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Transport Equations for the QCD Quark Wigner Operator

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Abstract:

We derive an exact gauge covariant equation of motion for the SU(N) quantum chromodynamic Wigner operator. That equation naturally splits up into an on-shell constraint equation and a transport equation. In the semiclassical limit the transport equation reduces to a Vlasov type equation with non-Abelian and spin-dependent terms. We show how quantum corrections can be calculated systematically. Under suitable conditions the transport equation reduces to a particularly simple set of Abelian equations in the Cartan basis of SU(N).

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1. Introduction

Interest in ultra-relativistic nuclear collisions stems from the possibility that in such collisions a quark-gluon plasma may be formed [1,2]. A central problem in this connection is to determine the formation time of the plasma and its evolution towards chemical and thermal equilibrium. For that purpose it is necessary to derive transport equations for quarks and gluons that can be applied even if equilibrium is never achieved. While there has been a considerable amount of work on deriving relativistic Abelian transport equations in the past [3]-[8], work on gauge covariant non-Abelian transport equations [9,10,11] has only recently been begun. In this paper we contribute to that development by deriving constraint and transport equations for the relativistic gauge covariant Wigner operator for fermions interacting via SU(N) gauge fields.

Based on recent investigations of the external field problem in QCD and of particle production by covariant constant fields in particular [12], one is led to conjecture that at least in certain limiting situations the transport equations for quarks and gluons may reduce to a particularly simple set of Abelian equations. As shown in ref.[12] for covariant constant fields, there exists a global gauge in which the external field tensor can be diagonalized and thus be written as $\langle F_a^{\mu\nu} t_a \rangle = \vec{F}^{\mu\nu} \cdot \vec{h}$, where $\vec{h} \equiv (h_1, \cdots, h_{N-1})$ are the generators of the Cartan (Abelian) subgroup of SU(N). The N different colored quarks, $\psi_{\vec{\epsilon}}$, couple to the $N-1$ Abelian fields with coupling $g\vec{\epsilon}_i \equiv g\vec{h}_{ii}$, and the $N(N-1)$ "charged" gluons, $W_{\vec{\eta}}^{\mu}$ couple with a strength $g\vec{\eta}_{ij} \equiv g(\vec{\epsilon}_i - \vec{\epsilon}_j)$. There are $N-1$ "neutral" gluons that do not couple to $\vec{F}^{\mu\nu}$. For such fields, the equations of motion for the quarks and gluons in the semiclassical limit reduce to a set of uncoupled Dirac's and vector field equations

$$(i\gamma_\mu D^\mu(\vec{\epsilon}) - m_f)\psi_{\vec{\epsilon}} = 0 \quad , \tag{1.1}$$

$$D_\mu(\vec{\eta})(D^\mu(\vec{\eta})W_{\vec{\eta}}^{\nu} - D^\nu(\vec{\eta})W_{\vec{\eta}}^{\mu}) - (W_{\vec{\eta}})_\mu [D^\mu(\vec{\eta}), D^\nu(\vec{\eta})] = 0 \quad , \tag{1.2}$$

where the effective Abelian covariant derivative that acts on a field with coupling $g\vec{q}$ is

$$D^\mu(\vec{q}) = \partial^\mu + ig\vec{q} \cdot \vec{A}^\mu \ . \tag{1.3}$$

This simplification of the equations of motion suggests that at least for slowly varying fields, the transport equations for quarks and gluons may also reduce to a set of Abelian Vlasov type equations:

$$(p_\mu \partial_x^\mu + g\vec{q} \cdot \vec{F}_{\mu\nu}(x) p^\nu \partial_p^\mu) f_{\vec{q}}(x,p) = C_{\vec{q}} + S_{\vec{q}} \ , \tag{1.4}$$

where $f_{\vec{q}}(x,p)$ is the phase space density for particles with effective charge $\vec{q}$, and $C_{\vec{q}}$ and $S_{\vec{q}}$ are collision and source terms. In this picture the self consistent fields $\vec{F}^{\mu\nu}$, $\vec{F}^{\mu\nu} \equiv \partial^\mu \vec{A}^\nu - \partial^\nu \vec{A}^\mu$, satisfy a set of Maxwell equations

$$\partial_\mu \vec{F}^{\mu\nu} = g\vec{J}_{\text{ext}}^{\nu} + \sum_{\vec{q}} g\vec{q} \int d^4p \ p^\mu f_{\vec{q}}(x,p) \ , \tag{1.5}$$

where $\vec{J}^{\mu}_{\rm ext}$ is an external source current and where the second term is the induced current in the plasma.

As in QED plasmas one expects infrared singularities in the collision terms [8] which must be regulated by including Debye screening via the Balescu-Lenard method [13] for example. A new feature of the above equations is the expected occurrence of source terms that result from pair production in the external fields. In QED such terms are never important in practical applications. However, in QCD the assumption of confinement requires that such terms appear to neutralize the color fields. In ref.[12] the total pair creation rate of quanta with charge $\vec{q}$ was calculated for covariant constant color electric fields, $\vec{F}^{30} = \vec{E}$ to be

$$\int d^4p\, S_{\vec{q}}(x,p) = \frac{\gamma g^2}{24\pi}(\vec{q}\cdot\vec{E})^2 \ , \tag{1.6}$$

where $\gamma = 1$ (1/2) for fermions (vector bosons) respectively. While the transverse momentum dependence of $S_{\vec{q}}$ is understood, there is considerable uncertainty as yet about the longitudinal momentum dependence [14]. In ref.[14] a plausible ansatz for that dependence was considered in the Abelian case.

The above chromo-transport equations represent the simplest and most natural generalization of relativistic QED plasma equations. However, it is far from clear whether and under what special conditions they may actually apply. For example, pair production is a quantum phenomenon and its inclusion via a source term requires us to understand systematic quantum corrections to classical transport theories. What restrictions are imposed by gauge invariance? How does spin enter into the transport equations? The purpose of the present study is to begin to answer some of these questions.

Usually, the operator that is expected to have the closest connection with the classical distribution functions is the Wigner operator [5,6,7]

$$\hat{W}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\bar{\psi}(x+\tfrac{1}{2}y)\otimes\psi(x-\tfrac{1}{2}y) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\bar{\psi}(x)e^{\frac{1}{2}y\cdot\partial_x^\dagger}\otimes e^{-\frac{1}{2}y\cdot\partial_x}\psi(x) \ , \tag{1.7}$$

where $\partial_x^\dagger \equiv \overleftarrow{\partial/\partial x^\mu}$ and $\partial_x \equiv \overrightarrow{\partial/\partial x^\mu}$ are the generators of translations acting to the left and to the right respectively. Unfortunately, the above definition cannot be correct for a gauge theory, since $\hat{W}$ does not transform covariantly under a gauge transformation. A gauge covariant definition can, however, be constructed by substituting the covariant derivative, $D^\mu \equiv \partial^\mu + igA^\mu$, and its adjoint in place of ∂^μ and its adjoint. Applying this minimal substitution rule leads to the following definition of the relativistic gauge covariant Wigner function for spin-1/2 particles:

$$\hat{W}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\bar{\psi}(x)e^{\frac{1}{2}y\cdot D_x^\dagger}\otimes e^{-\frac{1}{2}y\cdot D_x}\psi(x) \ . \tag{1.8}$$

The tensor product in eq.(1.8) implies that $\hat{W}$ is a matrix in spinor (4×4 components) as well as in color space ($N\times N$ components). Under a local gauge transformation, $S(x) \equiv \exp[i\theta_a(x)t_a]$, $\psi(x) \longrightarrow S(x)\psi(x)$ and $D^\mu \to S(x)D^\mu S^{-1}$, and the Wigner operator, eq.(1.8), transforms covariantly:

$$\hat{W}(x,p) \longrightarrow S(x)\hat{W}(x,p)S^{-1}(x) \ . \tag{1.9}$$

We shall see in Sec.3 how this definition of $\hat{W}$ is related to that proposed in ref.[10]. Note that the covariant derivative is the operator that corresponds to the ordinary *kinetic* momentum, $\hat{\pi}^\mu \equiv P^\mu - gA^\mu = iD^\mu$, where $P^\mu = i\partial_x^\mu$ is the canonical momentum conjugate to the position coordinate. The on-shell condition $\hat{\pi}\cdot\hat{\pi} = m^2$ applies to the kinetic rather than the conjugate momentum in the classical limit. Therefore, it is natural that the covariant rather than the ordinary derivative appears in the definition of the Wigner function in eq.(1.8).

