

UCLA

UCLA Previously Published Works

Title

Exploring dynamical gluon mass generation in three dimensions

Permalink

<https://escholarship.org/uc/item/9wn5301c>

Journal

Physical Review D, 93(2)

ISSN

2470-0010

Author

Cornwall, John M

Publication Date

2016-01-15

DOI

10.1103/physrevd.93.025021

Peer reviewed

Exploring dynamical gluon mass generation in three dimensions

John M. Cornwall*

Department of Physics and Astronomy, University of California, Los Angeles, California 90095, USA
(Received 14 October 2015; published 25 January 2016)

We reexamine the $d = 3$ dynamical gluon mass problem in pure-gluon non-Abelian $SU(N)$ gauge theories, paying particular attention to the observed (in Landau gauge) violation of positivity for the spectral function of the gluon propagator. This is expressed as a large bulge in the propagator at small momentum, due to the $d = 3$ avatar of asymptotic freedom. Mass is defined through $m^{-2} = \Delta(p = 0)$, where $\Delta(p)$ is the scalar function for the gluon propagator in some chosen gauge; it is not a pole mass and is generally gauge dependent, except in the gauge-invariant pinch technique (PT). We truncate the PT equations with a recently proposed method called the vertex paradigm that automatically satisfies the QED-like Ward identity relating the three-gluon PT vertex function with the PT propagator. The mass is determined by a homogeneous Bethe-Salpeter equation involving this vertex and propagator. This gap equation also encapsulates the Bethe-Salpeter equation for the massless scalar excitations, essentially Nambu-Goldstone fields, that necessarily accompany gauge-invariant gluon mass. The problem is to find a good approximate value for m and at the same time explain the bulge, which by itself leads, in the gap equation for the gluon mass, to excessively large values for the mass. Our point is not to give a high-accuracy determination of m but to clarify the way in which the propagator bulge and a fairly accurate estimate of m can coexist, and we use various approximations that illustrate the underlying mechanisms. The most critical point is to satisfy the Ward identity. In the PT we estimate a gauge-invariant dynamical gluon mass of $m \approx Ng^2/(2.48\pi)$. We translate these results to the Landau gauge using a background-quantum identity involving a dynamical quantity κ such that $m = \kappa m_L$, where $m_L^{-2} \equiv \Delta_L(p = 0)$. Given our estimates for m , κ , the relation is fortuitously well satisfied for $SU(2)$ lattice data.

DOI: 10.1103/PhysRevD.93.025021

I. INTRODUCTION: THE $d = 3$ AND $d = 4$ GLUON MASS PROBLEMS

This paper considers dynamical gluon mass generation in a pure-gluon $d = 3$ non-Abelian gauge theory (NAGT), based on the pinch technique (PT). Recall that the PT algorithm was introduced [1–6] to generate gauge-invariant Green's functions in non-Abelian gauge theories such as a NAGT, and it was later extended in $d = 4$ to an algorithm for Green's functions both gauge invariant and renormalization-group invariant (RGI) [7–9]. Of course, the gluon mass is not a pole mass, or we would see gluons in experiments; it is more in the nature of a screening mass, analogous to the polaron of condensed matter physics—an electron or a hole made heavy by coupling to the ionic background. In a NAGT the gluon couples to a background of other virtual gluons. We give a precise definition to the mass concept, defining a zero-momentum mass m by

$$m^2 \equiv \hat{\Delta}^{-1}(p = 0). \quad (1)$$

(Throughout this paper, hatted quantities are PT quantities.) We similarly define a Landau-gauge mass by

$$m_L^2 \equiv \Delta_L^{-1}(p = 0). \quad (2)$$

Here, $\hat{\Delta}(p)$ is the scalar function for the PT gluon propagator, and a similar situation for the Landau gauge.

The PT mass is the zero-momentum value of a running mass $m(p)$ that vanishes like $1/p^2$ at large momentum; see Sec. V. For technical reasons, in $d = 3$ it is much simpler to ignore the running of the mass, which we do throughout this paper. Clearly this definition of mass makes sense only if the right-hand side of Eqs. (1) and (2) is finite and positive. As we discuss at the end of this section, all lattice simulations in the Landau gauge show that this is indeed true. In Sec. VII we invoke a background-quantum identity showing that the zero-momentum Landau-gauge propagator $\Delta_L(p = 0)$ is a finite and positive multiple κ_L^2 of $\hat{\Delta}(p = 0)$:

$$\Delta_L(p = 0) = \kappa_L^2 \hat{\Delta}(p = 0), \quad (3)$$

and so $m = \kappa_L m_L$. Since $\kappa_L < 1$, these two masses are different. This is to be expected since the Landau-gauge propagator is gauge dependent and unphysical; the PT mass as defined in Eq. (1) is gauge invariant. By estimating κ_L and using the estimate of m^2 from the present work, we find a fortuitously close agreement between our resulting approximate value of m^2 and the Landau-gauge mass inferred from simulations. Or, conversely, we may take the simulation value m_L and infer m , again with fortuitously good

*cornwall@physics.ucla.edu

agreement, considerably better than the 20%–25% error that we are probably making in our approximate formulation.

One might think that $d = 3$ mass generation should be an easier problem than in $d = 4$, where renormalization is required. In contrast, a $d = 3$ NAGT is superrenormalizable, so that at infinite momentum the gluon coupling g^2 does not change from its value in the classical action. Nevertheless, we can define a running coupling $\bar{g}^2(p)$ without reference to a renormalization group and this running has important consequences for the gluon mass in $d = 3$.

A number of theoretical works on the gluon mass in $d = 3$ date from the 1990s [1–5,10–16]. These gave reasonable results for the gluon mass, but a closer analysis [11] of the theoretical gluon propagators seemed to be disappointing and unphysical for a reason that was not really appreciated at the time: nonpositivity of the propagator spectral function. This is manifested by a bulge in the Euclidean propagator, clearly evident in Landau-gauge lattice simulations. Nonpositivity is a consequence of $d = 3$ infrared slavery, inherited from the “wrong” sign of $d = 4$ asymptotic freedom (AF).

Aside from the works referenced above, there is also a decade-later work [17] using a special form of the PT and

oriented to later Landau-gauge lattice data. The general approach is similar to what is used here, including the addition of massless scalars to the three-gluon vertex [5]. The massless scalar fields are essentially Nambu-Goldstone (NG) excitations, and they must exist as bound states if the gluon mass is to be gauge invariant with no elementary Higgs fields. Using free vertices and free massive input propagators, the authors find important nonpositivity in the output propagator. However, there are significant differences from the present work in the treatment of mixing the massless scalars with gluons and of determination of the gluon mass. Moreover, there is no discussion of the effects of nonpositivity on the three-gluon vertex, which we estimate to be considerable and in the direction of canceling nonpositivity effects in the propagator when used in the gluon mass gap equation. Reference [17] and other later works benefited from good lattice data on the Landau-gauge propagator [18–20] that we will recap in Sec. II. The lattice evidence for dynamical generation of some sort of gluon mass is unequivocal in $d = 3$: The Landau-gauge inverse propagator is not zero at zero momentum, but finite and positive (see Fig. 1 in the next section). In $d = 4$ there is also an abundance of lattice work that confirms gluon

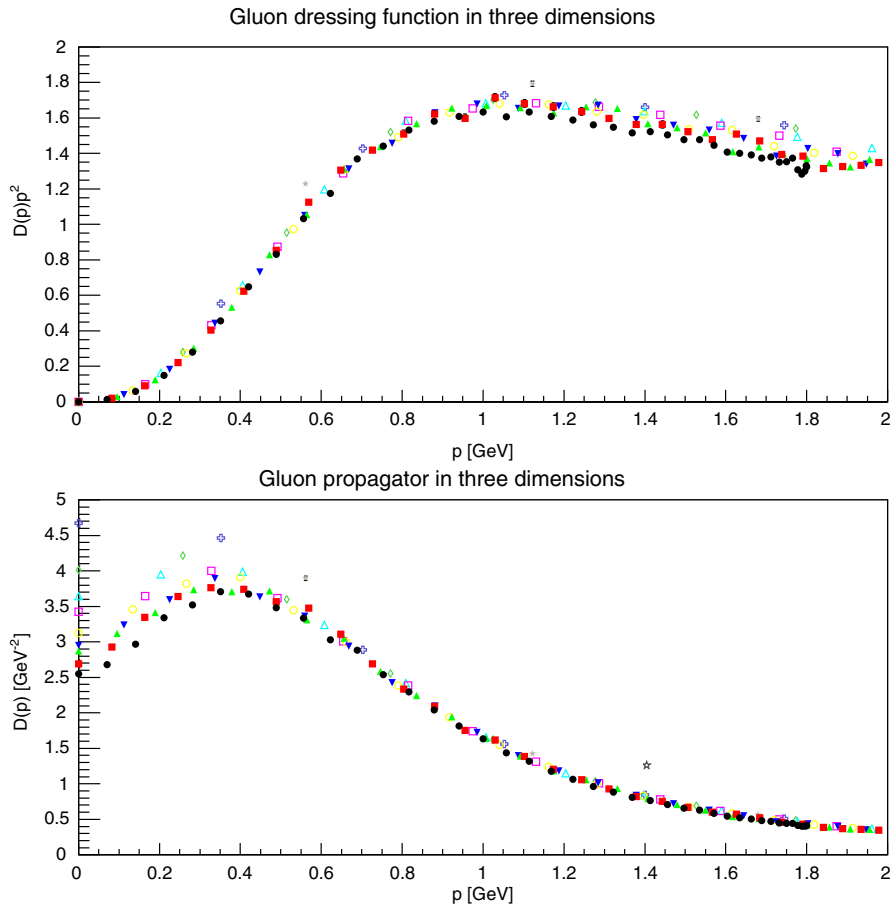


FIG. 1. The $d = 3$ Landau-gauge gluon propagator [$D(p)$, lower curve] as a function of momentum p for various lattice sizes. The filled circles are at the largest lattice size of 18 fm^3 . (The gluon dressing function in the upper graph is p^2 times the lower curve.)

mass generation. See, for example, [21–48]. Moreover, much is being done in continuum studies of the $d = 4$ problem, mostly by Papavassiliou and collaborators (for a discussion of work up to 2011, see [6]; later work can be traced from, for example, [49]).

In the present paper we argue that in the homogeneous Bethe-Salpeter (BS) equation governing the value of the dynamical mass, this positivity problem is largely ameliorated by a compensating dip in the three-gluon vertex, so that the predicted gluon mass value is much less affected by nonpositivity than the propagator itself is. That there must be a dip in the vertex that (partially) compensates the propagator bulge follows from the QED-like Ward identity (see Sec. III) relating them. Approximations not accounting for both the propagator bulge and the vertex dip can give gluons mass values far removed from reality. For this reason, it is particularly important that the Ward identity be satisfied, even in the face of approximations. The vertex paradigm that we use here is based on constructing an approximate vertex from which the propagator is extracted using the Ward identity. Much of the present work is devoted to the study of this complicated nonlinear problem. We can find an explicit Feynman-parameter integral for the approximate vertex at one-dressed loop, and it is easy to find the corresponding propagator from the Ward identity.

