Binary lattices

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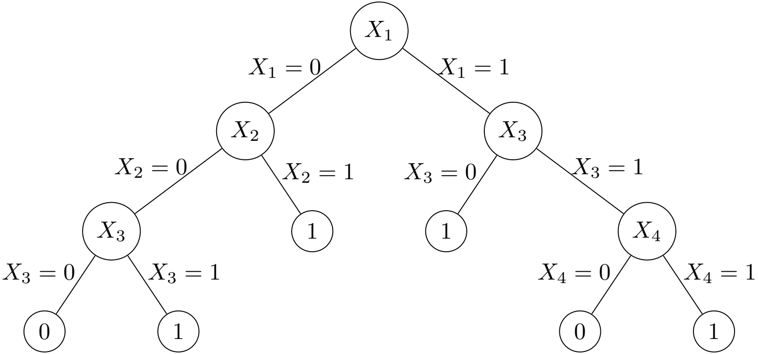
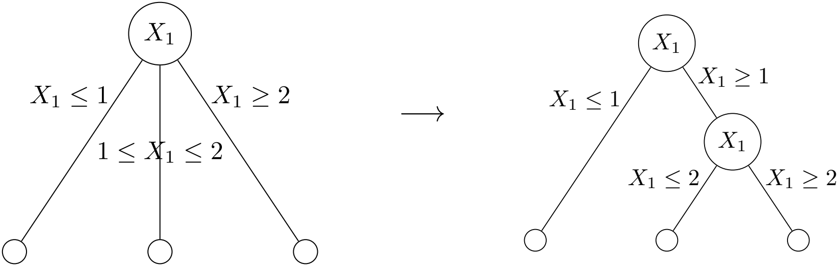
Decision trees



- easy to use,
- easy to understand,
- efficient.

Binary decision trees



Easier to :

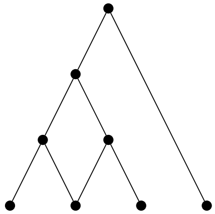
- build,
- understand,
- use.

Binary lattice

Definition (Binary lattice)

$(L, \leq)$ is *binary* if $\forall x \in L$

- x covers at most 2 elements,
- x is covered by at most 2 elements.



Poset embedding

Definition

$(L, \leq)$ can be *embedded* into $(L', \leq)$ if there exists $f : L \to L'$ such that :

$$\forall l_1, l_2 \in L, l_1 \leq l_2 \iff f(l_1) \leq f(l_2)$$

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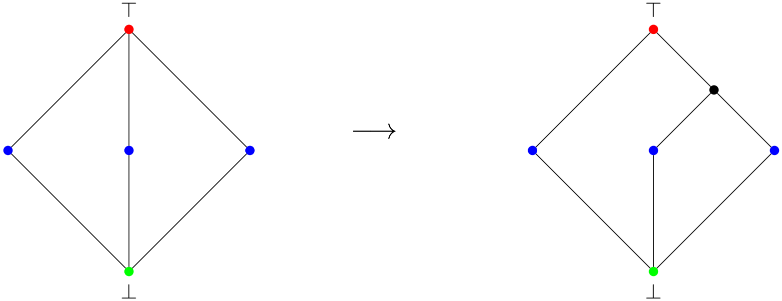

Binarizable lattice

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$(L, \leq)$ is *binarizable* if

$\exists (L', \leq)$ binary lattice such that $(L, \leq)$ can be embedded in $(L', \leq)$.

