A list of successes that has not yet changed the world
-or-
The importance of being stupid

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The Girard-Reynolds Isomorphism (2007)

A tale of Two Theorems

Girard's Representation Theorem

Every function that can be proved total in
second-order Peano arithmetic can be represented in
second-order lambda calculus.

projection : proofs $\rightarrow$ terms

Reynolds's Abstraction Theorem

Terms in second-order lambda calculus take
related arguments to related results,
for a suitable notion of logical relation.

embedding : terms $\rightarrow$ proofs

The Curry-Howard Isomorphism

$$\frac{\forall \quad \supset \quad \wedge \quad \vee \quad F}{\Pi \quad \rightarrow \quad \times \quad + \quad \bot}$$

The Girard-Reynolds Isomorphism

$$\frac{\forall \quad \forall^2 \quad \forall^1 \quad \longrightarrow}{\forall \qquad\qquad\qquad \longrightarrow}$$

Rather than enriching the type systems to match logic,
we impoverish logic to match the type structure.

— Daniel Leivant

Girard projection

$$\left(\dfrac{\begin{array}{c}[A]^{x}\\ \vdots\, u\\ B\end{array}}{A \to B}\to\mathrm{I}^{x}\right)^{\circ} \equiv \dfrac{\begin{array}{c}[x^{A^{\circ}}]\\ \vdots\\ u^{\circ\, B^{\circ}}\end{array}}{(\lambda x^{A^{\circ}}.\, u^{\circ})^{A^{\circ}\to B^{\circ}}}\to\mathrm{I}^{x}$$

$$\left(\dfrac{\begin{array}{c}\vdots\, s\\ A \to B\end{array} \quad \begin{array}{c}\vdots\, t\\ A\end{array}}{B}\to\mathrm{E}\right)^{\circ} \equiv \dfrac{\begin{array}{c}\vdots\\ s^{\circ\, A^{\circ}\to B^{\circ}}\end{array} \quad \begin{array}{c}\vdots\\ t^{\circ\, A^{\circ}}\end{array}}{(s^{\circ}\ t^{\circ})^{B^{\circ}}}\to\mathrm{E}$$

Reynolds embedding

$$\left(\dfrac{\begin{array}{c}[x^A]\\ \vdots \\ u^B\end{array}}{(\lambda x^A.\,u)^{A\to B}}\,\to\mathbf{I}^x\right)^* \equiv \dfrac{\dfrac{\dfrac{\begin{array}{c}[x\in A^*]^{x}\\ \vdots\ u^*\\ u\in B^*\end{array}}{(\lambda x^A.\,u)\;x\in B^*}\,\beta}{x\in A^*\to(\lambda x^A.\,u)\;x\in B^*}\,\to\mathbf{I}^{x}}{\forall x^A.\,x\in A^*\to(\lambda x^A.\,u)\;x\in B^*}\,\forall^1\mathbf{I}$$

$$\left(\dfrac{\begin{array}{c}\vdots\\ s^{A\to B}\end{array}\qquad\begin{array}{c}\vdots\\ t^A\end{array}}{(s\,t)^B}\,\to\mathbf{E}\right)^* \equiv \dfrac{\dfrac{\begin{array}{c}\vdots\ s^*\\ \forall x^A.\,x\in A^*\to s\,x\in B^*\end{array}}{t\in A^*\to s\,t\in B^*}\,\forall^1\mathbf{E}\qquad\begin{array}{c}\vdots\ t^*\\ t\in A^*\end{array}}{s\,t\in B^*}\,\to\mathbf{E}$$

Call-by-value is dual to Call-by-name (2003)

Gentzen 1935: Sequent Calculus

$$\&\text{–}IS\colon \quad \frac{\Gamma \to \Theta, \mathfrak{A} \qquad \Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \,\&\, \mathfrak{B}},$$

$$\&\text{–}IA\colon \quad \frac{\mathfrak{A}, \Gamma \to \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \to \Theta} \qquad \frac{\mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IA\colon \quad \frac{\mathfrak{A}, \Gamma \to \Theta \qquad \mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IS\colon \quad \frac{\Gamma \to \Theta, \mathfrak{A}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}},$$

$$\forall\text{–}IS\colon \quad \frac{\Gamma \to \Theta, \mathfrak{F}\mathfrak{a}}{\Gamma \to \Theta, \forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}},$$

$$\exists\text{–}IA\colon \quad \frac{\mathfrak{F}\mathfrak{a}, \Gamma \to \Theta}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}, \Gamma \to \Theta}.$$

Terms, Coterms, Statements

Term	M, N	$::=$	$x \mid \langle M, N\rangle \mid \langle M\rangle\text{inl} \mid \langle N\rangle\text{inr} \mid [K]\text{not} \mid (S).\alpha$
Coterm	K, L	$::=$	$\alpha \mid [K, L] \mid \text{fst}[K] \mid \text{snd}[L] \mid \text{not}\langle M\rangle \mid x.(S)$
Statement	S, T	$::=$	$M \bullet K$

