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Metric-like spaces, partial metric spaces and fixed points

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Abstract
By a metric-like space, as a generalization of a partial metric space, we mean a pair (X,σ), where X is a nonempty set and $\sigma : X \times X \to \mathbb{R}$ satisfies all of the conditions of a metric except that $\sigma(x,x)$ may be positive for $x \in X$. In this paper, we initiate the fixed point theory in metric-like spaces. As an application, we derive some new fixed point results in partial metric spaces. Our results unify and generalize some well-known results in the literature.

MSC: 47H10

Keywords: fixed point; metric-like space; partial metric space

1 Introduction and preliminaries

There exist many generalizations of the concept of metric spaces in the literature. In particular, Matthews [1] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. After that, fixed point results in partial metric spaces were studied by many other authors [2–11]. In this paper, we first introduce a new generalization of a partial metric space which is called a metric-like space. Then, we give some fixed point results in such spaces. Our fixed point theorems, even in the case of partial metric spaces, generalize and improve some well-known results in the literature.

In the rest of this section, we recall some definitions and facts which will be used throughout the paper.

Definition 1.1 A mapping $p : X \times X \to R^+$, where X is a nonempty set, is said to be a partial metric on X if for any $x, y, z \in X$, the following four conditions hold true:

(P1) $x = y$ if and only if $p(x,x) = p(y,y) = p(x,y)$;
(P2) $p(x,x) \leq p(x,y)$;
(P3) $p(x,y) = p(y,x)$;
(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is then called a *partial metric space*. A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$. A sequence $\{x_n\}$ of elements of X is called p-Cauchy if the limit $\lim_{m,n\to\infty} p(x_m,x_n)$ exists and is finite. The partial metric space (X,p) is called *complete* if for each p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there is some

$x \in X$ such that

$$\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{m,n\to\infty} p(x_m, x_n).$$

A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. For some other examples of partial metric spaces see [1–11] and references therein.

2 Main results

We first introduce the concept of a *metric-like space*.

Definition 2.1 A mapping $\sigma : X \times X \to R^+$, where X is a nonempty set, is said to be metric-like on X if for any $x, y, z \in X$, the following three conditions hold true:

(σ1) $\sigma(x, y) = 0 \Rightarrow x = y$;
(σ2) $\sigma(x, y) = \sigma(y, x)$;
(σ3) $\sigma(x, z) \le \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is then called a *metric-like space*. Then a metric-like on X satisfies all of the conditions of a metric except that $\sigma(x, x)$ may be positive for $x \in X$. Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x, \varepsilon) = \{y \in X : |\sigma(x, y) - \sigma(x, x)| < \varepsilon\}, \quad \text{for all } x \in X \text{ and } \varepsilon > 0.$$

Then a sequence $\{x_n\}$ in the metric-like space (X, σ) converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$.

Let (X, σ) and (Y, τ) be metric-like spaces, and let $f : X \to X$ be a continuous mapping. Then

$$\lim_{n\to\infty} x_n = x \quad \Rightarrow \quad \lim_{n\to\infty} f(x_n) = f(x).$$

A sequence $\{x_n\}_{n=0}^\infty$ of elements of X is called σ-Cauchy if the limit $\lim_{m,n\to\infty} \sigma(x_m, x_n)$ exists and is finite. The metric-like space (X, σ) is called *complete* if for each σ-Cauchy sequence $\{x_n\}_{n=0}^\infty$, there is some $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{m,n\to\infty} \sigma(x_m, x_n).$$

Every partial metric space is a metric-like space. Below we give another example of a metric-like space.

Example 2.2 Let $X = \{0, 1\}$, and let

$$\sigma(x, y) = \begin{cases} 2 & \text{if } x = y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then (X, σ) is a metric-like space, but since $\sigma(0, 0) \nleq \sigma(0, 1)$, then (X, σ) is not a partial metric space.

Remark 2.3 Let $X = \{0,1\}$, let $\sigma(x,y) = 1$ for each $x,y \in X$, and let $x_n = 1$ for each $n \in \mathbb{N}$. Then it is easy to see that $x_n \to 0$ and $x_n \to 1$, and so in metric-like spaces the limit of a convergent sequence is not necessarily unique.