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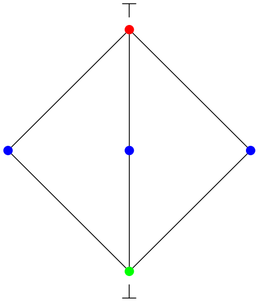

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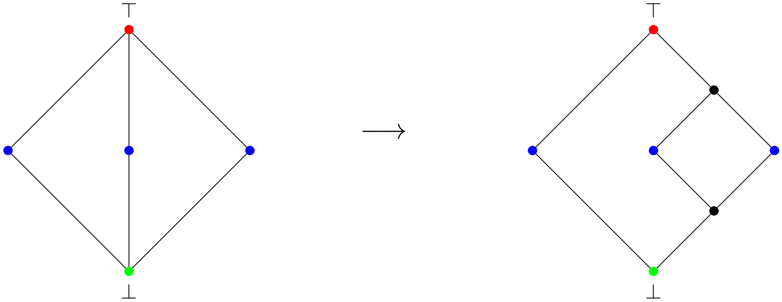
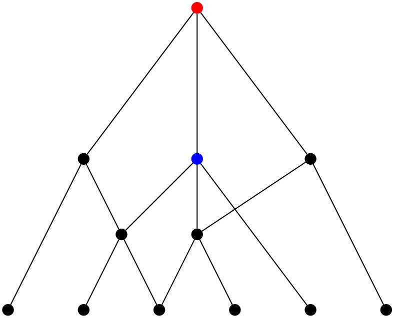


Example of binarization



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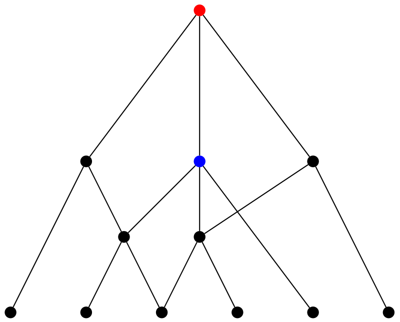

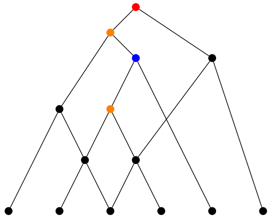

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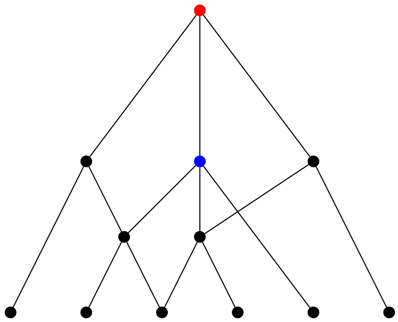
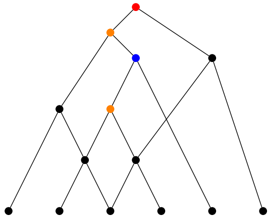
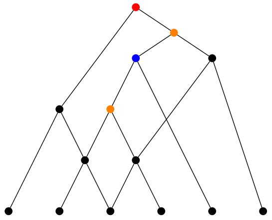


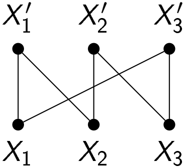




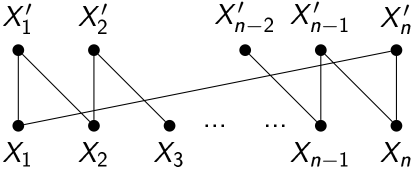
Definition

A *crown* is a poset $(X_1, X'_1, \ldots, X_n, X'_n)$ such that :

- $\forall i,\ X_i < X'_{i-1 \mod n},\ X_i < X'_i$,
- $\forall i, j, j \neq i, i-1 \mod n \implies X_i \parallel X'_j$.



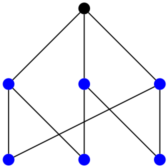
3-crown



n-crown

At most $\mathcal{O}(n^2)$ elements.

Binarizable $\implies$ crown-free



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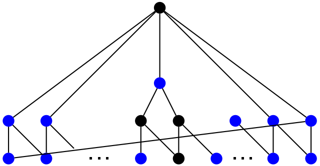

Binarization of a n-crown
$\implies$ (n-p)-crown

Equivalence with set systems

Definition

S is a *set system* on a set V if :

- $S \subseteq 2^V$,
- $A \in S, B \in S \implies A \cap B \in S$,
- S has a minimum and a maximum element.

