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Authors
Zimmerman, R.W.
Myer, L.R.
Cook, N.G.W.

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Grain and Void Compression in Fractured and Porous Rocks

Robert W. Zimmerman,[1] *Larry R. Myer,*[1] *and Neville G. W. Cook* [1,2]

[1]Earth Sciences Division
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

[2]Department of Materials Science and Mineral Engineering
University of California

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Introduction

One of the features of the mechanical behavior of crustal rocks that differs from that of most other materials is that rocks are typically subjected to both external stresses and internal pore pressures that arise from the fluids that permeate the cracks and pores. These cracks and pores may be dispersed throughout the rock (as in the case of a sandstone), may be in the form of planar fractures of various sizes, or may exist in some combination of these porosity types. In any case, the various physical properties of crustal rocks typically depend on both the externally applied "confining" stresses, as well as the pore pressure. The implications of having a pore pressure act throughout a rock, in addition to the usual confining stresses, has been the subject of much study, particularly with regard to the manner in which the pore pressure combines with the confining stresses in influencing, say, the volumetric strain or the permeability.

For the special case where the external stress increments are hydrostatic, which is to say that the three principal stress increments are all equal, a formalism has been developed by Geertsma [1] and others to predict the resulting changes in both bulk volume and pore volume. This theory, which is summarized and elaborated upon by Zimmerman et al. [2], is developed under the assumption that the solid rock-forming material is microscopically homogenous and isotropic. Despite the restriction of hydrostatic loading, this theory is nevertheless non-trivial, since macroscopic nonlinearity is introduced through the closure of highly-compressible cracks and fissures. In terms of the notation used in [2], four compressibility coefficients are needed to describe the relations between bulk volume V_b, pore volume V_p, confining pressure P_c, and pore pressure P_p (see Fig. 1). These compressibilities can be defined by

$$C_{bc} = \frac{-1}{V_b} \left[\frac{\partial V_b}{\partial P_c} \right]_{P_p}, \quad (1a)$$

$$C_{bp} = \frac{1}{V_b} \left[\frac{\partial V_b}{\partial P_p} \right]_{P_c} , \quad (1b)$$

$$C_{pc} = \frac{-1}{V_p} \left[\frac{\partial V_p}{\partial P_c} \right]_{P_p} , \quad (1c)$$

$$C_{pp} = \frac{1}{V_p} \left[\frac{\partial V_p}{\partial P_p} \right]_{P_c} . \quad (1d)$$

Under the assumption of grain isotropy and micro-homogeneity, which is only an approximation for most rocks, the following relations can be derived between the porous rock compressibilities [1,2]:

$$C_{bc} - C_{bp} = C_r , \quad (2a)$$

$$C_{pc} - C_{pp} = C_r , \quad (2b)$$

$$C_{bp} = \phi C_{pc} , \quad (2c)$$

where C_r is the compressibility of the rock-forming material, and ϕ is the porosity. Equations (2a-c) allow one to express any three of the porous rock compressibilities in terms of the fourth compressibility, along with C_r and ϕ. Hence only one of the four porous rocks compressibilities is independent, and it can conveniently be taken to be C_{bc}, since this coefficient plays a role analogous to the standard compressibility of a solid body. The numerical value of C_{bc}, and therefore also the other compressibilities, depends on C_r and ϕ, as well as on the micro-geometry of the void space. These values also depend on the Poisson ratio of the rock grains, ν_r, although this is not apparent from equations (1) and (2). From dimensional considerations, C_{bc} must depend on C_r linearly, which is to say $C_{bc} = C_r F(\nu_r, \phi, \text{pore geometry})$, where F is some dimensionless function. Since the derivation of equations (2) requires no assumption of macroscopic homogeneity, nor the existence of a statistically-meaningful number of pores, these equations apply equally well to a porous rock or, say, a rock sample containing a single fracture. Experimental data illustrating that equations (2a-c) are reasonably accurate when applied to the compression of consolidated sandstones can be found in [1,2].