Evaluating the many terms of this Feynman-parameter integral to find the vertex itself is a daunting task, never done even for the one-loop perturbative vertex. Even if it were evaluated precisely, it is an approximation that is not necessarily highly accurate. Consequently, we resort to other approximations based on $d = 5, 6$ scalar field theories to capture the essence of how the Ward identity relates a propagator bulge to a vertex dip [7–9]. These are useful because ϕ_6^3 is asymptotically free and its $d = 5$ descendant behaves much like a $d = 3$ NAGT; they are reviewed in Appendix B 1. The models are tweaked so that their one-dressed-loop Schwinger-Dyson equations (SDEs) resemble those of a NAGT as much as possible, and to this end some of the fields are endowed with an Abelian charge and a corresponding vertex with its Ward identity. It is uncomplicated to carry out the vertex-paradigm construction for these scalar field theories, and the results for the propagator are surprisingly close to previous approximations to NAGTs [9].

A. The vertex paradigm

The vertex paradigm was previously used in $d = 4$ for truncating the PT Schwinger-Dyson equations [7–9]. It begins with analytic tree-level approximations to the full PT inverse propagator and three-gluon PT vertex (the inputs) that are massive and therefore free of IR singularities. If the masses do not run, these inputs exactly satisfy the ghost-free QED-like Ward identity relating them. This is a critical point in showing that a one-dressed-loop output approximation to the three-gluon vertex using these input

Green’s functions satisfies the Ward identity. We then simply apply the Ward identity to find the output gluon propagator. There are a number of technical obstacles to overcome, in particular the treatment of bound-state massless scalar excitations, akin to NG fields, that are necessary if the gluon has mass. We give a road map of the vertex paradigm in the appendixes, and more details are found in [9]. In principle, the output Green’s functions can be recycled and used as input functions for another round, but nothing is known about what happens in this second stage.

In reality, the output gluon mass is a running mass, depending on momentum [2,50] in both $d = 3$ and $d = 4$. The author does not know how to guarantee the Ward identity with a momentum-dependent input mass, but it is much easier, although not trivial, if the mass does not run. In $d = 4$ using a constant mass prevents us from actually finding a value for the mass, which must vanish at infinite momentum for the gluon mass gap equation to be UV finite. (If a truly constant bare mass led to an UV-finite solution of the gap equation, then a NAGT with a mass term would be renormalizable in $d = 4$.) But in $d = 3$ we can still solve the gap equation with a constant mass. The error made in this approximation is small since the UV region contributes little to the gap equation. Ultimately, this is inconsistent because in $d = 3$ a constant mass input to the gap equation automatically leads to an output mass that runs to zero as $1/p^2$ at large momentum. An identically vanishing mass is not a solution to the gap equation, which becomes IR singular in this limit.

Unless otherwise specified, we carry out the vertex paradigm in Euclidean space with the usual Euclidean metric. We reserve a study of the properties of the dynamical mass in Minkowski space, where it is surely not a pole mass, for the future.

B. Organization of the paper

Section II brings up the critical properties, related to infrared confinement, that are the central themes of this paper. In Sec. III we introduce the vertex paradigm and the Ward identity, and we discuss the massless scalar poles necessary for gauge-invariant gluon mass generation. Section IV is a straightforward transcription of earlier $d = 4$ efforts in the vertex paradigm to construct the $d = 3$ pole-free vertex. Section V constructs the homogeneous BS equation whose solution is the running mass. Section VI and Appendix B 1 introduce an approximation to the output three-gluon vertex, based on IR confinement analogs found in IR confining scalar theories in $d = 5, 6$. This section also proposes, for heuristic purposes, a $d = 3$ running charge that is just another name for part of the three-gluon vertex, similar to what has been done in $d = 4$ [8,9]. Section VII estimates the function $1 + \hat{G}(p)$ that determines the ratio between the Landau-gauge lattice propagator Δ_L and the PT propagator $\hat{\Delta}$. Knowledge of

$1 + \hat{G}(p)$ allows us to compare the lattice-gauge Landau propagator to the PT propagator, and in particular the gluon masses (defined from the inverse propagators at zero momentum). The Landau-gauge mass m_L need not, and does not, agree with the PT mass m . Section VIII has a summary and conclusions, and Appendixes A, B, and C elaborate on the vertex-paradigm, scalar theories in $d = 5$, 6, and a useful regulator for certain divergent integrals, respectively.

II. CRITICAL PROPERTIES

Some properties that complicate the gluon mass problem hold in both the PT Green's functions and in the Landau-gauge lattice gluon propagator, and in both $d = 3$ and $d = 4$. The Green's functions have the properties that

- (1) The inverse propagator has zero-mass scalar poles, akin to Nambu-Goldstone poles. These are required by gauge invariance if the proper self-energy does not vanish at zero momentum. This nonvanishing is equivalent to a dynamical gluon mass, although not a simple pole mass. These NG poles cannot appear in physical quantities.
- (2) The propagator, although it obeys a spectral representation, does not have a strictly non-negative spectral function. This is the positivity problem. It is a $d = 3$ avatar of AF.
- (3) The PT three-gluon vertex has features related to the PT propagator nonpositivity through the QED-like Ward identity connecting them.

In an R_ξ gauge the inverse of the PT propagator has the form

$$\begin{aligned}\hat{\Delta}_{ij}^{-1}(p) &= P_{ij}(p)\hat{\Delta}^{-1}(p) + \frac{1}{\xi}p_i p_j, \\ \hat{\Delta}^{-1}(p) &= p^2 + \hat{\Pi}(p),\end{aligned}\quad (4)$$

where the transverse projector is

$$P_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{p^2}.\quad (5)$$

The PT proper self-energy $\hat{\Pi}(p)$ is independent of the gauge-fixing parameter ξ , whose coefficient in the propagator receives no physical corrections. We omit writing this gauge-fixing term in the following equations.

A. First critical property: A gluon mass

From Eq. (4) one sees that the first critical property, the NG-like poles, arises as long as $\hat{\Pi}(p=0) \neq 0$. This is equivalent to dynamical mass generation, which is signaled by an inverse propagator that is finite and positive at zero momentum. Through the QED-like Ward identity relating the three-gluon PT vertex to the inverse PT propagator, these NG-like poles have to be in the vertex, but these poles get projected out in Landau-gauge lattice studies. The

second critical property shows up in the lattice data cited above, most of it in Landau gauge, with clear evidence of nonpositivity in the propagator spectral function.

The first property, a gluon mass, is evident for the Landau gauge in Fig. 1, showing the $d = 3$ gluon propagator in the Landau gauge [20].

B. Second critical property: Nonpositivity

The second critical property is that the spectral function is negative in some regions. The scalar function $\hat{\Delta}$ has the spectral representation

$$\hat{\Delta}(p) = \frac{1}{\pi} \int_{\sigma_0}^{\infty} d\sigma \frac{\rho(\sigma)}{p^2 + \sigma}.\quad (6)$$

The same basic representation holds in, for example, the Landau gauge, but unlike the spectral representation for conventional gauge-dependent propagators, in the PT case there are no unphysical and gauge-dependent thresholds, such as would apply to ghosts, and σ_0 is strictly positive. If the spectral function is nowhere negative, it is apparent that the derivative with respect to p of the propagator can nowhere be positive, and equally apparent that this condition is violated in Fig. 1. The filled circles, data for the largest lattice, as well as data for smaller lattices, clearly show that the gluon propagator has a positive slope at zero momentum, which equally clearly shows that positivity is violated.

The positivity violation in $d = 4$ is not obvious just from a casual glance at the propagator. A little closer look shows that there is indeed nonpositivity in $d = 4$, but not as pronounced as for $d = 3$. See, for example, Fig. 1 of [19], comparing the two cases. In both $d = 3$ and $d = 4$ the cause of the bulge is wrong signs coming from IR confinement.

Finding the spectral function itself from lattice data is not straightforward, because only data in the Euclidean region are available and it is difficult to reconstruct the spectral function accurately just from knowledge of the propagator in this regime. For a brief review of these issues with references to original work, see [51].

The source of the bulge in $d = 3, 4$ is the IR confinement wrong sign. As has long been known [1,2,10], the $d = 3$ $SU(N)$ PT propagator in one-loop perturbation theory is

$$\hat{\Delta}^{-1}(p) = p^2 + \hat{\Pi}(p) = p^2 - \pi b g^2 p,\quad (7)$$

where g^2 is the $d = 3$ gauge coupling and b is the *gauge-invariant* number

$$b = \frac{15N}{32\pi}.\quad (8)$$

The minus sign in (7) comes directly from $d = 4$ AF. In Landau gauge 15 is replaced by 11; this suggests the degree

to which the Landau-gauge and the PT propagator differ, although both have the same so-called wrong sign. It is this sign in this inverse propagator that gives rise to an unphysical tachyonic pole in the perturbative PT propagator at a Euclidean momentum $p^2 = (\pi b g^2)^2$. This tachyon will be killed by a mass term and by massive internal gluon propagators, provided that the mass is large enough. But removing the tachyon is not enough; it leaves its mark behind in the propagator bulge of Fig. 1, which unambiguously reveals the nonpositivity problem. This means that the output propagator does not resemble a free massive propagator such as in Eq. (17) below, except in the extreme IR and UV.

C. Third critical property: An inverse bulge in the three-gluon vertex

There are no lattice data that are useful in understanding the PT three-gluon vertex, but this vertex has critical properties in the gluon gap equation. The Ward identity of Eq. (10) below, relating the divergence of the three-gluon Green's function to the inverse PT propagator, suggests that the positivity-violation bulge in the propagator is mitigated in dressed-loop graphs by an offsetting dip in the vertex. In the gap equation this can lead to substantial cancellation of nonpositivity effects between vertex and propagator.

This is a difficult issue to explore, since the output three-gluon vertex of the vertex paradigm is so complicated, and has wrong-sign problems of its own that could lead to unphysical tachyons. At the moment we can only address it heuristically and approximately. One element is to exploit AF in the scalar theory ϕ_6^3 , having strong analogs to $d = 4$ NAGTs, and its avatar in ϕ_3^3 , analogous to a $d = 3$ NAGT [8,9]. In particular, the propagator bulge can be well modeled, and it is related to a three-gluon vertex form factor by a QED-like Ward identity, just as in a NAGT. This suggests that our model for the form factor is also useful since it is from this form factor and the Ward identity that the propagator is derived.

In $d = 4$, the three-gluon form factor of interest is closely related to the usual running charge, which rises from the UV and saturates in the IR. We hope to make plausible here that in $d = 3$ there is also a running charge $\bar{g}(p)$ with the same properties, defined not through a renormalization group but through the PT three-gluon vertex of the NAGT and the Ward identity relating it to the PT propagator. See Appendix C for a few details. Because the renormalization group is not involved, this charge is defined in $d = 3$ as well as in $d = 4$ (where it agrees, through two loops, with the usual running charge in the UV). We define the running charge by one of the three-gluon vertex scalar form factors with one momentum set to zero [8,9]; let us call this $G(p, -p, 0)$. The running charge $\bar{g}^2(p)$ is related to the form factor by Eq. (C4), repeated here for convenience:

$$\bar{g}^2(p) = \frac{g^2}{G(p, -p, 0)}. \quad (9)$$

In the tweaked analog models we identify this form factor with the primary form factor of the Abelian current introduced above.

