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ON TRIGONOMETRIC INTERPOLATION IN AN EVEN NUMBER OF POINTS*

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Abstract. In contrast to odd-length trigonometric interpolants, even-length trigonometric interpolants need not be unique; this is apparent from the representation of the interpolant in the (real or complex) Fourier basis, which possesses an extra degree of freedom in the choice of the highest-order basis function in the even case. One can eliminate this degree of freedom by imposing a constraint, but then the interpolant may cease to exist for certain choices of the interpolation points. On the other hand, the Lagrange representation developed by Gauss always produces an interpolant despite having no free parameters. We discuss the choice Gauss's formula makes for the extra degree of freedom and show that, when the points are equispaced, its choice is optimal in the sense that it minimizes both the standard and 2-norm Lebesgue constants for the interpolation problem. For non-equispaced points, we give numerical evidence that Gauss's formula is no longer optimal and consider interpolants of minimal 2-norm instead. We show how to modify Gauss's formula to produce a minimal-norm interpolant and that, if the points are equispaced, no modification is necessary.

Key words. trigonometric interpolation, Lagrange interpolation, Lebesgue constant

AMS subject classifications. 41A05, 42A15, 65D05

1. Introduction.

Let $K \geq 2$ be an even integer, $K = 2N$. We consider the problem of interpolating typically real but potentially complex-valued data $f_0, \ldots, f_{K-1}$ given at points $x_0 < \cdots < x_{K-1}$ in $[0, 2\pi)$ by a trigonometric polynomial of degree at most N. That is, we seek a function t of the form

$$t(x) = \sum_{n=-N}^{N} c_n e^{inx} \tag{1.1}$$

or, equivalently, of the form

$$t(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + \sum_{n=1}^{N} b_n \sin(nx) \tag{1.2}$$

such that $t(x_k) = f_k$ for each k. We denote the space of trigonometric polynomials of degree at most N by $\mathscr{T}_N$.

We are immediately confronted with the following issue: there are $2N + 1$ unknown coefficients to be determined in (1.1) and (1.2), but there are only $2N$ interpolation conditions to constrain them. The interpolation problem is thus underdetermined and, accordingly, has an infinite number of solutions. To fix this, we must remove one degree of freedom from (1.1) and (1.2), and it seems natural to do this by imposing a restriction on the highest-order terms. The restriction most frequently employed in practice is the condition that (1.1) be *balanced* [5], which means that the highest-order coefficients are required to be equal:

$$c_{-N} = c_N.$$

In terms of (1.2), this is equivalent to requiring

$$b_N = 0,$$

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i.e., that the highest-order contribution be a pure cosine. The balanced condition is usually encountered when the interpolation points are taken to be the equispaced points

$$x_k = \frac{2\pi k}{K}, \qquad 0 \le k \le K-1. \tag{1.3}$$

For further discussion, see [14, Section 3].

Unfortunately, balanced trigonometric interpolants do not always exist for sets of points other than (1.3). As an example, if we instead use the shifted equispaced points

$$x_k = \frac{2\pi(k+\frac{1}{2})}{K}, \qquad 0 \le k \le K-1, \tag{1.4}$$

then Nx_k is an odd integer multiple of $\pi/2$, so $\cos(Nx_k) = 0$ for all k. It follows that the balanced trigonometric interpolation problem has no solution over this grid in general. For instance, if $N = 1$, then the interpolation points are $x_0 = \pi/2$ and $x_1 = 3\pi/2$. A balanced trigonometric interpolant would have the form $t(x) = a_0 + a_1 \cos(x)$, but since $\cos(x_0) = \cos(x_1) = 0$, it is only possible to interpolate constant data ($f_0 = f_1$), and for such data, the interpolant is not unique: the a_1 coefficient may be chosen arbitrarily.

If the balanced choice does not always work, how should one resolve the problem of the extra degree of freedom in (1.1) and (1.2) in general? For the shifted equispaced points (1.4), it is easily seen that requiring the highest-order term to be a pure sine, i.e., requiring

$$a_N = 0$$

or, equivalently,

$$c_{-N} = -c_N,$$

always yields a solvable problem. But like the balanced condition, this condition, which we will refer to as the *skew-balanced* condition, does not work universally: since $\sin(Nx)$ vanishes on the standard equispaced grid (1.3), skew-balanced trigonometric interpolation in (1.3) suffers from the same existence and uniqueness problems as does balanced trigonometric interpolation in (1.4). Moreover, balancing and skew-balancing are not the only possibilities. The imposition of *any* linear relation between c_{-N} and c_N (equivalently, a_N and b_N) other than the skew-balanced (respectively, balanced) condition will yield a solvable interpolation problem for (1.3) (respectively, (1.4)).

The balanced and skew-balanced conditions seem natural, but is there anything else that can be said in their favor? And are they still appropriate conditions when the interpolation points are no longer equispaced? In what follows, we answer these questions by analyzing the Lebesgue constants for the interpolation problems associated with the various conditions that can be imposed on the highest-order terms in t. For interpolation in the equispaced grid (1.3), we show that the balanced condition yields an interpolation problem that is optimally well-conditioned in the sense that both its ∞-norm and 2-norm Lebesgue constants are minimal. By (periodic) translation, we obtain the optimal conditions for interpolation in *any* equispaced grid; in particular, the skew-balanced condition is similarly optimal for interpolation in (1.4).

Our primary tool for conducting this analysis is the even-length trigonometric Lagrange interpolation formula of Gauss. We analyze this formula in detail and discuss how it handles the extra degree of freedom present in these problems. We show that, for equispaced interpolation, this formula imposes the optimal conditions naturally and thus provides a Lagrange representation of the optimal interpolant.

For points that are not equispaced, it is less clear how to select the linear relation on the highest-order terms in t to yield an interpolation problem with minimal Lebesgue constant (of either type). In this case, we propose instead to select t from $\mathscr{T}_N$ in such a way that its L^2-norm is minimal. We show how to accomplish this as well as how to modify Gauss's formula to yield a Lagrange representation of the minimal-norm interpolant. We also show that in the equispaced case, the interpolant produced by Gauss's formula (unmodified) is already of minimal norm, thus establishing its optimality in this third sense.

2. Gauss's formula and the general solution. In his treatise on interpolation, published posthumously in 1866, Gauss provides Lagrange-type formulae for trigonometric interpolants [9]. His formula for an interpolant in an even number of points can be expressed using the notation of Section 1 as

$$t(x) = \sum_{k=0}^{K-1} f_k \ell_k(x), \tag{2.1}$$

where

$$\ell_k(x) = \left[\prod_{\substack{j=0 \\ j\neq k}}^{K-1} \frac{\sin\left(\frac{x-x_j}{2}\right)}{\sin\left(\frac{x_k-x_j}{2}\right)} \right] \times \cos\left(\frac{x-x_k}{2}\right) \tag{2.2}$$

is the kth *trigonometric Lagrange basis function* for the interpolation grid. Since $\ell_k(x_k) = 1$ and $\ell_k(x_j) = 0$ for $j \neq k$, it is clear that $t(x_k) = f_k$ so that t interpolates the data. That ℓ_k (and, thus, t) belongs to $\mathscr{T}_N$ can be established using standard trigonometric identities or by using Euler's formula to rewrite ℓ_k in terms of complex exponentials.