Some slight modifications of the proof of Theorem 2.1 in [12] yield the following result which is a generalization of the well-known fixed point theorem of Ćirić [13].

Theorem 2.4 *Let (X,σ) be a complete metric-like space, and let $T : X \to X$ be a map such that*

$$\sigma(Tx,Ty) \le \psi\bigl(M(x,y)\bigr),$$

for all $x,y \in X$, where

$$M(x,y) = \max\bigl\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\sigma(x,Ty),\sigma(y,Tx),\sigma(x,x),\sigma(y,y)\bigr\},$$

where $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying

$$\psi(t) < t \quad \text{for all } t > 0, \qquad \lim_{s\to t^+} \psi(s) < t \quad \text{for all } t > 0 \quad \text{and} \quad \lim_{t\to\infty}\bigl(t - \psi(t)\bigr) = \infty.$$

Then T has a fixed point.

Proof Let $x_0 \in X$ be arbitrary, and let $x_{n+1} = Tx_n$ for $n \in \{0,1,2,\ldots\}$. Denote

$$O(x_0,n) = \{Tx_0,Tx_1,\ldots,Tx_n\} \quad \text{and} \quad O(x_0) = \{Tx_0,Tx_1,\ldots,Tx_n,\ldots\}.$$

First we show that $O(x_0)$ is a bounded set. We shall show that for each $n \in \mathbb{N}$,

$$\delta_n(x_0) = \operatorname{diam}\bigl(O(x_0,n)\bigr) = \sigma(Tx_0,Tx_k), \tag{1}$$

where $k = k(n) \in \{0,1,2,\ldots,n\}$. Suppose, to the contrary, that there are positive integers $1 \le i(n) = i \le j = j(n)$ such that

$$\delta_n(x_0) = \sigma(Tx_i,Tx_j) > 0.$$

From our assumption, we have

$$\begin{aligned}
&M(x_i,x_j)\\
&\quad= \max\bigl\{\sigma(x_i,x_j),\sigma(x_i,Tx_i),\sigma(x_j,Tx_j),\sigma(x_i,Tx_j),\\
&\qquad \sigma(x_j,Tx_i),\sigma(x_i,x_i),\sigma(x_j,x_j)\bigr\}\\
&\quad= \max\bigl\{\sigma(Tx_{i-1},Tx_{j-1}),\sigma(Tx_{i-1},Tx_i),\sigma(Tx_{j-1},Tx_j),\\
&\qquad \sigma(Tx_{i-1},Tx_j),\sigma(Tx_{j-1},Tx_i),\sigma(Tx_{i-1},Tx_{i-1}),\sigma(Tx_{j-1},Tx_{j-1})\bigr\}\\
&\quad\le \delta_n(x_0).
\end{aligned}$$

Thus, from the above and the contractive condition on T, we have

$$\begin{aligned}\delta_n(x_0) &= \sigma(Tx_i, Tx_j)\\ &\leq \psi\big(\max\big\{\sigma(x_i,x_j), \sigma(x_i,Tx_i), \sigma(x_j,Tx_j),\\ &\quad \sigma(x_i,Tx_j), \sigma(x_j,Tx_i), \sigma(x_i,x_i), \sigma(x_j,x_j)\big\}\big)\\ &\leq \psi\big(\delta_n(x_0)\big) < \delta_n(x_0),\end{aligned}$$

a contradiction. Thus, (1) holds. Since by the triangle inequality,

$$\sigma(Tx_0, Tx_k) \leq \sigma(Tx_0, Tx_1) + \sigma(Tx_1, Tx_k),$$

then from (1)

$$\delta_n(x_0) \leq \sigma(Tx_0, Tx_1) + \sigma(Tx_1, Tx_k). \tag{2}$$

From our assumption on T, we have

$$\sigma(Tx_1, Tx_k) \leq \psi\big(M(x_1,x_k)\big) \leq \psi\big(\delta_n(x_0)\big).$$