In the IR it is not possible to define a running charge uniquely, in $d = 3$ or in $d = 4$. But our definition is physically plausible and has useful properties. For example, we will take it that the squared running charge is, as its name suggests, positive. Moreover, our definition in $d = 3$ yields a running charge largest at zero momentum and monotonically decreasing toward the UV, where it approaches the fixed Lagrangian coupling g^2 at infinite momentum. The consequent properties for $G(p, -p, 0)$ imply a vertex-function dip that tends to offset the nonpositivity bulge in the gluon propagator. The final result is coexistence of the nonpositivity bulge in the PT (or Landau-gauge) propagator with a gluon mass that is consistent both with the gap equation and with Landau-gauge lattice data.

III. THE VERTEX PARADIGM, THE WARD IDENTITY, AND MASSLESS SCALAR POLES IN THE PT PROPAGATOR

We give here only an outline of the technically tedious steps in the one-loop vertex paradigm. Appendix A gives a very brief summary, and details are in [9].

The main point of the vertex paradigm, a truncation of the PT SDEs, is to construct successive approximations to the PT gluon proper self-energy and to the PT three-gluon vertex, following PT principles for constructing gauge-invariant Green's functions, that

- (1) Satisfy the QED-like (ghost-free) Ward identities of the PT.
- (2) Incorporate dynamical gluon mass in the IR.
- (3) Yield the correct perturbative results in the UV.

The potential advantage of the vertex paradigm, compared to other truncations the author knows about, is that, in principle at least, it yields a plausible semiquantitative candidate not only for the PT propagator but also for the PT three-gluon vertex. There is a strong connection between these Green's functions from the Ward identity relating them.

There are several obstacles to implementing the vertex paradigm:

- (1) Not every approximation to a three vertex will satisfy the Ward identity [see Eq. (10) below] structurally, that is, have a divergence that actually is the difference of two identical functions with different momenta as arguments.
- (2) A gluon mass requires poles in the inverse propagator and so, by the Ward identity of Eq. (10), also in the three vertex. But no such poles occur in a

one-loop vertex constructed with simple input propagators and vertices.

- (3) The method of successive approximations may show signs of nonconvergence.

The vertex paradigm [8,9] can handle the first two problems. We devote much of this paper to formulating a semiquantitative solution to the last problem, which arises because of nonpositivity. The trouble is that in $d = 3$ the one-loop output propagator does not at all resemble the input propagator.

In the PT, Ward identities are QED-like, with no ghost contributions. For example, the Ward identity relating the PT three-gluon vertex $\hat{\Gamma}_{ijk}$ to the PT inverse propagator is

$$p_{1i}\hat{\Gamma}_{ijk}(p_1, p_2, p_3) = \hat{\Delta}^{-1}(p_2)P_{jk}(p_2) - \hat{\Delta}^{-1}(p_3)P_{jk}(p_3). \quad (10)$$

Although this is a ghost-free and gauge-independent relation, the right-hand side is not a difference of inverse propagators (except in a ghost-free gauge). If $\hat{\Delta}^{-1}(p = 0)$ is not zero, there are poles in the right-hand side, and therefore such poles also exist in the vertex.

In the vertex paradigm, the vertex in the Ward identity is a sum of two pieces. The first vertex part is a simple Feynman integral and it does not have the poles required by the Ward identity to yield the poles of the massive inverse propagator in Eq. (4). So we add (see Sec. III A) a second vertex part [5], called V_{ijk} , which is the product of regular factors times terms with massless longitudinally coupled NG-like scalar excitations. It satisfies its own Ward identity that gives precisely the pole parts of the inverse propagator. We emphasize that these NG-like excitations do not imply symmetry breaking in dynamical gluon mass generation for a NAGT.

If we knew the vertex we could find the inverse propagator from the Ward identity. This seems like a circular statement since one needs the propagator to find the vertex. We try to avoid this circularity by using successive approximations, starting with a reasonable tree-level form for input propagators and vertices in one-dressed-loop graphs. The approximate output vertex is just the integral over the input propagator and vertices. Then the Ward identity gives the output inverse propagator. The hope is that this process of successive approximations will eventually converge.

An earlier truncation method, called the gauge technique and explained in [6], attempts the inverse problem: Given the propagator, find the vertex. This is a much easier problem with an algebraic solution (see [6] and Sec. III A), but it is not very accurate. It leads to SDEs written entirely in terms of the propagator. The gauge technique is based on the construction of Sec. III A below.

As is by now well known, the PT algorithm is equivalent to working in the background-field method Feynman gauge [6]. The first vertex piece, called G_{ijk} , is based on the PT

or background-field method (BFM)-Feynman approach [52–54] to the one-loop three-gluon vertex in perturbation theory, with massless gluons and ghosts. This one-loop vertex is quite complex and has never been evaluated fully, even in perturbation theory. Fortunately, its graphical construction makes it straightforward to verify the Ward identity from the momentum-space integrand of the one-loop vertex, without actually doing any integrals, and it is then easy to evaluate the vertex-paradigm PT proper self-energy.

In order to use the mass gap equation we need a semiquantitative approximation to this complicated output vertex. The vertex has some properties that are general consequences of IR confinement and a QED-Ward identity, so we will model these properties in one-loop graphs of ϕ_3^3 (see Appendix B 1). This can be at best semiquantitatively correct, but it serves to make the point about how non-positivity is subdued by cancellations between the product of propagator and vertex occurring in the mass gap equation.

A. The PT Ward identity and Nambu-Goldstone poles

The problem of finding V_{ijk} was solved in principle long ago [5]. The full vertex is the sum of these two parts:

$$\hat{\Gamma}_{ijk}(q, k_1, k_2) = G_{ijk}(q, k_1, k_2) + V_{ijk}(q, k_1, k_2). \quad (11)$$

The vertex V_{ijk} has the form, in $d = 3$:

$$\begin{aligned} V_{ijk}(p_1, p_2, p_3) = & \frac{-p_{1i}p_{2j}}{2p_1^2p_2^2}(p_1 - p_2)_a\Pi_{ak}^m(p_3) \\ & - \frac{p_{3k}}{p_3^2}[P_{ai}(p_1)\Pi_{aj}^m(p_2) \\ & - P_{aj}(p_2)\Pi_{ai}^m(p_1)] + \text{c.p.} \end{aligned} \quad (12)$$

We have expressed this vertex in terms of a special transverse self-energy Π_{ij}^m that is purely nonperturbative, and we define its scalar part by

$$\Pi_{ij}^m(p) = P_{ij}(p)\Pi^m(p) \equiv P_{ij}(p)m^2(p), \quad (13)$$

where $m(p)$ is the running mass. (In the earliest PT papers, this self-energy was assumed to be the full self-energy, yielding the gauge-technique truncation of the SDE.) In the constant-mass approximation we define $m = m(p = 0)$. Note that every term in V has not one, but two, massless scalar poles, but its Ward identity has only a single pole:

$$p_{1i}V_{ijk}(p_1, p_2, p_3) = \Pi^m(p_2)\frac{p_{2j}p_{2k}}{p_2^2} - \Pi^m(p_3)\frac{p_{3j}p_{3k}}{p_3^2}. \quad (14)$$

Observe that the Ward identity for the V vertex exactly satisfies the Ward identity necessary to accommodate the poles of the full inverse propagator. It follows that G_{ijk}

must also obey that Ward identity, but with an inverse propagator Δ^{-1} that has no poles. The full inverse propagator is the sum of this pole-free part plus the pole terms in Eq. (14).

B. The pole-free part of the Ward identity

We now come to the hard part [9]: To find an approximate and fairly simple form of the pole-free vertex and inverse propagator that exactly satisfies the same QED-like Ward identity,

$$p_{1i}G_{ijk}(p_1, p_2, p_3) = \Delta^{-1}(p_2)P_{jk}(p_2) - \Delta^{-1}(p_3)P_{jk}(p_3), \quad (15)$$

but with no poles either in the vertex or in the inverse propagator. Here, the inverse propagator Δ^{-1} is not the full inverse PT propagator $\hat{\Delta}^{-1}$, but only that part of it vanishing at zero momentum, thus yielding no poles. Similarly, G_{ijk} is not the full PT vertex, but its sum with V_{ijk} is the full vertex, as in Eq. (11), and the full inverse propagator is the sum of Δ^{-1} and the pole terms as in Eq. (16) below. In $d = 4$ an approximate one-loop pole-free three-gluon vertex that exactly satisfies the Ward identity is given in [9], with a construction based on a reasonably straightforward, if complicated in detail, extension of the one-loop perturbative three-gluon PT vertex [52]. IR singularities are removed with free massive propagators as in Eq. (17) below. Even in perturbation theory it would be a formidable job to do the integrals for the output three-gluon vertex explicitly [52–54], but in order to get the output PT propagator we need only use the QED-like Ward identity of Eq. (15) and the unintegrated vertex. Now we can add G_{ijk} to the pole vertex V_{ijk} , and similarly the pole part of the self-energy Π^m to the self-energy Π and form an approximation to the full PT inverse propagator:

$$\hat{\Delta}^{-1}(p) \equiv p^2 + \hat{\Pi}(p^2) = p^2 + \Pi(p^2) + \Pi^m(p^2). \quad (16)$$

The vertex sum of Eq. (11) obeys the full PT Ward identity.

IV. THE $d = 3$ VERTEX PARADIGM—STEP 1: THE POLE-FREE VERTEX

A. Inputs

The inputs for constructing the pole-free output Green's functions are essentially free massive propagators for gluons and ghosts:

$$\hat{\Delta}_{ij}^0(p) = \frac{\delta_{ij}}{p^2 + m^2}; \quad \hat{\Delta}_{gh} = \frac{1}{p^2 + m^2}, \quad (17)$$

plus free vertices. Their motivation (in particular, why the input ghost has the same mass as the gluon) and use in the PT are discussed a little further in Appendix A. These inputs yield outputs which (besides enforcing both gauge

invariance and RGI) are one-loop exact in the UV, are IR finite, and exactly satisfy the necessary Ward identity, as long as the mass is constant. Unfortunately, these outputs (with the pole terms added, as described in Sec. III A) do not much resemble the inputs because the output propagator has a distinct bulge and the input propagator does not.