If the assumption of microscopic homogeneity is relaxed, but the assumption of (incremental) linear elastic behavior is maintained, then equations (2a) and (2b) no longer hold. Equation (2c), on the other hand, follows from the reciprocal theorem of elasticity [3], and does not require an assumption of micro-homogeneity. In this more general development, C_r on the right-hand side of equations (2a,b) must be replaced with new parameters, C_s and C_ϕ, respectively. For a rock that consists of various different mineral components, there are no definite relationships between C_r, C_s, and C_ϕ, although some bounds are known [4].

Mean Stress in Rock Grains

Equations (1) and (2), and further equations that can be derived from them, such as those governing the total (non-incremental) bulk and pore strains (see [2]), are useful, for example, in petroleum reservoir calculations, in which the pore volume is an important parameter. Alternatively, we could develop equations in which V_r and ϕ were the basic kinematic variables, instead of V_b and V_p [3]. To derive the equations governing the changes in V_r and ϕ, under elastic, hydrostatic conditions, we start with the relations

$$V_r = V_b - V_p \,, \tag{3a}$$

$$\phi = V_p / V_b \,. \tag{3b}$$

Making use of the fact that $d\phi/\phi = d\ln\phi = d\ln V_p - d\ln V_b$, along with relations between the compressibilities given by equation (2), we find after some straightforward algebraic manipulations that

$$dV_r = \frac{-V_r C_r (dP_c - \phi dP_p)}{(1-\phi)} \,, \tag{4a}$$

$$d\phi = -[C_{bc}(1-\phi) - C_r](dP_c - dP_p) \,. \tag{4b}$$

The above equations show that the volume change in the rock grains is governed by the effective stress, $P_c - \phi P_p$, whereas the change in porosity is governed by the differential stress, $P_c - P_p$.

An interesting implication of equation (4a) can be found by recognizing that

$$dV_r = -V_r C_r d\bar{P} \,, \tag{5}$$

where $\bar{P}$ is the volumetric average value of the mean normal stress (i.e., pressure) in the rock grains. Equation (5) is obtained by integrating Hooke's law throughout the solid region of the rock, and contains no reference to the geometry of the pore space. This equation merely states that the average volumetric strain dV_r/V_r is related to the average stress $\bar{P}$ by the same Hooke's law that relates the local strain to the local stress at each point [see 5, p. 395]. Comparison of equations (4a) and (5) shows that the average value of the pressure in the rock grains is given by

$$\bar{P} = \frac{P_c - \phi P_p}{1-\phi}. \tag{6}$$

Equation (6) can be partially "verified" by noting that, in the special case where $P_p = P_c$, it reduces to $\bar{P} = P_c$, as one would expect. Equation (6) also correctly predicts that $\bar{P}$ reduces to P_c in the limit as the porosity goes to zero. We further note that, in the absence of a pore pressure (or an increment thereof), the presence of porosity has the effect of amplifying the confining pressure by a factor of $1/(1-\phi)$. This effect is completely independent of the shape of the pores, and depends only on the value of ϕ.

Although equation (6) was derived under the assumption of isotropic, micro-homogeneous rock grains, it is in fact completely general. A rigorous derivation of this equation is given in the Appendix, based only on the principle of conservation of linear momentum. The following non-rigorous derivation, which captures the fact that this result relies purely on considerations of static equilibrium, has been given by Greenwald [6], among others. Consider the idealized two-dimensional porous body in Fig. 2a, containing one pore, subjected to an external confining pressure P_c, and a pore pressure P_p. Now imagine a slice taken through the pore, and a free-body diagram constructed for the upper portion, as is done in elementary statics (see Fig. 2b). This slicing operation exposes the internal stress $\bar{P}$, which acts over the solid grain material. For an isotropic rock, the areal porosity of the exposed slice indicated by the dotted line in Fig. 2 must equal the volumetric porosity, on average; hence the ratio of the length of the exposed pore to the total length must equal ϕ.

