

# Convex Hull of Imprecise Points in $o(n \log n)$ Time after Preprocessing

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## Abstract

Motivated by the desire to cope with *data imprecision* [8], we study methods for preprocessing a set of planar regions such that whenever we are given a set of points, each of which lies on a distinct region, we can compute a specified structure on these points more efficiently than in “standard settings” (that is, without preprocessing).

In particular, we study the following problem. Given a set  $L$  of  $n$  lines in the plane, we wish to preprocess  $L$  such that later, upon receiving a set  $P$  of  $n$  points, each of which lies on a distinct line of  $L$ , we can construct the convex hull of  $P$  efficiently. We show that in quadratic time and space it is possible to construct a data structure on  $L$  that enables us to compute the convex hull of any such point set  $P$  in  $O(n\alpha(n)\log^*n)$  expected time. The analysis applies almost verbatim when  $L$  is a set of line-segments, and yields the same asymptotic bounds.

## 1 Introduction

Most studies in computational geometry rely on an unspoken assumption: whenever we are given a set of input points, their precise locations are available to us. Nowadays, however, the input is often obtained via sensors from the real world, and hence it comes with an inherent imprecision. Accordingly, an increasing effort is being devoted to achieving a better understanding of data imprecision and to developing tools to cope with it (see, e.g., [8] and the references therein). The notion of imprecise data can be formalized in numerous ways [8]. We consider a particular setting that has recently attracted considerable attention (see [2] and the references therein). We are given a set of planar regions, each of which represents an estimate about an input point, and the exact coordinates of the points arrive some time later and need to be processed quickly. This situation could occur, e.g., during a two-phase measuring process: first the sensors quickly obtain a rough estimate of the data, and then they invest considerably more time to find the precise locations. This raises the necessity to preprocess the preliminary (imprecise) locations of

the points, and store them in an appropriate data structure, so that when the exact measurements of the points arrive we can efficiently compute a pre-specified structure on them. In settings of this kind, we assume that for each input point its corresponding region is known (note that by this assumption we also avoid a point-location overhead). In light of the applications, this is a reasonable assumption, and it can be implemented by, e.g., encoding this information in the ordering of  $P$ .

**Data imprecision.** Previous work has mainly focused on computing a triangulation for the input points. Held and Mitchell [5] were the first to consider this framework, and they obtained optimal bounds for preprocessing disjoint unit disks for point set triangulations, a result that was later generalized by van Kreveld *et al.* [7] to arbitrary disjoint polygonal regions. For *Delaunay* triangulations, Löffler and Snoeyink [10] obtained an optimal result for disjoint unit disks (see also [4, 9]), which was later simplified and generalized by Buchin *et al.* [2] to *fat*<sup>1</sup> and possibly intersecting regions. The preprocessing phase typically takes  $O(n \log n)$  time and results in a linear size data structure; the time for finding the structure on the exact point set is usually linear or depends on the complexity (and the fatness) of the input regions.

Since the convex hull can be easily extracted from the *Delaunay* triangulation in linear time, the same bounds carry over. However, once the regions are not necessarily fat, the techniques in [2, 10] do not yield the aforementioned bounds anymore. In particular, if the regions consist of lines or line-segments, one cannot hope (under certain computational models) to construct the *Delaunay* triangulation of  $P$  in time  $o(n \log n)$ , regardless of preprocessing (see [10]). Nevertheless, if we are less ambitious and just wish to compute the *convex hull* of  $P$ , we can achieve better performance, which is the main problem studied in this paper.

**Convex hull.** Computing the convex hull of a planar  $n$ -point set is perhaps the most fundamental problem in computational geometry, and there are many algorithms available [1]. All these algorithms require  $\Theta(n \log n)$  steps, which is optimal in the algebraic computation tree model. However, there are numerous ways to exploit additional information to improve

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<sup>1</sup>A planar region  $o$  is said to be fat if there exist two concentric disks,  $D \subseteq o \subseteq D'$ , such that the ratio between the radii of  $D'$  and  $D$  is bounded by some constant.

this bound. For example, if the points are sorted along any fixed direction, Graham's scan takes only linear time [1]. If we know that there are only  $h$  points on the hull, the running time reduces to  $O(n \log h)$  [6]. Our work shows another setting in which additional information can be used to circumvent the theoretic lower bound.

