7 Pairwise Digit Sums and Concatenate

7.1 Problem definition

Erich Friedmann introduces a "Problem of the Month" in February 2000 which Eric Angelini revived recently in SeqFan, defined as follows.

For integer n, let f(n) be the concatenation of the sums of every pair of consecutive digits of n. For example, f(82671)=108138, since 8+2=10, 2+6=8, 6+7=13, and 7+1=8. If n is a single digit, define f(n)=0.

See *http://www2.stetson.edu/~efriedma/mathmagic/0200.html* for more information.

Several interesting questions can be asked regarding this problem, some of which are answered on the above Web site.

Additional investigations are presented in the following.

7.2 Some basics

Pairwise summing of digits gives 1-digit sums in 55 cases (45 cases for the first pair in a number since numbers do not start with 0):

```
00
01 10
02 11 20
03 12 21 30
04 13 22 31 40
05 14 23 32 41 50
06 15 24 33 42 51 60
07 16 25 34 43 52 61 70
08 17 26 35 44 53 62 71 80
09 18 27 36 45 54 63 72 81 90
```

And 2-digit sums in 45 cases (first digit is always 1):

```
19 28 37 46 55 64 73 82 91
29 38 47 56 65 74 83 92
39 48 57 66 75 84 93
49 58 67 76 85 94
59 68 77 86 95
69 78 87 96
79 88 97
89 98
99
```

For a n-digit number there are $n-1$ pairs to be summed. Each pair results in 1 or 2 digits.

Therefore the resulting number will have m digits, $n-1 \le m \le 2(n-1)$.

For a number to become shorter, all the pairwise sums need to be 1 digit. Obviously the chance of this happening diminishes with the number of digits in the number.

Given m, what is the range of n?

$n-1 \le m$ and $m \le 2(n-1)$

$n \le m+1$ and $floor(m/2)+1 \le n$ so

$floor(m/2)+1 \le n \le m+1$

If an *n*-digit number is transformed into a *m*-digit number then there have been $t=m-n+1$ pairs that have yielded a 2-digit result, $0 \le t \le n-1$.

When *f(2056) = 2511*, we say that *2511* is the successor of *2056*, and conversely, *2056* is the predecessor of *2511*.

All numbers have a successor.

Sometimes the successor iteration leads to one value being repeated indefinitely as a fixed point, for example *0*.

Other iterations leads to a set of values being repeated in a cycle, for example *991 --> 1810 --> 991*.

Yet other iterations leads to numbers that grow in size indefinitely.

Some numbers have no predecessors, for example *110*.

Numbers can have many predecessors, for example *9 <-- {18, 27, 36, 45, 54, 63, 72, 81, 90}*.

7.3 Cycles

Some successor sequences lead into cycles.

The following cycles have been found by Joseph DeVincentis and others:

```
99(a+1) --> 181a --> 99(a+1) [where 0≤a≤8]
3333(a+1) --> 666(a+4) --> 12121a -->3333(a+1) [where 0≤a≤5]
```

or, more explicitly:

```
L1          1810 --> 991 -->
L2          1811 --> 992 -->
L3          1812 --> 993 -->
L4          1813 --> 994 -->
L5          1814 --> 995 -->
L6          1815 --> 996 -->
L7          1816 --> 997 -->
L8          1817 --> 998 -->
L9          1818 --> 999 -->
M1          121210 --> 33331 --> 6664 -->
M2          121211 --> 33332 --> 6665 -->
M3          121212 --> 33333 --> 6666 -->
M4          121213 --> 33334 --> 6667 -->
M5          121214 --> 33335 --> 6668 -->
M6          121215 --> 33336 --> 6669 -->
```

There are 15 cycles containing a total of 36 values.

7.4 Expanding cycles

Some iterations do not terminate with 0 or one of the cycles, they go on forever.

Such iterations of course lead to numbers that grow beyond limit.

Some (all?) of these infinite iterations exhibit a cyclic pattern in the first few digits. When a cycle is completed, at least one more digit has been added to the right, otherwise we would have a proper cycle.