For this body to be in static equilibrium, the sum of the downward-acting forces must be balanced by the sum of all upward-acting forces. For a unit depth into the page, the net downward acting force is P_c, whereas the net upward force is $(1-\phi)\bar{P}+\phi P_p$. Here we use a theorem from elementary fluid statics which states that the resultant force in a given direction, due to a pressure acting over a curved surface, is equal to the magnitude of that pressure multiplied by the projection of the curved surface onto a plane perpendicular to the direction in question. Equating the upward and downward forces, and solving for the mean grain stress $\bar{P}$, yields equation (6).

If the rock is subjected to a state of pure shear on its outer boundary, a force-balance argument similar to that given above shows that the average shear stress in the rock grains is equal to the externally applied shear stress divided by the factor $(1-\phi)$. This can easily be seen by replacing the normal stress P_c with a shear stress τ in Fig. 2, in which case a force balance shows that the average internal stress must be a shear of magnitude $\tau/(1-\phi)$ in order for the piece of rock to be in equilibrium. Since the pore fluid cannot exert a shear stress along the walls of the pores, there is no "pore shear stress" analogous to the pore pressure. It also follows that an externally applied shear stress gives rise to no average hydrostatic stress in the rock grains. These results are actually true regardless of the constitutive (stress-strain) behavior of the rock, as is shown rigorously in the Appendix.

Compression of Rock Grains

Equation (6) for the mean pressure in the rock grains does not depend in any way on the geometry of the voids, but only on the relative ratio of grain volume to pore volume. This is perhaps unexpected, since the local stresses in the rock grains *are* highly dependent on the geometry. For example, singular stress concentrations will exist at any sharp corner of the grain/void interface, yet equation (6) implies that these singularities have no effect on the average stress in the rock. Another interesting result pertaining to the compression of a porous or fractured rock can be derived by considering equation (4a) under conditions of constant pore pressure. Since $V_r/(1-\phi) = V_b$, we have

$$dV_r = -V_b C_r dP_c \,, \tag{7}$$

whenever P_p is constant. Equation (7) shows that if the confining stress is varied, the change in volume of the rock grains is not only independent of the geometry of the void space, but is also independent of the *amount* of void space. This is in sharp contrast to the changes in bulk volume or pore volume, which depend very critically on both the geometry of the pore space and the total porosity [7, Part Two]. This result implies, for example, that if we consider a rock under fixed stress, and imagine that additional pores are cut out of this rock, the stresses and strains in the rock grains will redistribute themselves so as to maintain the same total volumetric dilatation. This result will be of use in developing conceptual models for the deformation of rock fractures, as discussed below. It may also be useful in analyzing processes such as stress dissolution, in which the porosity of a rock may change, while the stress it is subjected to remains fixed.

Implications for Fracture Compressibility Models

The fact that the volumetric change of the rock mineral phase in an externally-compressed porous or fractured rock is independent of the porosity should provide help in developing conceptual models of fracture deformation. If a rock specimen is compressed uniaxially, the existence of a single fracture lying perpendicular to the applied load will cause a drastic increase in the rock's apparent elastic modulus [8,9]. This effect is particularly pronounced at low stresses, at which the "excess" compressibility due to the fracture may be very much larger than the compressibility that would be measured in an intact rock. If the deformation that would exist in a similarly-shaped but *unfractured* specimen is subtracted from the measured deformation, the excess deformation can unambiguously be attributed to the presence of the fracture, and be used to define the "fracture compressibility" [10].