Our objective in this paper is to derive the equations of motion for $\hat{W}$ from Dirac's field equations. We show that they resemble their classical counterparts, a transport and a constraint equation respectively, however, modified by spin-dependent and specifically non-Abelian terms. Our work extends the pioneering work of Heinz [10] by deriving the exact quantum transport equations for spin-1/2 particles coupled to SU(N) gauge fields. We show that in the classical limit our equations reduce to those of ref.[10] and also how quantum and spin-dependent corrections to the classical equations can be calculated systematically. We note that a non-Abelian transport theory for scalar particles was studied by Winter [11]. A thorough study of scalar QED transport theory has been presented by Remler [3]. We reproduce the semiclassical transport equation for scalar particles derived in ref.[3] as a special case of our general transport equation. Extensive lists of references dealing with non-gauge-covariant Wigner functions can be found in refs.[5,6].

The plan of our paper is as follows. In Sec.2 we list definitions employed afterwards. In Sec.3 we introduce link operators which play a crucial role in our formalism and derive a general formula, which allows the easy calculation of derivatives of link operators. That formula is applied in Sec.4, where we derive a constraint and a proper quantum transport equation for the QCD Wigner operator. These equations are main new results of our work. In Sec.5 we perform the operator expansion of these equations which yields the semiclassical limit. We find that only for "slowly" varying fields and Wigner functions (in a generalized sense derived there) can the transport equations reduce to Vlasov's equation. Finally, in Sec.6 we show how at least the Vlasov part of the chromo-transport equation, eq.(1.4), arises from our general results in a model where the expectation value of the Wigner operator is assumed to be diagonal in the gauge that diagonalizes the field tensor. We close by pointing out directions for further studies and applications.

2. Definitions

We use the metric $g^{\mu\nu} \equiv diag(1,-1,-1,-1)$, $a\cdot b \equiv a_\mu b^\mu$, and choose units such that $\hbar = c = 1$ unless stated otherwise. The gauge field potential is a $N\times N$ matrix in color space defined by

$$A_\mu \equiv A_\mu^a t_a \ , \tag{2.1}$$

with the N^2-1 Hermitian generators of SU(N) in the fundamental representation satisfying $Tr\ t_a = 0$, $Tr\ t_a t_b = \delta_{ab}/2$, and $[t_a, t_b] = i f_{abc} t_c$. The covariant derivative,

$$D_\mu \equiv \partial_\mu + igA_\mu \quad , \tag{2.2}$$

is thus an $N \times N$ matrix in color space as is the field strength tensor, $F_{\mu\nu} \equiv [D_\mu, D_\nu]/(ig)$, which obeys the field equation

$$[D_\mu, F^{\mu\nu}] = gJ^\nu \quad . \tag{2.3}$$

The color current operator is given by

$$J^\mu \equiv \hat{\jmath}^\mu_a t_a \equiv t_a \bar{\psi}\gamma^\mu t_a \psi = \int d^4p\, t_a\, Tr\ \gamma^\mu t_a \hat{W}(x,p) \quad , \tag{2.4}$$

where the trace refers to spinor and color indices. Under a local gauge transformation S not only D_μ and $F^{\mu\nu}$ transform covariantly but also J^μ, i.e., $\hat{J} \to S(x)\hat{J}S^{-1}(x)$, because $t_a \cdot Tr(t_a S\hat{W}S^{-1}) = St_aS^{-1} \cdot Tr(t_a\hat{W})$.

The field equations for the Heisenberg operators ψ and $\bar{\psi}$ are

$$(i\gamma^\mu D_\mu - m)\psi = 0 = \bar{\psi}(i\gamma^\mu D^\dagger_\mu + m) \quad , \tag{2.5}$$

where henceforth $D^\dagger$ is defined to act always to the left, and $\bar{\psi} \equiv \psi^\dagger\gamma^0$, $\gamma^{\mu\dagger} \equiv \gamma^0\gamma^\mu\gamma^0$, with $\gamma^0 \equiv diag(1,1,-1,-1)$. We suppress quark flavor indices.

We will also use the quadratic form of Dirac's equation and its adjoint

$$\begin{aligned} (D_\mu D^\mu + \tfrac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu} + m^2)\psi &= 0 \ , \\ \bar{\psi}(D^\dagger_\mu D^{\dagger\mu} + \tfrac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu} + m^2) &= 0 \ , \end{aligned} \tag{2.6}$$

where $\sigma^{\mu\nu} \equiv \frac{1}{2}i[\gamma^\mu, \gamma^\nu]$ and $g^{\mu\nu} = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}$.

We note that reinstating $\hbar$ into any relation derived in units with $\hbar = 1$ is achieved as usual by inserting appropriate powers of $\hbar$ into it in order to convert all terms to the same units. Choosing units where all dimensionful quantities are expressed in terms of some length scale, L, is equivalent to setting $\hbar = 1$. In these units, g is dimensionless, p, m, and A^μ have dimensions L^{-1}, and $F^{\mu\nu}$ has dimensions L^{-2}.

3. Properties of Link Operators

The effect of the covariant translation operator in eq.(1.8) can be expressed in terms of a link operator as

$$e^{-y\cdot D_x}\psi(x) = U(x, x-y)\psi(x-y) \quad , \tag{3.1}$$

where U is given by the path ordered exponential of a line integral [15,16],

$$U(b,a) \equiv P\exp\left(-ig\int_a^b dz^\mu A^a_\mu(z)t_a\right) \quad , \tag{3.2}$$

and the path of integration is the *straight line* between the end points,

$$z(s) \equiv z(b,a,s) = a + (b-a)s \ , \ 0 \le s \le 1 \ . \tag{3.3}$$

These relations are derived by expressing

$$\begin{aligned} \exp(-y \cdot D_x) &= \lim_{n\to\infty} \left(1 - \Delta y(\partial_x + igA(x)) \right)^n \\ &= \lim_{n\to\infty} \left((1 - ig\Delta y \cdot A(x))\, e^{-\Delta y \cdot \partial_x} \, (1 + O(\Delta y^2)) \right)^n \\ &= \lim_{n\to\infty} (1 - ig\Delta y \cdot A(x)) \cdots (1 - ig\Delta y \cdot A(x - (n-1)\Delta y) \ \exp(-y \cdot \partial_x) \\ &= U(x, x-y) e^{-y \cdot \partial_x} \ , \end{aligned} \tag{3.4}$$

where $\Delta y \equiv y/n$.

Using the link operator U, we can thus express the gauge covariant Wigner operator in the form proposed in ref.[10],

$$\hat{W}(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \, \bar{\psi}(x + \tfrac{1}{2}y) U(x + \tfrac{1}{2}y, x) \otimes U(x, x - \tfrac{1}{2}y) \psi(x - \frac{1}{2}y) \ . \tag{3.5}$$

The advantage of our definition, eq.(1.8), is that the path defining $U(b,a)$ automatically turns out to be the straight line, cf.eqs.(3.1-3.4). Starting with eq.(3.5) instead raises the question of the choice of the path. A unique definition of the path is important, however, since the value of $U(b,a)$ depends on the path except in the trivial case, when $A^\mu = S\partial^\mu S^{-1}$ is a pure gauge field with $F^{\mu\nu} = 0$. Therefore, although $\hat{W}$ as given in eq.(3.5) also transforms covariantly, owing to the peculiar tranformation property of link operators [15,16],

$$U(b,a) \longrightarrow S(b)U(b,a)S^{-1}(a) \ , \tag{3.6}$$

the path ambiguity must be removed. The ambiguity is removed by requiring that the variable p in $\hat{W}(x,p)$ corresponds to the kinetic momentum in the classical limit. That requirement is what led to our definition of $\hat{W}$ given in eq.(1.8). We now show that for any other choice of the path eq.(3.4) does not hold and hence the interpretation of p would have to differ from what we want on physical grounds.