Now by (2),

$$\delta_n(x_0) \leq \sigma(Tx_0, Tx_1) + \psi\big(\delta_n(x_0)\big).$$

Hence,

$$(I-\psi)\big(\delta_n(x_0)\big) \leq \sigma(Tx_0, Tx_1), \tag{3}$$

where I is the identity map. Since the sequence $\{\delta_n(x_0)\}$ is nondecreasing, there exists $\lim_{n\to\infty}\delta_n(x_0)$. Suppose that $\lim_{n\to\infty}\delta_n(x_0) = \infty$. Then from (3), we get

$$\lim_{t\to\infty}\big(t-\psi(t)\big) = \lim_{n\to\infty}\big(\delta_n(x_0) - \psi\big(\delta_n(x_0)\big)\big) \leq \sigma(Tx_0, Tx_1) < \infty,$$

a contradiction. Therefore, $\lim_{n\to\infty}\delta_n(x_0) = \delta(x_0) < \infty$, that is,

$$\delta(x_0) = \operatorname{diam}\big(\{Tx_0, Tx_1, \ldots, Tx_n, \ldots\}\big) < \infty. \tag{4}$$

Now we show that $\{x_n\}$ is a σ-Cauchy sequence. Set

$$\delta(x_n) = \operatorname{diam}\big(\{Tx_n, Tx_{n+1}, \ldots\}\big).$$

Since $\delta(x_n) \leq \delta(x_0)$, then by (4) we conclude that $\{\delta(x_n)\}$ is a nonincreasing finite nonnegative number and so it converges to some $\delta \geq 0$. We shall prove that $\delta = 0$. Let $n \in \mathbb{N}$ be arbitrary, and let r, s be any positive integers such that $r,s \geq n+1$. Then $Tx_{r-1}, Tx_{s-1} \in \{Tx_n, Tx_{n+1}, \ldots\}$ and hence we conclude that $M(x_r,x_s) \leq \delta(x_n)$. Then

$$\sigma(Tx_r, Tx_s) \leq \psi\big(M(x_r,x_s)\big) \leq \psi\big(\delta(x_n)\big).$$

Hence, we get

$$\delta(x_{n+1}) = \sup\{\sigma(Tx_r, Tx_s) : r, s \geq n+1\} \leq \psi(\delta(x_n)).$$

Hence, as $\delta \leq \delta(x_n)$ for all $n \geq 0$, $\delta \leq \psi(\delta(x_n))$. Suppose that $\delta > 0$. Then we get

$$\delta \leq \lim_{n\to\infty} \psi(\delta(x_n)) = \lim_{s\to\delta^+} \psi(s) < \delta,$$

a contradiction. Therefore, $\delta = 0$. Thus, we have proved that

$$\lim_{n\to\infty} \operatorname{diam}(\{Tx_n, Tx_{n+1}, \ldots\}) = 0.$$

Hence, from the triangle inequality, we conclude that $\{x_{n+1} = Tx_n\}$ is a σ-Cauchy sequence. By the completeness of X, there is some $u \in X$ such that $\lim_{n\to\infty} Tx_n = u$, that is,

$$\lim_{n\to\infty} \sigma(Tx_n, u) = \sigma(u, u) = \lim_{m,n\to\infty} \sigma(Tx_n, Tx_m) = 0.$$

We show that $Tu = u$. Suppose, by way of contradiction, that $\sigma(Tu, u) > 0$. Then we have

$$\begin{aligned}\sigma(Tu, u) &\leq \sigma(u, Tx_{n+1}) + \sigma(Tu, Tx_{n+1}) \\ &\leq \sigma(u, Tx_{n+1}) + \psi(M(u, x_{n+1})),\end{aligned} \tag{5}$$

where

$$\begin{aligned}&M(u, x_{n+1}) \\ &\quad = \max\{\sigma(u, x_{n+1}), \sigma(u, Tu), \sigma(x_{n+1}, Tx_{n+1}), \\ &\qquad \sigma(u, Tx_{n+1}), \sigma(x_{n+1}, Tu), \sigma(u, u), \sigma(x_{n+1}, x_{n+1})\} \\ &\quad = \max\{\sigma(u, Tx_n), \sigma(u, Tu), \sigma(Tx_n, Tx_{n+1}), \\ &\qquad \sigma(u, Tx_{n+1}), \sigma(Tx_n, Tu), \sigma(u, u), \sigma(Tx_n, Tx_n)\}.\end{aligned}$$