Going one step further, we can write down the general solution to the even-length trigonometric interpolation problem. Since there are only $2N$ interpolation conditions but $\mathscr{T}_N$ is a $(2N+1)$-dimensional space, there is a trigonometric polynomial $\ell \in \mathscr{T}_N$ that vanishes on the interpolation grid but is not identically zero. Indeed, the *trigonometric node polynomial*,

$$\ell(x) = \prod_{k=0}^{K-1} \sin\left(\frac{x-x_k}{2}\right), \tag{2.3}$$

has this property. By elementary linear algebra, the general solution to the problem may be written as a combination of Gauss's particular solution and a multiple of ℓ:

$$t(x) = \sum_{k=0}^{K-1} f_k \ell_k(x) + \lambda \ell(x), \tag{2.4}$$

where λ is an arbitrary constant.

Gauss's formula (2.1)–(2.2), hereafter referenced as (2.1) only, produces a solution to the even-length trigonometric interpolation problem for any choice of the grid points x_k. This is remarkable in light of the discussion in Section 1. Two questions spring to mind. First, how does (2.1) eliminate the extra degree of freedom present in (1.1) and (1.2)—and how does it do so in a way that guarantees a solution exists? Second, what advantages, if any, does the solution via (2.1) enjoy over the infinitely many other solutions to the problem given by (2.4)?

3. The extra degree of freedom. We can answer the first question via direct computation, but first we introduce some notation. For $\theta \in \mathbb{R}$, let

$$\mathscr{T}_N^\theta = \mathscr{T}_{N-1} \oplus \operatorname{span}\{\cos(Nx - \theta)\}.$$

Since $\cos(Nx-\theta) = \cos(\theta)\cos(Nx)+\sin(\theta)\sin(Nx)$, the space $\mathscr{T}_N^\theta$ is the $2N$-dimensional subspace of $\mathscr{T}_N$ consisting of those members of $\mathscr{T}_N$ whose leading-order coefficients in the representation (1.2) obey $b_N/a_N = \tan(\theta)$.[1] In particular, $\mathscr{T}_N^0$ and $\mathscr{T}_N^{\pi/2}$ are the subspaces of $\mathscr{T}_N$ consisting of balanced and skew-balanced trigonometric polynomials, respectively. As θ varies, we obtain subspaces of $\mathscr{T}_N$ corresponding to all possible linear relations between b_N and a_N. Clearly, $\mathscr{T}_N^\theta = \mathscr{T}_N^{\theta+\pi}$ for any θ, so we could restrict θ to $[0,\pi)$; however, it will be convenient to allow θ to assume any real value. Finally, let

$$\sigma = \frac{1}{2}\sum_{k=0}^{K-1} x_k \qquad \text{and} \qquad \eta = \sum_{k=0}^{K-1} \gamma_k f_k,$$

where

$$\gamma_k = \prod_{\substack{j=0 \\ j\neq k}}^{K-1} \frac{1}{\sin\left(\frac{x_k - x_j}{2}\right)}.$$

These quantities will play important roles in what follows.

Identifying what choice (2.1) makes for the extra degree of freedom amounts to determining the subspace $\mathscr{T}_N^\theta$ of $\mathscr{T}_N$ to which the trigonometric polynomial defined by (2.1) belongs. We shall go further and identify this subspace not just for (2.1) but for the general interpolant (2.4), obtaining the result for (2.1) as a special case.

THEOREM 3.1. *The trigonometric interpolant defined by* (2.4) *belongs to* $\mathscr{T}_N^\theta$ *with*

$$\tan(\theta) = \frac{\lambda\sin(\sigma) - \eta\cos(\sigma)}{\lambda\cos(\sigma) + \eta\sin(\sigma)}. \tag{3.1}$$

Proof. We just need to compute the highest-order terms of t defined by (2.4). The required calculations appear in [5, Section 2] and [12]. Specifically, [5, equation (2.3)] gives, in our notation,

$$t(x) = a_N\cos(Nx) + b_N\sin(Nx) + \tilde{t}(x),$$

where $\tilde{t}$ belongs to $\mathscr{T}_{N-1}$ and

$$a_N = \frac{(-1)^N}{2^{2N-1}}\big(\lambda\cos(\sigma) + \eta\sin(\sigma)\big), \qquad b_N = \frac{(-1)^N}{2^{2N-1}}\big(\lambda\sin(\sigma) - \eta\cos(\sigma)\big).$$

The result now follows from the relation $\tan(\theta) = b_N/a_N$. ∎

COROLLARY 3.2. *The trigonometric interpolant defined by* (2.1) *belongs to* $\mathscr{T}_N^{\sigma+\pi/2}$.

Proof. Taking $\lambda = 0$ in (3.1) gives $\tan(\theta) = -\cot(\sigma) = \tan(\sigma+\pi/2)$. ∎

COROLLARY 3.3. *For the standard equispaced grid* (1.3)*, the trigonometric interpolant defined by* (2.1) *is balanced. For the shifted equispaced grid* (1.4)*, it is skew-balanced.*

Proof. With x_k as in (1.3), we have

$$\sigma = \frac{1}{2}\sum_{k=0}^{K-1}\frac{2\pi k}{K} = (K-1)\frac{\pi}{2} = N\pi - \frac{\pi}{2}.$$

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[1] In terms of the complex representation (1.1), this is equivalent to imposing $c_{-N}/c_N = e^{i2\theta}$. Note that by assuming that θ is real, we have restricted our attention to constraints on the highest-order terms such that b_N/a_N is real or, equivalently, such that $|c_{-N}/c_N| = 1$. In principle, one could consider more general constraints; however, this is unlikely to be interesting from a practical standpoint, as such constraints will yield interpolants t that are not real-valued even when the interpolation data f_k are.

By Corollary 3.2, the interpolant belongs to $\mathscr{T}_N^{N\pi} = \mathscr{T}_N^0$. Thus, the interpolant is balanced. On the other hand, with x_k as in (1.4),

$$\sigma = \frac{1}{2}\sum_{k=0}^{K-1} \frac{2\pi(k+\frac{1}{2})}{K} = K\frac{\pi}{2} = N\pi.$$

Thus, the interpolant belongs to $\mathscr{T}_N^{N\pi+\pi/2} = \mathscr{T}_N^{\pi/2}$ and so is skew-balanced. □

We interpret these results as saying that (2.1) and, more generally, (2.4) for a given λ, adaptively select the constraint on the highest-order terms in the interpolant in a way that at least ensures that a solution to the interpolation problem exists. Perhaps one reason to prefer (2.1) over (2.4) with $\lambda \neq 0$ is that (2.1) is the only variant of (2.4) that chooses the constraint in a way that depends only on the interpolation points x_k and not on the interpolation data f_k.

COROLLARY 3.4. *The subspace $\mathscr{T}_N^\theta$ to which the interpolant defined by* (2.4) *belongs is independent of the interpolation data f_k if and only if $\lambda = 0$.*

Proof. For θ to be independent of the f_k, we need the right-hand side of (3.1) to be independent of η. Differentiating (3.1) with respect to η, we obtain

$$\frac{\partial}{\partial\eta}\tan\theta = -\frac{\lambda}{(\lambda\cos(\sigma)+\eta\sin(\sigma))^2},$$

and this vanishes (independently of σ) if and only if $\lambda = 0$. □

4. Another viewpoint. In the previous section, we took the view that (2.1) and (2.4) were given and then found the subspace $\mathscr{T}_N^\theta$ of $\mathscr{T}_N$ from which they drew the interpolant. That is, we chose λ, and the parameter θ was then determined by the formula (3.1). Typically, the interpolation problem is posed the other way around: we first select θ, which determines the form of the interpolant—such as a balanced or skew-balanced form—and then find λ to give a formula for the interpolant of the desired form. Of course, it can happen that no such λ exists, in which case the interpolation problem has no solution.