**Our results.** We show that we can preprocess the input lines  $L$  such that given any set  $P$  of points, each of which lies on a distinct line of  $L$ , the convex hull  $\text{CH}(P)$  can be computed in expected time  $O(n\alpha(n)\log^*n)$ , where  $\alpha(\cdot)$  is the (slowly growing) inverse Ackermann function [12, Chapter 2.1]. Our data structure has quadratic preprocessing time and storage, and the convex hull algorithm is based on a batched randomized incremental construction similar to Seidel's tracing technique [11]. As part of the construction, we repeatedly trace the *zone* of (the boundary of) an intermediate hull in the *arrangement* of the input lines.<sup>2</sup> The fact that the complexity of the zone is only  $O(n\alpha(n))$  [12], and that it can be computed in the same asymptotic time bound (after having the arrangement at hand), is a key property of our solution. The analysis applies almost verbatim when  $L$  is a set of line-segments, and we obtain similar asymptotic bounds.

## 2 Convex Hulls

**Preliminaries** The input at the preprocessing stage is a set  $L$  of  $n$  lines in the plane. A *query* to the resulting data structure consists of any point set  $P$  such that each point lies on a distinct line in  $L$ , and for every point we are given its corresponding line. For simplicity, and without loss of generality, we assume that both  $L$  and  $P$  are in *general position* (see, e.g., [1, 12]). We denote by  $\text{CH}(P)$  the convex hull of  $P$ , and by  $E(P)$  the edges of  $\text{CH}(P)$ . We represent the vertices of  $\text{CH}(P)$  in clockwise order, and we direct each edge  $e \in E(P)$  such that  $\text{CH}(P)$  lies to its right. Given a subset  $Q \subset P$ , a point  $p \in P \setminus Q$ , and an edge  $e \in E(Q)$ , we say that  $e$  is in *conflict* with  $p$  if  $p$  lies to the left of the line supported by  $e$ . The set of all points in  $P \setminus Q$  in conflict with  $e$  is called the *conflict list*  $C_e$  of  $e$ , and the size of  $C_e$  is called the *conflict size*  $c_e$  of  $e$ .

### 2.1 The Construction

**Preprocessing.** We construct in  $O(n^2)$  time (and storage) the *arrangement*  $\mathcal{A}(L)$  of  $L$ , and produce its

<sup>2</sup>The arrangement of a set of planar lines is the decomposition of the plane into *vertices*, *edges* and *faces* (also called *cells*), each being a maximal connected set contained in the intersection of at most two lines and not meeting any other line.

*vertical decomposition*, that is, we erect an upward and a downward vertical rays through each vertex  $v$  of  $\mathcal{A}(L)$  until they meet some line of  $L$  (not defining  $v$ ), or else extend to  $\infty$ . The *complexity* of a face  $f$  in  $\mathcal{A}(L)$  is the number of edges incident to  $f$ . The *zone* of a curve  $\gamma$  consists of all faces that intersect  $\gamma$ , and the complexity of the zone is the sum of their complexities.

**Queries.** Given an exact point set  $P = \{p_1, \dots, p_n\}$  as described above, we obtain  $\text{CH}(P)$  through a *batched* randomized incremental construction as follows: Let  $P_1 \subseteq P_2 \subseteq \dots \subseteq P_{\log^*n} = P$  be a sequence of subsets, where  $P_{k-1}$  is a random sample of  $P_k$  of size  $z_{k-1} := \min\{\lfloor n/\log^{(k-1)}n \rfloor, n\}$ , for  $k = 2, \dots, \log^*n$ . Here,  $\log^{(i)}n$  is the  $i$ th iterated logarithm:  $\log^{(0)}n = n$  and  $\log^{(k)}n = \log(\log^{(k-1)}n)$ . This sequence of subsets is called a *gradation*. The idea is to construct  $\text{CH}(P_1)$ ,  $\text{CH}(P_2)$ ,  $\dots$ ,  $\text{CH}(P_{\log^*n})$  one by one. First, we have  $|P_1| = O(n/\log n)$ , so it takes  $O(n)$  time to find  $\text{CH}(P_1)$ , using, e.g., Graham's scan [1]. Then, for  $k = 2, \dots, \log^*n$ , we incrementally construct  $\text{CH}(P_k)$  by updating  $\text{CH}(P_{k-1})$ . This basic technique was introduced by Seidel [11] and it has later found many more applications.

