Adjoint Rupture Dynamics and Gradient-Based Dynamic Source Inversions

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Overview

Objective: Computationally efficient method to quantify sensitivity of dynamic rupture (or earthquake sequence) models to frictional parameters, initial stresses, etc. – only two simulations required for gradient calculation for arbitrary number of model parameters! Extends prior work (Kano et al., 2013, 2015) for BEM-discretized earthquake cycle model to continuum problem with inertia/waves.

Applications:

- Gradient-based inversions (determine parameters to minimize misfit with seismic, geodetic, other data + regularization/prior)
- Quantify uncertainty

Adjoint Method

Forward problem: Elasticity, boundary conditions, and

$$\bar{\tau} = \bar{F}(\bar{V}, \Psi, \sigma_n), \quad \text{rate-state friction,}$$
$$\dot{\Psi} = G(\bar{V}, \Psi, \sigma_n, \dot{\sigma}_n), \quad \text{state evolution,}$$
$$V_n = 0, \quad \text{no opening/interpenetration.}$$

Adjoint problem: Elasticity (with adjoint sources at receivers), boundary conditions, and

$$\tau_J^\dagger = F_J^\dagger(\bar{V}^\dagger, \Psi^\dagger) := \frac{\partial F_I}{\partial V_J} V_I^\dagger + \frac{\partial G}{\partial V_J}\Psi^\dagger, \quad \text{adjoint rate-state friction,}$$
$$-\dot{\Psi}^\dagger = G^\dagger(\bar{V}^\dagger, \Psi^\dagger) := \frac{\partial F_J}{\partial \Psi} V_J^\dagger + \frac{\partial G}{\partial \Psi}\Psi^\dagger, \quad \text{adjoint state evolution,}$$
$$V_n^\dagger = H^\dagger(\bar{V}^\dagger, \Psi^\dagger) \quad \text{nonzero opening/interpenetration.}$$
$$:= \left[\frac{\partial G}{\partial \dot{\sigma}_n}\frac{\partial F_J}{\partial \Psi} + \frac{\partial F_J}{\partial \sigma_n}\right] V_J^\dagger + \left[\frac{\partial G}{\partial \sigma_n} + \frac{\partial G}{\partial \dot{\sigma}_n}\frac{\partial G}{\partial \Psi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial G}{\partial \dot{\sigma}_n}\right)\right]\Psi^\dagger.$$

Adjoint friction is *linear in adjoint fields*, but with *coefficients that depend on forward solution* (varying in space and time).

Inversion (PDE-constrained optimization): Find parameters p that minimize misfit to data (e.g., velocity time series = seismograms)

$$\min_p \underbrace{\sum_n \frac{1}{2}\int_0^T \left(\dot{u}(\bar{x}_n, t; p) - d_n(t)\right)^2 \mathrm{d}t}_{\mathcal{F}(p)}$$

subject to *forward problem*.

Iterative algorithm for source inversions:
Start with initial guess for p. Loop:

1. Solve *forward problem*. Evaluate residuals and misfit.
2. Solve *time-reversed adjoint problem*: Residuals injected at receivers and back-propagated to fault, carrying stresses that drive adjoint rupture.
3. Compute *gradient* by convolving forward and adjoint solutions:
$$\frac{\delta \mathcal{F}}{\delta p} = -\left(V_J^\dagger, \frac{\partial F_J}{\partial p}\right)_T - \left(\Psi^\dagger, \frac{\partial G}{\partial p}\right)_T \quad (\mathrm{L}_2 \text{ inner product in time})$$
4. Update p using *gradient-based optimization algorithm* (e.g., gradient descent, L-BFGS, Hamiltonian Monte Carlo).

Numerics and Implementation

Solver: SBP-SAT finite difference method for wave equation in second-order form (displacements as unknowns), open-source MATLAB code sbplib (sourceforge.net/p/sbplib), L-BFGS optimizer (MATLAB's fmincon)

Dual-consistent discretization: Adjoint of discretized forward problem is consistent, high order accurate discretization of continuum adjoint problem – allowing use of exactly same code for forward and adjoint problems!