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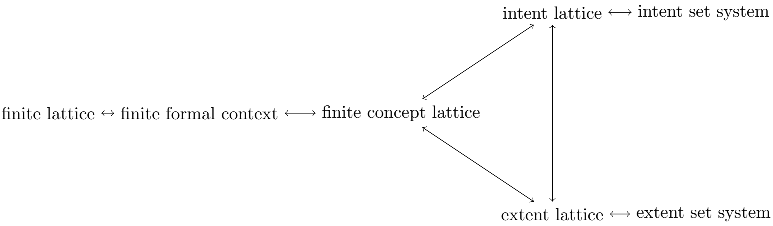
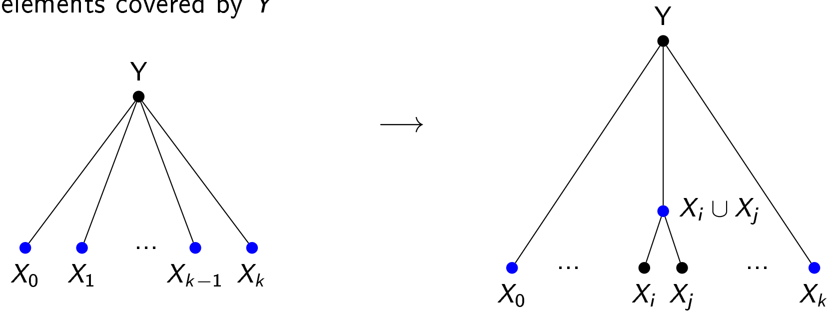


Binarization of an element of a set system

Choose X_i, X_j among the elements covered by Y

Create $X_i \cup X_j$



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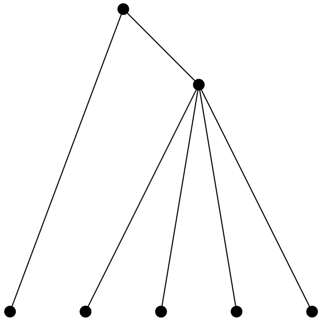
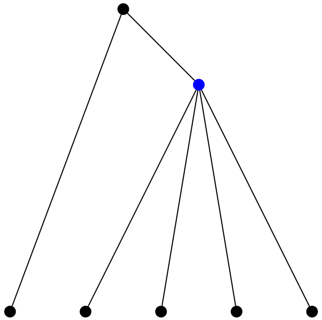


Binarization of a set system

For each element which covers more than two elements :

- While it covers more than two elements :



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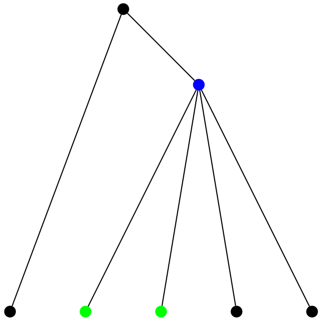
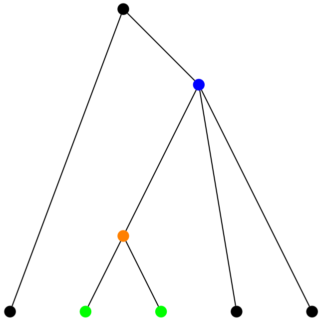


Binarization of a set system

For each element which covers more than two elements :

- While it covers more than two elements :
  - Chose two elements it covers
  - Create their union



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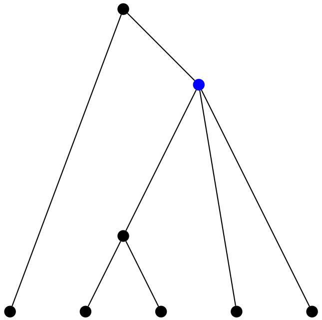

