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Conditions for Maximizing and Minimizing Diagonal Elements of a Hermitian Matrix

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PROBLEMS AND SOLUTIONS

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All problems and solutions should be sent, typewritten in duplicate, to Murray S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1. *An asterisk placed beside a problem number indicates that the problem was submitted without solution. Proposers and solvers whose solutions are published will receive* 5 *reprints of the corresponding problem section. Other solvers will receive just one reprint provided a self-addressed stamped (U.S.A. or Canada) envelope is enclosed. Proposers and solvers desiring acknowledgment of their contributions should include a self-addressed stamped postcard (no stamps necessary for outside the U.S.A. and Canada). Solutions should be received by March* 31, 1989.

PROBLEMS

Birthday Candles Blowout Problem

Problem 88-16, *by* M. PACHTER (National Research Institute for Mathematical Sciences, Pretoria, South Africa).

A straight-line high-pressure front approaches at constant speed and sweeps through a given array of n lit candles (supposedly arranged on top of a birthday cake). The candles are extinguished one at a time at the instant of passage through the moving front (see Fig. 1). The blowout sequence of the candles is observed (this is a permutation of n elements), and on the basis of this "measurement," it is required that we estimate the direction from which the front came. Obviously, the estimate consists of a confidence "interval" C, which is an angular sector.

For a given array of n candles, we have the following questions.

(a) From the numerical sequence information, how should we calculate the uncertainty C in the direction of arrival of the front?

(b) If there is no a priori information on the possible direction of arrival of the front, how should we arrange n available candles to minimize the maximal possible measure of uncertainty C in the estimated direction of movement of the front?

(c) If it is known that the possible direction of arrival of the front is confined to an a priori known angular sector of extent $2\alpha\pi$ radius, where $0<\alpha<1$, then how should n available candles be arranged to minimize the maximal possible measure of uncertainty C in the direction of movement of the front?

Conditions for Maximizing and Minimizing Diagonal Elements of a Hermitian Matrix

Problem 88-17*, *by* Y. HUA and T. K. SARKAR (Syracuse University).

Let $\omega_1, \omega_2, \cdots, \omega_M$ be distinct real numbers and let $e_1, e_2, \cdots, e_M$ be complex numbers satisfying $|e_1|=|e_2|=\cdots=|e_M|=1$. With $N \geqq M$, let Z be the $N\times M$ complex matrix whose (r, s) elements is $\exp(ir\omega_s)$. Set

$$Y=\begin{bmatrix} Z \\ ZE \end{bmatrix},$$

where $E=\operatorname{diag}[e_1, e_2, \cdots, e_M]$. Define the Hermitian matrix

$$J=Y^{+}\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}Y^{+H},$$

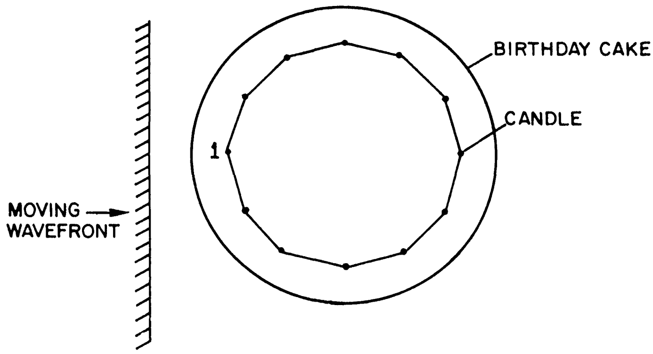

FIG. 1. *n birthday candles are arranged at the n vertices of the regular n-gon.*

where H denotes complex conjugate transpose, + denotes the Moore–Penrose generalized inverse, and R is an $N\times N$ Hermitian Toeplitz matrix with positive eigenvalues. (Since Y has full column rank, $Y^+=(Y^HY)^{-1}Y^H$.) Prove or disprove that, with respect to variations in $e_1, e_2, \cdots, e_M$, we have the following:

(1) All diagonal elements of J are maximized if and only if $e_1=e_2=\cdots=e_M$.

(2) For $M=2$, all diagonal elements of J are minimized if and only if $e_1=-e_2$.

To clarify (1), note that if $e_1=e_2=\cdots=e_M=\exp(i\theta)$ then J is constant for all θ. Statements (1) and (2) are our conjectures supported by the analysis in [1] and by many numerical examples. The problem arose from our perturbation analysis in [1] of an estimator of $\omega_1, \omega_2, \cdots, \omega_M$ from a short data sequence (y_k) where

$$y_k=\sum_{s=1}^{M} a_s \exp(ik\omega_s), \qquad k=1,2,\cdots,N.$$

REFERENCE

[1] Y. HUA AND T. K. SARKAR, *Perturbation analysis of TK method for harmonic retrieval problems*, IEEE Trans. Acoust. Speech Signal Process., ASSP-36 (1988), pp. 228–240.

A Probability Generating Function

Problem 88-18, *by* A. A. JAGERS (Universiteit Twente, Enschede, the Netherlands).

Let h_n denote the probability generating function of the maximum Z_n of n independent identically distributed geometric random variables $X_1, X_2, \cdots, X_n$ with $P\{X_j=k\}=pq^k$ $(k=0,1,2,\cdots; 0<p<1, p+q=1)$. It is shown in [1] that

$$h_n(z)=Q_n(z)\prod_{m=1}^{n}\frac{1-q^m}{1-q^m z}$$

where $Q_n(z)$ is a probability generating function as well. Prove that $Q_n(z)$ is the (unique) polynomial of degree $n-1$ such that for all $m=1,2,\cdots,n$,

$$Q_n(q^{-m})=q^{-m(m-1)/2}\binom{n}{m}\Big/\binom{n}{m}_q$$