Example: Dynamic Rupture Inversion

- 2D antiplane rupture on rough fault
- Rate-state friction, slip law for state evolution
- Ruptures velocity-weakening (VW) region, velocity-strengthening (VS) outside

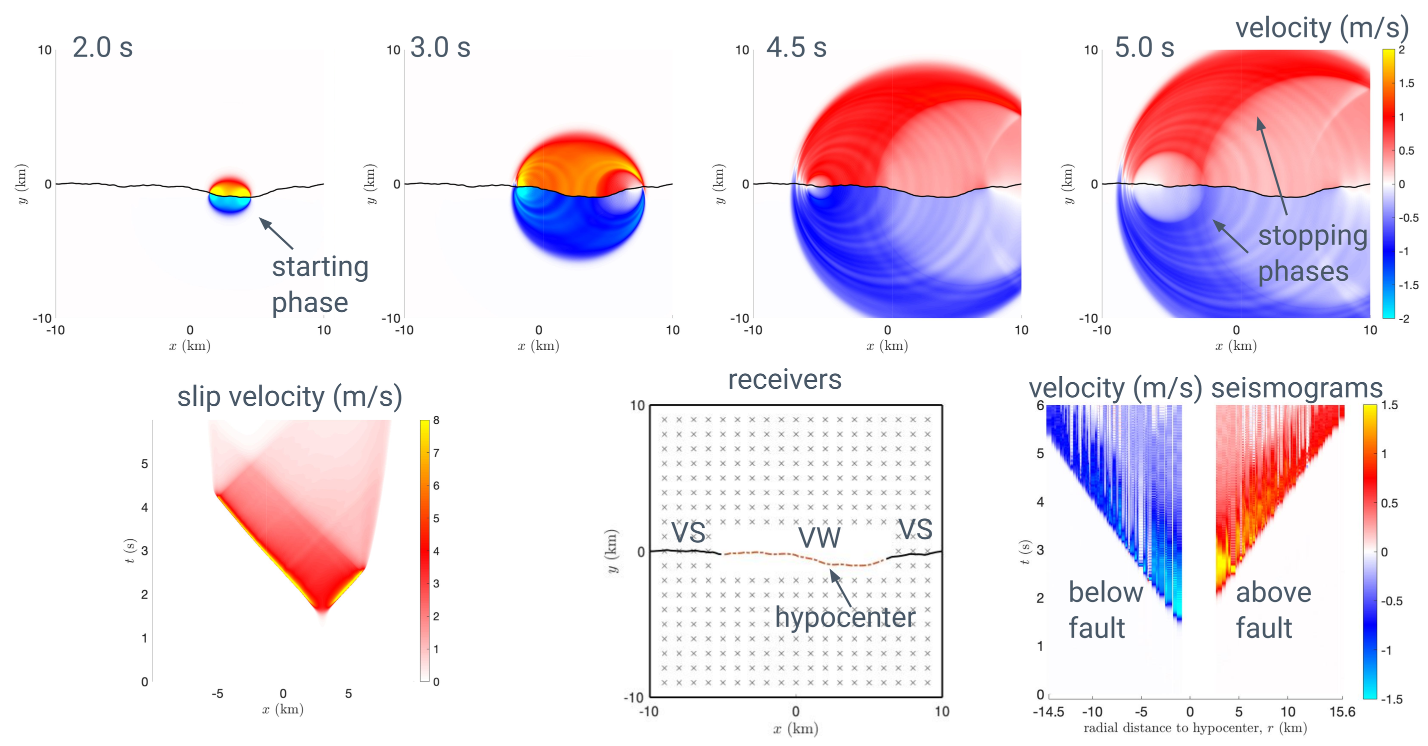

Figure 1: True solution to forward problem. Rupture arrests after entering VS region. Wavefield features sharp starting and stopping phases, as well as lower amplitude waves from rupture velocity fluctuations from interaction with roughness. Seismograms are non-oscillatory, limiting local minima in misfit from cycle skipping, a common issue in full waveform inversion (FWI).