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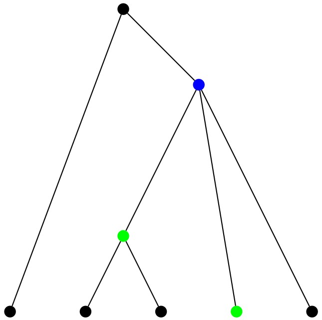

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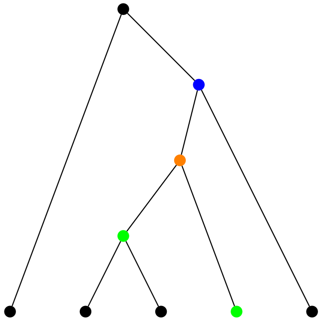
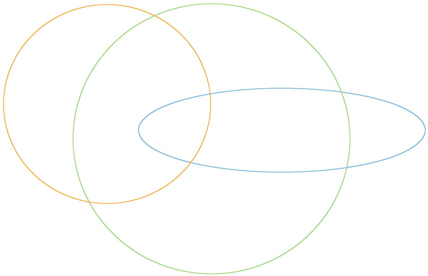


Choice of the elements to union

Definition (Maximal intersection elements)

X_i, X_j of *maximal intersection* if

$$\nexists k, \begin{cases} X_i \cap X_j \subsetneq X_i \cap X_k, \\ X_i \cap X_j \subsetneq X_j \cap X_k \end{cases}$$



$\Rightarrow$ Get similar objects together

Crown-free $\implies$ binarizable

L a crown-free set system
X_i, X_j of maximal intersection :

- $\Rightarrow$ $(X_i \cup X_j) \cap X_k \in L$
- $\Rightarrow$ $\forall Z, Z \cap (X_i \cup X_j) \in L \cup \{X \cup Y\}$
- $\Rightarrow$ resulting elements are still incomparable

Equivalence

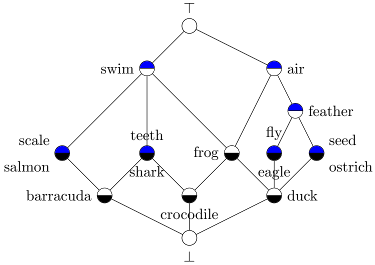
Theorem (C., Brucker, Préa)

Let $(L, \leq)$ finite lattice. $(L, \leq)$ is binarizable iff $(L, \leq)$ is crown-free.

Formal context example

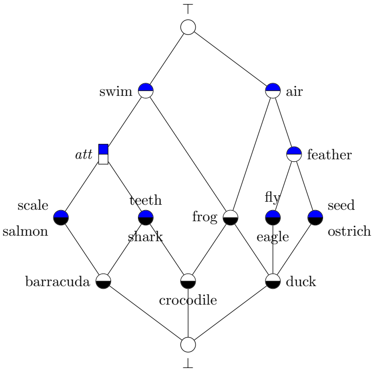
	scale	teeth	swim	fly	seed	feather	air
salmon	×		×				
shark		×	×				
barracuda	×	×	×				
frog			×				×
crocodile		×	×				×
eagle				×		×	×
ostrich					×	×	×
duck			×	×	×	×	×

Formal context example

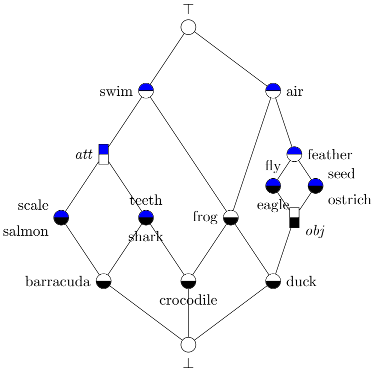


Associated concept lattice

Example of binarization



Lower-binarized concept lattice



Binarized concept lattice

Thank you for your attention