The additional digits for each cycle makes it an expanding cycle.

Of course, there might be more digits to the right of the cycling numbers which may or may not expand by themselves.

The following 24 expanding cycles have been found, each has been given a name.

The information inside () are the extra digits generated with each cycle. When the count of these digits exceeds 9, the count is instead given inside (<>).

```
A1 18141 --> 9955 --> 18141(0)
A2 18146 --> 9951(0) --> 18146(1)
A3 18151 --> 9966 --> 18151(2)
A4 18157 --> 9961(2) --> 18157(3)
A5 18161 --> 9977 --> 18161(4)
A6 18168 --> 9971(4) --> 18168(5)
A7 18171 --> 9988 --> 18171(6)
A8 18179 --> 9981(6) --> 18179(7)
A9 18181 --> 9999 --> 18181(8)

B1 66661 --> 1212127 --> 333339 --> 66661(2)
B2 66666 --> 1212121(2) --> 333333(3) --> 66666(6)
B3 66678 --> 1212131(5) --> 333344(6) --> 66678(10)
B4 66691 --> 1212151(0) --> 333366(1) --> 66691(27)

C1 18121 --> 9933 --> 18126 --> 9938 --> 18121(1)
C2 18124 --> 9936 --> 18129 --> 9931(1) --> 18124(2)

D1 18111 --> 9922 --> 18114 --> 9925 --> 18117 --> 9928 --> 18111(0)
D2 18113 --> 9924 --> 18116 --> 9927 --> 18119 --> 9921(0) --> 18113(1)
D3 18131 --> 9944 --> 18138 --> 9941(1) --> 18135(2) --> 9948(7) --> 18131(215)

E1 66642 --> 1212106 --> 333316 --> 66647 --> 1212101(1) --> 333311(2) --> 66642(3)
E2 66681 --> 1212149 --> 333351(3) --> 66686(4) --> 1212141(410) --> 333355(551) -->
66681(010106)

F  222222 --> 44444 --> 8888 --> 161616 --> 77777 --> 14141414 --> 5555555 --> 101010101010 --
> 11111111111 --> 222222(2222)

G  66651 --> 1212116 --> 333327 --> 66659 --> 1212111(4) --> 333322(5) --> 66654(7) -->
1212119(11) --> 333321(0102) --> 66653(1112) --> 1212118(4223) --> 333329(12645) -->
66651(11038109)

H  18101 --> 9911 --> 18102 --> 9912 --> 18103 --> 9913 --> 18104 --> 9914 --> 18105 --> 9915
--> 18106 --> 9916 --> 18107 --> 9917 --> 18108 --> 9918 --> 18109 --> 9919 --> 18101(0)

I  101088 --> 111816 --> 22997 --> 411181(6) --> 52299(7) --> 741118(16) --> 115229(97) -->
26741(11816) --> 813115(22997) --> 94426(74111816) --> 13868(<10>) --> 411141(<14>) -->
52255(<18>) --> 74710(<25>) --> 111181(<30>) --> 22299(<40>) --> 441118(<50>) --> 85229(<68>)
--> 137411(<82>) --> 4101152(<112>) --> 511267(<141>) --> 62381(<188>) --> 85119(<241>) -->
13621(<318>) --> 4983(<403>) --> 13171(<532>) --> 4488(<680>) --> 81216(<881>) -->
9337(<1123>) --> 12610(<1453>) --> 3871(<1865>) --> 11158(<2407>) --> 22613(<3078>) -->
4874(<3961>) --> 121511(<5108>) --> 33662(<6540>) --> 69128(<8454>) --> 151031(<10813>) -->
66134(<14019>) --> 12747(<17945>) --> 391111(<23196>) --> 1210222(<29697>) --> 331244(<38401>)
--> 64368(<49279>) --> 10791(<63825>) --> 17161(<81773>) --> 8877(<105797>) -->
16151(<135743>) --> 7766(<175542>) --> 141312(<225282>) --> 55443(<291102>) -->
10987(<373857>) --> 19171(<483295>) --> 101088(<620681>)
```