To construct  $\text{CH}(P_k)$  from  $\text{CH}(P_{k-1})$ , we use the data structure from the preprocessing to quickly construct the conflict lists of the edges in  $E(P_{k-1})$  with respect to  $P_k$ . In the standard Clarkson-Shor randomized incremental construction [3] it takes  $O(n \log n)$  time to maintain the conflict lists. However, once we have the arrangement  $\mathcal{A}(L)$  at hand, this can be done significantly faster. In fact, we use a refinement of the conflict lists: we shoot an upward vertical ray from each point on the upper hull of  $P_{k-1}$ , and a downward vertical ray from each point on the lower hull. Furthermore, we erect vertical walls through the leftmost and the rightmost points of  $\text{CH}(P_{k-1})$ . This partitions the complement of  $\text{CH}(P_{k-1})$  into vertical slabs  $S(e)$ , for each edge  $e \in E(P_{k-1})$ , and two boundary slabs  $S(v_l)$ ,  $S(v_r)$ , associated with the respective leftmost and rightmost vertices  $v_l$  and  $v_r$  of  $\text{CH}(P_{k-1})$ . The *refined conflict list* of  $e$ ,  $C_e^*$ , is defined as  $C_e^* := (P_k \setminus P_{k-1}) \cap S(e)$ . We add to this collection the sets  $C_{v_l}^* := (P_k \setminus P_{k-1}) \cap S(v_l)$  and  $C_{v_r}^* := (P_k \setminus P_{k-1}) \cap S(v_r)$ , which we call the refined conflict lists of  $v_l$  and  $v_r$ , respectively. Note that  $C_e^* \subseteq C_e$ , for every  $e \in E(P_{k-1})$ . Moreover,  $C_{v_l}^*$  (resp.,  $C_{v_r}^*$ ) is contained in  $C_{e_1} \cup C_{e_2}$ , where  $e_1, e_2 \in E(P_{k-1})$  are the two respective edges emanating from  $v_l$  (resp.,  $v_r$ ); see Figure 1(a). We now state a key property of the conflict lists  $C_e$  (this property is fairly standard and follows from related studies [3]):

**Lemma 1** *Let  $Q$  be a planar  $m$ -point set,  $r$  a positive integer satisfying  $1 \leq r \leq m$ , and  $R \subseteq Q$  a random subset of size  $r$ . Suppose that  $f(\cdot)$  is a monotone non-decreasing function, so that  $f(x)/x^c$  is decreasing, for*

some constant  $c > 0$ . Then  $\mathbf{Exp}[\sum_{e \in E(R)} f(c_e)] = O(r \cdot f(\frac{m}{r}))$ , where  $c_e$  is the number of points  $p \in Q \setminus R$  in conflict with  $e \in E(R)$ .  $\square$

**Constructing the refined conflict lists.** We next present how to construct the refined conflict lists at the  $k$ -th round of the algorithm. We first construct, in a preprocessing step, the refined conflict lists  $C_{v_l}^*$ ,  $C_{v_r}^*$  in overall  $O(z_k)$  time. For the sake of the analysis, we eliminate these points from  $P_k$  for the time being, and continue processing them only at the final step of the construction—see below.

Let  $\text{UH}(P_{k-1})$  be the upper hull of  $P_{k-1}$ , and let  $\text{LH}(P_{k-1})$  be its lower hull. Having these structures at hand, we construct the zones of  $\text{UH}(P_{k-1})$  and  $\text{LH}(P_{k-1})$  in  $\mathcal{A}(L)$ . This takes overall  $O(n\alpha(n))$  time, using the vertical decomposition of  $\mathcal{A}(L)$  and the fact that the zone complexity of a convex curve in a planar arrangement of  $n$  lines is  $O(n\alpha(n))$ ; see [12, Theorem 5.11].

As soon as we have the zones as above, we can determine for each line  $\ell \in L$  the edges  $e \in E(P_{k-1})$  that  $\ell$  intersects (if any). Let  $L_1$  be the lines that intersect  $\text{CH}(P_{k-1})$ , and put  $L_2 := L \setminus L_1$ . (At this stage of the analysis, we ignore all lines corresponding to the points in  $P_k$  that were eliminated at the time we processed  $C_{v_l}^*$ ,  $C_{v_r}^*$ .)

Next, we wish to find, for each point  $p \in P_k \setminus P_{k-1}$  the edges in  $E(P_{k-1})$  in conflict with  $p$ . If  $p$  lies inside  $\text{CH}(P_{k-1})$ , there are no conflicts. Otherwise, we efficiently find an edge  $e_p \in E(P_{k-1})$  visible from  $p$ , whence we search for the slab  $S(e_p^*)$  containing  $p$ —see below.