The general expression for the link operator valid for any path can be built up from an ordered product of infinitesimal link operators

$$l(z, dz) \equiv U(z + dz, z) = 1 - igA(z) \cdot dz \ , \tag{3.7}$$

as

$$U(b,a) = Pexp\left\{ -ig \int_0^1 ds \frac{dz^\mu}{ds} A_\mu(z(s)) \right\} = P \prod_{s=0}^{1} l(z(s), dz(s)) \ , \tag{3.8}$$

where the path is parametrized by a function $z(s)$ with $0 \le s \le 1$ such that $z(0) = a$ and $z(1) = b$. The product is defined by dividing the interval $[0,1]$ into a large number n of intervals ($ds = 1/n$) so that $dz = (dz/ds)ds$ and taking the limit $n \to \infty$ at the end. The transformation property (3.6) follows from noting that $A^\mu \to SA^\mu S^{-1} + S\partial^\mu S^{-1}/(ig)$, and thus, $l(z,dz) \to S(z+dz)l(z,dz)S^{-1}(z)$ to first order in dz.

The link operators are unitary (since $A = A^\dagger$) and have a group property,

$$U(b,c) \cdot U(c,a) = U(b,a) \ . \tag{3.9}$$

However, eq.(3.9) holds in general only, if c lies on the path $z(b,a,s)$ and if the path $z(z_2,z_1,s)$ defining $U(z_2,z_1)$ for arbitrary $z_1 \equiv z(b,a,s_1)$, $z_2 \equiv z(b,a,s_2)$ on the path between a and b coincides with $z(b,a,s)$ in the range $s_1 \leq s \leq s_2$ for all a,b. These conditions certainly hold for straight line paths, cf. eq.(3.3). We will see that the group property greatly simplifies the calculation of the derivatives of link operators with respect to the end points.

Such derivatives are needed in the derivation of the equation of motion for $\hat{W}$. Therefore, we now derive an equation for the first order variation δU of a link operator, where the variation is considered to be due to an infinitesimal variation of the path of integration in eq.(3.8), $z(s) \longrightarrow z(s) + \delta(s)$, $0 \leq s \leq 1$. The variation of the path is illustrated for the straight line case in Fig. 1. The variation of the link operator is given by

$$\delta U(b,a) \equiv U_\delta(b,a) - U(b,a) \ , \tag{3.10}$$

where U_δ is again given by eq.(3.8), but with an integration along the varied path. The trick to calculate U_δ consists of writing it as a product of all possible consecutive contributions of the kind depicted in Fig. 2,

$$U_\delta(b,a) = l(b,\delta(1))\ P\left(\prod_{s=0}^{1} l(z(s),dz(s))\ l_\Box(z(s),\delta(s),ds)\right)\ l^\dagger(a,\delta(0)) \ , \tag{3.11}$$

where $l_\Box(z,\delta,ds)$ denotes the product of infinitesimal link operators going around a closed loop (plaquette) starting and ending at z as shown in Fig. 2. The product in eq.(3.11) is path ordered so that terms with $s = s_1$ appear always to the rigth of terms with $s = s_2$ if $s_2 > s_1$ for all s_1, s_2. Note that all of the extra contributions from parallel displacements off the varied path cancel exactly due to the group property of link operators, eq.(3.9). Also eq.(3.11) contains appropriate corrections for the edges at $s = 0$ and $s = 1$.

Using eq.(3.7) one obtains

$$\begin{aligned} l_\Box(z(s),\delta(s),ds) &= l(z(s+ds),-dz(s)) \times l(z(s+ds)+\delta(s+ds),-\delta(s+ds)) \\ &\quad \times l(z(s)+\delta(s),dz(s)+d\delta(s)) \times l(z(s),\delta(s)) \\ &= [1+igA(z(s+ds))\cdot dz(s)] \\ &\quad \times [1+igA(z(s+ds)+\delta(s+ds))\cdot\delta(s+ds)] \\ &\quad \times [1-igA(z(s)+\delta(s))\cdot d(z(s)+\delta(s))] \\ &\quad \times [1-igA(z(s))\cdot\delta(s)] \end{aligned}$$

$$= 1 + ig\left(\partial_\mu A_\nu(z) - \partial_\nu A_\mu(z) + ig[A_\mu(z), A_\nu(z)]\right)\frac{dz^\mu}{ds}\delta^\nu(s)ds$$

$$= 1 + igF_{\mu\nu}(z(s))\frac{dz^\mu(s)}{ds}\delta^\nu(s)ds \ , \tag{3.12}$$

where a consistent expansion up to first order either in $\delta(s)$ or ds or the product of both has been performed (only terms $\propto \delta(s)ds$ contribute, while the others cancel). The last line in eq.(3.12) follows from the definition of the field tensor. Inserting the result of eq.(3.12) into eq.(3.11) and expanding the product to the same order in the same quantities yields the desired result for δU according to eq.(3.10):

$$\begin{aligned}\delta U(b,a) &= \Big[\,(1 - igA(b)\cdot\delta(1)) \times \Big\{\prod_{s=0}^{1} l(z(s),dz(s)) \\ &\quad + ig\int_0^1 ds\left(\prod_{s'=s}^{1} l(z(s'),dz(s'))\right)F_{\mu\nu}(z(s))\frac{dz^\mu(s)}{ds}\delta^\nu(s)\left(\prod_{s''=0}^{s} l(z(s''),dz(s''))\right)\Big\} \\ &\quad \times (1 + igA(a)\cdot\delta(0))\,\Big] \ - \ U(b,a) \\ &= -igA(b)\cdot\delta(1)U(b,a) + igU(b,a)A(a)\cdot\delta(0) \\ &\quad + ig\int_0^1 dsU(b,z(s))F_{\mu\nu}(z(s))U(z(s),a)\frac{dz^\mu(s)}{ds}\delta^\nu(s) \ .\end{aligned} \tag{3.13}$$

The last line in particular follows only if the link operators obey the the group property, eq.(3.9), which allows us to write

$$U(z(s_2),z(s_1)) = \prod_{s=s_1}^{s_2} l(z(s),dz(s)) \ . \tag{3.14}$$

Otherwise the partial products of infinitesmal link operators could not be simply expressed in terms of the U's again.

Note that $\delta U \neq 0$ even for variations of the path leaving the end points fixed. That is why eq.(3.4) only holds for straight line paths. Thus, the path defining U in eq.(3.5) must be chosen to be a straight line if p is required to correspond to the kinetic momentum.

For the interesting case of straight line paths, eq.(3.3), $dz/ds = b - a$ and $\delta(s) \equiv da + (db - da)s$ is the variation of the path of integration due to infinitesimal shifts of the endpoints. Thus we obtain

$$\begin{aligned}\delta U(b,a) &= -igA(b)\cdot db\, U(b,a) + igU(b,a)A(a)\cdot da \\ &\quad + ig\int_0^1 ds\, U(b,z(s))F_{\mu\nu}(z(s))U(z(s),a)(b-a)^\mu(da + (db-da)s)^\nu \ .\end{aligned} \tag{3.15}$$

It follows for example that

$$D_\mu(b)\; U(b,a) = -ig(b-a)^\nu \int_0^1 ds\; s\; U(b,z(s))\; F_{\mu\nu}(z(s))\; U(z(s),a) \quad , \tag{3.16}$$

with $z(s) \equiv a + (b-a)s$.

Higher order derivatives can be calculated by repeated application of the above formulas. Eq.(3.15) was also derived in ref.[11] by considering the differential equation which a link operator has to obey "parallel to its path", and deriving and solving a similar one for δU. However, we believe that our approach is more intuitive and transparent (cf. Fig.1 and Fig.2). Also, as discussed after eq.(3.5), we are able to remove any ambiguity associated with the choice of the path.

