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New Lyapunov function and extra information on membership functions for improving stability conditions of TS systems ⋆

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Abstract: This paper presents an improved stability condition for Takagi-Sugeno (TS) fuzzy systems based on a new fuzzy Lyapunov function. This new fuzzy Lyapunov function aggregates more information with respect to the membership function variation (time-derivative) via an augmented state vector. This new fuzzy Lyapunov function is not just parametrized by the membership functions variations but it is also based on a polynomial combination of them. To derive LMI based stability condition using the proposed new fuzzy Lyapunov function two ideas are invoked: *i*) the inclusion of the membership function time-derivative information using a finite number of vectors and, *ii*) the use of the so-called null term. Two numerical examples are presented to illustrate the improvements.

Keywords: Takagi-Sugeno fuzzy systems, Lyapunov theory, Nonlinearity, LMIs.

1. INTRODUCTION

Over the last two decades, control design and stability issues for Takagi-Sugeno (TS) fuzzy systems (Takagi and Sugeno, 1985) have been widely handled as convex optimization problems that describe Lyapunov stability constraints, formulated either as LMIs (Tanaka et al., 1996) or as SOS problems (Tanaka et al., 2009). Quadratic stability has been of paramount importance in this field since the polytopic nature of TS systems can be easily explored to get LMI conditions, whereas some time-varying features are left aside. Recently, Montagner et al. (2009) presented quadratic stability conditions that are sufficient with a certificate of convergence to necessity, as a parameter increase. However, as pointed out by Sala (2009), a major source of conservativeness in the Lyapunov approach is the unsuitable choice of the candidate function. This motivated the appearance of alternative types of candidates functions: piecewise (Johansson et al., 1999) and fuzzy (Jadbabaie, 1999) combination of quadratic forms; integral over conservative fields (Rhee and Won, 2006); k-sample variation approach (Kruszewski et al., 2008); polynomial dependency on the states (Tanaka et al., 2009). Among those, the fuzzy Lyapunov function is quite interesting because carries into the LMI formulation information regarding the membership functions which can better characterize the time-varying nature of TS systems. As shown in Mozelli et al. (2009b) and Souza et al. (2009) this information can potentially decrease conservativeness in stability analysis.

As far as the authors' concern, this paper presents three novelties for stability analysis of TS systems. First, a new Lyapunov function for stability analysis of TS systems is proposed. An augmented state vector is considered which brings more information about the membership functions variation, following the ideas proposed by Ebihara et al. (2005), providing a fuzzy Lyapunov function with polynomial dependency on the membership functions. Second, the systematic rationale proposed by Mozelli et al. (2009a) is improved to cope with this new function and to allow that more information regarding derivatives is available in LMI conditions. Finally, to include the time-derivative of the membership functions in a non-conservative way the strategy discussed in Geromel and Colaneri (2006) and Chesi et al. (2007) is adopted. Those three steps combined lead to a new LMI stability condition. Numerical examples are performed to illustrate the main features of the proposed condition and to compare it with recent results reported in the literature

Notation: Uppercase and lowercase indicate matrices and vectors, respectively; superscript ($'$) is for transpose; $M > 0$ (< 0) means that M positive (negative) definite; $\bullet$ denote the transposed term in symmetric matrix; $\mathrm{He}(M)$ indicates $M + M'$; $M_{(i,j)}$ stands for the element (i, j) in matrix M; $\mathrm{co}\{\cdot\}$ denotes the convex hull; the subsets $\{1, 2, \ldots, r\} \subset \mathbb{N}^*$, $\{1, 2, \ldots, 5\} \subset \mathbb{N}^*$ e $\{1, 2, \ldots, m\} \subset \mathbb{N}^*$ are denoted by $\mathcal{R}$, $\mathcal{F}$ and $\mathcal{M}$, respectively.

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2. PRELIMINARIES

In this paper the stability analysis of Takagi-Sugeno (TS) fuzzy systems is investigated, which are described by:

$$\dot{x} = A(h)x, \quad A(h) \triangleq \sum_{i=1}^{r} h_i A_i, \tag{1}$$

where h_i are the normalized membership functions and the model is obtained trough a center-average defuzzifier, product inference and singleton fuzzifier (Tanaka et al., 1996). The membership functions satisfy the following properties

$$h_i \in [0,1], \ \sum_{i=1}^{r} h_i = 1, \ \sum_{i=1}^{r} \dot{h}_i = 0. \tag{2}$$

