\neg_r pre_R Q)'$

**from** *a3* **have**  $c: \forall (s_0, t_0) \cdot \langle t_0 \rangle \leq_u \langle t \rangle$

$\wedge [\$st \mapsto_s \langle s \rangle, \$st' \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t_0 \rangle] \dagger post_R P$   
 $\implies [\$st \mapsto_s \langle s_0 \rangle, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \langle t \rangle - \langle t_0 \rangle] \dagger pre_R Q'$

**by** (*simp add: wp-rea-circus-form-alt[of post\_R P pre\_R Q] closure assms unrest usubst*)  
*(rel-simp)*

**from**  $c$   $b(2-4)$  **show** *False*

**by** (*rel-auto*)

**qed**

**show**  $\exists t_1 t_2.$

$t = t_1 @ t_2 \wedge$   
 $(\exists s_0. \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg] \dagger \text{pre}_R P \wedge$   
 $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg] \dagger \text{post}_R P' \wedge$   
 $\text{'}[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger \text{pre}_R Q \wedge$   
 $[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger \text{post}_R Q' \wedge$   
 $\neg \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (\neg_r \text{pre}_R P) \text{' } \wedge$   
 $(\forall t_0 s_1. \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger \text{pre}_R P \wedge$   
 $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger \text{post}_R P' \longrightarrow$   
 $t_0 \leq t_1 @ t_2 \longrightarrow \neg \text{'}[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t_0 \gg] \dagger (\neg_r$   
 $\text{pre}_R Q) \text{'})$   
**apply** (*rule-tac*  $x=tr_0$  **in**  $exI$ )  
**apply** (*rule-tac*  $x=(t - tr_0)$  **in**  $exI$ )  
**apply** (*auto*)  
**using**  $tr_0$  **apply** *auto*[1]  
**apply** (*rule-tac*  $x=s_0$  **in**  $exI$ )  
**apply** (*auto intro:wpr simp add: taut-conj preP preQ postP postQ ndiv wpr t-minus-tr0*)  
**done**  
**qed**

**show**  $?rhs \subseteq ?lhs$

**proof** (*rdes-expand cls: assms, simp add: traces-def divergences-def rdes closure assms rdes-def unrest rpred usubst, auto*)

**fix**  $t_1 t_2 :: 'e \text{ list}$  **and**  $s_0 s' :: 's$

**assume**

$a1: \neg \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (\neg_r \text{pre}_R P) \text{'}$  **and**  
 $a2: \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg] \dagger \text{pre}_R P' \text{'}$  **and**  
 $a3: \text{'}[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 \gg] \dagger \text{post}_R P' \text{'}$  **and**  
 $a4: \text{'}[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger \text{pre}_R Q' \text{'}$  **and**  
 $a5: \text{'}[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger \text{post}_R Q' \text{'}$  **and**  
 $a6: \forall t s_1. \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger \text{pre}_R P \wedge$   
 $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger \text{post}_R P' \longrightarrow$   
 $t \leq t_1 @ t_2 \longrightarrow \neg \text{'}[\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t \gg] \dagger (\neg_r \text{pre}_R Q) \text{'}$

**from**  $a1$  **have**  $\text{pre}P: \text{'}[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (\text{pre}_R P) \text{'}$

**by** (*simp add: taut-not unrest-all-circus-vars-st assms closure unrest, rel-auto*)

**have**  $\text{'}[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg] \dagger \text{post}_R Q' \text{'}$

**proof** –

**have**  $[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger \text{post}_R Q =$

$[\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg] \dagger [\$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_2 \gg] \dagger \text{post}_R Q$

**by** *rel-auto*

**also have**  $\dots = [\$st \mapsto_s \ll s_0 \gg, \$st' \mapsto_s \ll s' \gg] \dagger [\$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg] \dagger \text{post}_R Q$

**by** (*simp add: R2-subst-tr assms closure, rel-auto*)

**finally show**  $?thesis$  **using**  $a5$

**by** (*rel-auto*)

**qed**

**with**  $a3$

**have**  $\text{post}PQ: \text{'}[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (\text{post}_R P ;; \text{post}_R Q) \text{'}$

**by** (*rel-auto, meson Prefix-Order.prefixI*)

**have**  $\text{'}[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg] \dagger \text{pre}_R Q' \text{'}$

**proof** –

**have**  $[\$st \mapsto_s \ll s_0 \gg, \$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg] \dagger \text{pre}_R Q =$