The next result describes (2.4) from the latter viewpoint. While it should be viewed as a corollary to (or restatement of) Theorem 3.1, we label it as a theorem to emphasize the change in perspective.

THEOREM 4.1. *The interpolant defined by* (2.4) *belongs to $\mathscr{T}_N^\theta$ if and only if*

$$\lambda = \eta\cot(\sigma-\theta). \tag{4.1}$$

Thus, the even-length trigonometric interpolation problem has a unique solution in $\mathscr{T}_N^\theta$ if and only if $\theta \neq \sigma + n\pi$ for any $n \in \mathbb{Z}$.

Proof. Just solve (3.1) for λ:

$$\lambda = \eta\frac{\cos(\sigma)+\tan(\theta)\sin(\sigma)}{\sin(\sigma)-\tan(\theta)\cos(\sigma)} = \eta\frac{1+\tan(\theta)\tan(\sigma)}{\tan(\sigma)-\tan(\theta)} = \eta\cot(\sigma-\theta),$$

the third equality following from the standard identity for the tangent of a difference. □

As a corollary, we recover the following result due to Salzer [12]. (See also [5, equation (2.4)] for the balanced case.)

COROLLARY 4.2. *The interpolant defined by* (2.4) *is balanced if and only if*

$$\lambda = \eta \cot(\sigma).$$

The balanced interpolation problem has a unique solution if and only if $\sigma \neq n\pi$ for any $n \in \mathbb{Z}$. Likewise, the interpolant is skew-balanced if and only if

$$\lambda = -\eta \tan(\sigma).$$

The skew-balanced problem has a unique solution if and only if $\sigma \neq n\pi/2$ for any odd $n \in \mathbb{Z}$.

Proof. Take $\theta = 0$ and $\theta = \pi/2$ in (4.1). □

Suppose the interpolation problem were posed from the outset in $\mathscr{T}_N^{\sigma+\pi/2}$. This choice of space looks strange *a priori*. But unlike the standard choices of balanced and skew-balanced spaces, Theorem 4.1 guarantees that the interpolant in $\mathscr{T}_N^{\sigma+\pi/2}$ always exists. Moreover, (4.1) yields $\lambda = 0$, so the Lagrange interpolation formula is Gauss's formula (2.1).

There are many other ways to choose θ as a function of σ that ensure the existence of an interpolant in $\mathscr{T}_N^{\theta}$; according to (4.1), setting $\theta = \sigma + \varphi$, where φ is any real number that is not an integer multiple of π will work. But note that the value of λ is independent of the interpolation data f_k if and only if $\theta = \sigma + n\pi/2$ for some odd $n \in \mathbb{Z}$. This gives one reason to prefer the choice $\theta = \sigma + \pi/2$. Some other reasons that apply when the data are equispaced will be given in the sections that follow.

5. Lebesgue constants.

We now turn to the second question raised at the end of Section 2 above: is there a reason to prefer the interpolant given by Gauss's formula (2.1) to the many other interpolants of the general form (2.4)? Alternatively, adopting the viewpoint of Section 4, is there an advantage—besides the guarantee that a solution exists—to posing the interpolation problem in $\mathscr{T}_N^{\theta}$ with $\theta = \sigma + \pi/2$ instead of some other value of θ?

We can address these questions by studying the *Lebesgue constant* Λ^θ for interpolation in $\mathscr{T}_N^{\theta}$, defined as the ∞-norm of the operator that maps the interpolation data f_k, viewed as a vector $f = (f_0, \ldots, f_{K-1})$ in $\mathbb{C}^K$, to the interpolant t:

$$\Lambda^\theta = \max_{\|f\|_\infty = 1} \|t\|_\infty. \tag{5.1}$$

The Lebesgue constant quantifies the effect that perturbations to the interpolation data may have on the size of the interpolant. A scheme with a large Lebesgue constant is unlikely to be useful in practice, as it will tend to amplify small errors in the data, such as those due to roundoff, into large errors in the interpolant. Perhaps the most well-known example of this occurs with polynomial interpolation in equispaced points, for which the Lebesgue constants are so large that the method is almost entirely useless [13, Chapter 15]. In contrast, equispaced points yield an optimal (minimal) Lebesgue constant for trigonometric interpolation [7].

The Lebesgue constant for an interpolation scheme can be analyzed with the aid of the associated *Lebesgue function*. For interpolation in $\mathscr{T}_N^{\theta}$, taking $\lambda = \eta\cot(\sigma - \theta)$ in (2.4) gives the Lagrange representation

$$t(x) = \sum_{k=0}^{K-1} f_k \ell_k^\theta(x),$$

where

$$\ell_k^\theta(x) = \ell_k(x) + \gamma_k \cot(\sigma - \theta)\ell(x). \tag{5.2}$$

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The Lebesgue function is just the sum of the absolute values of the ℓ_k^θ,

$$L^\theta(x) = \sum_{k=0}^{K-1} |\ell_k^\theta(x)|.$$

The reason L^θ is important in the study of Λ^θ is given by the following theorem.

THEOREM 5.1. *The Lebesgue constant* (5.1) *for interpolation in* $\mathscr{T}_N^\theta$ *satisfies*

$$\Lambda^\theta = \max_{0\le x\le 2\pi} L^\theta(x).$$

The proof of Theorem 5.1 is given by a standard argument that can be found in most texts on approximation theory, e.g., [13, Chapter 15]. We omit the details.

We use this characterization of Λ^θ to prove the following result, which shows that when the points are equispaced, Λ^θ is minimized for the choice $\theta = \sigma + \pi/2$. Thus, Gauss's formula gives an optimal solution to the even-length equispaced interpolation problem in the sense that it constrains the highest-order terms to give a problem that is "best-conditioned".

THEOREM 5.2. *For equispaced interpolation points* x_k*,*

$$\Lambda^\theta \ge \Lambda^{\sigma+\pi/2}.$$

Proof. We will establish the result assuming that the x_k are given by (1.3). To handle other equispaced grids such as (1.4), we can reduce the argument to this case by periodic translation. Since $\mathscr{T}_N^{\sigma+\pi/2} = \mathscr{T}_N^0$ for this grid, we must show that $\Lambda^\theta \ge \Lambda^0$, and we can do this by studying L^θ.