Difficulties in interpreting and modeling fracture compressibility tests arise when one attempts to attribute some of the excess compressibility to "void deformation", and some portion of it to deformation of the rock minerals, often referred to in this context as "asperity deformation". Consider an experimental set-up such as shown in Fig. 3, in which a fractured rock specimen of length L and cross-sectional area A is subjected to a longitudinal stress P. For the purposes of this analysis, we assume that the pore pressure is held constant throughout the experiment. As a fracture will not "interact" with principal normal stresses that act parallel to its nominal plane [11,12], we can assume that the load P acts hydrostatically, not just uniaxially, so that the conclusions presented above can be invoked.

During a compression test of a fractured sample, one typically measures the overall bulk deformation, and then attempts to relate this bulk deformation to that of the void space. Since $V_b = V_r + V_p$, the measured bulk volumetric deformation is the sum of the volume change of the rock material and that of the void space. But according to equation (7), dV_r has exactly the same value as it would have in the absence of the fracture. Hence, any measured bulk volume change over and above that which would occur in an intact rock *must* occur within the void space of the fracture. An equivalent conclusion was reached by Cook [13] based on an analysis of a planar fracture in an otherwise infinite rock: "Therefore, any apparent reduction in volume of material in each half space as a result of asperity deformation must be compensated by corresponding expansion into the void space between the surfaces".

The deformed boundary of the rock, after an increment of confining stress is applied, is indicated by the dotted line in Fig. 3. Two phenomena can be said to occur: the asperities compress, and the rock bulges into the void space [14, Fig. 4]. Some micromechanical analyses that attempt to predict the fracture compressibility focus on one, or the other, of these phenomena. For example, the "bed-of-nails" model [8] essentially assumes that *all* of the excess fracture deformation occurs in the asperities, whereas Tsang and Witherspoon [9] consider only the compression of the crack-like void spaces. The analysis which led to equation (7) shows that the deformation of the crack-like voids does in fact exactly equal the excess bulk deformation of the fractured rock. It would seem to be problematic, however, to attempt to calculate the excess bulk deformation due to the fracture by analyzing the deformation of the asperities. This is because the asperity deformation is compensated by expansion of the rock into the void space, so to speak, and the net "excess" deformation of the rock grains is zero. The deformation of the material in the asperities represents, by definition, only part of the total deformation of the rock material. Moreover, equation (7) shows that there is no connection between the total volumetric dilatation of the rock grains, and the volumetric dilatation of the void space. In this regard we mention the work of Xu and King [14], who found that the asperity deformation calculated using a "bed-of-nails" model was typically two orders of magnitude less than the void deformation calculated using an ellipsoidal crack model. The analysis given above implies that the asperity deformation, although negligibly small, should nevertheless not be added to the void compression term when calculating the overall bulk deformation of the rock.

Summary and Conclusions

Aside from a brief review of certain definitions and results concerning the deformation of fluid-saturated porous rocks, this note contains two main results. The first is a rigorous proof of equation (6), which relates the volumetric average of the mean normal stress (i.e., the pressure) in the rock grains to the confining pressure, the pore pressure, and the porosity. It is a general relation, in the sense that it applies to any porous material, regardless of the pore structure. This relation may be useful, for example, in studies of stress dissolution or other thermodynamic processes, since $\bar{P}$ plays the role of the thermodynamic pressure [15].

Our other main result is equation (7), which shows that the total volume change within the mineral grains, due to an increment of confining pressure, is completely independent of the pore structure or the amount of porosity. This perhaps surprising result can be verified, for example, for the classic problem of a hollow spherical shell. Consider a hollow sphere made of an elastic material of compressibility $C = 1/K$, with inner radius a and outer radius b, subjected to external pressure P. Simple calculations based on the solution for the displacements [see 5, p. 344] show that the volume change in the material comprising the shell itself is equal to $-4\pi b^3 P/3K$, regardless of the value of the inner radius a.

The implication of equation (7) is that any bulk volumetric strain that occurs in an externally pressurized porous rock, over and above that which would occur in a similarly-shaped non-porous rock composed of the same minerals, must exactly equal the total change in pore volume. This result can also be applied to tests in which a rock that contains a single fracture is loaded by normal stress. In this case, the excess volume change of the rock, due to the existence of the fracture, is exactly equal to the reduction in void volume within the fracture. This result should be useful in developing models for fracture compressibility and fracture permeability.