2.1 On the time-derivative of membership functions

Usually when considering fuzzy Lyapunov functions, including the time-derivative of the membership functions it is done by taking its upper bounds, as in Tanaka et al. (2003). However, since the time-derivative of the membership functions are related by the equation

$$\sum_{i=1}^{r} \dot{h}_i = 0, \tag{3}$$

this approach tends to be conservative, because all derivatives assuming its maximum is an impossible scenario. As noticed in Geromel and Colaneri (2006) and in Chesi et al. (2007) the time-derivative of the membership functions are confined to the polytope

$$\begin{aligned} \Omega &\triangleq \text{co}\{v^1, v^2, \ldots, v^m\} \\ &= \{v^j \in \mathbb{R}^r | -\phi_k \leq v_k^j \leq \phi_k, c'v^j = 0\}, \end{aligned} \tag{4}$$

with $c' = [1 \ 1 \ \ldots \ 1] \in \mathbb{R}^r$, k is the kth coordinate of v_k^j and $|\dot{h}_k| \leq \phi_k$. Thus it is possible to include the time-derivative information using a finite number of vectors as in Geromel and Colaneri (2006). To accumulate this set of vectors the following matrix is employed

$$V_\Omega \triangleq \begin{bmatrix} v^1 \ v^2 \ \cdots \begin{pmatrix} v_1^j \\ v_2^j \\ \vdots \\ v_r^j \end{pmatrix} \cdots \ v^m \end{bmatrix}. \tag{5}$$

Remark 1. The price to be paid by considering the time-derivative of the membership functions as in Geromel and Colaneri (2006) is an exponential growth in the number of constraints, since for $r = 2 \rightarrow V_\Omega \in \mathbb{R}^{2\times 2}$, $r = 3 \rightarrow V_\Omega \in \mathbb{R}^{3\times 6}$, and so on.

2.2 Lyapunov Function

The new fuzzy Lyapunov function proposed is:

$$V(x,\dot{x},h) \triangleq [x' \ \dot{x}'] \sum_{i=1}^{r} h_i \begin{bmatrix} P_{1i} & P'_{2i} \\ P_{2i} & P_{3i} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \tag{6}$$

where P_{2i} are any $n\times n$ matrices and P_{1i}, P_{3i} are symmetric of appropriated order. As proposed in Ebihara et al. (2005) this function takes into account the time-derivative of the state by using an augmented vector. In the context of TS systems this function is parametrized by a polynomial combination of the membership functions. It is easily checked replacing $\dot{x}$ by $A(h)x$ in (6) and rewriting it in a quadratic form:

$$\begin{aligned} V(x,\dot{x},h) = x' \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i h_j h_k \times \\ \left[P_{1i} + A'_j P_{2i} + P'_{2i} A_j + A'_j P_{3i} A_k\right] x. \end{aligned} \tag{7}$$

Notice that third degree monomials appear combining the matrices of this function, instead of an affine combination as occurs in Jadbabaie (1999); Tanaka et al. (2003) and Mozelli et al. (2009b).

The time-derivative of this function is given in a compact form by

$$\begin{aligned} \dot{V}(x,\dot{x},h) = [x' \ \dot{x}' \ \ddot{x}'] \sum_{i=1}^{r} h_i \begin{bmatrix} 0 & P_{1i} & P'_{2i} \\ P_{1i} & P_{2i} + P'_{2i} & P_{3i} \\ P_{2i} & P_{3i} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} \\ + [x' \ \dot{x}' \ \ddot{x}'] \sum_{i=1}^{r} \dot{h}_i \begin{bmatrix} P_{1i} & P'_{2i} & 0 \\ P_{2i} & P_{3i} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}. \end{aligned} \tag{8}$$

Notice that the information concerning the time-derivative of the membership functions appears twice. Explicitly, as occurs in Jadbabaie (1999); Tanaka et al. (2003) and Mozelli et al. (2009b) and also implicitly due to the presence of $\ddot{x}$ since

$$\ddot{x} = A(h)\dot{x} + \dot{A}(h)x = \sum_{i=1}^{r} h_i A_i \dot{x} + \sum_{i=1}^{r} \dot{h}_i A_i x. \tag{9}$$

2.3 Null Terms

In Mozelli et al. (2009a) the following null term was successfully used to reduce simultaneously conservatism and complexity of LMIs in stability analysis:

$$2\left[x'M_1 + \dot{x}'M_2\right] \times [\dot{x} - A(h)x] = 0. \tag{10}$$

This approach is also suitable for the new fuzzy Lyapunov function as well and it is shown in Mozelli et al. (2010) that it is equivalent with Finsler's lemma and the descriptor approach for stability analysis purposes. Nevertheless, in this paper this strategy is further improved and two new null terms are coined

$$2\left[x'M_3 + \dot{x}'M_4 + \ddot{x}'M_5\right] \times \left[\ddot{x} - \dot{A}(h)x - A(h)\dot{x}\right] = 0 \tag{11}$$

and

$$
\begin{aligned}
0 &= 2\left[x'\sum_{i=1}^{r}\dot{h}_iX_i + \dot{x}'\sum_{i=1}^{r}\dot{h}_iY_i\right]\times[\dot{x}-A(h)x]\\
&= 2\left[x'\dot{X}(h)+\dot{x}'\dot{Y}(h)\right]\times[\dot{x}-A(h)x]\,. \qquad (12)
\end{aligned}
$$