The number of iterations until the cycle is closed is

```
A1 2
A2 2
A3 2
A4 2
A5 2
A6 2
A7 2
A8 2
A9 2
B1 3
B2 3
B3 3
B4 3
C1 4
```

```
C2 4
D1 6
D2 6
D3 6
E1 6
E2 6
F  9
G  12
H  18
I  53
```

Berend Jan van der Zwaag states 23 expanding cycles, he forgot to mention A3 (the value 9966 in his enumeration).

7.5 Successor statistics

Given a number, the successor iteration ends up in 0, one of the cycles (L-M), or one of the expanding cycles (A-I).

For all numbers up to 10^6 the outcome has been determined for the 10 intervals of size 10^5 resulting in the following table:

Interval	1	2	3	4	5	6	7	8	9	10
ZERO	20822	1475	1288	1064	918	781	637	548	447	320
L1	23	1	0	2	0	1	1	1	1	0
L2	27	1	0	2	0	1	1	0	1	1
L3	33	1	0	2	0	1	1	0	1	1
L4	60	3	1	3	0	3	2	0	2	1
L5	71	4	5	2	1	3	0	0	2	2
L6	81	4	4	4	2	4	0	0	2	2
L7	106	8	8	7	7	5	1	0	4	2
L8	79	4	6	3	4	4	1	1	0	1
L9	19	1	1	0	1	1	1	0	0	0
M1	19	1	1	1	1	1	1	0	0	0
M2	20	1	2	1	1	0	1	0	0	0
M3	21	2	2	2	1	0	1	0	1	0
M4	22	2	3	2	2	0	1	0	1	0
M5	30	3	4	2	2	2	1	0	1	0
M6	29	3	2	3	2	2	2	0	1	0
A1	9753	12964	11447	11849	10948	10176	11331	12278	13125	12828
A2	293	390	455	480	457	398	305	230	123	602
A3	858	1288	1247	1208	970	962	1221	882	873	1466
A4	2633	3438	3403	3901	4248	4740	3408	2693	2901	2668
A5	13599	17598	15508	16018	14980	16854	16515	18464	19115	16617
A6	735	833	612	747	901	1109	916	896	750	1038
A7	1785	3214	1907	2478	3243	2410	2138	1861	1955	2712
A8	59	94	64	58	43	43	30	31	20	210
A9	968	1306	1420	1185	1235	1331	1042	1085	1055	1661
B1	464	591	526	540	533	582	684	807	705	588
B2	561	279	299	401	403	349	901	985	1127	1086
B3	70	41	48	66	144	124	121	120	120	120
B4	115	112	135	146	193	168	232	286	240	202
C1	12112	15739	16582	14827	14048	14516	15655	15793	17271	16787
C2	100	162	147	122	26	34	43	30	41	445
D1	172	213	245	172	107	85	12	75	69	759
D2	103	198	159	107	58	69	41	55	51	362

D3	2145	1989	2233	2730	2475	2634	3272	3753	1731	2549
E1	130	124	131	205	125	129	218	206	189	195
E2	50	30	40	30	30	140	142	133	110	100
F	2609	2092	2765	3385	3919	2963	2782	3265	4431	4794
G	70	20	20	120	120	120	120	110	100	100
H	842	622	608	729	703	692	750	898	960	2162
I	28311	35149	38672	37396	39149	38563	37469	34514	32474	29619

The count of numbers ending in each cycle and expanding cycle varies greatly.

Zeros decrease as numbers increase.

Cycles are scarce overall and seems to get more scarce as the numbers increase.

All iterations that go off to infinity are one of the expanding cycles. No iteration with "random" digit pattern has been found for numbers < 10^6.

This suggests a few challenges:

1) Prove that the cycles L-M are the only possible ones.

2) Prove that the expanding cycles A-I are the only possible ones.