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Appendix: Mean Hydrostatic Stress in Rock Grains

In this appendix we give a more rigorous and general derivation of equation (6), which will show that the result $\bar{P} = (P_c - \phi P_p)/(1-\phi)$ does not require the assumption of homogeneity, isotropy, or even elastic behavior. The result is therefore *independent* of the constitutive behavior of the rock grains.

Imagine (see Fig. 1) a piece of a porous body V, with an outer boundary denoted by ∂V_b. If this body is a chunk of rock that we imagine to be "carved out" of a larger rock mass, its "outer" boundary may be its interface with neighbouring pieces of rock. The inner boundaries of the body, consisting of the surfaces of the pores, are denoted by $\{\partial V_1, \partial V_2, ... \partial V_n\}$. Collectively these boundaries can be denoted by $\{\partial V_k\}$, where k is an index. Let the components of the local stress tensor in the grains be denoted by as τ_{ij}. In the following derivation, we will use the notation $F_{,j}$ to denote $\partial F / \partial x_j$, as well as the convention of implied summation over repeated indices, i.e., $\tau_{ii} = \tau_{11} + \tau_{22} + \tau_{33}$. We will also make use of the Kronecker delta tensor, δ_{ij}, which is defined to be equal to 1 if $i = j$, and to be equal to 0 if $i \neq j$. In the terminology of matrix algebra, δ_{ij} is the 3×3 identity matrix. The x_3 coordinate will be oriented in the vertical direction, parallel to the gravitational gradient.

The law of conservation of linear momentum then states, independently of the stress-strain relationship of the grains, that $\tau_{ij,j} + \rho g \delta_{i3} = 0$, where ρ is the local density of the rock grain, and g is gravitational acceleration [5]. In this analysis we can neglect the gravitational body force, $\rho g \delta_{i3}$, since in general it will be negligible compared to the local stress gradients. To see this, consider the case where the chunk of rock shown in Fig. 1 is at a depth D below the surface, in which case the stresses in each grain will be on the order of ρGD. If the mean grain size (or distance between pores) is d, the stress gradient terms appearing in $\tau_{ij,j}$ will be on the order of $\rho gD/d$, and so the ratio of magnitudes of the stress gradient terms and the gravitational body force will be $\rho gD/\rho gd = D/d \gg 1$.

Using the product rule for derivatives, we can say that

$$(\tau_{ij}x_i)_{,j} = \tau_{ij,j}x_i + \tau_{ij}x_{i,j} \tag{A1}$$

But $x_{i,j} = \partial x_i / \partial x_j = \delta_{ij}$, so $\tau_{ij}x_{i,j} = \tau_{ij}\delta_{ij} = \tau_{ii}$. Also, $\tau_{ij,j} = 0$ in equation (A1) by conservation of momentum, so

$$(\tau_{ij}x_i)_{,j} = \tau_{ii} \ . \tag{A2}$$

To find the average hydrostatic stress in the grains, we use equation (A2) to average τ_{ii} over the volume occupied by the grains (shaded area in Fig. 1), which will be denoted by V_r, and eventually make use of the definition $P = -\tau_{ii}/3$. First, using equation (A2) and the definition of the averaging process,

$$<\tau_{ii}> = \frac{1}{V_r}\int_{V_r}(\tau_{ij}x_i)_{,j}\,dV \ . \tag{A3}$$

Using the divergence theorem, we can transform this integral into an integral over the surface of the region V_r, as follows:

$$<\tau_{ii}> = \frac{1}{V_r}\int_{\partial V_r}\tau_{ij}x_i n_j\,dA \ , \tag{A4}$$

where n_i is the outward unit normal vector to the surface, and ∂V_r denotes the total surface of the region V_r, which consists of the outer surface ∂V_b, and the pore surfaces ∂V_k. Hence

$$<\tau_{ii}> = \frac{1}{V_r}\left[\int_{\partial V_b} \tau_{ij} x_i n_j dA + \sum_{k=1}^{n} \int_{\partial V_k} \tau_{ij} x_i n_j dA\right] . \qquad \text{(A5)}$$