Adjoints, Time-Reversed Imaging, and Back-projection

Adjoint problem closely related to time-reversed imaging (e.g., Larmat et al., 2006) and back-projection method (e.g., Kiser & Ishii, 2017). Time-reversed imaging injects recorded seismograms as point sources at receivers. Waves coalesce (constructively interfere) around source location and time at which they were radiated. Back-projection is similar but usually simpler (e.g., just time-shifting seismograms). For adjoint method, residuals (instead of recorded seismograms) are injected as point sources at receivers. Waves coalesce around portions of fault where rupture process needs to be updated to better match data. Stresses carried by these waves provide tractions on fault that drive adjoint rupture – which creates its own wavefield and stress changes that affect adjoint rupture. Ability of fault to slip in response to this loading also depends on forward solution through coefficients of adjoint friction law.

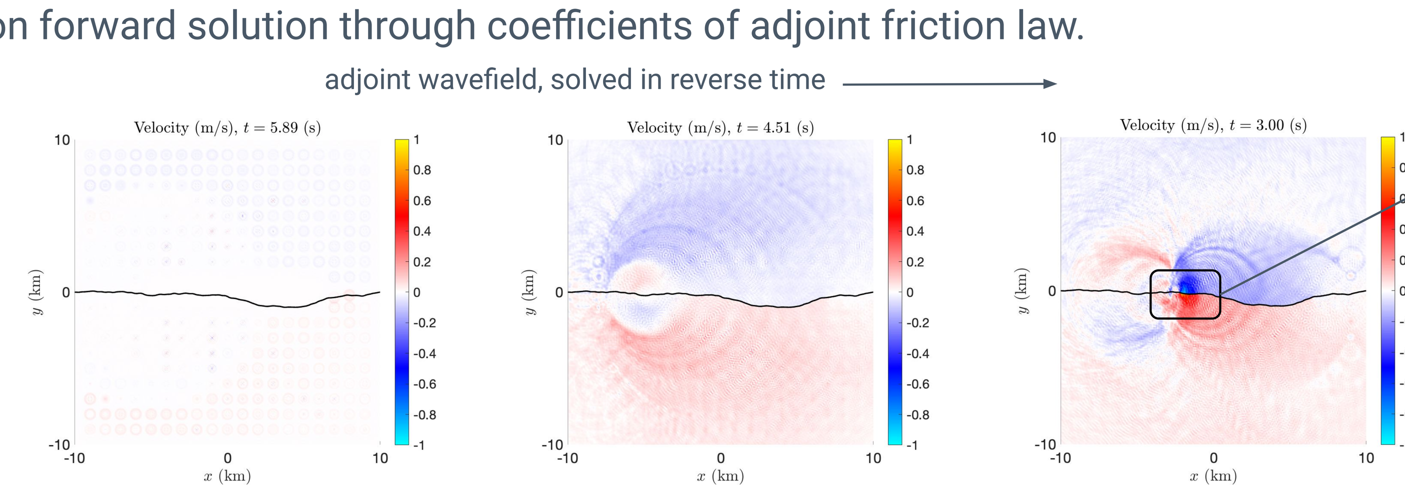

Figure 4: Adjoint wavefield from iteration 20 of τ_0 inversion.

Discussion and Next Steps

- Application to real earthquakes requires regularization to handle ill-posedness (e.g., from inadequately sampled, spatially aliased wavefield), non-uniqueness, and parameter trade-offs.
- Gradient-accelerated Bayesian methods like Hamiltonian Monte Carlo.
- Extension to slip-weakening friction possible but challenging due to locked-slipping inequality constraint and non-smooth friction (slip) relation.
- General 3D adjoint problem features fault opening/interpenetration and non-collinear shear traction and slip velocity, requiring modification to existing codes, but adjoint problem is well-posed iff forward problem is.
- Theory/method applies also to earthquake cycle modeling, where same adaptive time steps in forward problem can be used in adjoint problem.

Lots of exciting work to do!

preprint →

Inversion for Initial Stress τ_0

Start with initial guess for τ_0 that is too low to allow rupture. Iterations increase τ_0, starting around hypocenter and gradually expanding to full rupture extent. No constraints on τ_0 outside slipped region (except low enough to prevent slip).