B. Pole-free outputs

We will not give the tedious algebra [9] needed to find the pole-free three-vertex and inverse propagator. They follow from a straightforward adaptation of the $d = 4$ vertex-paradigm results [9] to $d = 3$. The result for the full inverse PT propagator including the pole terms is

$$\begin{aligned} \hat{\Delta}_{ij}^{-1}(p) &\equiv \Delta_{ij}^{-1}(p) + P_{ij}(p)\Pi^m(p) = P_{ij}(p)p^2 \\ &- \frac{Ng^2}{(2\pi)^3} \int \frac{d^3k}{(k^2 + m^2)[(p+k)^2 + m^2]} \\ &\times \left[P_{ij}(p)(4p^2 + m^2) + \frac{1}{2}(2k+p)_i(2k+p)_j \right] \\ &+ \frac{Ng^2}{2(2\pi)^3} \delta_{ij} \int \frac{d^3k}{k^2 + m^2} + P_{ij}(p)\Pi^m(p), \quad (18) \end{aligned}$$

where we have omitted irrelevant gauge-fixing terms and $P_{ij}(p)$ is the transverse projector. This is the sum, as in Eq. (16), of a pole-free term Δ^{-1} and the new term with Π^m . Only this new term contributes to the pole parts of the vertex and inverse propagator.

In the next section we will deal with the determination of the mass function $\Pi^m(p)$ multiplying the massless scalar poles [see Eq. (12)] that are necessary for generating a gluon mass in NAGTs. This term is the analog, in a NAGT, of the gluonic self-energy part that couples to the massless scalar excitations in a simple Abelian model of dynamical gluon mass generation given long ago [55]. The massless pole in the Π^m term comes from the $p_i p_j / p^2$ term in the transverse projector. By gauge invariance Π^m also occurs in the nonpole terms, to complete the transverse projector.

For the first evaluation of the vertex-paradigm output propagator we use a constant mass m everywhere, including in the seagull term. It turns out that for constant m the explicit seagull and the term with numerator $\sim(2k+p)_i(2k+p)_j$ in Eq. (18) add up to a term that is transverse and vanishes at zero momentum, although each separately is nontransverse and nonvanishing at zero momentum. The reason that a p -independent seagull restores transversality to a p -dependent integral is that the divergence of the integral only depends on the value of the integral at $p = 0$, as one easily checks by taking the divergence of the term with this numerator. The calculations require the regulator formula of Eq. (D1) in Appendix D. Applied to the usual seagull integral—with a constant mass—the regulator yields

$$\frac{1}{(2\pi)^3} \int \frac{d^3k}{k^2 + m^2} \rightarrow -\frac{2}{(2\pi)^3} \int \frac{d^3km^2}{(k^2 + m^2)^2} = -\frac{m}{4\pi}. \quad (19)$$

The regulator replaces an integral with a single propagator by an integral with two propagators, which is why the seagull can cancel out the (zero-momentum part of) another term that has two propagators. In $d = 4$ the regulated integral still diverges (logarithmically) for constant mass, and there is no useful regulation. The logarithmic divergence can only be removed if the dynamical gluon mass vanishes at infinite momentum in $d = 4$. But this is not required in $d = 3$, and the same techniques that in $d = 3$ allow us to estimate quantitatively the gluon mass only lead to lower bounds in $d = 4$ [9].

Again, all of these considerations would be much more complicated with a running mass.

C. First step in evaluating the vertex-paradigm output propagator

From now on we take the pole mass function $\Pi^m(p^2)$ to have the constant value $m^2 \equiv \Pi^m(p = 0)$. Then the basic scalar integral has the value

$$\frac{1}{(2\pi)^3} \int \frac{d^3k}{(k^2 + m^2)[(p+k)^2 + m^2]} = \frac{1}{4\pi p} \arctan\left(\frac{p}{2m}\right). \quad (20)$$

The remaining steps in the vertex paradigm lead to a self-consistent value for m^2 .

With these remarks, we write the output inverse propagator as

$$\hat{\Delta}^{-1}(p) = p^2 - 2bg^2 p \arctan\left[\frac{p}{2m}\right] + \frac{8bg^2 m^2}{15p} \arctan\left[\frac{p}{2m}\right] - \frac{4bg^2 m}{15} + m^2. \quad (21)$$

If one drops the final m^2 [that is, the $\Pi^m(p = 0)$ term], the remainder of this inverse propagator vanishes at zero momentum. In these equations, b is the parameter of Eq. (8).

The next step is to find the dynamical BS equation that governs both the value of the mass, in terms of g^2 , and the existence of the NG-like massless scalars. The self-consistency requirement of the vertex paradigm is that the output mass in Eq. (21) equals the input mass, and the consequences of this consistency condition come from the BS equation.

V. THE $d = 3$ VERTEX PARADIGM—STEP 2: FINDING THE GAP EQUATION

For gauge-invariant gluon mass generation there must be an adjoint multiplet of massless scalars, derivatively

coupled to the gluons. These are the NG-like particles spoken of earlier. They furnish the poles in the vertex and inverse propagator.

A. From the mixing amplitude to the gluon propagator

The effective action

$$S_{\text{mix}} = \int d^3x m \text{Tr}(A_i \partial_i \phi) \quad (22)$$

describes the mixing. Here the coupling mass m is the running dynamical gluon mass at zero momentum, and ϕ is the composite NG field.

Self-consistency of the successive approximation scheme requires that we define the zero-momentum value of the inverse propagator to be the same as the input squared mass m^2 :

$$\hat{\Delta}^{-1}(p = 0) = \Pi^m(p = 0) \equiv m^2. \quad (23)$$

[One could then consider $\Pi^m(p) - \Pi^m(0)$ as part of the pole-free self-energy, but we will keep $\Pi^m(p)$ as a separate entity.] This self-energy comes from a strictly nonperturbative amplitude that mixes the longitudinal part of the gluon with the NG-like particle.

How does the mixing amplitude enter into the gluon propagator? It must generate the pole term in the inverse propagator, of the form

$$\hat{\Delta}_{ij}^{-1}(p) = -m^2 \frac{p_i p_j}{p^2} + \dots \quad (24)$$

A few minutes' play with Feynman diagrams shows that if two particles A and B have a linear mixing term such as the action of Eq. (22) with strength λ , the AA inverse propagator has the form

$$D_{AA}^{-1}(p) = p^2 + \Pi_{A;1PI} - \lambda D_{BB} \lambda, \quad (25)$$

where $\Pi_{A;1PI}$ is the A proper self-energy that is one-particle irreducible (1PI) with respect to A, and D_{BB} is the B propagator that is 1PI with respect to A. For dynamic gluon mass generation, there is no term S_{mix} in the original action and λ should be replaced by a BS form with one or more loops, as in Fig. 2 below. Take particle A to be the gluon and B to be the NG boson; comparison of Eqs. (24) and (25) then shows how the mixing amplitude enters.

B. The mixing amplitude

This amplitude obeys a homogeneous gap equation, much like the equation for a quark constituent mass coming from chiral symmetry breakdown (CSB), as shown in Fig. 3. If there is a solution to this homogeneous equation, then there is CSB and spontaneous fermion mass generation. But at the same time, the gap equation is the



FIG. 2. The dynamical gluon mass equation for the mixing amplitude described by the action in Eq. (22) and indicated by the black circles.

zero-momentum Bethe-Salpeter equation for a massless triplet of pions, so the Nambu-Goldstone mechanism works for composite NG bosons.

There are some critical differences for dynamical gluon mass generation. First, there is no symmetry being broken, and second, the gap equation refers to a mixing amplitude between particles of very different character: the gluon and the NG particle. However, although not usually thought of as such, a dynamical mass for a chirally symmetric quark is a mixing process between different particles, coupling left-handed and right-handed quarks. In the case of NAGT gluons, the mixing process adds the third longitudinal polarization state needed for a massive gluon.

The steps to follow are familiar indeed in related contexts, but technically more involved because of the proliferation of spin indices on gluons and the fact (see [55] and the references therein) that the scalar pole, just like a NG particle, cannot occur in a physical amplitude. Since the NG particle is in the adjoint representation, it has a standard gauge coupling to the gauge boson, with strength g . Elementary symmetry considerations show that there is no coupling of one NG particle to two gluons. In consequence, the one-loop BS equation describing the NG field has the graphical representation of Fig. 2. There is also a seagull graph that enforces gauge invariance, which we do not show.

At first glance, this equation seems to violate, because of the massless internal line in the figure, the well-known principle that NG particles cannot occur in the S matrix or other physical quantities, such as the running mass. But in fact there is no pole for this line because of a cancellation brought about in the numerator. In Fig. 2 let q be the momentum of the internal NG line and p_i be the external momentum. Then the graph in the figure has a kinematic factor of p_i multiplying a scalar graph. The momentum dependence of the numerator of the graph comes out to be

$$-2(p_i p \cdot q - q_i p^2) - 2(q^2 p_i - q_i p \cdot q). \quad (26)$$

The first term is orthogonal to p_i for all q , and hence contributes zero, since the graph itself must be proportional

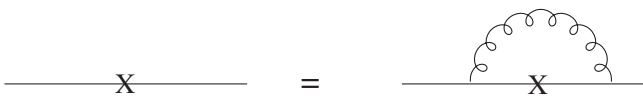


FIG. 3. A standard gap equation for the running mass of a quark with CSB. The cross indicates the insertion of the running mass into the propagator.

to p_i . The second term is orthogonal to q_i and vanishes for the component of q along p . This suggests, and calculation confirms, that the second term in this numerator can be replaced by

$$-\frac{4}{3}q^2 p_i \quad (27)$$

since only two of three directions of q can contribute. Now one sees that the q^2 in the numerator cancels the NG pole, and what remains is a one-loop self-energy graph for two scalars of mass m , as a function of p . Note that there is always a solution for a running mass $m(p)$ in this equation because g^2 has the dimensions of mass. Of course, the solution may or may not be reasonably accurate. We can find the leading term in $m(p)$ at large momentum by evaluating this scalar self-energy with constant mass, and it leads to $m^2(p) \sim 1/p^2$, as the operator product expansion dictates [50].

As we said in the beginning, one of the virtues of $d = 3$ is that it is possible to find a description of dynamical gluon mass generation with a mass m that does not run, which is the running mass $m(p)$ evaluated at zero momentum. The simplest equation for m comes from evaluating the BS equation at $p = 0$, using the input propagators of Eq. (17):

$$1 = \frac{4Ng^2}{3(2\pi)^3} \int \frac{d^3q}{(q^2 + m^2)((p+q)^2 + m^2)} \Big|_{p=0} = \frac{Ng^2}{6\pi m}. \quad (28)$$

Taken as it stands, this equation yields

$$m = \frac{Ng^2}{6\pi}, \quad (29)$$

which is a factor of 2 or 3 less than in other works.

In fact, the actual mass ratio m/g^2 could possibly be very different because of nonpositivity effects in the three-gluon vertex. In Sec. VI below we show that a simple approximation to the three-gluon vertex leads to a tachyonic pole in the propagator unless the gluon mass is large enough, and for the same wrong-sign reason leading to nonpositivity. But in Sec. VID we cure this pole through a dispersion relation.

VI. THE $d = 3$ VERTEX PARADIGM—STEP 3: DRESSING THE GAP EQUATION

A. Dressed propagator

The next step in understanding the gap equation is to dress the lines and vertices in Eq. (28). First, we evaluate the integral in the gap equation with the output propagator of Eq. (21) and bare vertices. Using a dressed propagator with bare vertices is a common approximation in dealing with gap equations.