4. Derivation of Equations of Motion

The Wigner operator as given by eqs.(1.8, 3.5) is related to a gauge covariant density matrix,

$$\hat\rho(x+\tfrac{y}{2}, x-\tfrac{y}{2}) \equiv \bar\psi(x+\tfrac{y}{2})U(\tfrac{y}{2},x) \otimes U(x,x-\tfrac{y}{2})\psi(x-\tfrac{y}{2}) \quad , \tag{4.1}$$

via

$$\hat W(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ipy}\; \hat\rho(x+\tfrac{y}{2}, x-\tfrac{y}{2}) \quad . \tag{4.2}$$

The equation of motion for the density matrix follows from Dirac's equation with

$$\bar\psi(x_2)U(x_2,x) \otimes U(x,x_1)\,(i\gamma^\mu D_\mu(x_1) - m)\,\psi(x_1) = 0 \quad , \tag{4.3}$$

where $x_1 \equiv x - \frac{1}{2}y$ and $x_2 \equiv x + \frac{1}{2}y$. This will become the so-called *constraint equation* to be interpreted later on. A similar one could be obtained, of course, by applying the adjoint Dirac's equation to the left. However, we will find it convenient to consider a linear combination of the quadratic Dirac's equation and its adjoint instead:

$$\begin{aligned}&\bar\psi(x_2)U(x_2,x) \otimes U(x,x_1)\left(D_\mu(x_1)D^\mu(x_1) + \tfrac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu}(x_1)\right)\psi(x_1)\\ &- \bar\psi(x_2)\left(D^\dagger_\mu(x_2)D^{\dagger\mu}(x_2) + \tfrac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu}(x_2)\right)U(x_2,x) \otimes U(x,x_1)\psi(x_1) \;= 0 \quad .\end{aligned} \tag{4.4}$$

While the equation resulting from eq.(4.4) is not independent from the constraint equation deduced from eq.(4.3), we will see that it corresponds more closely to the form of a *transport equation* that we expect for the Wigner operator. In order to convert eq.(4.3) and eq.(4.4) into equations for the Wigner operator, the tedious task of moving the derivative operators to the left or right in these equations (i.e. commuting them with link operators) has to be performed.

In the following we need a shorthand notation keeping track of the endpoints in more complicated expressions of the type encountered in the integral in eq.(3.15). Therefore, we define

$$F_{\mu\nu}(b;z;a) \iff F_{\mu\nu}(z)\;,\; z \equiv z(s) \equiv a + (b-a)s\;,\; 0 \le s \le 1 \quad . \tag{4.5}$$

4.1 The Constraint Equation

We begin to manipulate eq.(4.3) by rewriting it more explicitly as

$$\begin{aligned} 0 &= -m\hat{\rho}(x_2,x_1) + i\gamma_\mu\bar{\psi}(x_2)U(x_2,x)\otimes U(x,x_1)igA^\mu(x_1)\psi(x_1) \\ &+i\gamma_\mu\partial^\mu_{x_1}\hat{\rho}(x_2,x_1) \\ &-i\gamma_\mu\bar{\psi}(x_2)U(x_2,x)\otimes[\partial^\mu_{x_1}U(x,x_1)]\psi(x_1) \\ &-i\gamma_\mu\bar{\psi}(x_2)[\partial^\mu_{x_1}U(x_2,x)]\otimes U(x,x_1)\psi(x_1) \ , \end{aligned} \tag{4.6}$$

where γ_μ acts on the right spinor, and where we used the definition of covariant derivatives. Next, we calculate the derivatives of link operators. Employing eq.(3.15) we obtain:

$$\begin{aligned} \partial^\mu_{x_1}U(\tfrac{1}{2}(x_1+x_2),x_1) &= -\tfrac{1}{2}igA^\mu(x)U(x,x_1) + igU(x,x_1)A^\mu(x_1) \\ +ig\int_0^1 ds\ U(x,z)F_\nu{}^\mu(x;z;x_1)U(z,x_1)&\cdot\tfrac{1}{2}(x_2-x_1)^\nu\cdot(1-\tfrac{1}{2}s) \ , \end{aligned} \tag{4.7}$$

and,

$$\begin{aligned} \partial^\mu_{x_1}U(x_2,\tfrac{1}{2}(x_1+x_2)) &= +\tfrac{1}{2}igU(x_2,x)A^\mu(x) \\ +ig\int_0^1 ds\ U(x_2,z)F_\nu{}^\mu(x_2;z;x)U(z,x)&\cdot\tfrac{1}{2}(x_2-x_1)^\nu\cdot\tfrac{1}{2}(1-s) \ , \end{aligned} \tag{4.8}$$

where we also used the definition (4.5) to indicate the respective paths of integration and their parametrization, and $z\equiv z(s)$ always. Before we use these results, eq.(4.7) and eq.(4.8), in connection with eq.(4.6), we define the product of any operator $\hat{O}$ times $\hat{\rho}$ by

$$\begin{aligned} \hat{O}\hat{\rho}(x_2,x_1) &\equiv \bar{\psi}(x_2)U(x_2,x)\otimes\hat{O}U(x,x_1)\psi(x_1) \ , \\ \hat{\rho}(x_2,x_1)\hat{O} &\equiv \bar{\psi}(x_2)U(x_2,x)\hat{O}\otimes U(x,x_1)\psi(x_1) \ . \end{aligned} \tag{4.9}$$

Thus, for example, under a gauge transformation S the density matrix transforms covariantly with respect to its midpoint:

$$\hat{\rho}(x+\tfrac{y}{2},x-\tfrac{y}{2}) \longrightarrow S(x)\hat{\rho}(x+\tfrac{y}{2},x-\tfrac{y}{2})S^{-1}(x) \ . \tag{4.10}$$

Eqs.(4.9) allow us to rewrite the result of inserting eq.(4.7) and eq.(4.8) back into eq.(4.6) in a compact way:

$$\begin{aligned} &i\gamma_\mu\left(\partial^\mu_{x_1}\hat{\rho}(x_2,x_1)+\tfrac{1}{2}ig[A^\mu(x),\hat{\rho}(x_2,x_1)]\right)-m\hat{\rho}(x_2,x_1) \\ &= -\tfrac{1}{2}g(x_2-x_1)^\nu\gamma^\mu\left(\int_0^1 ds\ (1-\tfrac{1}{2}s)U(x,z)F_{\nu\mu}(x;z;x_1)U(z,x_1)U(x_1,x)\hat{\rho}(x_2,x_1)\right. \\ &\left.+\hat{\rho}(x_2,x_1)\int_0^1 ds\ \tfrac{1}{2}(1-s)U(x,x_2)U(x_2,z)F_{\nu\mu}(x_2;z;x)U(z,x)\right) \ , \end{aligned} \tag{4.11}$$

where we also used the group property of link operators, eq.(3.9), to express all terms as functionals of $\hat{\rho}$.

We remark here that the above definition, eqs.(4.9), is quite essential for keeping track of the operator ordering, since for interacting quantum fields e.g. $[(F_{\mu\nu})_{ab}, U_{cd}] \neq 0$. For the special case of a purely classical external field ordinary matrix multiplication rules suffice, since we must only keep track of the sequence of matrix indices, while the order of matrix elements is irrelevant. In general those matrices are field operators and the ordering in eqs.(4.9) is important.

Next, we note that the pairs of link operators standing side by side in eq.(4.11) can be reexpressed by a single one using their group property again. The resulting terms typically are of the form $U(x,z)\ F_{\mu\nu}(b;z;a)\ U(z,x)$; interestingly, this can be interpreted as a well-defined local gauge transformation of the field strength tensor, $F_{\mu\nu}(z) \rightarrow S(z)F_{\mu\nu}(z)S^{-1}(z)$ with $S(z) \equiv U(x,z)$. For an arbitrary gauge transformation S, however, respecting the transformation property of link operators, eq.(3.6), we obtain:

$$U(x,z)F_{\mu\nu}(z)U(z,x) \longrightarrow S(x)U(x,z)F_{\mu\nu}(z)U(z,x)S^{-1}(x) \ . \tag{4.12}$$

Defining the covariant derivative of a second-rank tensor $\mathcal{T}$ by

$$\mathcal{D}(x)\mathcal{T}(x) \equiv \partial_x\mathcal{T}(x) + ig[A(x),\mathcal{T}(x)] \ , \tag{4.13}$$

one may show by a similar proof as in eq.(3.4) that

$$e^{-y\cdot\mathcal{D}(x)}\ \mathcal{T}(x) \ = \ U(x,x-y)\mathcal{T}(x-y)U(x-y,x) \ , \tag{4.14}$$

provided the gauge potential involved in the definition of $\mathcal{D}$ is differentiable and $\mathcal{T}$ is differentiable to all orders. We will employ this useful relation in the following to simplify our results.