$[\$st \mapsto_s \ll s_0 \gg] \dagger [\$tr \mapsto_s \ll t_1 \gg, \$tr' \mapsto_s \ll t_1 \gg + \ll t_2 \gg] \dagger \text{pre}_R Q$

by *rel-auto*  
 also have ... =  $[\$st \mapsto_s \ll s_0 \gg] \dagger [\$tr \mapsto_s 0, \$tr' \mapsto_s \ll t_2 \gg] \dagger pre_R Q$   
 by (*simp add: R2-subst-tr assms closure*)  
 finally show *?thesis using a4*  
 by (*rel-auto*)  
 qed

from *a6*  
 have *a6'*:  $\bigwedge t s_1. \llbracket t \leq t_1 @ t_2; ' [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger pre_R P'; ' [\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t \gg] \dagger post_R P' \rrbracket \implies$   
 $' [\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t \gg] \dagger pre_R Q'$   
 apply (*subst (asm) taut-not*)  
 apply (*simp add: unrest-all-circus-vars-st assms closure unrest*)  
 apply (*rel-auto*)  
 done

have *wpR*:  $' [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (post_R P wp_r pre_R Q)'$   
 proof –  
 have  $\bigwedge s_1 t_0. \llbracket t_0 \leq t_1 @ t_2; ' [\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P' \rrbracket$   
 $\implies ' [\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t_0 \gg] \dagger pre_R Q'$   
 proof –  
 fix *s1 t0*  
 assume *c*:  $t_0 \leq t_1 @ t_2$   $' [\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger post_R P'$   
  
 have *preP'*:  $' [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P'$   
 proof –  
 have  $(pre_R P) \llbracket 0, \ll t_0 \gg / \$tr, \$tr' \rrbracket \sqsubseteq (pre_R P) \llbracket 0, \ll t_1 @ t_2 \gg / \$tr, \$tr' \rrbracket$   
 by (*simp add: RC-prefix-refine closure assms c*)  
 hence  $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_0 \gg] \dagger pre_R P \sqsubseteq [\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger pre_R P$   
 by (*rel-auto*)  
 thus *?thesis*  
 by (*simp add: taut-refine-impl preP*)  
 qed

with *c a3 preP a6'* [of *t0 s1*] show  $' [\$st \mapsto_s \ll s_1 \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll (t_1 @ t_2) - t_0 \gg] \dagger pre_R Q'$   
 by (*simp*)  
 qed

thus *?thesis*  
 apply (*simp-all add: wp-rea-circus-form-alt assms closure unrest usubst rea-impl-alt-def*)  
 apply (*simp add: R1-def usubst tcontr-alt-def*)  
 apply (*auto intro!: taut-shAll-intro-2*)  
 apply (*rule taut-impl-intro*)  
 apply (*simp add: unrest-all-circus-vars-st-st' unrest closure assms*)  
 apply (*rel-simp*)  
 done

qed  
 show  $' ([\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger pre_R P \wedge$   
 $[\$st \mapsto_s \ll s \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (post_R P wp_r pre_R Q)) \wedge$   
 $[\$st \mapsto_s \ll s \gg, \$st' \mapsto_s \ll s' \gg, \$tr \mapsto_s \langle \rangle, \$tr' \mapsto_s \ll t_1 @ t_2 \gg] \dagger (post_R P ;; post_R Q)'$   
 by (*auto simp add: taut-conj preP postPQ wpR*)

qed  
qed

**lemma** *Cons-minus* [*simp*]:  $(a \# t) - [a] = t$   
by (*metis append-Cons append-Nil append-minus*)

**lemma** *traces-prefix*:  
assumes *P* is NCSP  
shows  $tr\llbracket\langle a \rangle\rrbracket \rightarrow_C P\llbracket s \rrbracket = \{(a \# t, s') \mid t s'. (t, s') \in tr\llbracket P \rrbracket s\}$   
apply (*auto simp add: PrefixCSP-def traces-seq traces-do divergences-do lit.rep-eq assms closure Healthy-if trace-divergence-disj*)  
apply (*meson assms trace-divergence-disj*)  
done

### 10.3 Deadlock Freedom

The following is a specification for deadlock free actions. In any intermediate observation, there must be at least one enabled event.

**definition** *CDF* ::  $(s, e)$  action where  
[*rdes-def*]:  $CDF = \mathbf{R}_s(\text{true}_r \vdash (\prod (s, t, E, e) \cdot \mathcal{E}(\langle\langle s \rangle\rangle, \langle\langle t \rangle\rangle, \langle\langle \text{insert } e \ E \rangle\rangle)) \diamond \text{true}_r)$

**lemma** *CDF-NCSP* [*closure*]: *CDF* is NCSP  
apply (*simp add: CDF-def*)  
apply (*rule NCSP-rdes-intro*)  
apply (*simp-all add: closure unrest*)  
apply (*rel-auto*)  
done

**lemma** *Skip-deadlock-free*:  $CDF \sqsubseteq \text{Skip}$   
by (*rdes-refine*)

end

## 11 Meta-theory for Stateful-Failure Reactive Designs

**theory** *utp-sf-rdes*  
**imports**  
  *utp-sfrd-core*  
  *utp-sfrd-rel*  
  *utp-sfrd-healths*  
  *utp-sfrd-contracts*  
  *utp-sfrd-ectchoice*  
  *utp-sfrd-prog*  
  *utp-sfrd-recursion*  
  *utp-sfrd-fdsem*  
**begin end**

## References

- [1] S. Foster, F. Zeyda, and J. Woodcock. Unifying heterogeneous state-spaces with lenses. In *ICTAC*, LNCS 9965. Springer, 2016.



- [2] M. V. M. Oliveira. *Formal Derivation of State-Rich Reactive Programs using Circus*. PhD thesis, Department of Computer Science - University of York, UK, 2006. YCST-2006-02.