Now imagine that the outer boundary ∂V_b is subjected to a hydrostatic pressure P_c, while the inner boundaries are subjected to a pore pressure P_p. (It may be convenient to suppose that the volume V is small, in the sense that, say, $V^{1/3} \ll D$, so that the confining pressure at the outer boundary is nearly uniform.) Since $\tau_{ij} n_j$ represents the traction vector on the surface, and a hydrostatic stress always acts normal to a surface, on the outer boundary we have $\tau_{ij} n_j = -P_c n_i$, while on the inner pore boundaries $\tau_{ij} n_j = -P_p n_i$. Hence equation (A5) can be written as

$$<\tau_{ii}> = \frac{1}{V_r}\left[-P_c \int_{\partial V_b} x_i n_i dA - P_p \sum_{k=1}^{n} \int_{\partial V_k} x_i n_i dA\right] . \qquad \text{(A6)}$$

We now use the divergence theorem in the opposite direction, so to speak, to transform the surface integrals back to volume integrals. In this case, however, we note that the region interior to ∂V_b is the entire volume V, not just the region occupied by the grains. Similarly, the interior of each pore surface ∂V_k is the region occupied by the kth pore. However, the unit normal vectors on the pore surfaces that were pointing *outward* with respect to the grain volume V_r are now pointing *inward* with respect to the pore volumes V_k, so a minus sign appears in the integrals over V_k. Hence, using $x_{i,i} = \delta_{ii} = 3$, we have

$$<\tau_{ii}> = \frac{1}{V_r}\left[-P_c \int_{V_b} x_{i,i} dV + P_p \sum_{i=1}^{n} \int_{V_k} x_{i,i} dV\right]$$

$$= \frac{1}{V_r}\left[-P_c \int\limits_{V_b} 3dV + P_p \sum_{k=1}^{n} \int\limits_{V_k} 3dV\right]$$

$$= \frac{1}{V_r}\left[-3P_c V_b + 3P_p \sum_{k=1}^{n} V_k\right] . \tag{A7}$$

We now divide through by 3, and use the facts that $V_r = (1-\phi)V_b$, and $\sum V_k = \phi V_b$, to arrive at

$$\frac{<\tau_{ii}>}{3} = \frac{1}{V_b(1-\phi)}\left[-P_c V_b + P_p \phi V_b\right] . \tag{A8}$$

But $-<\tau_{ii}>/3$ is simply the average pressure in the grains, $\bar{P}$, so after cancelling out V_b we arrive at

$$\bar{P} = \frac{P_c - \phi P_p}{1-\phi} . \tag{A9}$$

This completes the proof, which required no assumptions concerning elastic behavior, grain isotropy, etc.

Although the above result was derived for the case where the external stress was hydrostatic, it continues to hold if the external boundary of the rock is also subjected to a uniform shear (deviatoric) stress, as can be shown by a slight generalization of the proof given above. In other words, if a uniform shear stress is applied along the outer boundary of the rock, the incremental mean pressure in the mineral grains will be zero. (If the rock grains are also assumed to be microscopically homogeneous and elastic, equation (5) then shows that a uniform external shear stress will cause no change in the mineral grain volume.) To prove that an external shear gives rise to no mean pressure, we return to the step between equations (A5) and (A6), but let $\tau_{ij}n_j = \tau^o_{ij}n_j$ on the outer boundary, where τ^o_{ij} is any uniform, deviatoric stress tensor. The traction on the inner pore walls can be taken to be zero, since the pore fluid pressure will always cause normal tractions, not shear tractions. Hence the equation analogous to equation (A5) for the shear case will be

$$<\tau_{ii}> = \frac{1}{V_r}\int_{\partial V_b} \tau^o_{ij} x_i n_j dA \ . \qquad \text{(A10)}$$