Right sequent	$\Gamma \rightarrow \Theta \mid M : A$
Left sequent	$K : A \mid \Gamma \rightarrow \Theta$
Center sequent	$\Gamma \mid S \longmapsto \Theta$

Logical rules

$$\frac{\Gamma \to \Theta \mid M : A \qquad \Gamma \to \Theta \mid N : B}{\Gamma \to \Theta \mid \langle M, N \rangle : A \,\&\, B}\ \&R$$

$$\frac{K : A \mid \Gamma \to \Theta}{\mathrm{fst}[K] : A \,\&\, B \mid \Gamma \to \Theta} \qquad \frac{L : B \mid \Gamma \to \Theta}{\mathrm{snd}[L] : A \,\&\, B \mid \Gamma \to \Theta}\ \&L$$

$$\frac{\Gamma \to \Theta \mid M : A}{\Gamma \to \Theta \mid \langle M \rangle \mathrm{inl} : A \vee B} \qquad \frac{\Gamma \to \Theta \mid N : B}{\Gamma \to \Theta \mid \langle N \rangle \mathrm{inr} : A \vee B}\ \vee R$$

$$\frac{K : A \mid \Gamma \to \Theta \qquad L : B \mid \Gamma \to \Theta}{[K, L] : A \vee B \mid \Gamma \to \Theta}\ \vee L$$

$$\frac{K : A \mid \Gamma \to \Theta}{\Gamma \to \Theta \mid [K]\mathrm{not} : \neg A}\ \neg R \qquad \frac{\Gamma \to \Theta \mid M : A}{\mathrm{not}\langle M \rangle : \neg A \mid \Gamma \to \Theta}\ \neg L$$

Structural rules

$$\frac{}{x : A \rightarrow \mid x : A}\ \mathrm{IdR} \qquad \frac{}{\alpha : A \mid \rightarrow \alpha : A}\ \mathrm{IdL}$$

$$\frac{\Gamma \mid S \mapsto \Theta, \alpha : A}{\Gamma \rightarrow \Theta \mid (S).\alpha : A}\ \mathrm{RI} \qquad \frac{x : A, \Gamma \mid S \mapsto \Theta}{x.(S) : A \mid \Gamma \rightarrow \Theta}\ \mathrm{LI}$$

$$\frac{\Gamma \rightarrow \Theta \mid M : A \qquad K : A \mid \Delta \rightarrow \Lambda}{\Gamma, \Delta \mid M \bullet K \mapsto \Theta, \Lambda}\ \mathrm{Cut}$$

Call-by-value

Value	V, W	$::=$	$x \mid \langle V, W\rangle \mid \langle V\rangle\mathrm{inl} \mid \langle W\rangle\mathrm{inr} \mid [K]\mathrm{not}$
Term context	E	$::=$	$\langle\{\ \}, M\rangle \mid \langle V, \{\ \}\rangle \mid \langle\{\ \}\rangle\mathrm{inl} \mid \langle\{\ \}\rangle\mathrm{inr}$

$(\beta\&)$	$\langle V, W\rangle \bullet \mathrm{fst}[K]$	$\longrightarrow_v$	$V \bullet K$
$(\beta\&)$	$\langle V, W\rangle \bullet \mathrm{snd}[L]$	$\longrightarrow_v$	$W \bullet L$
$(\beta\vee)$	$\langle V\rangle\mathrm{inl} \bullet [K, L]$	$\longrightarrow_v$	$V \bullet K$
$(\beta\vee)$	$\langle W\rangle\mathrm{inr} \bullet [K, L]$	$\longrightarrow_v$	$W \bullet L$
$(\beta\neg)$	$[K]\mathrm{not} \bullet \mathrm{not}\langle M\rangle$	$\longrightarrow_v$	$M \bullet K$
$(\beta\mathrm{L})$	$V \bullet x.(S)$	$\longrightarrow_v$	$S\{V/x\}$
$(\beta\mathrm{R})$	$(S).\alpha \bullet K$	$\longrightarrow_v$	$S\{K/\alpha\}$
$(\eta\mathrm{L})$	K	$\longrightarrow_v$	$x.(x \bullet K)$
$(\eta\mathrm{R})$	M	$\longrightarrow_v$	$(M \bullet \alpha).\alpha$
(ς)	$E\{M\}$	$\longrightarrow_v$	$(M \bullet x.(E\{x\} \bullet \beta)).\beta$