We require an expression for $\ell_k^\theta(x)$. For $0 \le k \le K-1$, define

$$\tilde{\ell}_k(x) = \prod_{\substack{j=0\\ j\ne k}}^{K-1} \sin\left(\frac{x-x_k}{2}\right).$$

Then, $\ell_k(x) = \gamma_k \tilde{\ell}_k(x)\cos\big((x-x_k)/2\big)$ and $\ell(x) = \tilde{\ell}_k(x)\sin\big((x-x_k)/2\big)$. In the proof of Corollary 3.3, we computed $\sigma = N\pi - \pi/2$. Thus, $\cot(\sigma-\theta) = \tan(\theta)$, and so by (5.2),

$$\ell_k^\theta(x) = \gamma_k\tilde{\ell}_k(x)\left[\cos\left(\frac{x-x_k}{2}\right) + \tan(\theta)\sin\left(\frac{x-x_k}{2}\right)\right].$$

Pulling a factor of $1/\cos(\theta)$ out of the bracketed expression and using the addition formula for cosine, we rewrite this as

$$\ell_k^\theta(x) = \gamma_k\tilde{\ell}_k(x)\frac{\cos\left(\theta - \frac{x-x_k}{2}\right)}{\cos(\theta)}.$$

It follows that the Lebesgue function for this grid is

$$L^\theta(x) = \sum_{k=0}^{K-1}\left|\gamma_k\tilde{\ell}_k(x)\frac{\cos\left(\theta-\frac{x-x_k}{2}\right)}{\cos(\theta)}\right|. \tag{5.3}$$

In [5, Proposition 1], it is shown that the γ_k for this grid all have the same absolute value; see also (A.3). Denoting the common value by γ, we have, by Theorem 5.1,

$$\Lambda^\theta = \frac{\gamma}{|\cos(\theta)|} \max_{0 \le x \le 2\pi} \sum_{k=0}^{K-1} \left|\tilde{\ell}_k^\theta(x)\right|, \tag{5.4}$$

where $\tilde{\ell}_k^\theta(x) = \tilde{\ell}_k(x)\cos\bigl(\theta - (x - x_k)/2\bigr)$.

Now, observe that for $0 \le k \le N-1$,

$$\left|\tilde{\ell}_k^\theta(x)\right| + \left|\tilde{\ell}_{k+N}^\theta(x)\right| = \left[\prod_{\substack{j=0 \\ j\ne k, k+N}}^{K-1} \left|\sin\left(\frac{x - x_j}{2}\right)\right|\right] \times A_k(x,\theta),$$

where

$$A_k(x,\theta) = \left|\cos\left(\theta - \frac{x - x_k}{2}\right)\sin\left(\frac{x - x_{k+N}}{2}\right)\right| + \left|\sin\left(\frac{x - x_k}{2}\right)\cos\left(\theta - \frac{x - x_{k+N}}{2}\right)\right|.$$

Using $x_k = 2\pi k/K$ and applying standard trigonometric identities, we find

$$\cos\left(\theta - \frac{x - x_k}{2}\right)\sin\left(\frac{x - x_{k+N}}{2}\right) = -\frac{1}{2}\cos(\theta) - \frac{1}{2}\cos(\theta - x + x_k)$$

and

$$\sin\left(\frac{x - x_k}{2}\right)\cos\left(\theta - \frac{x - x_{k+N}}{2}\right) = \frac{1}{2}\cos(\theta) - \frac{1}{2}\cos(\theta - x + x_k)$$

for $0 \le k \le N-1$. Thus, since $|a+b|/2 + |a-b|/2 = \max(|a|,|b|)$ for real a, b,

$$A_k(x,\theta) = \max\bigl(|\cos(\theta)|, |\cos(\theta - x + x_k)|\bigr).$$

Noting that $A_k(x,0) = 1$, we have

$$|\tilde{\ell}_k^\theta(x)| + |\tilde{\ell}_{k+N}^\theta(x)| \ge \bigl(|\tilde{\ell}_k^0(x)| + |\tilde{\ell}_{k+N}^0(x)|\bigr)|\cos(\theta)|.$$

The theorem now follows by summing both sides of this inequality from $k = 0$ to $k = N-1$, taking the maximum over $0 \le x \le 2\pi$, and applying (5.4). ◻

6. 2-norm Lebesgue constants. In the last section, we measured the size of the interpolation operator using the ∞-norm. We can also measure it using the 2-norm. We call the resulting operator norm for interpolation in $\mathscr{T}_N^\theta$ the *2-norm Lebesgue constant* and denote it by Λ_2^θ. That is, we define

$$\Lambda_2^\theta = \max_{\|f\|_2 = 1} \|t\|_2, \tag{6.1}$$

where $\|f\|_2$ is the Euclidean norm of $f = (f_1, \ldots, f_{K-1})$ in $\mathbb{C}^K$,

$$\|f\|_2 = \left(\sum_{k=0}^{K-1} |f_k|^2\right)^{1/2},$$

and $\|t\|_2$ is the L^2-norm of the interpolant t in $\mathscr{T}_N^\theta$,

$$\|t\|_2 = \left(\int_0^{2\pi} |t(x)|^2 \, dx \right)^{1/2}.$$

The latter is induced by the L^2-inner product on $\mathscr{T}_N^\theta$,

$$\langle u, v \rangle = \int_0^{2\pi} u(x)\overline{v(x)} \, dx. \tag{6.2}$$

The bar over $v(x)$ in (6.2) denotes complex conjugation.

The 2-norm Lebesgue constant admits a simple characterization as the largest eigenvalue of the *Gram matrix* G^θ associated with the Lagrange basis functions (5.2) for the inner product (6.2); this is the Hermitian positive-definite matrix with (j,k)-entry $G^\theta_{jk} = \left\langle \ell^\theta_j, \ell^\theta_k \right\rangle$. We state this result as a theorem for later reference.

THEOREM 6.1. *The* 2*-norm Lebesgue constant* (6.1) *for interpolation in* $\mathscr{T}_N^\theta$ *is*

$$\Lambda_2^\theta = \sqrt{\lambda_{\max}(G^\theta)},$$

where $\lambda_{\max}(G^\theta)$ *is the largest eigenvalue of* G^θ.

This result is merely an instance of the familiar fact that the 2-norm of a linear operator is its largest singular value, so we omit the proof. For details within an interpolation context, see [1, Section 3.4.1].

Remarkably, when the points are equispaced, we can calculate Λ_2^θ exactly.

THEOREM 6.2. *For equispaced interpolation points* x_k*,*

$$\Lambda_2^\theta = \sqrt{\frac{\pi}{N} \max\left(\frac{1}{2}\csc(\sigma - \theta)^2, 1 \right)}.$$

Proof. As before, we lose no generality in assuming that the x_k are given by (1.3); the result for other equispaced grids can be obtained by periodic translation.

We must compute G^θ. From (5.2), we have

$$\left\langle \ell^\theta_j, \ell^\theta_j \right\rangle = \langle \ell_j, \ell_k \rangle + \left(\gamma_k \langle \ell_j, \ell \rangle + \gamma_j \langle \ell, \ell_k \rangle \right) \cot(\sigma - \theta) + \gamma_j \gamma_k \cot(\sigma - \theta)^2 \langle \ell, \ell \rangle .$$

Using the explicit formulae for ℓ_k, ℓ, and γ_k for this grid given in Appendix A, we calculate

$$\langle \ell_j, \ell_k \rangle = \begin{cases} (-1)^{j+k+1} \frac{\pi}{4N^2} & \text{if } j \neq k \\ (4N-1)\frac{\pi}{4N^2} & \text{if } j = k \end{cases} \tag{6.3}$$

and

$$\langle \ell_k, \ell \rangle = \langle \ell, \ell_k \rangle = 0 \tag{6.4}$$

and

$$\gamma_j \gamma_k \langle \ell, \ell \rangle = (-1)^{k+j} \frac{\pi}{4N^2}. \tag{6.5}$$

These identities may be established using residue calculus; details are given in Appendix B. Letting $g = \left(1, -1, 1, \ldots, (-1)^{K-1}\right)$ in $\mathbb{C}^K$, written as a column vector, it follows that

$$G^\theta = \frac{\pi}{N} I + \frac{\pi}{4N^2}\big(\cot(\sigma-\theta)^2 - 1\big) g g^*,$$

where I is the $K \times K$ identity matrix and g^* is the conjugate transpose of g.