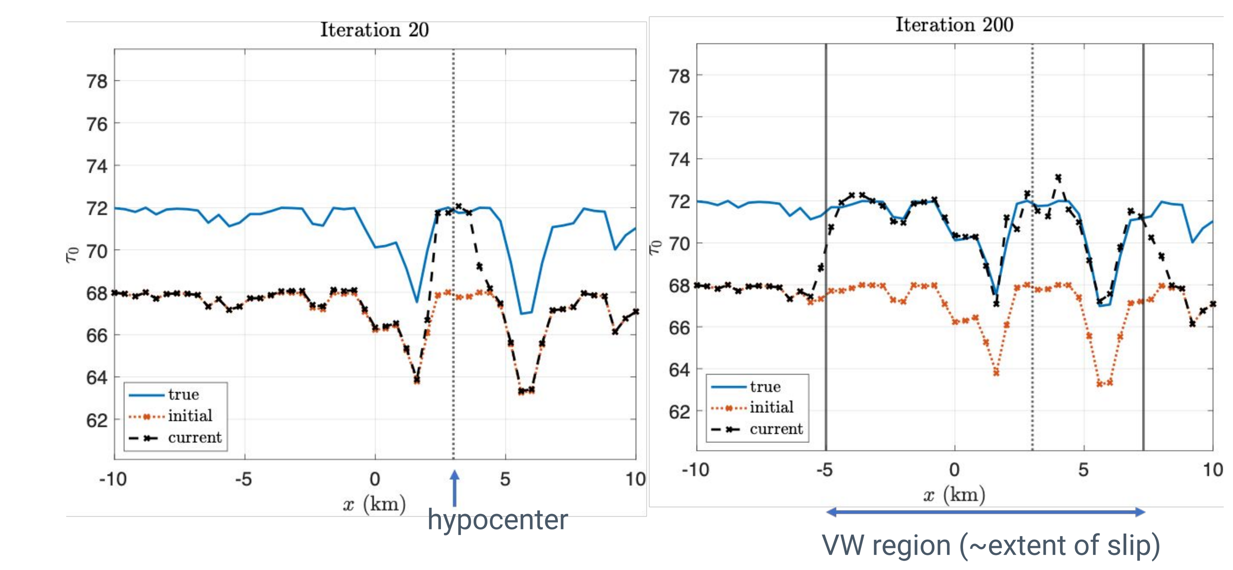

Figure 2: Initial shear stress on fault, after 20 and 200 L-BFGS iterations. True values and initial guess obtained by resolving spatially uniform "regional" stress tensor onto nonplanar fault.