Call-by-name

Covalue	P, Q	$::=$	$\alpha \mid [P, Q] \mid \text{fst}[P] \mid \text{snd}[Q] \mid \text{not}\langle M \rangle$
Coterm context	F	$::=$	$[\{\ \}, K] \mid [P, \{\ \}] \mid \text{fst}[\{\ \}] \mid \text{snd}[\{\ \}]$

$$
\begin{array}{llll}
(\beta\vee) & \langle M \rangle \text{inl} \bullet [P, Q] & \longrightarrow_n & M \bullet P \\
(\beta\vee) & \langle N \rangle \text{inr} \bullet [P, Q] & \longrightarrow_n & N \bullet Q \\
(\beta\&) & \langle M, N \rangle \bullet \text{fst}[P] & \longrightarrow_n & M \bullet P \\
(\beta\&) & \langle M, N \rangle \bullet \text{snd}[Q] & \longrightarrow_n & N \bullet Q \\
(\beta\neg) & [K]\text{not} \bullet \text{not}\langle M \rangle & \longrightarrow_n & M \bullet K \\
(\beta R) & (S).\alpha \bullet P & \longrightarrow_n & S\{P/\alpha\} \\
(\beta L) & M \bullet x.(S) & \longrightarrow_n & S\{M/x\} \\
(\eta R) & M & \longrightarrow_n & (M \bullet \alpha).\alpha \\
(\eta L) & K & \longrightarrow_n & x.(x \bullet K) \\
(\varsigma) & F\{K\} & \longrightarrow_n & y.((y \bullet F\{\alpha\}).\alpha \bullet K)
\end{array}
$$

Duality

$$
\begin{aligned}
(X)^\circ &\equiv X \\
(A \mathbin{\&} B)^\circ &\equiv A^\circ \vee B^\circ \\
(A \vee B)^\circ &\equiv A^\circ \mathbin{\&} B^\circ \\
(\neg A)^\circ &\equiv \neg A^\circ
\end{aligned}
$$

$$
\begin{aligned}
(x)^\circ &\equiv x^\circ & (\alpha)^\circ &\equiv \alpha^\circ \\
(\langle M, N\rangle)^\circ &\equiv [M^\circ, N^\circ] & ([K, L])^\circ &\equiv \langle K^\circ, L^\circ\rangle \\
(\langle M\rangle \mathrm{inl})^\circ &\equiv \mathrm{fst}[M^\circ] & (\mathrm{fst}[K])^\circ &\equiv \langle K^\circ\rangle \mathrm{inl} \\
(\langle N\rangle \mathrm{inr})^\circ &\equiv \mathrm{snd}[M^\circ] & (\mathrm{snd}[L])^\circ &\equiv \langle K^\circ\rangle \mathrm{inr} \\
([K]\mathrm{not})^\circ &\equiv \mathrm{not}\langle K^\circ\rangle & (\mathrm{not}\langle M\rangle)^\circ &\equiv [M^\circ]\mathrm{not} \\
((S).\alpha)^\circ &\equiv \alpha^\circ.(S^\circ) & (x.(S))^\circ &\equiv (S^\circ).x^\circ
\end{aligned}
$$

$$
(M \bullet K)^\circ \equiv K^\circ \bullet M^\circ
$$

Well-typed programs can't be blamed (2009) (with Robby Findler)

and Amal Ahmed, Jacob Matthews, Jeremy Siek, and Peter Thiemann

Subtyping

Subtype

$$\frac{}{\texttt{Dyn} <: \texttt{Dyn}}$$

$$\frac{S' <: S \qquad T <: T'}{S \to T <: S' \to T'}$$

$$\frac{s \text{ implies } t}{\{x : B \mid s\} <: \{x : B \mid t\}}$$

Naive

$$\frac{}{S <:_n \texttt{Dyn}}$$

$$\frac{S <:_n S' \qquad T <:_n T'}{S \to T <:_n S' \to T'}$$

$$\frac{s \text{ implies } t}{\{x : B \mid s\} <:_n \{x : B \mid t\}}$$

Positive

$$\frac{}{S <:^+ \texttt{Dyn}}$$

$$\frac{S' <:^- S \qquad T <:^+ T'}{S \to T <:^+ S' \to T'}$$

$$\frac{s \text{ implies } t}{\{x : B \mid s\} <:^+ \{x : B \mid t\}}$$

Negative

$$\frac{}{\texttt{Dyn} <:^- T}$$

$$\frac{S' <:^+ S \qquad T <:^- T'}{S \to T <:^- S' \to T'}$$

$$\frac{}{\{x : B \mid s\} <:^- \{x : B \mid t\}}$$

$$\frac{s \text{ sf } p \qquad S <:^{+} T}{\langle T \Leftarrow S \rangle^{p}\, s \text{ sf } p}$$

$$\frac{s \text{ sf } p \qquad S <:^{-} T}{\langle T \Leftarrow S \rangle^{\bar{p}}\, s \text{ sf } p}$$

$$\frac{s \text{ sf } p \qquad p \neq q \qquad \bar{p} \neq q}{\langle T \Leftarrow S \rangle^{q}\, s \text{ sf } p}$$

The Blame Theorem

Preservation

If s sf p and $s \longrightarrow t$ then t sf p.