The second matrix in the sum for G^θ has rank 1 and therefore possesses $K-1$ zero eigenvalues. The remaining eigenvalue, denoted λ_0, is

$$\lambda_0 = \frac{\pi}{4N^2}\big(\cot(\sigma-\theta)^2 - 1\big)\|g\|_2^2 = \frac{\pi}{2N}\big(\cot(\sigma-\theta)^2 - 1\big).$$

Since adding a multiple of I to a matrix adds that multiple to the eigenvalues, it follows that G^θ has $K-1$ eigenvalues equal to π/N and one eigenvalue equal to $\pi/N + \lambda_0$. Thus,

$$\lambda_{\max}(G^\theta) = \max\left(\frac{\pi}{N} + \lambda_0, \frac{\pi}{N}\right) = \frac{\pi}{N}\max\left(\frac{1}{2}\big(\cot(\sigma-\theta)^2+1\big), 1\right).$$

The result now follows from Theorem 6.1 and the identity $1 + \cot(\alpha)^2 = \csc(\alpha)^2$. ∎

COROLLARY 6.3. *For equispaced interpolation points x_k,*

$$\min_{\theta\in\mathbb{R}} \Lambda_2^\theta = \sqrt{\frac{\pi}{N}}.$$

The minimum is attained when $\theta = \sigma + (2n-1)\pi/2 + \alpha$ with $n \in \mathbb{Z}$ and $|\alpha| \le \pi/4$. In particular, the minimum is attained for $\theta = \sigma + \pi/2$, and hence

$$\Lambda_2^\theta \ge \Lambda_2^{\sigma+\pi/2}.$$

Proof. For such θ, $\csc(\sigma-\theta) = -\csc\big((2n-1)\pi/2+\alpha\big) = (-1)^{n+1}\sec(\alpha)$. Since $|\alpha| \le \pi/4$, we have $\csc(\sigma-\theta)^2 = \sec(\alpha)^2 \le 2$, and so $\max\big(\csc(\sigma-\theta)^2/2, 1\big) = 1$. ∎

Thus, Gauss's formula (2.1) is optimal for equispaced interpolation in a second sense: it selects θ to yield a minimal value of Λ_2^θ. Of course, as the statement of the corollary makes clear, this choice is not unique.

7. Non-equispaced points. When the interpolation points are not equispaced, the Lebesgue constants—of both varieties—are considerably more difficult to analyze.[2] While the formula (5.3) for the Lebesgue function L^θ remains valid, the expression (5.4) for Λ^θ does not because $|\gamma_k|$ is no longer independent of k. Moreover, our analysis of Λ_2^θ relied on simple explicit expressions for ℓ_k, ℓ, and γ_k that are not available for general grids.

Nevertheless, we can gain some insight into the non-equispaced case through numerical computation. Using the Chebfun software package for MATLAB [8], we can build a machine-precision-accurate piecewise polynomial approximation to L^θ for any grid of our choosing and compute its maximum to find Λ^θ. By varying θ, we can determine which choice of θ yields an optimal (minimal) value of Λ^θ.

A plot of Λ^θ as a function of θ for the equispaced grid (1.3) with $K = 6$ is shown in the solid blue line of Figure 7.1a. The dashed orange line marks the location of $\theta = \sigma$, reduced modulo π, which, for this grid, is $\pi/2$. The singularity present in the blue curve at this point reflects the fact that a unique interpolant in $\mathscr{T}_N^{\pi/2}$ does not exist for this grid, as proved in Theorem 4.1 and Corollary 4.2. Similarly, the dashed gold line marks the location of $\theta = \sigma + \pi/2$, reduced modulo π, which is 0 here; recall (Corollary 3.2) that this is the choice

[2] One can sometimes make progress by assuming something about the nature of the nonuniform spacing; this is done, for instance in [1, Chapter 3] [2, 15], which study the Lebesgue constants for odd-length trigonometric interpolation in perturbed equispaced grids. It is likely that similar techniques can be applied to our even-length setting, but we do not attempt to pursue this here.

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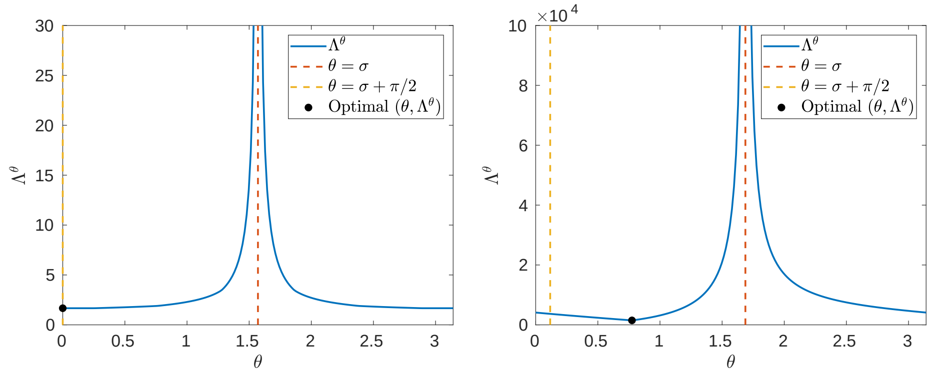

(a) (b)

FIG. 7.1. *Lebesgue constant Λ^θ (solid blue lines) as a function of θ for* (a) *the equispaced grid* (1.3) *with $K = 6$ and* (b) *a 6-point non-equispaced grid described in the text. The values of θ in $[0, \pi)$ corresponding to $\theta = \sigma$ (dashed orange lines) and $\theta = \sigma + \pi/2$ (dashed gold lines) are marked, as are the optimal (minimal) values of Λ^θ (black dots).*

of θ made by Gauss's formula (2.1). The black dot marks the minimal value of Λ^θ.[3] We see that the choice made by Gauss's formula minimizes Λ^θ, in accordance with Theorem 5.2.

Figure 7.1b displays a similar plot but for a non-equispaced grid with interpolation points at approximately 0.584, 0.618, 0.647, 4.126, 4.522, and 5.443. Again, there is a singularity present at $\theta = \sigma$, but more importantly, it is evident that the minimal value of Λ^θ is not attained for the Gauss choice $\theta = \sigma + \pi/2$. That is, Gauss's formula does *not* select θ to minimize the Lebesgue constant in this case. Thus, we conclude that Gauss's formula is not optimal for non-equispaced grids in general.

Similar plots can be made for the 2-norm Lebesgue constant Λ_2^θ. As they are qualitatively similar to those of Figure 7.1, we omit them.