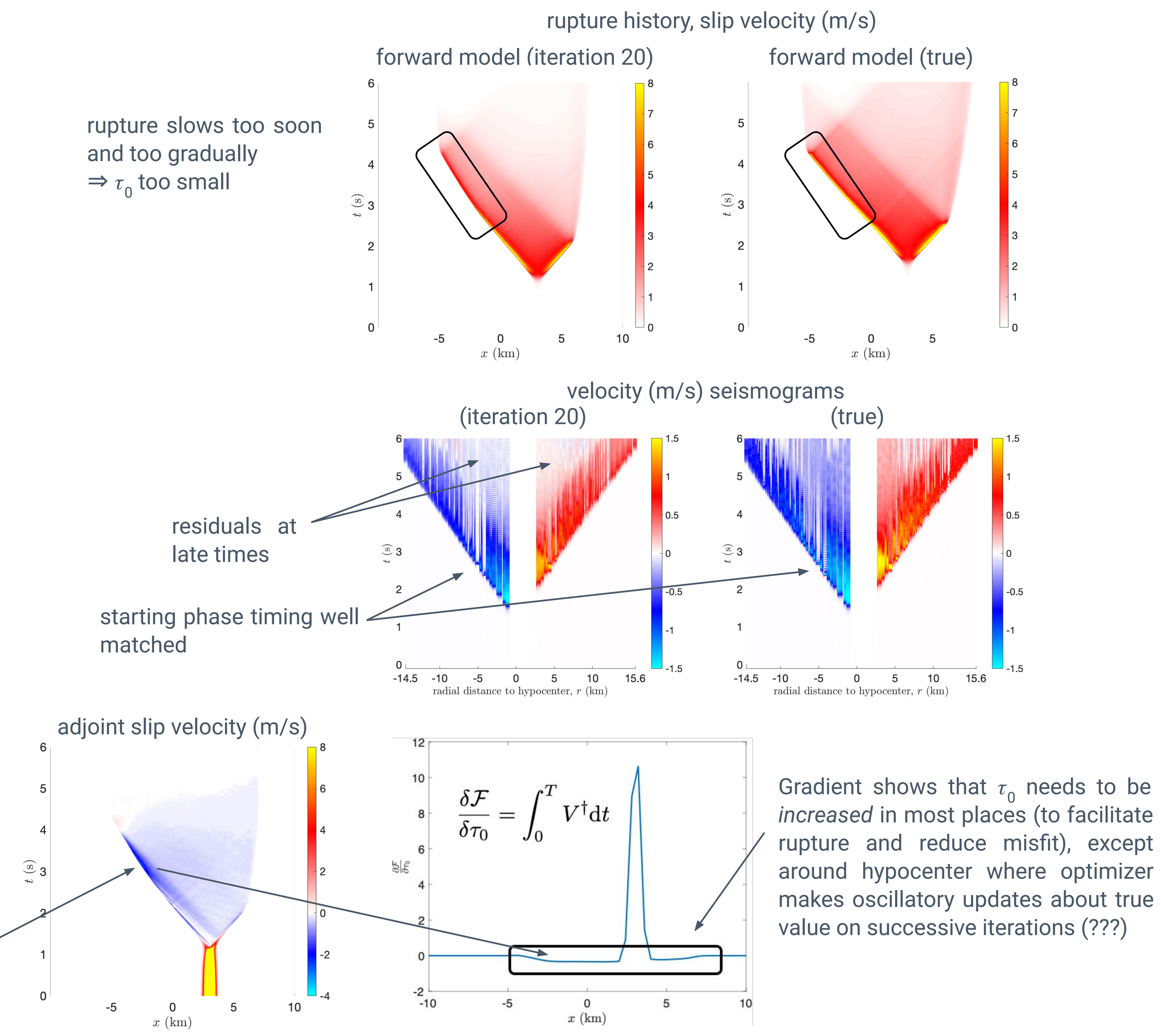

Figure 3: Gradient calculation for iteration 20. Rupture timing is almost correct by this iteration, but rupture velocity and timing of arrest require adjustments.

References

Larmat, C., Montagner, J. P., Fink, M., Capdeville, Y., Tourin, A., & Clévédé, E. (2006). Time-reversal imaging of seismic sources and application to the great Sumatra earthquake. Geophysical Research Letters, 33(19), doi:10.1029/2006GL026336.

Kano, M., Miyazaki, S. I., Ito, K., & Hirahara, K. (2013). An adjoint data assimilation method for optimizing frictional parameters on the afterslip area. Earth, Planets and Space, 65, 1575-1580, doi:10.5047/eps.2013.08.002.

Kano, M., Miyazaki, S. I., Ishikawa, Y., Hiyoshi, Y., Ito, K., & Hirahara, K. (2015). Real data assimilation for optimization of frictional parameters and prediction of afterslip in the 2003 Tokachi-oki earthquake inferred from slip velocity by an adjoint method. Geophysical Journal International, 203(1), 646-663, doi:10.1093/gji/ggv289.

Kiser, E., & Ishii, M. (2017). Back-projection imaging of earthquakes. Annual Review of Earth and Planetary Sciences, 45(1), 271-299, doi: 10.1146/annurev-earth-063016-015801.

Stiernström, V., M. Almquist, E. M. Dunham, Adjoint-based inversion for stress and frictional parameters in earthquake modeling, in review, doi:10.48550/arXiv.2310.12279.

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