Progress

If s sf p then $s \not\longrightarrow$ `blame` p.

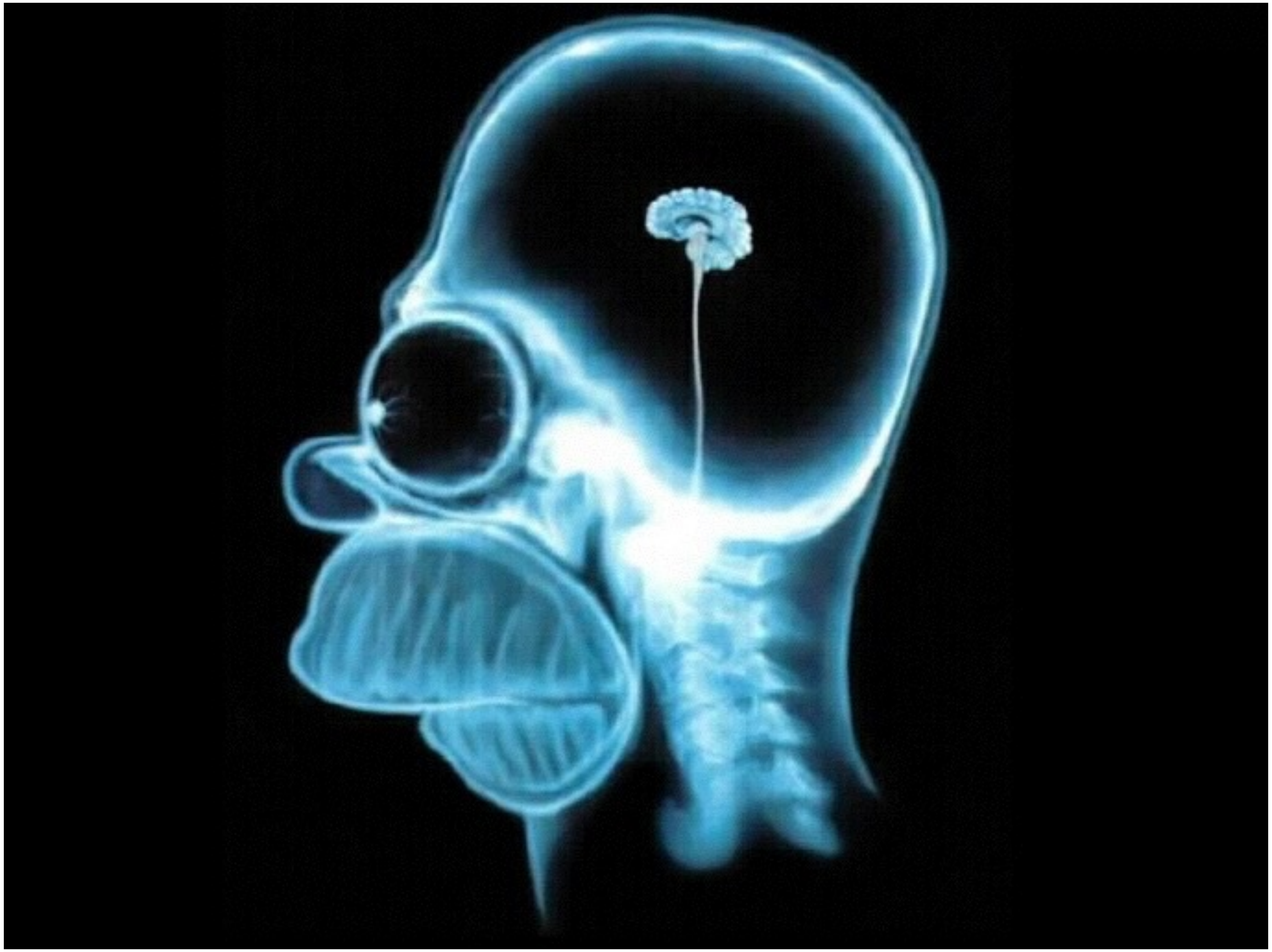
The First Tangram Theorem

$$S <: T \text{ if and only if } S <:^{+} T \text{ and } S <:^{-} T$$

The Second Tangram Theorem

$$S <:_n T \text{ if and only if } S <:^{+} T \text{ and } T <:^{-} S$$

Conclusion:
The importance of being stupid



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- Call-by-value is dual to call-by-name. Philip Wadler. *ICFP*, August 2003

- Well-typed programs can't be blamed. Philip Wadler and Robert Bruce Findler. *ESOP*, March 2009. (See also: A complement to blame. Philip Wadler. *SNAPL*, May 2015.)