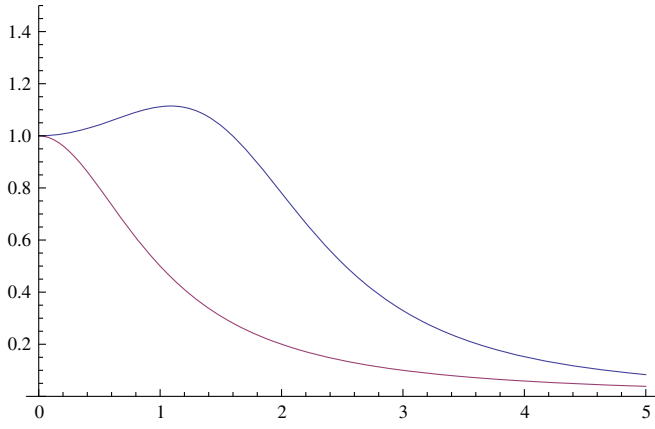


FIG. 4. The lower (red) curve is the free massive propagator and the upper curve is the vertex-paradigm output propagator, for the mass given in the text.

Unfortunately, the dressed propagator in Eq. (21) is very different from the input propagator. A comparison of the input propagator and the output propagator, shown in Fig. 4, is a measure of the potential discrepancy. This figure makes the comparison for the specific mass value $m = Ng^2/(2.48\pi)$, which is, as we will see, the final estimated value of the self-consistent mass. Clearly, the output propagator does not resemble the input because of the bulge in the output propagator. The bulge is evident not only in the Landau-gauge propagator but also in the vertex-paradigm output. It is also clear that the dressed output propagator must give a much larger value to the gap-equation integral than the input propagator will give. In fact, the resulting integral with the dressed propagator is a factor of 6.68 larger than quoted in Eq. (28) and the estimated mass is larger than that given in Eq. (29) by the same factor. Such wild swings would not encourage one to believe that one is making progress on the $d = 3$ gluon mass problem by successive approximation methods. We now show that dressing the three-gluon vertex substantially mitigates the problem.

B. Approximate dressed vertex

Our task now is to estimate the effect of nonpositivity on the gap equation both from the propagator and from the vertex. One should not make the natural supposition that finding the propagator from the Ward identity for the vertex means that we know the vertex explicitly. Given the three-gluon vertex as a momentum integral over tree-level propagators, it is a much simpler problem to find the results of a one-loop Ward identity than it is to evaluate that integral [52]. In fact, that has never been done even in one-loop perturbation theory. So for this exploratory study of gluon mass with nonpositivity, we will ultimately come to a heuristic and qualitative form for the three-gluon vertex that someday must be supplanted by more accurate calculations.

Even without full knowledge of the dressed vertex there is a nonpositivity effect in the vertex that ameliorates the effect from the dressed propagator and that can be qualitatively appreciated directly from the Ward identity of Eq. (10): The bigger the propagator, the smaller the vertex. The BS equation (28) effectively has the product of two gluon propagators and two three-gluon vertices in it, and so it is considerably less sensitive to nonpositivity effects than either of the pieces is. We simplify this exploratory study by omitting the massless pole parts of the vertex since they cannot appear in a physical amplitude, and we approximate the remaining pole-free part as described in Sec. VID below. Because we are using the approximation of a nonrunning mass $m \equiv m(p = 0)$, we remove the momentum dependence from the BS equation by evaluating it at zero momentum. Then each vertex is the scalar function $G(q, -q, 0)$, where G is (an approximation to) the appropriate scalar vertex function. For want of a better approximation, we take these functions from the model described in Appendix B 1. It is based on an extension of previous work [7–9] that models $d = 4$ NAGT effects on the asymptotically free scalar theory ϕ_6^3 , and that is described briefly in Appendix B. The extension uses a tweaked version of ϕ_5^3 . The approximate (scalar) BS equation becomes

$$1 = \frac{4Ng^2}{3(2\pi)^3} \int d^3q G^2(q, -q, 0) \hat{\Delta}^2(q). \quad (30)$$

In this equation the scalar function G is intended to model the scalar form factor of the $d = 3$ three-gluon vertex, also called G and defined in Eq. (B5). As such, it appears in the Ward identity (10).

Appendix B 1 gives a first approximate form for this form factor:

$$G(p_1, p_2, p_3) = 1 - 2bg^2 \int \frac{[dz]}{(D + m^2)^{1/2}}, \quad (31)$$

where the factor $(D + m^2)^{-1/2}$ is the denominator of the $d = 5$ equal-mass scalar triangle graph, as well as the appropriate factor for $d = 3$ NAGTs with massive input propagators. The negative sign is inherent in ϕ_5^3 , but the value $2bg^2$ is chosen so that the $d = 3$ Ward identity is valid to $\mathcal{O}(g^2)$ in the massless (large-momentum) limit. That is, the term of this order in G corresponds to the same term in the output propagator of Eq. (7). The Ward identity tells us that G behaves inversely to the propagator $\hat{\Delta}$, so the propagator bulge ends up being a vertex dip. The minus sign responsible for the vertex dip expresses the $d = 3$ realization of IR confinement, just as it is responsible for the propagator bulge; see Eq. (7).

The vertex G of Eq. (31) is not usable as it stands because, for the self-consistent mass value, the vertex has a zero in the Euclidean region. Appendix C gives a way

around this problem, using a dispersion relation for $G^{-1}(p, -p, 0)$, and an argument first given [8] for the $d = 4$ three-gluon vertex that relates the inverse vertex to the squared running charge $\bar{g}^2(p)$. The running charge is not defined through a renormalization group or a beta function, and the definition applies to $d = 3$ as well as $d = 4$. It is not essential for subsequent calculations that the inverse of a vertex function with one momentum zero is a running charge squared because, in the end, everything is expressed in terms of $G(p, -p, 0)$. However, the idea that there is a relation such as (32) below leads us to insist that $G(p, -p, 0)$ be positive.

C. The running charge concept in $d = 3$

The Ward identity (10) provides a path to a running charge $\bar{g}^2(p)$ that agrees with the PT running charge to two loops at high momenta, is well defined and physically reasonable at all momenta, and does not rely upon a renormalization group or beta function for its definition. The last feature makes it possible to use it in $d = 3$. According to the reasoning of Appendix C, this running charge is

$$\bar{g}^2(p) = \frac{g^2}{G(p, -p, 0)}. \quad (32)$$

The above definition equates $G^{-1}(p, -p, 0)$ to an ostensibly positive quantity, the square of a running charge. However, IR confinement interferes. Based on (31) the corresponding approximation to $G(p, -p, 0)$ would be

$$G(p, -p, 0) \approx 1 - \frac{2bg^2}{p} \arctan\left(\frac{p}{2m}\right), \quad (33)$$

clearly not positive in general, and, as said above, this nonpositivity comes from IR confinement in $d = 3$. If this formula has a zero it completely spoils its interpretation as an inverse running charge, which would not only have a tachyonic pole but would also change sign from positive to negative.

It could happen, but does not, that the self-consistent mass m is so large that $G(p, -p, 0)$ is nevertheless positive for all Euclidean momenta. Unfortunately, for the self-consistent mass used in Fig. 4, the approximate $G(p, -p, 0)$ does have a Euclidean zero, which the Ward identity translates into an unwanted tachyonic propagator pole. While this is not necessarily fatal since the pole need not appear in the S matrix (because the vertex zero cancels it), it is unnecessary; it also results in the unphysical result $\bar{g}^2(p) < 0$ for a finite range of momentum.

We proceed to a second step in modeling the vertex dip that removes the zero of the approximate vertex by postulating a dispersion relation for $G^{-1}(p, -p, 0)$ (or, equivalently, the running charge).

D. A dispersion relation for the vertex form factor

The type of dispersion relation we use here is sometimes invoked under the name of analytic perturbation theory [56], but our use of it has nothing to do with this subject.

The building block of the dispersion relation is the simple formula

$$\int_{4m^2}^{\infty} \frac{d\sigma}{\sqrt{\sigma}(\sigma + p^2)} = \frac{2}{p} \arctan\left(\frac{p}{2m}\right), \quad (34)$$

with a manifestly positive spectral function. Now construct a dispersion relation for $\bar{g}^2(p)$, based on the approximate form of Eq. (33) and this building block:

$$\bar{g}^2(p) = g^2 \left[1 + \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{d\sigma}{\sigma + p^2} \text{Im}\bar{g}^2(\sigma) \right], \quad (35)$$

with

$$\text{Im}\bar{g}^2(\sigma) = \frac{\pi b g^4}{Q(\sigma)\sqrt{\sigma}} \quad (36)$$

and

$$Q(\sigma) = \left[1 - b g^2 \mathcal{P} \int \frac{d\sigma'}{\sqrt{\sigma'}(\sigma' - \sigma)} \right]^2 + \left[\frac{\pi b g^2}{\sqrt{\sigma}} \right]^2. \quad (37)$$

There is a subtraction at infinity, corresponding to the appearance of 1 in Eq. (33) for G , and by hypothesis there is no other subtraction that would yield a pole in the running charge. Just as in $d = 4$, this running charge is non-negative and monotone decreasing with momentum from a finite positive value at $p = 0$. But in $d = 3$ it approaches the value g^2 at infinity.

With this form for the running charge, we now use the formula (32) for $G(p, -p, 0)$, inverse to $\bar{g}^2(p)$. It approaches 1 at infinity and is less than 1 for all finite momenta, but by construction it has no zeros. Its spectral function is negative, as it must be if it is inverse to a function with a positive spectral function. The precise analytic expression of this dispersion integral is complicated and unnecessary for our purposes. As it happens, a simple modification of the original approximate formula (33) is sufficiently accurate as a stand-in for the dispersive integral:

$$\frac{g^2}{\bar{g}^2(p)} = G(p, -p, 0) = 1 - \frac{0.95 b g^2}{p} \arctan\left(\frac{p}{2m}\right), \quad (38)$$

in which the coefficient is $0.95 b g^2$ instead of $2 b g^2$. This misstates the vertex in the deep UV, but this is not of concern since mass generation is purely an IR issue and UV contributions are not as important. For the self-consistent mass there is no Euclidean zero, and it is numerically reasonably close to the dispersive form. There is no reason to suppose that the coefficient 0.95 is highly accurate;

depending on how the stand-in vertex is fit to the numerical vertex, this coefficient might change by $\pm 20\%$.

As advertised, the corrected vertex tends to offset the propagator bulge. At the self-consistent mass $m = Ng^2/(2.48\pi)$ it has the zero-momentum value 0.5–0.6, a reduction below the constant-vertex value of one. The zero-momentum running charge is inverse to these numbers, that is, between 1.66 and 2. Just as in a $d = 4$ NAGT, the running charge grows in the IR.