Even though the plot of Figure 7.1b reveals that the choice $\theta = \sigma + \pi/2$ is not optimal in general, we observe that it is "not far" from optimal for this example in the sense that $\Lambda^{\sigma+\pi/2}$ is not significantly larger than the minimal value of Λ^θ. It is natural to wonder whether this near-optimality holds for arbitrary sets of interpolation points. Were that true, there would be little to gain by pursuing the true optimal value of θ; one could use Gauss's formula as-is in confidence, knowing that no other choice for θ is significantly superior to the one it makes.

While we shall not attempt to answer this question completely, we can gain some insight by considering the following simple inequality.

PROPOSITION 7.1. *For any choice of the interpolation points x_k and the parameter θ,*

$$|\Lambda^{\sigma+\pi/2} - \Lambda^\theta| \le C|\cot(\sigma - \theta)|,$$

where C is a constant that does not depend on θ.

Proof. By (5.2) and the triangle inequality,

$$L^\theta(x) \le L^{\sigma+\pi/2}(x) + |\ell(x)\cot(\sigma - \theta)| \sum_{k=0}^{K-1} |\gamma_k|.$$

[3] As our proof of Theorem 5.2 suggests, this point is not unique. Computationally, we find that the minimum is attained for approximately $0 \le \theta \le 0.253518$ and $\pi - 0.253518 \le \theta < \pi$.

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Taking the maximum over $0 \le x \le 2\pi$ yields

$$\Lambda^{\theta} \le \Lambda^{\sigma+\pi/2} + C|\cot(\sigma-\theta)|,$$

where

$$C = \left[\max_{0\le x\le 2\pi} |\ell(x)|\right] \sum_{k=0}^{K-1} |\gamma_k|$$

does not depend on θ. A similar argument yields the same inequality with the roles of Λ^{θ} and $\Lambda^{\sigma+\pi/2}$ reversed. Combining these two inequalities, we obtain the result. □

COROLLARY 7.2. *For equispaced interpolation points x_k,*

$$\Lambda^{\sigma+\pi/2} \le \Lambda^{\theta} \le \Lambda^{\sigma+\pi/2} + |\cot(\sigma-\theta)|.$$

Proof. The first inequality is given by Theorem 5.2. The second follows from Proposition 7.1 upon using the formulae in Appendix A to show that the expression for the constant C that appears in the proof evaluates to 1. □

Observe that $\Lambda^{\sigma} = \infty$, a consequence of the fact that the interpolation problem may not have a solution when $\theta = \sigma$. Because Λ^{θ} is a continuous function of θ, it follows that the optimal value of Λ^{θ} cannot occur for θ near σ. It thus seems natural to look for the optimal θ at points far from σ, and Gauss's formula takes this intuition to the extreme: modulo π, $\theta = \sigma + \pi/2$ is the furthest point in $[0, \pi)$ from $\theta = \sigma$.

As Figure 7.1b shows, this intuition does not always lead to the optimum; however, letting θ_* denote a value of θ in $[0, \pi)$ for which Λ^{θ} is minimal, Proposition 7.1 shows that the difference between $\Lambda^{\sigma+\pi/2}$ and Λ^{θ_*} is at most $C|\cot(\sigma-\theta_*)|$. Thus, $\Lambda^{\sigma+\pi/2}$ is "nearly optimal" provided that neither C nor $|\cot(\sigma-\theta_*)|$ is large. Bounding either of these quantities may be difficult in general. Nevertheless, the intuition developed in the previous paragraph suggests that $|\sigma-\theta_*|$ (with σ taken modulo π) ought to be bounded well away from 0; hence, $|\cot(\sigma-\theta)|$ ought not to be very large. Moreover, potential-theoretic considerations [13, Ch. 11] lead us to expect that the constant C will be large only if the distribution of the interpolation points is so far from equispaced that the problem of computing the interpolant is too ill-conditioned—even for $\theta = \theta_*$—for the scheme to be of any real use.

Thus, we expect (but certainly have not proved) that the choice made by Gauss's formula will be "nearly optimal" for all "practical" choices of the interpolation points, even if they are not equispaced. We leave the development of a more precise version of this statement as a matter for future work.

8. Minimal-norm solutions. In the preceding sections, we took the perspective that one should select the constraint on the highest-order terms in the interpolant to minimize a norm of the interpolation operator, thereby obtaining an interpolant that is least sensitive to perturbations as measured by that norm. An alternative way to proceed is to take a cue from the linear algebra of underdetermined systems and select the interpolant t from $\mathscr{T}_N$ to minimize $\|t\|_2$. As the 2-norm is the only norm we consider, we will refer to these minimal-2-norm solutions simply as minimal-norm solutions for brevity.

Remarkably, we can give a simple formula for a minimal-norm interpolant even when the points are not equispaced. Observe that the general solution (2.4) to the interpolation problem may be expressed as

$$t(x) = \sum_{k=0}^{K-1} f_k \left[\ell_k(x) + \mu_k \ell(x)\right], \tag{8.1}$$

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where the μ_k are free parameters. This representation is misleading in that it seems that t has K degrees of freedom when it is really only the value of $\lambda = \sum_{k=0}^{K-1} f_k\mu_k$ that matters. Nevertheless, it is useful because, unlike (2.4), it is a Lagrange-type formula for the interpolant. We view (8.1) as a modification for $\lambda \neq 0$ of Gauss's Lagrange formula (2.1).

We now show how to select the μ_k so that (8.1) gives a minimal-norm solution to the interpolation problem. We require the following fact from linear algebra.

LEMMA 8.1. *Let $A \in \mathbb{C}^{m\times n}$, nonzero $u \in \mathbb{C}^m$, and $f \in \mathbb{C}^n$ be given. The quantity $\|(A + uv^*)f\|_2$, where $v \in \mathbb{C}^n$, is minimal for*

$$v = -\frac{A^*u}{u^*u}.$$

Proof. By orthogonal projection, the value of α for which $\|Af + \alpha u\|_2$ is minimal is $\alpha = -u^*Af/u^*u$. We just need to select v so that $v^*f = \alpha$, and a direct calculation shows that the given choice of v accomplishes this. □

The choice of v given by Lemma 8.1 is essentially unique. Although we can add to the given v any vector orthogonal to f and still obtain a minimal value of $\|(A + uv^*)f\|_2$, the value of v^*f remains the same, and therefore so does the vector $(A + uv^*)f$ that attains the minimum.

We use Lemma 8.1 to prove the following result, which says that a choice for the μ_k that yields a minimal-norm solution can be found by projecting the Lagrange basis functions ℓ_k orthogonally onto the subspace spanned by the trigonometric node polynomial ℓ.

THEOREM 8.2. *A minimal-norm solution to the even-length trigonometric interpolation problem is given by* (8.1) *with*

$$\mu_k = -\frac{\langle \ell_k, \ell\rangle}{\langle \ell, \ell\rangle}.$$

Proof. By standard results from finite-dimensional linear algebra, there is a vector space isomorphism (a bijective linear map) $\varphi : \mathscr{T}_N \to \mathbb{C}^{2N+1}$ that is isometric in the sense that $\langle s, t\rangle = \varphi(t)^*\varphi(s)$ for any $s, t \in \mathscr{T}_N$. Abusing notation, we identify members of $\mathscr{T}_N$ with their images under φ and use (8.1) to write

$$t = (L + \ell\mu^*)f$$

in $\mathbb{C}^{2N+1}$, where $L = \begin{bmatrix}\ell_0 & \cdots & \ell_{K-1}\end{bmatrix}$ and $\mu = (\mu_0, \ldots, \mu_{K-1})$ and $f = (f_0, \ldots, f_{K-1})$, both written as column vectors. By Lemma 8.1, the choice of μ that minimizes $\|t\|_2$ is

$$\mu = -\frac{L^*\ell}{\ell^*\ell}.$$

Using φ to translate this from a statement about objects in $\mathbb{C}^{2N+1}$ to one about objects in $\mathscr{T}_N$, we obtain the result. □

Note that the values of the μ_k given by Theorem 8.2 depend only on the interpolation points x_k and not on the interpolation data f_k. That is, the modified Gauss formula (8.1) with these μ_k yields a minimal-norm solution for *any* f_k.