Changing now variables from x_2, x_1 to the midpoint and relative distance variables $x \equiv \frac{1}{2}(x_1+x_2)$ and $y \equiv x_2 - x_1$, which implies $\partial_{x_1} = \frac{1}{2}\partial_x - \partial_y$, and using eqs.(4.13, 4.14), we obtain from eq.(4.11):

$$\begin{aligned} &-i\gamma_\mu\left(\partial_y^\mu - \tfrac{1}{2}\mathcal{D}^\mu(x)\right)\hat{\rho}(x+\tfrac{1}{2}y, x-\tfrac{1}{2}y) - m\hat{\rho}(x+\tfrac{1}{2}y, x-\tfrac{1}{2}y) \\ &= \ -\tfrac{1}{2}gy^\nu\gamma^\mu\left(\int_0^1 ds\ (1-\tfrac{1}{2}s)\ [e^{-\frac{1}{2}(1-s)y\cdot\mathcal{D}(x)}\ F_{\nu\mu}(x)]\ \hat{\rho}(x+\tfrac{1}{2}y, x-\tfrac{1}{2}y)\right. \\ &\qquad \left.+\hat{\rho}(x+\tfrac{1}{2}y, x-\tfrac{1}{2}y)\int_0^1 ds\ \tfrac{1}{2}(1-s)\ [e^{\frac{1}{2}sy\cdot\mathcal{D}(x)}\ F_{\nu\mu}(x)]\right) \ . \end{aligned} \tag{4.15}$$

In order to obtain an equation for the Wigner operator from eq.(4.15), we apply the identity

$$\int \frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}f(y)g(y) = f(i\partial_p)\int \frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}g(y) \ , \tag{4.16}$$

which holds, provided that the function f can be expanded in a Taylor series. Performing a partial integration of the first term on the l.h.s. of eq.(4.15) after integrating the whole equation with $\int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}$, we obtain finally:

$$\left(\gamma^\mu p_\mu - m + \tfrac{1}{2} i\gamma^\mu \mathcal{D}_\mu(x)\right) \hat{W}(x,p)$$

$$= -\tfrac{1}{2} ig\partial_p^\nu \gamma^\mu \left(\int_0^1 ds \, (1 - \tfrac{1}{2}s) \, [e^{-\frac{1}{2}i(1-s)\partial_p \cdot \mathcal{D}(x)} \, F_{\nu\mu}(x)] \, \hat{W}(x,p) \right.$$

$$\left. + \hat{W}(x,p) \int_0^1 ds \, \tfrac{1}{2}(1-s) \, [e^{\frac{1}{2}is\partial_p \cdot \mathcal{D}(x)} \, F_{\nu\mu}(x)] \right) \; , \qquad (4.17)$$

where ∂_p always acts on $\hat{W}$. The *gauge covariant operator equation*, eq.(4.17), presents the result we were looking for in this subsection, its semiclassical limit will be discussed in Sec. 5 .

In order to gain some understanding of the structure of this equation, we consider the special case of a *constant Abelian field* $F^{\mu\nu}$. In that case $\mathcal{D}_\lambda F_{\mu\nu} = 0$, and thus eq.(4.17) reduces to

$$(\gamma \cdot K - m)\hat{W}(x,p) = 0 \; , \qquad (4.18)$$

where

$$K^\mu \equiv p^\mu + \tfrac{1}{2} i(\partial_x^\mu - gF^\mu{}_\nu \partial_p^\nu) \; . \qquad (4.19)$$

Multiplying eq.(4.18) by $(\gamma \cdot K + m)$ gives rise to a *complex* equation:

$$\left(p^2 - m^2 - \tfrac{1}{4}(\partial_x - gF\cdot\partial_p)^2 + i(p\cdot\partial_x - gp\cdot F\cdot\partial_p) - \tfrac{1}{2} g\sigma^{\mu\nu}F_{\mu\nu} \right) \hat{W}(x,p) = 0 \; . \qquad (4.20)$$

We can isolate the Hermitian or anti-Hermitian part of the equation by adding or subtracting its adjoint. This leads to the following *two* equations:

$$\left(p^2 - m^2 - \tfrac{1}{4}(\partial_x - gF\cdot\partial_p)^2 \right) \hat{W}(x,p) - \tfrac{1}{4} gF_{\mu\nu}\{\sigma^{\mu\nu}, \, \hat{W}(x,p)\} = 0 \; , \qquad (4.21)$$

$$(\, p\cdot\partial_x - gp\cdot F\cdot\partial_p \,) \, \hat{W}(x,p) + \tfrac{1}{4} igF_{\mu\nu}[\sigma^{\mu\nu}, \, \hat{W}(x,p)] = 0 \; , \qquad (4.22)$$

where we used the property $\hat{W}^\dagger = \gamma^0 \hat{W} \gamma^0$. The first equation generalizes the classical mass-shell condition and may be called a *constraint equation* (for a study of its consequences in a non-gauge-covariant formulation of the usual Wigner function approach see ref.[6]). The second equation generalizes the Vlasov equation to include spin-dependent effects, and thus may be called a *transport equation*. From eqs.(4.21, 4.22) one sees directly that p plays the role of a kinetic momentum variable, as we had intended by the construction of eq.(1.8).

We therefore conclude that eq.(4.17) contains both a constraint and a transport equation. In the general case it is, however, not so simple to uncover the transport equation from it. In effect, the transport equation followed by applying the Dirac operator twice. To obtain the general transport equation, we found it more convenient to start with the quadratic Dirac's equation in the form of eq.(4.4).

4.2 The Transport Equation

Using the same technique and steps demonstrated in detail in the preceding section, where we derived the general constraint equation (4.17), we turn now to the much more tedious derivation of a transport equation for the Wigner operator.

The idea is to start from eq.(4.4) and to move the derivatives out of the respective pairs of field operators, $\bar{\psi}(x_2)\cdots\partial_{x_1,x_2}\cdots\psi(x_1)=\partial_{x_1,x_2}\bar{\psi}(x_2)\cdots\psi(x_1)+corrections$ etc., until one finally obtains an equation which can be rewritten in terms of the density matrix $\hat{\rho}$ defined in eq.(4.1). The terms involving the $\sigma^{\mu\nu}$ matrices are easy to deal with, since they involve no derivatives at all. However, looking at eq.(4.4) one notices that there are also second order derivatives which then lead to the necessity to calculate second order derivatives of link operators; these calculations are straightforward to carry out by using eq.(3.15) twice. The main difficulty is that about an order of magnitude more terms arise due to the necessity of calculating second derivatives of the link operators.

Having managed this bookkeeping problem, one then changes variables from x_2, x_1 to the midpoint and relative distance variables $x\equiv\frac{1}{2}(x_1+x_2)$ and $y\equiv x_2-x_1$ as before, so that $\partial_{x_1}=\frac{1}{2}\partial_x-\partial_y$, $\partial_{x_2}=\frac{1}{2}\partial_x+\partial_y$, and $\partial_{x_1}\cdot\partial_{x_1}-\partial_{x_2}\cdot\partial_{x_2}=-(\partial_x\cdot\partial_y+\partial_y\cdot\partial_x)\neq 2\partial_x\cdot\partial_y$. It is particularly important to observe the latter inequality when calculating derivatives of link operators.

The final step in the derivation of the transport equation consists of integrating the equation for the density matrix obtained in the way outlined above with $\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}$. One also uses eq.(4.14) and then eq.(4.16), again replacing $y\to i\partial_p$ and also $\partial_y\to ip$ (via partial integration), wherever it is necessary in order to express all terms as functionals of the Wigner operator as defined in eq.(1.8). The result of the lengthy calculation is:

$$
\begin{aligned}
0 \;&=\; p\cdot\mathcal{D}(x)\,\hat{W}(x,p)\\
&+\;\tfrac{1}{2}gp^{\mu}\partial_p^{\nu}\int_0^1 ds\,\Big(\,[e^{-\frac{1}{2}i(1-s)\triangle}F_{\nu\mu}(x)]\,\hat{W}(x,p)+\hat{W}(x,p)\,[e^{\frac{1}{2}is\triangle}F_{\nu\mu}(x)]\,\Big)\\
&-\;\tfrac{1}{4}ig\mathcal{D}^{\mu}(x)\partial_p^{\nu}\int_0^1 ds\,\Big(\,(s-1)[e^{-\frac{1}{2}i(1-s)\triangle}F_{\nu\mu}(x)]\,\hat{W}(x,p)+s\,\hat{W}(x,p)\,[e^{\frac{1}{2}is\triangle}F_{\nu\mu}(x)]\,\Big)\\
&+\;\tfrac{1}{4}ig\partial_p^{\nu}\int_0^1 ds\,\Big(\,(s-1)[e^{-\frac{1}{2}i(1-s)\triangle}\mathcal{D}^{\mu}(x)F_{\nu\mu}(x)]\,\hat{W}(x,p)+s\,\hat{W}(x,p)\,[e^{\frac{1}{2}is\triangle}\mathcal{D}^{\mu}F_{\nu\mu}(x)]\,\Big)\\
&-\;\tfrac{1}{8}ig^2\partial_p^{\mu}\partial_p^{\nu}\int_0^1 ds\int_0^1 d\tilde{s}\,(s-1+\tilde{s})[e^{-\frac{1}{2}i(1-s)\triangle}F_{\mu}^{\ \lambda}(x)]\,\hat{W}(x,p)\,[e^{\frac{1}{2}i\tilde{s}\triangle}F_{\nu\lambda}(x)]\\
&-\;\tfrac{1}{16}ig^2\partial_p^{\mu}\partial_p^{\nu}\int_0^1 ds\int_0^1 d\tilde{s}\,\Big(\,(s^2(1+\tilde{s})-2s)[e^{-\frac{1}{2}i(1-s)\triangle}F_{\mu}^{\ \lambda}(x)]\,[e^{-\frac{1}{2}i(1-s\tilde{s})\triangle}F_{\nu\lambda}(x)]\,\hat{W}(x,p)\\
&\qquad\qquad+(1-s)^2(\tilde{s}-2)[e^{-\frac{1}{2}i(1-s)(1-\tilde{s})\triangle}F_{\mu}^{\ \lambda}(x)]\,[e^{-\frac{1}{2}i(1-s)\triangle}F_{\nu\lambda}(x)]\,\hat{W}(x,p)
\end{aligned}
$$

$$+ s^2(1+\tilde{s})\,\hat{W}(x,p)\,[e^{\frac{1}{2}is\triangle}F_\mu{}^\lambda(x)]\,[e^{\frac{1}{2}is\tilde{s}\triangle}F_{\nu\lambda}(x)]$$

$$+\ (1-s)(2s+(1-s)\tilde{s})\,\hat{W}(x,p)\,[e^{\frac{1}{2}i(s+(1-s)\tilde{s})\triangle}F_\mu{}^\lambda(x)]\,[e^{\frac{1}{2}is\triangle}F_{\nu\lambda}(x)]\,\Big)$$

$$+\ \tfrac{1}{4}ig\,\Big(\,\sigma^{\mu\nu}[e^{-\frac{1}{2}i\triangle}F_{\mu\nu}(x)]\hat{W}(x,p)\ -\ \hat{W}(x,p)\sigma^{\mu\nu}[e^{\frac{1}{2}i\triangle}F_{\mu\nu}(x)]\,\Big)\ , \tag{4.23}$$

where $\triangle \equiv \partial_p \cdot \mathcal{D}(x)$ and the small brackets, [...], delimit where the $\mathcal{D}$-derivative from $\triangle$ acts; ∂_p always acts on $\hat{W}$. Eq.(4.23) presents the *quantum transport equation for the QCD Wigner operator* as defined in eq.(1.8); according to our knowledge it has been derived here for the first time. It is a *gauge covariant operator equation.* Note that the first and second term on the r.h.s. of eq.(4.23) have a structure similar to that of the classical Abelian Vlasov's equation. The transport equation given by Heinz [10] corresponds to the $\triangle \to 0$ and $\sigma^{\mu\nu} \to 0$ limit of eq.(4.23). In the next section we show how a classical non-Abelian Vlasov type equation can be derived from eq.(4.23) including a spin-dependent term and also how quantum corrections can be systematically calculated.

5. The Semiclassical Limit

In our units where all quantities are measured in terms of a length scale L, $\hbar$ did not appear in the equations. However, we see from eqs.(4.17, 4.23) that $F^{\mu\nu}$ is always acted on by an operator of the form

$$\hat{O}(\triangle) = \int ds\ f(s)\ e^{g(s)\,\triangle} = \sum_{n=0}^{\infty} a_n\,\triangle^n\ , \tag{5.1}$$

where a_n are dimensionless constants and $\triangle = \partial_p \cdot \mathcal{D}$. If we would have chosen ordinary classical units for p in terms of some momentum scale, while retaining length units for x, then obviously $\triangle$ would have to be replaced by

$$\triangle \longrightarrow \hbar\triangle\ . \tag{5.2}$$

In the semiclassical limit, we therefore need to retain only the leading powers of $\triangle$ in eq.(5.1).

Carrying out the above expansion in the constraint equation (4.17) leads to

$$(\gamma^\mu p_\mu - m)\,\hat{W}(x,p)\ =$$

$$-\ \tfrac{1}{2}i\gamma^\mu \mathcal{D}_\mu(x)\,\hat{W}(x,p)\ +\ \tfrac{1}{2}ig\partial_p^\mu\gamma^\nu\ (\,F_{\mu\nu}(x)\,\hat{W}(x,p)\ -\ \tfrac{1}{4}\big[\,F_{\mu\nu}(x),\hat{W}(x,p)\,\big]\,)$$

$$+\ O(\partial_p\cdot\triangle F\cdot\gamma\hat{W})\ . \tag{5.3}$$

In the case of an *external Abelian gauge potential* eq.(5.3) reduces to

$$(\gamma^\mu p_\mu - m)\,\hat{W}(x,p)\ =$$

$$-\ \tfrac{1}{2}i\gamma_\mu\partial_x^\mu\,\hat{W}(x,p) +\ \tfrac{1}{2}ig\partial_p^\mu\gamma^\nu F_{\mu\nu}(x)\,\hat{W}(x,p)\ +\ O(\partial_p\cdot\triangle F\cdot\gamma\hat{W})\quad , \tag{5.4}$$

which for constant field tensors coincides with eq.(4.18).

We see that the on-shell condition, $p^2 = m^2$, can hold for quarks only, when the *covariant derivative* of $\hat{W}$ is small compared to p (in the reference frame of a specified ensemble) *and* the *field strength* is also small. By small we mean that the ensemble averaged terms in eq.(5.3) satisfy:

$$\langle p\hat{W}\rangle \gg \langle \mathcal{D}\hat{W}\rangle \ , \tag{5.5}$$

$$\langle p\hat{W}\rangle \gg \langle gF\,\partial_p\hat{W}\rangle \ . \tag{5.6}$$

These conditions constrain the possible ensembles for which a simple semiclassical picture may hold. Whether such ensembles exist and correspond to interesting physical situations cannot, of course, be guaranteed ahead of time. We return to this point in the next section. Note that for non-Abelian fields, the smallness of the covariant derivative means not only that the spatial gradients in the system are sufficiently small, but also that A and $\hat{W}$ approximately commute, i.e.

$$\langle p\hat{W}\rangle \gg \langle [gA, \hat{W}]\rangle \ . \tag{5.7}$$

When the above conditions are not satisfied, then significant off-shell corrections must be applied to the transport theory of quarks.

For covariant constant fields [12] satisfying

$$\mathcal{D}_\lambda F_{\mu\nu} = 0 \ , \tag{5.8}$$

the higher order corrections in $\triangle F$ vanish. However, in general $O(\triangle^n F)$ corrections must be calculated. Only for slowly varying fields, in the sense that

$$\langle F\hat{W}\rangle \gg \langle \mathcal{D}F\,\partial_p\hat{W}\rangle \ , \tag{5.9}$$

can an expansion in powers of $\triangle$ possibly converge rapidly.

In the general case for strong or rapidly varying fields the full quantum equation, eq.(4.17), must be solved. This would be equivalent to solving the field equations, eqs.(2.5), i.e. hopeless at this time. Only under the rather restrictive conditions, eqs.(5.5, 5.6, 5.9), can we expect that the transport theory for quarks reduces to a simpler, more manageable form.