E. Final results

After numerical integration of the formulas of the last section, the Bethe-Salpeter self-consistency relation that replaces the original of Eq. (28) is

$$1 = \frac{1.9 \times 2Ng^2}{3\pi^2 m}, \quad (39)$$

provided that these integrals are evaluated with the mass value

$$m = \frac{Ng^2}{2.48\pi}. \quad (40)$$

This is numerically consistent with Eq. (39). The BS integral has been enhanced from the bare integral of Eq. (28), but not nearly as much as if bare vertices and the output propagator (shown in Fig. 4) were used in the BS integral. The reduction comes from the decreasing value of G as the momentum decreases.

As a result, the mass value coming from the bare equation (29) of $m = Ng^2/(6\pi)$ is considerably changed. We might compare it to other published mass values by introducing a number ζ , with

$$m = \frac{Ng^2}{\zeta\pi}. \quad (41)$$

Our present value for ζ is 2.48. Various authors have given mass values, but not necessarily the mass as defined by us, as related to the propagator at zero momentum. Values estimated with gauge-invariant techniques include Ref. [10], giving $\zeta = 1.68$; [11] gives $\zeta = 2.57$; [16] has $\zeta = 2.00$; and [17] claims $\zeta = 2.18$. The authors of [12,13] invoke a Higgs field, and Ref. [11] attempts to remove the Higgs mechanism by taking the Higgs mass to infinity, leaving only the NG bosons that occur both in the Higgs mechanism and in the PT. The result is that $\zeta = 2.24$, which in principle would be gauge invariant. For whatever it is worth, the average of these numbers is $\zeta = 2.19$, and the spread around the mean is roughly 20%.

Although it is gauge dependent and need not agree with the PT mass, we can define a Landau-gauge mass m_L and the corresponding ζ_L as

$$\Delta_L(p=0) \equiv 1/m_L^2; \quad m_L = (Ng^2)/(\zeta_L\pi). \quad (42)$$

For numerics, we take $\Delta_L(p)$ as the $d = 3$ Landau-gauge propagator shown in Fig. 1, for which $\zeta_L = 2.05$. Next, we discuss a relation between m and m_L and evaluate their ratio approximately.

VII. FROM THE PT TO THE LANDAU GAUGE

If lattice information were available on the PT propagator and vertex, we could stop here. Unfortunately, such data do not yet exist, but extensive propagator data are available in Landau gauge. It is, in fact, possible for us to use these data along with Eq. (46) below to give another estimate of the PT mass, not dependent on the gap equation that yielded $\zeta \approx 2.48$. To do this, use a background-quantum identity (used somewhat differently in [17]), reviewed in [6], that relates the Landau-gauge propagator Δ_L to the PT propagator $\hat{\Delta}$:

$$\Delta_L(p) = (1 + \hat{G}(p))^2 \hat{\Delta}(p). \quad (43)$$

[We have used the simpler notation

$$\kappa_L = [1 + \hat{G}(0)] \quad (44)$$

in Eq. (3).]

The function \hat{G} is, in principle, computable in terms of Landau-gauge Green's functions involving ghosts, but these obey their own SDEs that are not elementary to solve. Instead, we will use a simple approximation, in the spirit of the approximations already made for the three-gluon vertex. The asymptotic UV behavior is easy to find since it comes from one-loop perturbation theory, and by comparing one-loop results in the PT, given in Eq. (7), and in the Landau gauge, we find the UV behavior

$$1 + \hat{G}(p) \rightarrow 1 - \frac{2\pi b g^2}{15p}. \quad (45)$$

Not unexpectedly, this has a forbidden tachyonic pole in the Euclidean regime. We cure it by the simple expedient of the replacement of the massless perturbative one-loop integral by the massive one, and we propose

$$1 + \hat{G}(p) \approx 1 - \frac{4bg^2}{15p} \arctan\left(\frac{p}{2m}\right); \quad 1 + \hat{G}(0) \approx 1 - \frac{\zeta}{16}. \quad (46)$$

Unlike the original approximation for the three-gluon vertex of Eq. (31), for the estimated value of m/g^2 in Eq. (40) this expression is nonsingular in the Euclidean regime of a real positive p and we will use it as it stands. (Using the dispersion-relation approach as we did for the three-gluon vertex makes little difference in the Euclidean

regime.) Then with $\zeta = 2.48$ we find $1 + \hat{G}(p=0) = 0.845$, and of course $\hat{G}(p=\infty) = 1$. [At $p=0$ the authors of [17] replace the $4/15$ in Eq. (46) by $32/45$.] The variation of $\hat{G}(p)$ is so little in the IR that it is not worth plotting; the major IR effect is just rescaling the propagator from the PT to the Landau gauge.

We use the approximation in Eq. (46) to relate our PT estimates and Landau-gauge lattice data in two ways. The rescaling of Eq. (43) leads at zero momentum to the relation

$$m = [1 + \hat{G}(0)]m_L, \quad (47)$$

which, with the aid of Eq. (46), becomes the quadratic equation

$$\frac{\zeta_L}{\zeta} = 1 - \frac{\zeta}{16}. \quad (48)$$

If now we use the gap-equation value $\zeta = 2.48$ of Eq. (39) to solve for ζ_L , we find $\zeta_L = 2.10$, to be compared to the lattice value of 2.05. If, on the other hand, we ignore the gap equation and use the lattice data for ζ_L to estimate the PT value ζ , the relevant root of the quadratic equation (48) is $\zeta = 2.41$, compared to the gap-equation value of 2.48. This unnaturally close agreement can only be a coincidence, given the roughness of our approximations. Further and more accurate work is necessary.

VIII. SUMMARY AND CONCLUSIONS

Landau-gauge lattice simulations of the gluon propagator in $d=3$ show a quantitatively important positivity violation that could be a serious problem for studies of nonperturbative effects such as dynamical gluon mass generation with the gauge-invariant PT. We study this effect with the vertex paradigm, a method for truncating the PT Schwinger-Dyson equations that is, in principle, more accurate for the three-gluon vertex than some other truncation schemes and that has so far been implemented only at the one-dressed-loop level. We complete the vertex paradigm with a homogeneous Bethe-Salpeter equation describing a set of composite NG bosons that are essential to describing gluon mass generation gauge invariantly; these NG bosons cancel out of all physical quantities. However, implementing the vertex paradigm and the BS equation with successive approximations could lead to serious quantitative errors from nonpositivity. These are substantially mitigated by the fact that, in the crucial BS equation, nonpositivity effects in the output three-gluon vertex are in the opposite direction from those in the gluon propagator, as the QED-like Ward identity of the PT shows, and significant cancellation can occur.

In the successive-approximation scheme, the input ghost and gluon propagators are free propagators with mass m , and the input vertices are the usual tree-level ones. There are two parts to the vertex: a pole-free part constructed by

modification of the perturbative one-loop three-gluon vertex and a part containing the NG massless poles. The part with NG poles algebraically satisfies the Ward identity relating its divergence to the poles of the inverse PT propagator, as in Eq. (4). The full vertex satisfies the Ward identity of Eq. (10).

It is straightforward, if tedious, to write the momentum-space integral for the output pole-free three-gluon vertex, but it is far from easy to compute the complete vertex, which in fact has not even been done for the one-loop three-gluon vertex in perturbation theory. But it is not difficult, given the momentum-space integral, to use the Ward identity to express the inverse propagator as the divergence of the three-gluon vertex, provided that the mass m is nonrunning and the same for all gluons. In view of the difficulty in actually calculating the three-gluon vertex itself, we approximate it using a tweaked version of ϕ_5^3 . This scalar theory is a descendant of the asymptotically free theory ϕ_6^3 in one higher dimension, and it can be used as a heuristic model for $d=3$ NAGT phenomena, including nonpositivity effects. From this model plus a dispersion relation that guarantees the absence of unphysical tachyons coming from wrong signs, we construct an approximate model for one of the scalar form factors of the three-gluon vertex when one of its momenta vanishes. As noted in the main text, gluon mass generation can tame the completely unphysical tachyonic pole in the propagator down to a nonpositivity bulge. Absence of these singularities allows us to argue (as previously suggested for $d=4$ NAGTs [8,9]) that the Ward identity suggests an interpretation for the scalar form factor as the square of a running charge that is defined without reference to a renormalization group or beta function. As the Ward identity (10) suggests, a bulge in the propagator results in a dip in the three-gluon vertex.

Then we use this approximate three-gluon vertex in the gap equation in the constant-mass approximation, and we note that there is a certain amount of cancellation between the propagator bulge and the vertex dip. In terms of a parameter ζ , we define the PT mass as $m = Ng^2/(\zeta\pi)$. With no accounting for nonpositivity, ζ has the unacceptably large value of 6. The results of the present paper, taking into account nonpositivity and the Ward identity between vertex and propagator, give the lower value $\zeta = 2.48$. To compare this to published data on the Landau-gauge propagator, we use a background-quantum identity telling us that the Landau-gauge propagator and the PT propagator differ at zero momentum by a scale factor $[1 + G(0)]^2 < 1$. We make a simple estimate of this scale factor and use this estimate plus the gap-equation mass value to estimate the lattice Landau-gauge mass parameter as $\zeta_L = 2.05$, compared to the simulation value of 2.10. Conversely, using lattice data and the scale factor, but not the gap equation, leads to a PT mass parameter estimate of $\zeta = 2.41$, as compared to the PT estimate of 2.48. The closeness of the comparisons is undoubtedly fortuitous.

APPENDIX A: A BRIEF REVIEW OF THE VERTEX PARADIGM

The following is a road map, not a complete exposition. The main point is to suggest how to organize matters so that the NG-like massless poles are canceled as much as possible before carrying out any serious calculations. This leads us to the pole-free vertex discussed in the main text.

The first step is to construct a tree-level model that has gluon and ghost masses in it. The action is the usual NAGT action plus a gauged nonlinear sigma model mass term, plus gauge-fixing terms.

Pinching is greatly simplified in the Feynman gauge, which we use for the gauged nonlinear sigma model. In this model the gluon propagator is

$$\Delta_{ij}^0(p) = \frac{\delta_{ij}}{p^2 + m^2} + \frac{p_i p_j}{p^2} \left[\frac{1}{p^2} - \frac{1}{p^2 + m^2} \right]. \quad (\text{A1})$$

The last term $\sim p_i p_j$, a difference of massless and massive scalar propagators, suggests that the massless ghosts are canceled out and replaced by ghosts of mass m , as in the Feynman–Fujikawa–Lee–Sanda gauge [57], and this is indeed what happens. Some such replacement must occur in the PT. In general, the ghost mass is gauge dependent, and since the PT propagator has only physical thresholds, the massless ghost poles get replaced by poles at m^2 , in much the same way as it happens in the PT for electroweak theory [6]. Or one may simply argue that the PT propagator can be constructed in a ghost-free gauge because it is the same in any gauge. So for practical purposes we use the tree-level propagators of Eq. (17), repeated here for convenience,

$$\hat{\Delta}_{ij}^0(p) = \frac{\delta_{ij}}{p^2 + m^2}; \quad \hat{\Delta}_{gh} = \frac{1}{p^2 + m^2}, \quad (\text{A2})$$

that have no massless poles.