3) Is there a number above which no number goes to 0?

7.6 Predecessor statistics

There are 165 numbers < 10^3 with no predecessor:

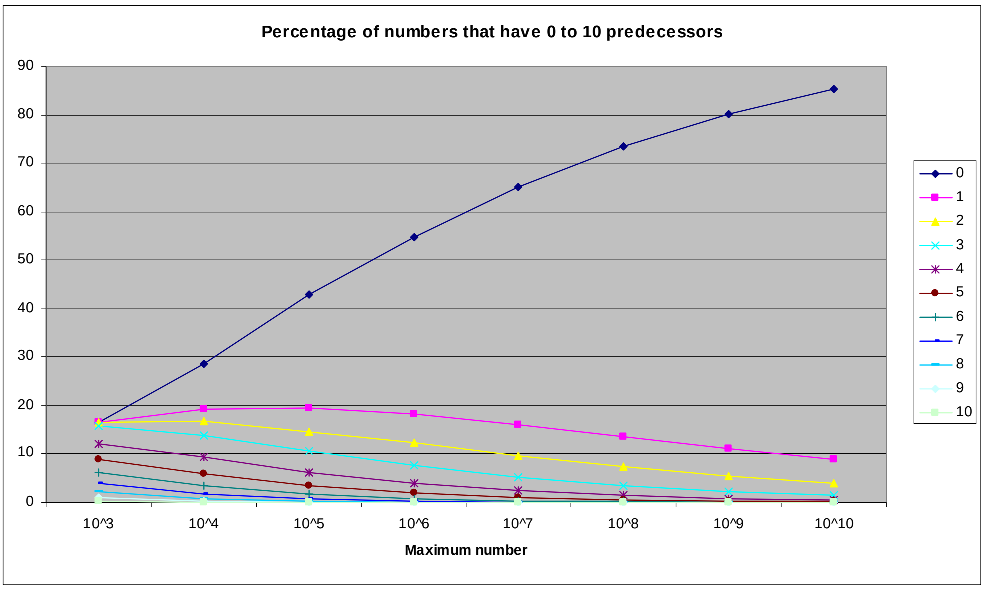
```
110
120
121
130
131
132
...
```

The requirement is that the middle digit is at least as big as the sum of the other two (Berend Jan van der Zwaag).

Using an algorithm that determines all predecessors to a given number, we get the following table.

Up to->	**10^3**	**10^4**	**10^5**	**10^6**	**10^7**	**10^8**	**10^9**	**10^10**
# predecessors (below)								
0	165	2867	42907	548144	6520193	73543012	801726786	8528432789
1	164	1915	19359	181374	1595745	13495392	110380431	881047611
2	164	1668	14622	122298	961085	7283293	53447981	383324517
3	158	1378	10570	76789	528707	3527594	22879751	145514909
4	121	929	6269	40189	243655	1432601	8187936	45923129
5	89	591	3446	19090	100242	511013	2536968	12372762
6	62	348	1716	8081	36128	157222	667483	2789252
7	40	184	748	2929	10946	40031	143455	508212
8	23	83	271	866	2683	8229	24994	75694
9	11	29	76	199	525	1390	3701	9892
10	2	7	15	40	90	222	513	1233
Sum	*999*	*9999*	*99999*	*999999*	*9999999*	*99999999*	*999999999*	*9999999999*