The minimal-norm interpolant of Theorem 8.2 is essentially unique for the same reason that the vector v of Lemma 8.1 is essentially unique. Although there are other choices of the μ_k that work, they all yield the same value of $\lambda = \sum_{k=0}^{K-1} f_k\mu_k$ and thus the same interpolant. If the points are equispaced, then selecting μ_k according to Theorem 8.2 causes (8.1) to reduce to Gauss's original formula.

COROLLARY 8.3. *For equispaced interpolation points x_k, Gauss's formula* (2.1) *gives a minimal-norm solution to the even-length trigonometric interpolation problem.*

Proof. If the points are given by (1.3), Theorem 8.2 and (6.4) show that $\mu_k = 0$ gives a minimal-norm solution. For other equispaced grids, use periodic translation. ∎

In particular, the balanced solution is a minimal-norm solution to an interpolation problem posed in the points (1.3). For interpolation in (1.4), the skew-balanced solution is of minimal norm. It is tempting to state further that Gauss's formula will give a minimal-norm solution for any set of interpolation points for which $\langle \ell_k, \ell \rangle = 0$ for all k; however, the following result shows that this statement is not more general than what we have already proved.

PROPOSITION 8.4. *We have $\langle \ell_k, \ell \rangle = 0$ for all k if and only if the interpolation points x_k are equispaced.*

Proof. The "if" part is a consequence of (6.4) and periodic translation. For the converse, suppose that $\langle \ell_k, \ell \rangle = 0$ for all k. Since $\mathscr{T}_{N-1} \subset \operatorname{span}\{\ell_0, \ldots, \ell_{K-1}\}$, we must have $\langle t, \ell \rangle = 0$ for all $t \in \mathscr{T}_{N-1}$. And since $\ell \in \mathscr{T}_N$, this can only happen if $\ell \in \operatorname{span}\{\cos(Nx), \sin(Nx)\}$. Thus, $\ell(x) = A\cos(Nx - \varphi)$ for some $A \in \mathbb{R}$, $A \neq 0$, and $\varphi \in [0, 2\pi)$, and so ℓ has $2N$ equispaced zeros in $[0, 2\pi)$. But by (2.3), the zeros of ℓ in $[0, 2\pi)$ are precisely the interpolation points x_k. ∎

9. Conclusion. The present work was born of the author's desire to understand 1) how Gauss's formula (2.1) resolves the issue of the extra degree of freedom present in even-length trigonometric interpolants and 2) what rationale, if any, could be given to support the balancing condition, which is discussed often in the literature [5, 10, 12] but seems to have little to recommend it over simply using Gauss's formula by itself, especially when interpolating in arbitrary points.

For 1), we have analyzed Gauss's formula in detail and precisely identified the choice it makes for the constraint on the highest-order terms in the interpolant. We have shown that for equispaced interpolation, this choice is optimal in every sense of the word: it constrains the extra degree of freedom in a way that minimizes the standard ∞-norm and 2-norm Lebesgue constants and, moreover, produces interpolants of minimal 2-norm.

Regarding 2), we have shown that the balancing condition is optimal for the equispaced grid (1.3), and Gauss's formula imposes this condition naturally. For other grids, nothing in our analysis gives any reason to prefer the balancing condition. Accordingly, if a Lagrange representation is desired, we recommend using Gauss's formula without modifying it to force a balanced interpolant.[4]

When the interpolation points are not equispaced, the choice made by Gauss's formula ceases to be optimal. We have demonstrated this numerically for the ∞-norm Lebesgue constant and proved it rigorously for minimal-norm interpolants by identifying the optimal choice in that case.

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[4]More specifically, for computational efficiency and to avoid numerical instabilities, we recommend using the *barycentric* version of Gauss's formula; see [3, 5, 6, 10, 11, 12] for details and [4] for a recent application to trigonometric rational approximation.

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Appendix A. Formulae for ℓ_k, ℓ, and γ_k for the grid (1.3).

In the proofs of our theorems regarding interpolation in the points (1.3), we require formulae for ℓ_k, ℓ, and γ_k for this grid. In [10, Section 2], it is shown that

$$\ell_k(x) = \frac{(-1)^k}{2N}\sin(Nx)\cot\left(\frac{x-x_k}{2}\right). \tag{A.1}$$

From [5, equation (2.2)] and the fact that $\sigma = N\pi - \pi/2$ for this grid (see the proof of Corollary 3.3), we infer

$$\ell(x) = -\frac{1}{2^{2N-1}}\sin(Nx). \tag{A.2}$$

Finally, we determine γ_k from these results and the relation $\ell_k(x) = \gamma_k \ell(x)\cot\big((x-x_k)/2\big)$:

$$\gamma_k = (-1)^{k+1}\frac{2^{2(N-1)}}{N}. \tag{A.3}$$

Appendix B. Proofs of identities used in proving Theorem 6.2. In this appendix, we employ residue calculus to establish the identities (6.3), (6.4), and (6.5) used in the proof of Theorem 6.2. Recall that the residue of a function $f : \mathbb{C} \to \mathbb{C}$ at a point $c \in \mathbb{C}$ at which it has a pole of order n may be calculated by the formula

$$\operatorname*{Res}_{z=c} f(z) = \frac{1}{(n-1)!}\lim_{z\to c}\frac{d^{n-1}}{dz^{n-1}}\big[(z-c)^n f(z)\big]. \tag{B.1}$$

We use this formula to determine the residues at $z = 0$ of a pair of functions that will arise in our computations.

LEMMA B.1. *For $a \in \mathbb{C}$, $a \neq 0$, and integer $n \geq 1$,*

$$\operatorname*{Res}_{z=0} \frac{1}{z^n}\frac{z+a}{z-a} = \begin{cases} -1 & \text{if } n = 1, \\ -\frac{2}{a^{n-1}} & \text{if } n \geq 2. \end{cases}$$

Proof. For $n = 1$, the result follows from the computation

$$\lim_{z\to 0} z\frac{1}{z}\frac{z+a}{z-a} = \frac{a}{-a} = -1.$$

For $n \geq 2$, by repeatedly differentiating the expansion

$$\frac{z+a}{z-a} = 1 + \frac{2a}{z-a},$$

we obtain

$$\frac{d^{n-1}}{dz^{n-1}}\left[\frac{z+a}{z-a}\right] = (-1)^{n-1}(n-1)!\frac{2a}{(z-a)^n},$$

and the result now follows from (B.1). ∎

LEMMA B.2. *For $a \in \mathbb{C}$, $a \neq 0$, and integer $n \geq 1$,*

$$\operatorname*{Res}_{z=0} \frac{1}{z^n}\left(\frac{z+a}{z-a}\right)^2 = \begin{cases} 1 & \text{if } n = 1, \\ \frac{4}{a^{n-1}}(n-1) & \text{if } n \geq 2. \end{cases}$$