The second step is to follow [52] and write down the sum of one-loop graphs that give the S-matrix element for the scattering of three external quarks. Figure 5 shows these graphs. Of course, some of these graphs, e.g., (g), are not vertex parts, but they contain vertex parts that are extracted by using tree-level Ward identities that pinch out the internal parts of quark lines.

Satisfying the PT Ward identity at one-loop level is a matter of satisfying it at tree level. The form quoted in the text [Eq. (10)], although ghost free, is not really useful at tree level because of its massless poles. Instead [6], we write the usual tree-level vertex as the sum of two parts:

$$\Gamma_{ijk}(p_1, p_2, p_3) = \Gamma^F ijk(p_1, p_2, p_3) + \Gamma_{ijk}^P(p_1, p_2, p_3), \quad (\text{A3})$$

where Γ^F is the BFM-Feynman-gauge vertex. It has one line (called the background line) singled out; say it is p_1 . The background lines are those attached directly to quark vertices in Fig. 5. The remaining part, Γ^P , has only longitudinal terms $\sim p_{2j}, p_{3k}$ that trigger pinch parts. On the p_1 line, Γ^F satisfies a simple Ward identity with no massless poles:

$$p_{1i} \Gamma_{ijk}^F(p_1, p_2, p_3) = [\hat{\Delta}_{ij}^0(p_2)]^{-1} - [\hat{\Delta}_{ij}^0(p_3)]^{-1}. \quad (\text{A4})$$

It obeys this Ward identity even for the massive propagator of Eq. (17), provided that the mass m does not run with

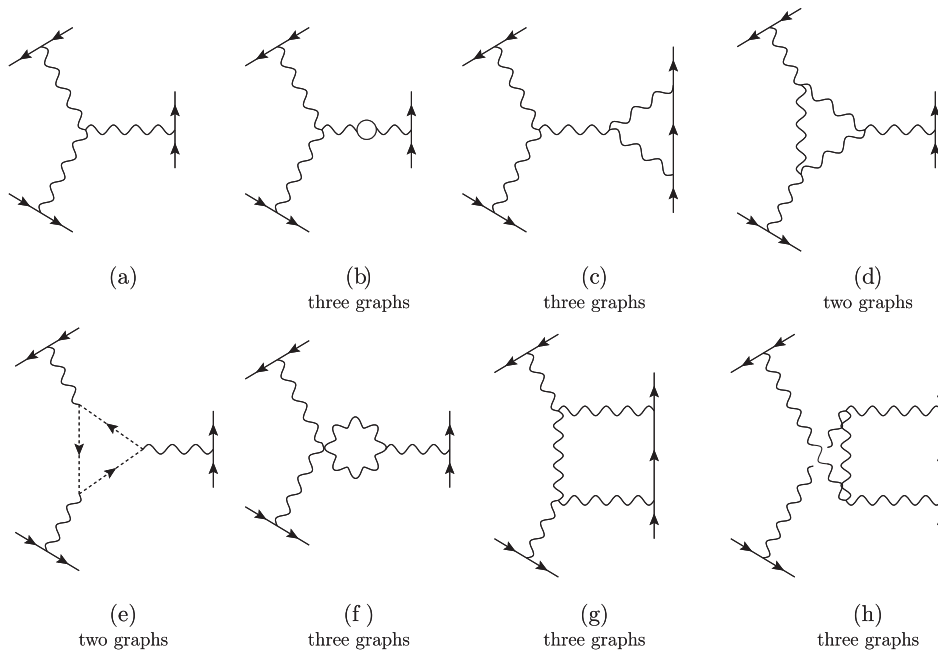


FIG. 5. The one-loop S-matrix element for finding the PT three-gluon vertex. Solid lines represent quarks.

momentum and is the same for all gluons and ghosts. It certainly would not satisfy this identity for a running mass. This is the fundamental reason that we use the constant-mass approximation in the $d = 3$ problem. Satisfying the tree-level Ward identity is 90% of the task of satisfying the one-loop Ward identity with massive tree-level propagators; for the one-loop massless perturbative vertex, it is 100%. The interested reader can study [9] for the other 10% that arises from other complications, such the appearance of uncanceled m^2 in the numerator arising from the pinch process and dealing with seagull terms.

APPENDIX B: TWEAKED ϕ_3^5 IS ANALOGOUS TO A $d = 3$ NAGT

We can get a qualitative—even semiquantitative—understanding of how the full vertex can help to tame the bulge in the propagator by turning to some higher-dimension theories having no spin complications because they refer to scalar fields. One of them, ϕ_6^3 , has AF analogous to that of $d = 4$ NAGTs, and the other, ϕ_5^3 scalar theory in $d = 5$, inherits certain “wrong-sign” properties just as does a $d = 3$ NAGT. These higher-dimension scalar models are to be used only at the one-dressed-loop level, and with graphical coefficients adjusted to yield appropriate results for gauge theories in two fewer dimensions. Taken seriously, the scalar theories do not exist because there is no stable vacuum, but that will not concern us here. Just as for NAGTs, we do not allow a bare mass term, but radiative corrections induce a mass m , assumed to be the same for all fields.

There are two ways to remove the power-law divergences of the self-energies of these theories. One is a regulation scheme [2] already used in $d = 3, 4$. Appendix D gives this scheme for higher dimensions. The other way is to calculate the self-energy from a Ward identity, by introducing an Abelian charge for two of the fields, and we will concentrate on that here. We use the corresponding Abelian vertex to find a simple approximation to one of the scalar functions occurring in the pole-free vertex G_{ijk} , as an integral over Feynman parameters. This scalar function multiplies the Born kinematics, and its prefactor is taken not as prescribed by the $d = 5$ model, but from the requirement that the Ward identity yields the correct one-loop propagator in perturbation theory, given in Eq. (7).

1. A vertex approximation coming from the tweaked models

As shown in earlier works [7–9], the asymptotically free six-dimensional theory ϕ_6^3 can be slightly modified to lead to a qualitatively reasonable approximation for the one-dressed-loop three-gluon vertex and gluon proper self-energy of a $d = 4$ NAGT. The modifications involve introducing an Abelian charge for the scalars

(two of which carry equal and opposite charge) and a corresponding one-loop current vertex, as well as supplying one-loop graphs “by hand” with coefficients taken from the NAGT. We call this model tweaked ϕ_6^3 . The QED-like Ward identity for the tweaked-model current vertex yields the ϕ proper self-energy, which turns out to be practically identical to what the vertex paradigm yields for $d = 4$ NAGTs.

Similarly, tweaked ϕ_5^3 bears a close resemblance to a $d = 3$ NAGT, as one might expect because the trilinear coupling g has the mass dimension $1/2$ in both theories. We introduce an Abelian current and find a finite propagator from its Ward identity. In $d = 6$, the Ward identity reduces the self-energy divergence from the quadratic divergence of ϕ_6^3 to the logarithmic one of NAGTs. In $d = 5$, the Ward identity removes completely the linear self-energy divergence. The same reduction in divergences comes from the regulator of Appendix D below.

In $d = 5$, the one-loop Abelian current vertex is

$$G_i(p_i) = (p_2 - p_3)_i - 2b \int [dz] \frac{[p_2(1 - 2z_3) - p_3(1 - 2z_2)]_i}{(D + m^2)^{1/2}}, \quad (\text{B1})$$

with b from the z_j representing Feynman parameters and

$$\int [dz] = 2 \int_0^1 dz_1 dz_2 dz_3 \delta\left(1 - \sum z_i\right);$$

$$D = p_1^2 z_2 z_3 + p_2^2 z_3 z_1 + p_3^2 z_1 z_2. \quad (\text{B2})$$

The factor $2b$ (but not the minus sign, which comes from $d = 6$ AF) is chosen by hand so as to give the correct perturbative correction to the propagator, as determined by the QED-like Ward identity of Eq. (B7).

The current vertex should obey the Ward identity

$$p_{1i} G(p_i) = \Delta^{-1}(p_3) - \Delta^{-1}(p_2). \quad (\text{B3})$$

Since we are given the current vertex, this equation can be used to define the inverse propagators, provided that it has the correct structural form to be the difference of two inverse propagators, one with momentum p_3 and the other with p_2 . This is the case because

$$p_1 \cdot [p_2(1 - 2z_3) - p_3(1 - 2z_2)] = \left[\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_3} \right] [D + m^2] \quad (\text{B4})$$

and the integrals over the z_i give only end-point contributions that are of the needed functional form as in Eq. (B3).

In order to bridge from this Abelian one-gluon vertex to the needed three-gluon NAGT vertex, we define the scalar function G in the $d = 3$ NAGT by

$$G_{ijk}(p_1, p_2, p_3) = \Gamma_{ijk}^0(p_1, p_2, p_3)G(p_1, p_2, p_3), \quad (\text{B5})$$

where Γ^0 is the Born vertex and $\sum p_i = 0$. Similarly, we define a scalar form factor from the Abelian ϕ_5^3 model as the coefficient of $(p_2 - p_3)_i$:

$$G(p_1, p_2, p_3) = 1 - 2bg^2 \int \frac{[dz]}{(D + m^2)^{1/2}}, \quad (\text{B6})$$

and as the notation suggests we use the Abelian scalar form factor of this equation as an approximation to the non-Abelian scalar form factor.

Now check to see that the coefficient $2b$ in (B1) is correctly chosen. At zero mass, the Ward identity yields

$$\begin{aligned} \Delta^{-1}(p) &= p^2 - 2bg^2 \int_0^1 dz(1-z)[p^2 z(1-z)]^{1/2} \\ &= p^2 - \pi bg^2 p, \end{aligned} \quad (\text{B7})$$

as one-loop PT perturbation theory requires [see Eq. (7)]. Another feature of the approximation is that the vertex integrand $(D + m^2)^{-1/2}$ is correct for the equal-mass triangle graph in $d = 3$. The numerator, however, is not correct for $d = 3$ NAGTs.

A potential problem with the expression (B6) is that it might have a zero in the Euclidean region if the mass is small enough. Generally, such a zero leads to a tachyonic pole in the inverse propagator, from the Ward identity (15), and there is no evidence for this pole in lattice data. Such a coincident pole and vertex zero does not appear in the S matrix. One way to remove the tachyon is to use a dispersion relation for a squared running charge, quite analogous to a dispersion relation used for a squared running charge proposed earlier [7–9] in $d = 4$. This necessarily positive quantity is related, in $d = 4$, to the vertex form factor $G(p, -p, 0)$ introduced above, which cannot have any tachyon. Of course, any running charge in $d = 3$ is not related to a beta function or a renormalization group; the point is that it can be defined from a plausible interpretation of the Ward identity.

APPENDIX C: A RUNNING CHARGE IN THREE DIMENSIONS

In $d = 4$ there is another use for this Abelian vertex that, at least qualitatively, links it to NAGTs. It was argued [8,9] that the current-vertex form factor that multiplies the Born kinematics yields a running charge defined at all momenta down to zero that agrees in the UV with the usual running charge based on the renormalization group for one and two loops. This running charge has the usual UV properties, and due to mass generation it is well defined in the IR as well. It is a physically well-motivated function in the IR, but it is not unique. In $d = 4$ the renormalization-group properties of

PT Green's functions suggest writing the PT propagator, multiplied by g^2 , as

$$g^2 \hat{\Delta}(p) = \bar{g}^2(p)H(p). \quad (\text{C1})$$

Provided that both the PT propagator and g^2 are renormalized at the same renormalization point, their product is renormalization-group invariant [8,9]. By definition each of the factors in (C1) is also renormalization-group invariant.