From the triangle inequality, we have

$$|\sigma(Tu, Tx_{n+1}) - \sigma(Tu, u)| \leq \sigma(u, Tx_{n+1}) \to 0 \quad \text{as } n \to \infty.$$

Thus, $\lim_{n\to\infty} \sigma(Tu, Tx_{n+1}) = \sigma(Tu, u)$. Since $\lim_{n\to\infty} \sigma(u, Tx_n) = 0$, $\lim_{n\to\infty} \sigma(Tx_n, Tu) = \sigma(Tu, u)$, for large enough n, we have

$$M(u, x_{n+1}) = \max\{\sigma(u, Tu), \sigma(Tx_n, Tu)\}.$$

If $M(u, x_{n+1}) = \sigma(u, Tu)$, then from (5), we get

$$\sigma(Tu, u) \leq \sigma(u, Tx_{n+1}) + \psi(\sigma(Tu, u)).$$

Letting n tend to infinity, we get

$$0 < \sigma(Tu,u) \leq \psi\big(\sigma(Tu,u)\big) < \sigma(Tu,u),$$

a contradiction. If $M(u,x_{n+1}) = \sigma(Tx_n, Tu)$, then we have

$$\sigma(Tx_n, Tu) = M(u,x_{n+1}) \geq \sigma(Tu,u),$$

and so $\sigma(Tx_n, Tu) \to \sigma(Tu,u)^+$. Then from (5) and our assumptions on ψ, we get $\sigma(Tu,u) < \sigma(Tu,u)$, a contradiction. Thus, $\sigma(Tu,u) = 0$ and so $Tu = u$. □

Example 2.5 Let $\psi_1(t) = kt$ for each $t \in [0,\infty)$, where $k \in [0,1)$, and let $\psi_2(t) = t - \ln(1+t)$ for each $t \in [0,\infty)$. Then ψ_1 and ψ_2 satisfy the conditions of Theorem 2.4.

Now we illustrate our previous result by the following example.

Example 2.6 Let $X = \{0,1,2\}$. Define $\sigma : X \times X \to \mathbb{R}_+$ as follows:

$$\sigma(0,0) = 0, \qquad \sigma(1,1) = 3, \qquad \sigma(2,2) = 1, \qquad \sigma(0,1) = \sigma(1,0) = 7,$$
$$\sigma(0,2) = \sigma(2,0) = 3, \qquad \sigma(1,2) = \sigma(2,1) = 4.$$

Then (X,σ) is a complete metric-like space. Note that σ is not a partial metric on X because

$$\sigma(0,1) \nleq \sigma(0,2) + \sigma(2,1) - \sigma(2,2).$$

Define the map $T : X \to X$ by

$$T0 = 0, \qquad T1 = 2, \quad \text{and} \quad T2 = 0.$$

Then

$$\sigma(Tx, Ty) \leq \frac{3}{4}\sigma(x,y) \leq \frac{3}{4}M(x,y),$$

for each $x,y \in X$. Then all the required hypotheses of Theorem 2.4 are satisfied. Then T has a unique fixed point.

Theorem 2.7 *Let (X,σ) be a complete metric-like space, and let $T : X \to X$ be a map such that*

$$\sigma(Tx, Ty) \leq \sigma(x,y) - \varphi\big(\sigma(x,y)\big),$$

for all $x,y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function such that $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof Let $x_0 \in X$ and define $x_{n+1} = Tx_n$ for $n \geq 0$. Then by our assumption,

$$\sigma(x_{n+1}, x_{n+2}) = \sigma(Tx_n, Tx_{n+1}) \leq \sigma(x_n, x_{n+1}) - \varphi\big(\sigma(x_n, x_{n+1})\big), \tag{6}$$

for each $n \in \mathbb{N}$. Then $\{\sigma(x_n, x_{n+1})\}$ is a nonnegative nonincreasing sequence and hence possesses a limit $r_0 \geq 0$. Since φ is nondecreasing, then from (6), we get

$$\sigma(x_{n+1}, x_{n+2}) \leq \sigma(x_n, x_{n+1}) - \varphi(r_0)$$

for each $n \in \mathbb{N}$. Then $r_0 \leq r_0 - \varphi(r_0)$ and so $r_0 = 0$. Therefore,