We now take the uniform tensor τ^o_{ij} outside of the integral, and again use the divergence theorem to transform the surface integral into a volume integral, which yields

$$<\tau_{ii}> = \frac{\tau^o_{ij}}{V_r}\int_{V_b} x_{i,j} dV \ . \qquad \text{(A11)}$$

But $x_{i,j} = \delta_{ij}$, so the integral in equation (A11) gives $\delta_{ij}V_b$, in which case we have

$$<\tau_{ii}> = \tau^o_{ij}\delta_{ij}\frac{V_b}{V_r} = \frac{\tau^o_{ii}}{1-\phi} = 0 \, , \qquad \text{(A12)}$$

where in the last step we use the fact that the trace τ^o_{ii} of the deviatoric tensor τ^o_{ij} is zero.

Figure Captions

Fig. 1. Schematic diagram of a porous rock. The shaded region denotes the solid mineral phase, whose total volume is V_r. The solid boundary encloses the bulk volume V_b, and the dotted lines enclose the pores, whose volume is V_p.

Fig. 2. Diagram used in non-rigorous, but essentially correct, derivation of the equation for the average pressure in the mineral grains. A force balance on the upper portion of the rock, in the vertical direction, leads to $\bar{P} = (P_c - \phi P_p)/(1-\phi)$.

Fig. 3. Schematic diagram of normal compression of a fractured rock. The curved solid line denotes the boundary of the rock material before deformation, and the dotted line denotes the boundary after an increase in the confining stress. If the bottom face of the rock specimen is taken as the datum, the fracture asperities (see point marked =>) appear to be compressed. If the nominal fracture plane is taken as the datum, the rock material appears to expand into the void space.

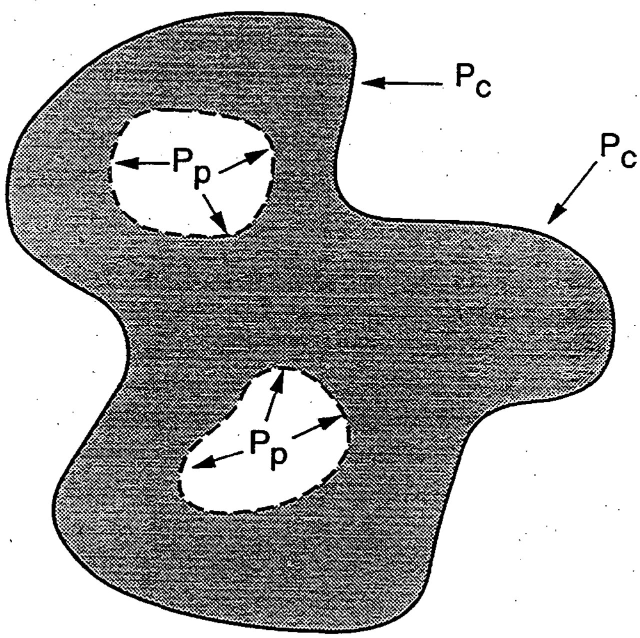

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Fig. 1. Schematic diagram of a porous rock. The shaded region denotes the solid mineral phase, whose total volume is V_r. The solid boundary encloses the bulk volume V_b, and the dotted lines enclose the pores, whose volume is V_p.