Although $\hat{\Delta}$ is unique, its factorization is not. To be definitive, in both the $d = 3$ and $d = 4$ cases we *define* $H(p)$ as a standard massive propagator with a running mass:

$$H(p) = \frac{1}{p^2 + m^2(p)}, \quad (\text{C2})$$

where the running mass is finite at zero momentum and vanishes in perturbation theory. In $d = 4$ perturbation theory, this definition for \bar{g} in the UV is the usual running charge to two-loop order.

All of this has its analogs in $d = 3$, except for the fact that there is no renormalization group in this dimension. In $d = 3$ we write the pole-free propagator Δ as

$$\Delta_{ij}^{-1}(p) = \frac{p^2}{\bar{g}^2(p)} P_{ij}(p) + \dots \quad (\text{C3})$$

(omitted terms are irrelevant). Then simple manipulations of the pole-free Ward identity (15) with one vertex momentum set to zero express the so-defined running charge in the form of Eq. (32), repeated here for convenience,

$$\bar{g}^2(p) = \frac{g^2}{G(p, -p, 0)}, \quad (\text{C4})$$

where G is the scalar coefficient of the Born term in the three-gluon vertex. Our introduction of the $d = 3$ running charge is just another way of speaking of this vertex form factor, but we use it heuristically to argue that the vertex $G(p, -p, 0)$ is everywhere positive in the Euclidean region.

APPENDIX D: REGULATING THE TWEAKED MODELS

If ϕ_6^3 is a decent model of $d = 4$ NAGT, does ϕ_3^3 resemble $d = 3$ NAGT, with the characteristic signs inherited from $d = 4$? At first sight this seems impossible since ϕ_5^3 is not obviously a superrenormalizable theory; it has linear UV divergences in the proper self-energy, as calculated directly from a Feynman graph. But because the three-point Abelian current vertex is finite, the Ward identity

yields a finite proper self-energy. The regulator given here also yields a finite self-energy.

In addition, one might worry that seagull graphs in $d = 3, 5$ are divergent. But our regulator [2,4] gets rid of power-law divergences, as of course dimensional regularization also does, but the alternative regulator does not introduce (as dimensional regularization does) potential IR divergences by the introduction of terms with massless poles. This regulator was originally used in $d = 3, 4$, but it can be constructed for any dimension $d \geq 3$. The regulator rule is

$$\int d^d k F(k^2) \rightarrow -\frac{2}{d-2} \int d^d k \left(1 + k^2 \frac{\partial}{\partial k^2}\right) F(k^2). \quad (\text{D1})$$

This replacement is an identity when the integrals in question converge, and it gets rid of power-law divergences when they do not. The effect of the regulator is to reduce divergent integrals by two space-time dimensions. It is an alternative to using the Abelian current vertex to define the proper self-energy, but we do not pursue that subject further here.

-
- [1] J. M. Cornwall, Centre de Physique Théorique Report No. CPT-81/P-1345, 1982.
- [2] J. M. Cornwall, *Phys. Rev. D* **26**, 1453 (1982).
- [3] J. M. Cornwall, W. S. Hou, and J. E. King, *Phys. Lett.* **153B**, 173 (1985).
- [4] J. M. Cornwall and B. Yan, *Phys. Rev. D* **53**, 4638 (1996).
- [5] J. M. Cornwall and W. S. Hou, *Phys. Rev. D* **34**, 585 (1986).
- [6] J. M. Cornwall, J. Papavassiliou, and D. Binosi, *The Pinch Technique and Its Applications to Non-Abelian Gauge Theories* (Cambridge University Press, Cambridge, England, 2011).
- [7] J. M. Cornwall, *Proc. Sci.*, QCD-TNT-11 (2011) 010 [arXiv:1111.0322].
- [8] J. M. Cornwall, arXiv:1211.2019.
- [9] J. M. Cornwall, arXiv:1410.2214.
- [10] G. Alexanian and V. P. Nair, *Phys. Lett. B* **352**, 435 (1995).
- [11] J. M. Cornwall, *Phys. Rev. D* **57**, 3694 (1998).
- [12] W. Buchmuller and O. Philipsen, *Nucl. Phys.* **B443**, 47 (1995).
- [13] W. Buchmuller and O. Philipsen, *Phys. Lett. B* **397**, 112 (1997).
- [14] R. Jackiw and S. Y. Pi, *Phys. Lett. B* **368**, 131 (1996).
- [15] R. Jackiw and S. Y. Pi, *Phys. Lett. B* **403**, 297 (1997).
- [16] D. Karabali and V. P. Nair, *Nucl. Phys.* **B464**, 135 (1996); *Phys. Lett. B* **379**, 141 (1996); D. Karabali, C. j. Kim, and V. P. Nair, *Nucl. Phys.* **B524**, 661 (1998); *Phys. Lett. B* **434**, 103 (1998).
- [17] A. C. Aguilar, D. Binosi, and J. Papavassiliou, *Phys. Rev. D* **81**, 125025 (2010).
- [18] A. Cucchieri, T. Mendes, and A. R. Taurines, *Phys. Rev. D* **67**, 091502 (2003).
- [19] A. Cucchieri and T. Mendes, *Proc. Sci.*, QCD-TNT-09 (2009) 026 [arXiv:1001.2584].
- [20] A. Maas, *Phys. Rev. D* **91**, 034502 (2015).
- [21] C. Alexandrou, P. de Forcrand, and E. Follana, *Phys. Rev. D* **63**, 094504 (2001).
- [22] C. Alexandrou, P. de Forcrand, and E. Follana, *Phys. Rev. D* **65**, 114508 (2002).
- [23] V. G. Bornyakov, V. K. Mitrjushkin, and M. Müller-Preussker, *Phys. Rev. D* **79**, 074504 (2009).
- [24] Ph. Boucaud *et al.*, arXiv:hep-ph/0507104.
- [25] P. O. Bowman, U. M. Heller, D. B. Leinweber, M. B. Parappilly, and A. G. Williams, *Phys. Rev. D* **70**, 034509 (2004).
- [26] Ph. Boucaud *et al.*, arXiv:hep-lat/0602006.
- [27] A. Cucchieri, T. Mendes, O. Oliveira, and P. J. Silva, *Phys. Rev. D* **76**, 114507 (2007).
- [28] O. Oliveira and P. J. Silva, *Eur. Phys. J. A* **31**, 790 (2007).
- [29] I. L. Bogolubsky, E. M. Ilgenfritz, M. Müller-Preussker, and A. Sternbeck, *Proc. Sci.*, LAT2007 (2007) 290.
- [30] I. L. Bogolubsky, E. M. Ilgenfritz, M. Müller-Preussker, and A. Sternbeck, *Phys. Lett. B* **676**, 69 (2009).
- [31] A. Sternbeck, L. von Smekal, D. B. Leinweber, and A. G. Williams, *Proc. Sci.*, LAT2007 (2007) 340.
- [32] I. L. Bogolubsky, V. G. Bornyakov, G. Burgio, E. M. Ilgenfritz, M. Müller-Preussker, P. Schemel, and V. K. Mitrjushkin, *Proc. Sci.*, LAT2007 (2007) 318.
- [33] V. G. Bornyakov, V. K. Mitrjushkin, and R. N. Rogalyov, *Phys. Rev. D* **89**, 054504 (2014).
- [34] O. Oliveira and P. J. Silva, *Phys. Rev. D* **86**, 114513 (2012).
- [35] O. Oliveira, P. J. Silva, E. M. Ilgenfritz, and A. Sternbeck, *Proc. Sci.*, LAT2007 (2007) 323.
- [36] P. J. Silva and O. Oliveira, *Proc. Sci.*, ConfinementX2012 (2012) 045 [arXiv:1301.4094].
- [37] P. O. Bowman, U. M. Heller, D. B. Leinweber, M. B. Parappilly, A. Sternbeck, L. von Smekal, A. G. Williams, and J. Zhang, *Phys. Rev. D* **76**, 094505 (2007).
- [38] E. M. Ilgenfritz, M. Müller-Preussker, A. Sternbeck, A. Schiller, and I. L. Bogolubsky, *Braz. J. Phys.* **37**, 193 (2007).
- [39] A. Cucchieri and T. Mendes, *Proc. Sci.*, LAT2007 (2007) 297.
- [40] A. Cucchieri and T. Mendes, *Phys. Rev. D* **81**, 016005 (2010).
- [41] A. Cucchieri and T. Mendes, *Phys. Rev. Lett.* **100**, 241601 (2008).
- [42] A. Cucchieri and T. Mendes, *Phys. Rev. D* **78**, 094503 (2008).
- [43] Ph. Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Pene, and J. Rodriguez-Quintero, *J. High Energy Phys.* **06** (2008) 099.

- [44] R. Aouane, F. Burger, M. Müller-Preussker, E.-M. Ilgenfritz, and A. Sternbeck, *Proc. Sci.*, LATTICE2013 (2014) 454 [[arXiv:1312.1914](#)].
- [45] S. Gongyo, T. Iritani, and H. Suganuma, *Phys. Rev. D* **86**, 094018 (2012).
- [46] P. J. Silva, O. Oliveira, D. Dudal, P. Bicudo, and N. Cardoso, *Acta Phys. Pol. B Proc. Suppl.* **8**, 119 (2015).
- [47] J. M. Pawłowski, D. Spielmann, and I.-O. Stamatescu, *Nucl. Phys.* **B830**, 291 (2010).
- [48] Y.-B. Zhang, J.-L. Ping, X.-F. Lu, and H.-S. Zong, *Commun. Theor. Phys.* **50**, 125 (2008).
- [49] A. C. Aguilar, D. Binosi, and J. Papavassiliou, *Phys. Rev. D* **89**, 085032 (2014).
- [50] M. Lavelle, *Phys. Rev. D* **44**, R26 (1991).
- [51] J. M. Cornwall, *Mod. Phys. Lett. A* **28**, 1330035 (2013).
- [52] J. M. Cornwall and J. Papavassiliou, *Phys. Rev. D* **40**, 3474 (1989).
- [53] M. Binger and S. J. Brodsky, *Phys. Rev. D* **74**, 054016 (2006).
- [54] N. Ahmadinia and C. Schubert, *Nucl. Phys.* **B869**, 417 (2013).
- [55] J. M. Cornwall and R. E. Norton, *Phys. Rev. D* **8**, 3338 (1973).
- [56] D. V. Shirkov and I. L. Solovtsov, *Theor. Math. Phys.* **150**, 132 (2007).
- [57] K. Fujikawa, B. W. Lee, and A. I. Sanda, *Phys. Rev. D* **6**, 2923 (1972).