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0.$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. Fix $\varepsilon > 0$ and choose N such that

$$\sigma(x_n, x_{n+1}) < \min\left\{\frac{\varepsilon}{2}, \varphi\left(\frac{\varepsilon}{2}\right)\right\} \quad \text{for } n \geq N.$$

We show that if $\sigma(x, x_N) \leq \varepsilon$, then $\sigma(Tx, x_N) \leq \varepsilon$. To show the claim, let us assume first that $\sigma(x, x_N) \leq \frac{\varepsilon}{2}$. Then

$$\begin{aligned}\sigma(Tx, x_N) &\leq \sigma(Tx, Tx_N) + \sigma(Tx_N, x_N)\\ &\leq \sigma(x, x_N) - \varphi\big(\sigma(x, x_N)\big) + \sigma(x_{N+1}, x_N) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Now we assume that $\frac{\varepsilon}{2} < \sigma(x, x_N) \leq \varepsilon$. Then $\varphi(\sigma(x, x_N)) \geq \varphi(\frac{\varepsilon}{2})$. Therefore, from the above, we have

$$\begin{aligned}\sigma(Tx, x_N) &\leq \sigma(x, x_N) - \varphi\big(\sigma(x, x_N)\big) + \sigma(x_{N+1}, x_N)\\ &\leq \sigma(x, x_N) - \varphi\left(\frac{\varepsilon}{2}\right) + \varphi\left(\frac{\varepsilon}{2}\right)\\ &= \sigma(x, x_N) \leq \varepsilon.\end{aligned}$$

Since $\sigma(x_{N+1}, x_N) \leq \varepsilon$, then from the above, we deduce that $\sigma(x_n, x_N) \leq \varepsilon$ for each $n \geq N$. Since $\varepsilon > 0$ is arbitrary, we get $\lim_{m,n\to\infty} \sigma(x_m, x_n) = 0$ and so $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is some $u \in X$ such that $\lim_{n\to\infty} x_n = u$, that is,

$$\lim_{n\to\infty} \sigma(x_n, u) = \sigma(u, u) = \lim_{m,n\to\infty} \sigma(Tx_n, Tx_m) = 0. \tag{7}$$

Since

$$\sigma(x_{n+1}, Tu) = \sigma(Tx_n, Tu) \leq \sigma(x_n, u) - \varphi\big(\sigma(x_n, u)\big) \tag{8}$$

and φ is continuous, then from (7) and (8), we have

$$\lim_{n\to\infty} \sigma(x_n, Tu) = 0. \tag{9}$$

Since

$$\sigma(u, Tu) \leq \sigma(x_n, u) + \sigma(x_n, Tu)$$

then by (7) and (9), we infer that $\sigma(u, Tu) = 0$ and so $Tu = u$. To prove the uniqueness, let v be another fixed point of T, that is, $Tv = v$. Then

$$\sigma(u, v) = \sigma(Tu, Tv) \leq \sigma(u, v) - \varphi\big(\sigma(u, v)\big),$$

which gives $\varphi(\sigma(u, v)) = 0$ and so $u = v$. □

Example 2.8 Let $X = [0, \infty)$ and $\sigma(x, y) = \max\{x, y\}$. Then (X, σ) is a complete metric-like space. Take $\varphi(t) = \frac{t}{1+t}$ for $t \in [0, \infty)$. Let $Tx = \frac{x^2}{1+x}$ for each $x \in X$. Take $x, y \in X$, without loss of generality, we may assume that $y \leq x$. Then

$$\sigma(Tx, Ty) = Tx = x - \varphi(x) = \sigma(x, y) - \varphi\big(\sigma(x, y)\big).$$

Then T satisfies the hypothesis of Theorem 2.7 and so T has a fixed point ($x = 0$ is the unique fixed point of T). Now since $\lim_{t\to\infty} \varphi(t) = 1 < \infty$, we cannot invoke Theorem 2.1 of [9] to show the existence of fixed point of T.

The following corollary improves Theorem 1 in [2].