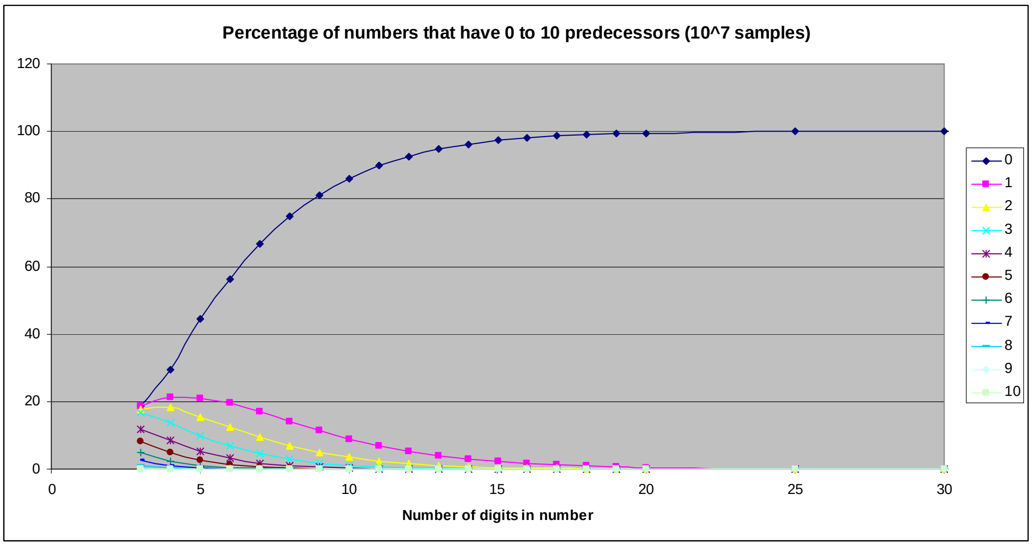
Numbers with no predecessors become more frequent as numbers grow larger, whereas ones with 1..10 predecessors seem to reach a maximum and then recede as illustrated graphically below.



What is surprising is that no number (below 10^10) has more than 10 predecessors. (The reverse algorithm used here is more complicated but requires less time and memory than using the forward algorithm for this purpose. The data in the above diagram up to 10^8 has been verified by using the forward algorithm.)

More predecessors maybe will appear for numbers larger than 10^10.

Using 10^7 random samples each of numbers up to 30 digits, the following graph was obtained.



This diagram is similar to the previous one.

No number with more that 10 predecessors was found, but it may be that 11 or more is extremely rare and needs a very large number of digits.

So there are two more challenges:

4) Find a number that has 11 or more predecessors, or

5) Prove that no number can have more than 10 predecessors.

7.7 Numbers with 10 predecessors

Numbers with 10 predecessors has been calculated for all numbers < 10^10 (see previous section).

The count of occurrences numbers less than different powers of 10 is:

10^3	10^4	10^5	10^6	10^7	10^8	10^9	10^10
2	7	15	40	90	222	513	1233

A few of the first entries are

Number	Predecessors...									
10	100	19	28	37	46	55	64	73	82	91
109	1009	190	281	372	463	554	645	736	827	918
1010	191	282	373	464	555	646	737	828	9100	919
1011	10010	192	283	374	465	556	647	738	829	9101
1099	10090	1909	2818	3727	4636	5545	6454	7363	8272	9181
1110	10100	1019	291	382	473	564	655	746	837	928
9910	18100	1819	2728	3637	4546	5455	6364	7273	8182	9091
10109	1918	2827	3736	4645	5554	6463	7372	8281	91009	9190
10119	100109	1927	2836	3745	4654	5563	6472	7381	8290	91018
10910	10091	28100	2819	3728	4637	5546	6455	7364	8273	9182
10911	10092	19010	28101	3729	4638	5547	6456	7365	8274	9183
10999	100909	19090	28181	37272	46363	55454	64545	73636	82727	91818
11109	101009	10190	2918	3827	4736	5645	6554	7463	8372	9281
98910	18091	27182	36273	45364	54455	63546	72637	81728	908100	90819
99109	181009	18190	27281	36372	45463	54554	63645	72736	81827	90918

From this and from the complete data set in the file *Pred10.txt* one can see some recurring patterns.

Maybe these patterns are a clue to solving the 10-predecessor problem.

A few more observations:

- Numbers start with 1 or 9.
- Numbers contain only digits 0, 1, 2, 8, 9, therefore no number contains any of the digits 3-7.
- Last digit is 0, 1 or 9.
- Last two digits are 09, 10, 11, 19 or 99.