Turning now to the transport equation, eq.(4.23), we obtain by the $\triangle$ expansion the *semiclassical transport equation for the QCD Wigner operator*:

$$\begin{aligned} & p\cdot\mathcal{D}(x)\,\hat{W}(x,p) \ + \ \tfrac{1}{2}gp^\mu\partial_p^\nu\Big\{\ F_{\nu\mu}(x), \hat{W}(x,p)\ \Big\} \ + \ \tfrac{1}{4}ig\Big[\ \sigma^{\mu\nu}F_{\mu\nu}(x), \hat{W}(x,p)\ \Big] \\ &= -\ \tfrac{1}{8}ig\partial_p^\mu\Big[\ F_{\mu\nu}(x), (\mathcal{D}^\nu(x)\,\hat{W}(x,p))\ \Big] \ - \ \tfrac{1}{16}ig^2\partial_p^\mu\partial_p^\nu\Big[\ F_\mu{}^\lambda(x)F_{\nu\lambda}(x), \hat{W}(x,p)\ \Big] \\ &\quad + O(\triangle F) \ . \end{aligned} \tag{5.10}$$

For example, the lowest order corrections to the Vlasov and spin terms on the l.h.s. of eq.(5.10) appearing among the $O(\triangle F)$ terms are given by

$$+\ \tfrac{1}{8}igp^\mu\partial_p^\nu\partial_p^\lambda\Big[\ (\mathcal{D}_\lambda(x)F_{\nu\mu}(x)), \hat{W}(x,p)\ \Big] \ - \ \tfrac{1}{8}g\partial_p^\lambda\Big\{\ \sigma^{\mu\nu}(\mathcal{D}_\lambda(x)F_{\mu\nu}(x)), \hat{W}(x,p)\ \Big\} \ . \tag{5.11}$$

Eq.(5.10) clearly generalizes Vlasov's equation in the following sense: i) it is still an operator equation; ii) the first and second term on the l.h.s. of eq.(5.10) present the usual combination of phase space variables and derivatives, however, modified by (anti)commutators, which can only be simplified for the external Abelian field problem (cf. eq.(5.12)); iii) explicit spin-dependent corrections arise; iv) quantum corrections can be systematically calculated via expansion of eq.(4.23) in powers of the $\triangle$ operator.

We remark here that the purely classical transport equation for quarks given in ref.[10] corresponds to the sum of the first two terms on the l.h.s. of eq.(5.10) being set equal to zero and decomposing them in color space. Furthermore, adding the corrections from (5.11) to the r.h.s. of eq.(5.10) and dropping all $\sigma^{\mu\nu}$-dependent terms, we reproduce the result for scalar quarks obtained in ref.[11], where, however, there is a sign error in the Vlasov term and correspondingly in the $O(\triangle)$ correction to it.

As a final limiting form, we consider the transport equation which follows from eq.(4.23) for the case of an *external Abelian gauge potential*:

$$
\begin{aligned}
&p\cdot\partial_x \hat{W}(x,p) \;+\; gp^\mu\partial_p^\nu F_{\nu\mu}(x)\,\hat{W}(x,p) \;+\; \tfrac{1}{4}igF_{\mu\nu}(x)\left[\,\sigma^{\mu\nu},\hat{W}(x,p)\,\right] \\
&= -\tfrac{1}{12}\left(\,g\partial_p\cdot[\partial_x F_{\mu\nu}(x)]\partial_p^\mu\partial_x^\nu \;+\; g^2\partial_p\cdot[\partial_x F_\mu{}^\lambda(x)]F_{\nu\lambda}(x)\partial_p^\mu\partial_p^\nu\,\right)\hat{W}(x,p) \\
&\quad -\tfrac{1}{8}g\partial_p\cdot[\partial_x F_{\mu\nu}(x)]\left\{\,\sigma^{\mu\nu},\hat{W}(x,p)\,\right\} \;+\; O(\partial_x^2 F\partial_p^2\hat{W}) \quad . \qquad (5.12)
\end{aligned}
$$

This is Vlasov's equation for the Wigner operator including spin-dependent (quantum) corrections. It generalizes the result obtained by Remler [3] for the case of scalar QED to spinor QED and reduces to eq.(4.22) for constant field tensors.

Finally, we remark again that the expressions *classical* or *semiclassical* employed in this section only refer to the approximate expansion of operators in powers of $\triangle$ acting on F. To relate the operator equations to the transport equations presented in the introduction, expectation values of the equations with respect to a suitable ensemble must be calculated as we discuss in the next section.

6. Discussion

The constraint and transport operator equations are completely equivalent to the original Dirac's field equation, eq.(2.5). They provide, however, a systematic means to study the semiclassical limit of QCD as expressed in terms of the Wigner function $\langle\hat{W}\rangle$. The choice of the ensemble over which $\hat{W}$ is to be averaged is, of course, dictated by the particular physical situation under study. Unfortunately, at present it is not known which field configurations are most relevant in actual physical processes such as nuclear collisions. This uncertainty is due to the unsolved confinement problem in QCD. To make further progress it is necessary to make an *ansatz*, i.e. a bold and unjustified guess, about the characteristics of that ensemble. We thus proceed in the spirit of the MIT Bag model and the color flux tube models [12,14] and assume that it is sensible to talk about quarks, as if they obeyed some effective plasma transport equation at least in some finite volume $V = L^3$.

Because of the asymptotic freedom property of QCD, we expect that the effective coupling, $g^2/4\pi \sim 1/\log(1/L\Lambda)$, where $\Lambda \sim 200$ MeV is the scale of QCD, vanishes as $L \to 0$. Thus in the limit of very small confined plasmas the transport equations are expected to reduce to the trivial free gas equation, $p\partial_x\langle\hat{W}\rangle = 0$. From the renormalization group approach we expect [1,2] furthermore that for high temperature and/or density plasmas the running coupling becomes small for any L. In fact, the strength of the coupling is always expected to be controlled by the largest of the parameters $M = T, \mu, L^{-1}$, or$(F^{0i})^{1/2}$ with dimensions of momentum that pertain to the problem via $g^2/4\pi \sim 1/\log(M/\Lambda)$. This inherent dependence of the coupling on the physical situation greatly complicates the analysis of the transport equations for quarks. As in the MIT Bag model we must therefore treat g as a phenomenological parameter. In the color flux tube models, for example, gF^{01} is parameterized in terms of the effective string tension.

In general, we found that the transport equations can only be simplified if rather restrictive conditions, eqs.(5.5, 5.6, 5.9), apply for the relevant physical ensemble. As a final exercise we now show that at least in the context of the color flux tube models the reduction can be carried further and that the plasma transport equations become the simple intuitive ones discussed in the introduction. We emphasize that this choice of the ensemble is only an *ansatz* motivated by previous phenomenological successes of such models (see references in [12]). Ultimately, comparison of results with high energy nuclear collision experiments will be necessary to establish the range of validity and limitations of this picture of the reaction dynamics.

With the above reservations clearly in mind, we consider an ensemble that corresponds to a quark plasma in a finite flux tube in which quarks are subject to a covariant constant or at most slowly varying mean field, $\langle F^{\mu\nu}(x)\rangle$. There exists a gauge where $\langle F^{\mu\nu}(x)\rangle$ is diagonal. Since every traceless $N \times N$ matrix can be expanded in terms of the $N-1$ commuting generators, h_i, of SU(N) defined by [12]

$$h_j \equiv (2j(j+1))^{-\frac{1}{2}} diag(1, \cdots, 1, -j, 0, \cdots, 0) \quad , \tag{6.1}$$

with $-j$ appearing in the $j+1$ column, we can always write

$$\langle F_{\mu\nu}(x)\rangle \equiv S(x)\vec{F}_{\mu\nu}(x) \cdot \vec{h}S^{-1}(x) \quad , \tag{6.2}$$

where $S(x)$ is a particular gauge transformation.

If we now make the *ansatz* that the ensemble in the flux tube is such that $\hat{W}$ is diagonal in the *same gauge* where $\langle F_{\mu\nu}(x)\rangle$ is diagonal, then we can express

$$\langle\hat{W}\rangle \equiv S(x)\left(\sum_{j=1}^{N-1} W^j h_j + W^0 \underline{1}\right) S^{-1}(x) \quad , \tag{6.3}$$

in terms of N Wigner *functions* depending on (x, p). We suppress presently irrelevant spinor indices, but a similar decomposition in spinor space in terms of the Clifford algebra can be performed [5]. Note that the property $\hat{W}^\dagger = \gamma^0\hat{W}\gamma^0$ insures that the N diagonal elements in color space $\langle\hat{W}\rangle^{ii}$ are real numbers after taking appropriate traces in spinor space.

Eq.(6.3) is a strong model assumption. It is, however, plausible that for slowly varying fields it is satisfied if the initial conditions for the plasma are assumed to correspond to $\langle\hat{W}\rangle = 0$, as in flux tube models [12,14].