Corollary 2.9 *Let (X, p) be a complete partial metric space, and let $T : X \to X$ be a map such that*

$$p(Tx, Ty) \leq \psi\big(\max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx), p(x, x), p(y, y)\}\big),$$

for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying

$$\psi(t) < t \quad \text{for all } t > 0, \qquad \lim_{s\to t^+} \psi(s) < t \quad \text{for all } t > 0 \quad \text{and} \quad \lim_{t\to\infty} \big(t - \psi(t)\big) = \infty.$$

Then T has a unique fixed point.

Proof The existence of a fixed point follows immediately from Theorem 2.4. To prove the uniqueness, let us suppose that x and y are fixed points of T. Then from our assumption on T, we get

$$p(x, x) = p(Tx, Ty) \leq \psi\big(\max\{p(x, y), p(x, x), p(y, y)\}\big) = \psi\big(p(x, y)\big).$$

Thus, $p(x, y) = 0$ and $x = y$. □

The following corollary improves Corollary 1 and Theorem 2 in [2] and the main fixed point result of Matthews [1].

Corollary 2.10 *Let (X,p) be a complete partial metric space, and let $T : X \to X$ be a map such that*

$$p(Tx, Ty) \le \lambda \max\{p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx), p(x,x), p(y,y)\},$$

for all $x, y \in X$, where $\lambda \in [0,1)$. Then T has a unique fixed point.

Proof Let $\psi(t) = \lambda t$ for each $t \in [0,\infty)$ and apply Corollary 2.9. □

Now, we present the following version of Rakotch's fixed point theorem [14] in metric-like spaces.

Theorem 2.11 *Let (X,σ) be a complete metric-like space, and let $T : X \to X$ be a mapping satisfying*

$$\sigma(Tx, Ty) \le \alpha\big(\sigma(x,y)\big)\sigma(x,y),$$

for each $x, y \in X$ with $x \neq y$, where $\alpha : [0,\infty) \to [0,1)$ is nonincreasing. Then T has a unique fixed point.

Proof Fix $x \in X$ and let $x_n = T^n x$ for each $n \in \mathbb{N}$. Following the lines of the proof of the Theorem 3.6 in [15], we get that

$$\lim_{m,n\to\infty} \sigma\big(T^n x, T^m x\big) = 0,$$

and so $\{x_n\}$ is a σ-Cauchy sequence. Since (X,σ) is complete, then there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} \sigma\big(T^n x, x_0\big) = \sigma(x_0,x_0) = \lim_{m,n\to\infty} \sigma\big(T^n x, T^m x\big) = 0.$$

From our assumption, we have

$$\sigma\big(T^n x, Tx_0\big) \le \alpha\big(\sigma\big(T^{n-1}x, x_0\big)\big)\sigma\big(T^{n-1}x, x_0\big),$$

which yields $\lim_{n\to\infty} \sigma(T^n x, Tx_0) = 0$. Also, notice that $\sigma(Tx_0, Tx_0) \le \sigma(x_0,x_0) = 0$ and hence $\sigma(Tx_0, Tx_0) = 0$. Thus,

$$\lim_{n\to\infty} \sigma\big(T^n x, Tx_0\big) = \sigma(Tx_0, Tx_0) = \lim_{m,n\to\infty} \sigma\big(T^n x, T^m x\big) = 0.$$

By the triangle inequality, we have

$$\sigma(x_0, Tx_0) \le \sigma\big(T^n x, x_0\big) + \sigma\big(T^n x, Tx_0\big) \to 0 \quad \text{as } n \to \infty,$$

and so $\sigma(x_0, Tx_0) = 0$, that is, $Tx_0 = x_0$. The uniqueness easily follows from our contractive condition on T. □

The following corollary is another new extension of Matthews's fixed point result [1].

Corollary 2.12 *Let (X,p) be a complete partial metric space, and let $T : X \to X$ be a mapping satisfying*

$$p(Tx, Ty) \leq \alpha\big(p(x,y)\big)p(x,y)$$

for each $x, y \in X$ with $x \neq y$, where $\alpha : [0,\infty) \to [0,1)$ is nonincreasing. Then T has a unique fixed point.

Competing interests
The authors declare that they have no competing interests.

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