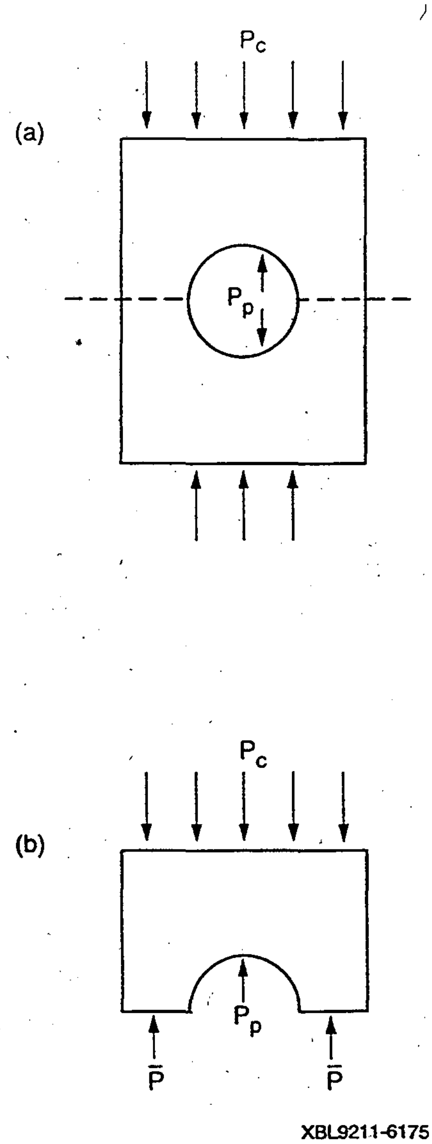

Fig. 2. Diagram used in non-rigorous, but essentially correct, derivation of the equation for the average pressure in the mineral grains. A force balance on the upper portion of the rock, in the vertical direction, leads to $\overline{P} = (P_c - \phi P_p)/(1-\phi)$.

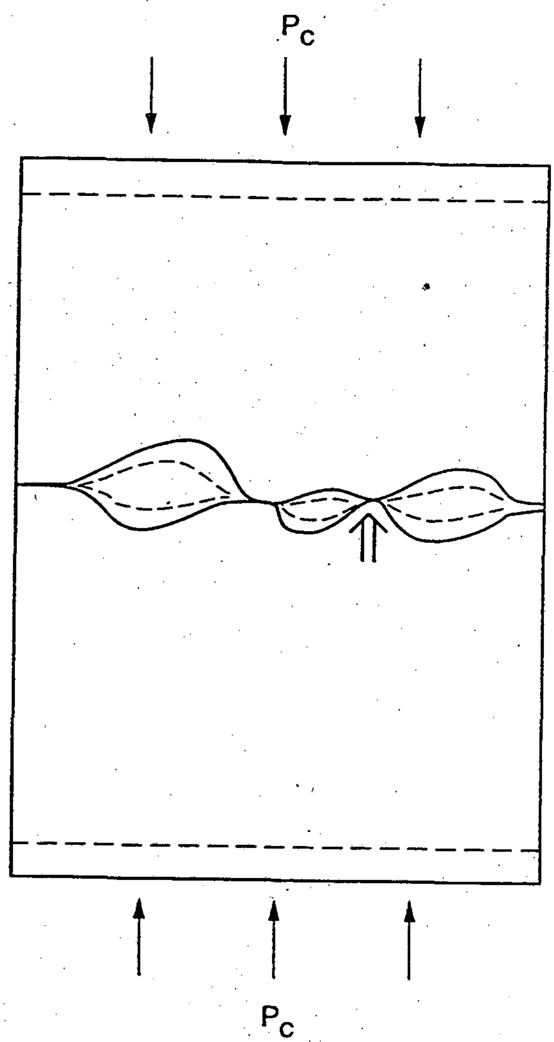

Fig. 3. Schematic diagram of normal compression of a fractured rock. The curved solid line denotes the boundary of the rock material before deformation, and the dotted line denotes the boundary after an increase in the confining stress. If the bottom face of the rock specimen is taken as the datum, the fracture asperities (see point marked =>) appear to be compressed. If the nominal fracture plane is taken as the datum, the rock material appears to expand into the void space.

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
TECHNICAL INFORMATION DEPARTMENT
BERKELEY, CALIFORNIA 94720