If (6.3) holds, then it is obvious that the most convenient gauge to work in is the $S(x)$ gauge. In that gauge by assumption

$$\begin{aligned}\langle\hat{W}\rangle_{ij} &= (\vec{W}\cdot\vec{h}+W^0)_{ij} \\ &= \delta_{ij}(\vec{W}\cdot\vec{\epsilon}_j+W^0) \equiv \delta_{ij}f_j \ ,\end{aligned} \tag{6.4}$$

with the "charges" $\vec{\epsilon}_j$ given by

$$\vec{\epsilon}_j \equiv (\vec{h})_{jj} = ((h_1)_{jj},\cdots,(h_{N-1})_{jj}) \ . \tag{6.5}$$

These are just the elementary weight vectors of SU(N). Eq.(6.4) provides the model dependent relation between the Wigner operator and the classical quark distribution functions discussed in the introduction.

The semiclassical transport equations for the $f_j(x,p)$ in this model are obtained by taking the expectation value of eq.(5.10) in this ensemble. Using eqs.(6.2-6.5), the color structure of the semiclassical transport equation, eq.(5.10), is simplified considerably:

$$\begin{aligned}&(p\cdot\partial_x + g\vec{\epsilon}_j\cdot\vec{F}_{\mu\nu}p^\nu\partial_p^\mu)\ f_j(x,p) \\ &= -\tfrac{1}{4}ig\vec{\epsilon}_j\cdot\vec{F}_{\mu\nu}[\,\sigma^{\mu\nu},f_j(x,p)\,] \ + \ \tilde{C}_j(x,p) \ ,\end{aligned} \tag{6.6}$$

where $\tilde{C}_j$ represents correlation terms of the form

$$\tilde{C}_j(x,p) = -\tfrac{1}{2}gp^\mu\partial_p^\nu\left(\langle\{F_{\nu\mu},\hat{W}(x,p)\}\rangle - 2\langle F_{\nu\mu}\rangle\langle\hat{W}(x,p)\rangle\right)_j + \cdots \ . \tag{6.7}$$

It is known [6] that the Boltzmann collision terms are contained in such correlations. Those correlation terms correspond to expectation values of two body operators. Just as in the BBGKY hierarchy, n-body expectation values can be related to expectation values of (n+1)-body operators. The Boltzmann *ansatz* corresponds to a truncation of that hierarchy at the one body level. Thus correlation terms are expressed as non-linear combinations of one body expectation values. We shall make no attempt in this paper to decompose the important correlation terms. Rather, we stop here with having shown how the Vlasov terms arise in a particular phenomenological model from the underlying quantum theory. Note that all the non-Abelian commutator terms dropped out for the model *ansatz*, eq.(6.3).

The model transport equation, eq.(6.6), confirms some of the expectations discussed in Sec.1. In particular, we observe the effective coupling constant $g\vec{\epsilon}_j$ entering the above set of N Abelian-like equations for the components f_j of $\langle\hat{W}\rangle$ (cf. eq.(6.4)). In the absence of correlations the equations would simply decouple. This equation seems to be as close as one can get to the classical Vlasov's equation starting from the Wigner operator defined in eq.(1.8). It is valid for fermions with positive or negative energy. For $p^0 < 0$ we obtain the equation for antiquarks by replacing p^μ by $-p^\mu$ everywhere, which effectively reverses the sign of g in eq.(6.6) as expected. We see the explicit spin-dependence as well. The important quantum corrections, especially pair production, are buried in the uncalculated correlation terms.

For an electron-positron plasma the transport equation derived in this paper may be useful as well. To our knowledge spin and systematic quantum corrections to that equation have not been derived previously. For a U(1) gauge symmetry one only has to replace $\vec{\epsilon}_j \rightarrow 1$ and the matrix transport equation becomes a single Vlasov's equation including corrections.

Obviously a great deal of work remains in the development of the transport theory for quarks and gluons. The extraction of Debye screened collision terms and color neutralizing pair production terms from the correlation terms needs to be performed. There remains the challenge of defining a gauge covariant Wigner function for gluons and the derivation of the gluon transport equation from the field equations. The question of the proper physical basis for the ensemble averaging also is a crucial area for study. Can the *ansatz* (6.3) be justified? These and other problems will have to be solved in order to connect upcoming nuclear collision data with the properties of quark-gluon plasmas.

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Figure captions

*Fig.*1 Variation of a straight line path due to infinitesimal shifts of its endpoints.

*Fig.*2 The infinitesimal closed loop (plaquette) employed in eq.(3.11).

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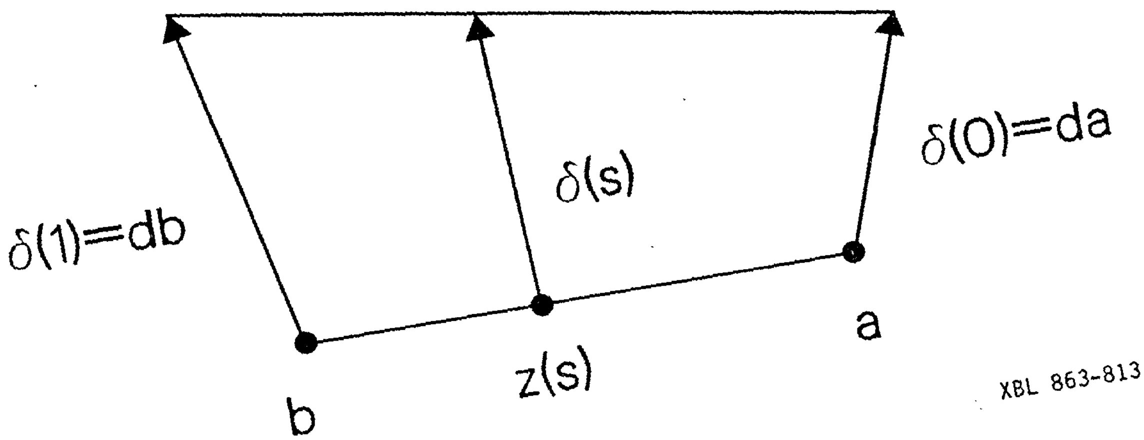

Figure 1

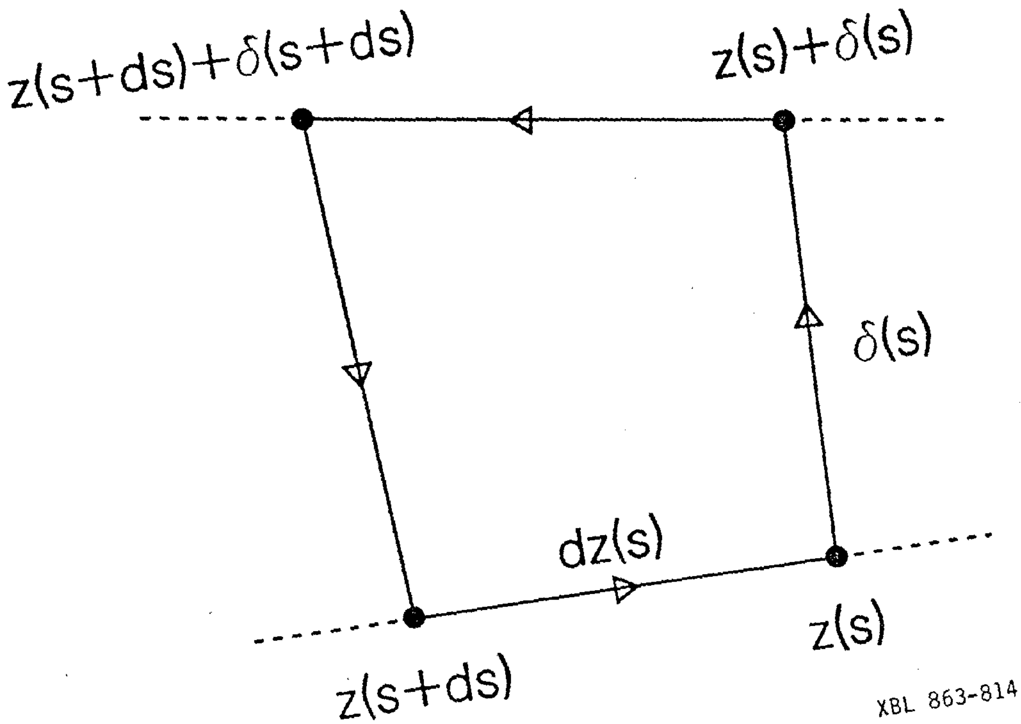

Figure 2

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