Let us first consider the points on the lines in  $L_1$ . Fix a line  $\ell \in L_1$ , let  $p \in P$  be the point on  $\ell$ , and let  $q_1, q_2$  be the intersections between  $\ell$  and the boundary of  $\text{CH}(P_{k-1})$ . The points  $q_1, q_2$  subdivide  $\ell$  into two rays  $\rho_1, \rho_2$ , and the line segment  $\overline{q_1 q_2}$ . By convexity,  $\overline{q_1 q_2} \subseteq \text{CH}(P_{k-1})$  and the rays  $\rho_1, \rho_2$  lie outside  $\text{CH}(P_{k-1})$ . Hence, if  $p$  lies on  $\overline{q_1 q_2}$ , it must be contained in  $\text{CH}(P_k)$ . Otherwise,  $p$  sees an edge of  $E(P_{k-1})$  that meets one of the rays  $\rho_1, \rho_2$ , and we thus set  $e_p$  to be this edge (which can be determined in constant time); see Figure 1(b).

We next process the lines in  $L_2$ . Note that all points on the lines in  $L_2$  conflict with at least one edge in  $E(P_{k-1})$ , since no line in  $L_2$  meets  $\text{CH}(P_{k-1})$ . To find these edges we determine for each  $\ell \in L_2$  a vertex  $p_\ell$  on the boundary of  $\text{CH}(P_{k-1})$  that is extreme for  $\ell$ .<sup>3</sup> This can be done in total time  $O(n)$  by ordering  $E(P_{k-1})$  and  $L_2$  according to their slopes (the latter being performed during preprocessing), and then merging these two lists in linear time. Next, fix such a line  $\ell \in L_2$ , and let  $p \in \ell$  be a query point, then  $p$

must see one of the two edges in  $E(P_{k-1})$  incident to  $p_\ell$  (which can be determined in constant time given  $p_\ell$ ), and we thus set  $e_p$  to be the corresponding edge; see Figure 1(c).

We are now ready to determine, for each point  $p \in P_k$  outside  $\text{CH}(P_{k-1})$ , the slab  $S(e_p^*)$  that contains it (note that  $e_p^*$  must be vertically visible from  $p$ ). If  $e_p$  is vertically visible from  $p$ , we set  $e_p^* := e_p$ . Otherwise, we walk along (the boundary of)  $\text{CH}(P_{k-1})$ , starting from  $e_p$  and progressing in an appropriate direction (uniquely determined by  $p$  and  $e_p$ ), until the appropriate slab is found. Using cross pointers between the edges and the points, we can thus easily compute  $C_e^*$  for each  $e \in E(P_{k-1})$ . By construction, all traversed edges are in conflict with  $p$ , and thus the overall time for this procedure is proportional to the total size of the conflict lists  $C_e$ . Recalling that  $c_e = |C_e|$ , we obtain

$$\mathbf{Exp}\left[\sum_{e \in E(P_{k-1})} c_e\right] = O(z_k) = O(n),$$

by Lemma 1 with  $f : m \mapsto m$ . This concludes the construction of the refined conflict lists.

**Computing  $\text{CH}(P_k)$ .** We next describe how to construct the upper hull of  $P_k$ , the analysis for the lower hull is analogous. Let  $(e_1, \dots, e_s)$  be the edges along the upper hull of  $P_{k-1}$ , ordered from left to right. For each  $e_i$ , we sort the points in  $C_{e_i}^*$  according to their  $x$ -order, using, e.g., merge sort, as well as the points in  $C_{v_l}^*$ ,  $C_{v_r}^*$ . We then concatenate the sorted lists  $C_{v_l}^*, C_{e_1}^*, C_{e_2}^*, \dots, C_{e_s}^*, C_{v_r}^*$ , and merge the result with the vertices of the upper hull of  $P_{k-1}$ . Call the resulting list  $Q$ , and use Graham's scan to find the upper hull of  $Q$  in time  $O(|Q|)$ . This is also the upper hull of  $P_k$ . Applying once again Lemma 1 with  $f : m \mapsto m \log m$ , and putting  $c_e^* := |C_e^*|$ ,  $c_{v_l}^* := |C_{v_l}^*|$ ,  $c_{v_r}^* := |C_{v_r}^*|$ , and  $A > 0$  an absolute constant, the overall expected running time of this step is bounded by

$$\begin{aligned} & \mathbf{Exp}\left[A \cdot (c_{v_l}^* \log c_{v_l}^* + c_{v_r}^* \log c_{v_r}^* + \sum_{e \in E(P_{k-1})} c_e^* \log c_e^*)\right] \\ & \leq \mathbf{Exp}\left[3A \cdot \sum_{e \in E(P_{k-1})} c_e \log c_e\right] \\ & = O(z_k \log(z_k/z_{k-1})) = O(n), \end{aligned}$$

because by definition  $C_e^* \subseteq C_e$ , so  $c_e^* \leq c_e$ , and  $c_{v_l}^* \leq c_{e_1} + c_{e_2}$  for two edges  $e_1, e_2$  (and similarly for  $c_{v_r}^*$ ). In total, we obtain that the expected time to construct  $\text{CH}(P_k)$  given  $\text{CH}(P_{k-1})$  is  $O(n\alpha(n))$ , and since there are  $\log^* n$  iterations, the total running time is  $O(n\alpha(n) \log^* n)$ . We have thus shown:

**Theorem 2** *Using  $O(n^2)$  space and time, we can preprocess a set  $L$  of  $n$  lines in the plane, such that given*

<sup>3</sup>By this we mean that  $p_\ell$  is extremal in the direction of the outer normal of the halfplane that is bounded by  $\ell$  and contains  $\text{CH}(P_{k-1})$ .

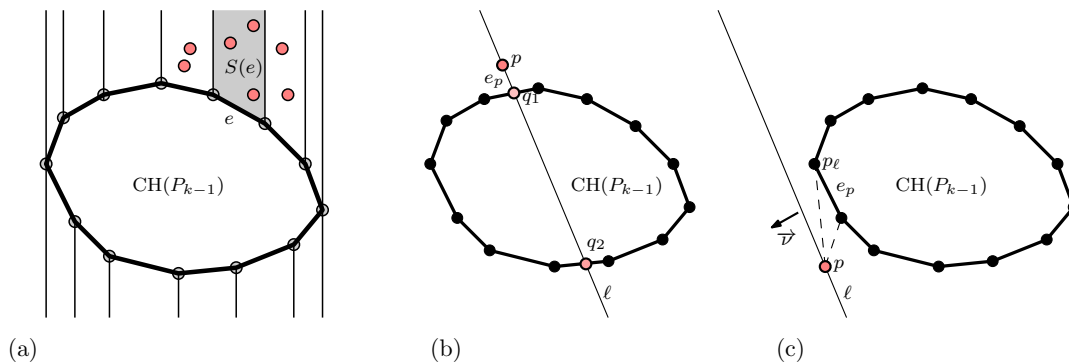


Figure 1: (a) The conflict list  $C_e$  of the edge  $e \in E(P_{k-1})$  contains all the lightly-shaded points, whereas the refined conflict list  $C^*(e)$  has only those points in the vertical slab  $S(e)$ ; (b–c) The edge  $e_p$  of  $E(P_{k-1})$  is visible to  $p$  when (b)  $\ell$  intersects  $\text{CH}(P_{k-1})$ , or (c)  $\ell$  does not meet  $\text{CH}(P_{k-1})$ . In this case  $p_\ell$  is an extreme vertex in the direction  $\vec{v}$ , and the two dashed lines depict the visibility lines between  $p$  and the two respective endpoints of  $e_p$ .

any point set  $P$  with each point lying on a distinct line in  $L$ , we can construct  $\text{CH}(P)$  in expected time  $O(n\alpha(n)\log^*n)$ .

We note that the analysis proceeds almost verbatim when  $L$  is just a set of *line-segments* in the plane. In this case, we preprocess the lines containing the input segments, and proceed as in the original problem. Omitting the straightforward details, we obtain:

**Corollary 3** *If  $L$  is a set of  $n$  line-segments, we can preprocess  $L$  in  $O(n^2)$  space and time, such that given a point set  $P$  with each point lying on a distinct segment of  $L$ , we can construct  $\text{CH}(P)$  in expected time  $O(n\alpha(n)\log^*n)$ .*

**Further results.** In the full version of the paper, we show that under the “obliviousness assumption” we can improve the expected running time of our algorithm to  $O(n\alpha(n))$ , while leaving the preprocessing and storage time unchanged. The obliviousness assumption states that the random choices of the preprocessing phase are unknown to the adversary and that the inputs are chosen in a manner oblivious of the current data structure. We also show how to modify the algorithm to make it output sensitive, and how to produce trade-off bounds for storage and running time.

We can extend our result to computing  $k$ -sets, and we achieve an improved running time as long as  $k = o(\log n)$ . Finally, we provide lower bounds in the algebraic computation tree model for related problems, such as preprocessing lines in the plane for sorting a point set according to its  $x$  order, where each point in the set lies on a distinct line; and preprocessing a set of planes in  $\mathbb{R}^3$  for computing the convex hull of a point set with each point incident to a distinct plane. In all these cases preprocessing does not reduce the  $\Theta(n \log n)$  time bound.

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