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Proof. For $n = 1$, we have

$$\lim_{z\to 0} z\frac{1}{z}\left(\frac{z+a}{z-a}\right) = \left(\frac{a}{-a}\right)^2 = 1.$$

For $n \geq 2$, we repeatedly differentiate

$$\left(\frac{z+a}{z-a}\right)^2 = 1 + \frac{4a}{z-a} + \frac{4a^2}{(z-a)^2}$$

to find

$$\frac{d^{n-1}}{dz^{n-1}}\left[\left(\frac{z+a}{z-a}\right)^2\right] = (-1)^{n-1}(n-1)!\frac{4a}{(z-a)^n} + (-1)^{n-1}n!\frac{4a^2}{(z-a)^{n+1}}$$

and apply (B.1) again. ∎

These results enable us to evaluate a pair of trigonometric integrals with the aid of the residue theorem.

LEMMA B.3. *With x_k as in (1.3),*

$$\int_0^{2\pi} \sin(Nx)^2 \cot\left(\frac{x-x_k}{2}\right)\, dx = 0.$$

Proof. Using Euler's identity, we rewrite the integrand in terms of complex exponentials and then change variables according to $z = e^{ix}$, $dz = iz\, dx$, giving

$$\int_0^{2\pi} \sin(Nx)^2 \cot\left(\frac{x-x_k}{2}\right)\, dx = -\frac{1}{4}\int_{\mathbb{T}} (z^N - z^{-N})^2 \frac{z+z_N^k}{z-z_N^k}\frac{dz}{z},$$

where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ and $z_N = e^{i\pi/N}$ so that $z_N^k = e^{ix_k}$. Noting that the singularity of the integrand at $z = z_N^k$ is removable, the value of the integral will be determined by the residue of the integrand at $z = 0$, where it has a pole of order $2N + 1$. Expanding

$$(z^N - z^{-N})^2 \frac{z+z_N^k}{z-z_N^k}\frac{1}{z} = z^{2N-1}\frac{z+z_N^k}{z-z_N^k} - \frac{2}{z}\frac{z+z_N^k}{z-z_N^k} + \frac{1}{z^{2N+1}}\frac{z+z_N^k}{z-z_N^k}$$

and applying Lemma B.1, we obtain

$$\operatorname*{Res}_{z=0}(z^N - z^{-N})^2 \frac{z+z_N^k}{z-z_N^k}\frac{1}{z} = 0 + 2 - \frac{2}{z_N^{2Nk}} = 0,$$

since $z_N^{2Nk} = 1$. Thus, by the residue theorem,

$$\int_0^{2\pi} \sin(Nx)^2 \cot\left(\frac{x-x_k}{2}\right)\, dx = -\frac{1}{4}2\pi i \operatorname*{Res}_{z=0}(z^N - z^{-N})^2 \frac{z+z_N^k}{z-z_N^k}\frac{1}{z} = 0,$$

as claimed. ∎

LEMMA B.4. *With x_k as in (1.3),*

$$\int_0^{2\pi} \sin(Nx)^2 \cot\left(\frac{x-x_k}{2}\right)^2\, dx = (4N-1)\pi.$$

Proof. Proceeding as in the proof of Lemma B.3, we rewrite the integrand in terms of complex exponentials and change variables to obtain

$$\int_0^{2\pi} \sin(Nx)^2 \cot\left(\frac{x-x_k}{2}\right)^2\, dx = -\frac{i}{4}\int_{\mathbb{T}} (z^N - z^{-N})^2 \left(\frac{z+z_N^k}{z-z_N^k}\right)^2 \frac{dz}{z}.$$

Once again, the singularity in the integrand at $z = z_N^k$ is removable, and the only other singularity is a pole of order $2N+1$ at $z=0$. We expand

$$(z^N - z^{-N})^2 \left(\frac{z+z_N^k}{z-z_N^k}\right)^2 \frac{1}{z} = z^{2N-1}\left(\frac{z+z_N^k}{z-z_N^k}\right)^2 - \frac{2}{z}\left(\frac{z+z_N^k}{z-z_N^k}\right)^2 + \frac{1}{z^{2N+1}}\left(\frac{z+z_N^k}{z-z_N^k}\right)^2$$

and apply Lemma B.2 to find

$$\operatorname*{Res}_{z=0}(z^N - z^{-N})^2 \left(\frac{z+z_N^k}{z-z_N^k}\right)^2 = 0 - 2 + \frac{8N}{z_N^{2Nk}} = 8N-2.$$

Thus, by the residue theorem,

$$\int_0^{2\pi} \sin(Nx)^2 \cot\left(\frac{x-x_k}{2}\right)^2\, dx = -\frac{i}{4}2\pi i \operatorname*{Res}_{z=0}(z^N - z^{-N})^2 \left(\frac{z+z_N^k}{z-z_N^k}\right)^2 = (4N-1)\pi,$$

as was to be shown. ◻

We can now establish the claimed identities. From (A.1), we have

$$\langle \ell_j, \ell_k\rangle = \frac{(-1)^{j+k}}{4N^2}\int_0^{2\pi} \sin(Nx)^2 \cot\left(\frac{x-x_j}{2}\right)\cot\left(\frac{x-x_k}{2}\right) dx.$$

For $j \neq k$, use the standard identity for the cotangent of a difference to write

$$\cot\left(\frac{x-x_j}{2}\right)\cot\left(\frac{x-x_k}{2}\right) = \cot\left(\frac{x_j-x_k}{2}\right)\left[\cot\left(\frac{x-x_k}{2}\right) - \cot\left(\frac{x-x_k}{2}\right)\right] - 1.$$

By Lemma B.3, the integral over $[0, 2\pi]$ of the product of $\sin(Nx)^2$ and the terms in square brackets on the right-hand side vanishes, leaving

$$\langle \ell_j, \ell_k\rangle = -\frac{(-1)^{j+k}}{4N^2}\int_0^{2\pi} \sin(Nx)^2\, dx = (-1)^{j+k+1}\frac{\pi}{4N^2},$$

which is the formula given the first clause of (6.3). For $j = k$, we have

$$\langle \ell_k, \ell_k\rangle = \frac{1}{4N^2}\int_0^{2\pi} \sin(Nx)^2 \cot\left(\frac{x-x_k}{2}\right)^2 dx = (4N-1)\frac{\pi}{4N^2}$$

by Lemma B.4, yielding the second clause of (6.3). For (6.4), we compute

$$\langle \ell_k, \ell\rangle = \frac{(-1)^{k+1}}{2^{2N}N}\int_0^{2\pi} \sin(Nx)^2 \cot\left(\frac{x-x_k}{2}\right) dx = 0$$

by (A.1) and (A.2) and Lemma B.3. Finally, using (A.2) and (A.3), we have

$$\gamma_j\gamma_k \langle \ell, \ell\rangle = (-1)^{j+k}\frac{2^{4(N-1)}}{2^{4N-2}N^2}\int_0^{2\pi} \sin(Nx)^2\, dx = (-1)^{j+k}\frac{\pi}{4N^2},$$

giving (6.5).

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