3. STABILITY ANALYSIS CONDITION

Considering the previous discussion, in this section a new stability analysis condition for TS fuzzy systems based on LMIs is presented.

Theorem 2. Consider $\dot{h}_i \in \mathcal{C}^1$, $|\dot{h}|_i \le \phi_i$. The TS fuzzy system (1) is stable if there exist symmetric matrices $P_{1i}, P_{3i} \in \mathbb{R}^{n\times n}$, any matrices $P_{2i}, X_i, Y_i, M_p \in \mathbb{R}^{n\times n}$, $i \in \mathcal{R}, p \in \mathcal{F}$ such that the following LMIs are satisfied

$$
\begin{bmatrix} P_{1i} & \bullet \\ P_{2i} & P_{3i}\end{bmatrix} > 0,_{\;i\in\mathcal{R}} \qquad (13)
$$

$$
\Theta_i + P_i + \sum_{k=1}^{r} V_{\Omega_{(k,j)}}\left(\breve{\Theta}_{(k,i)} + \breve{P}_k\right) < 0,_{\;i\in\mathcal{R}, j\in\mathcal{M}} \qquad (14)
$$

where

$$
\Theta_i \triangleq \qquad (15)
$$

$$
\begin{bmatrix} -\mathrm{He}(M_1A_i) & \bullet & \bullet \\ -A_i'M_3 - M_2A_i + M_1' & \mathrm{He}(M_2) - \mathrm{He}(M_4A_i) & \bullet \\ M_3' & M_4' - M_5A_i & M_5 + M_5' \end{bmatrix},
$$

$$
\breve{\Theta}_{(k,i)} \triangleq \begin{bmatrix} -M_3A_k - A_k'M_3' - X_kA_i - A_i'X_k' & \bullet & \bullet \\ -M_4A_k - Y_kA_i + X_k' & Y_k + Y_k' & \bullet \\ -M_5A_k & 0 & 0 \end{bmatrix}, \qquad (16)
$$

$$
P_i \triangleq \begin{bmatrix} 0 & \bullet & \bullet \\ P_{1i} & P_{2i} + P_{2i}' & \bullet \\ P_{2i} & P_{3i} & 0 \end{bmatrix},\ \breve{P}_i \triangleq \begin{bmatrix} P_{1i} & \bullet & \bullet \\ P_{2i} & P_{3i} & \bullet \\ 0 & 0 & 0 \end{bmatrix}. \qquad (17)
$$

Proof.

Due to the convexity in h_i, LMI (13) is sufficient to guarantee the positiveness of the candidate Lyapunov function (6). Then the time-derivative of the candidate function (8) is taken and the null terms in (10)-(12) are added with it. Since the time-derivatives of the membership functions lie on the polytope given in (4), it suffices to look its vertices acumulated in matrix V_Ω. Thus the terms multiplied by $\sum_{k=1}^{r}\dot{h}_k$ are replaced by j terms with $\sum_{k=1}^{r}V_{\Omega_{(k,j)}}$, where j is the number of columns in V_Ω, thus $j \in \mathcal{M}$. Once again, relying on the convexity in h_i, the LMI (14) is sufficient, completing the proof.

4. EXAMPLES

In this section numerical examples are explored to illustrate the new stability conditions.

4.1 Example 1

Consider the following nonlinear system (Tanaka et al., 2003, 2009):

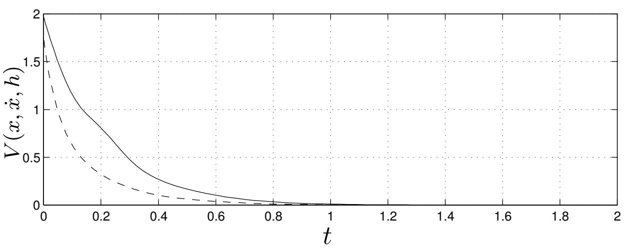

Fig. 1. Time evolution of the Lyapunov function for some initial conditions: $x = [-1\ 1]$ (dashed line) and $x = [1\ 1]$ (solid line).

$$
\begin{aligned}
\dot{x}_1 &= -\left(\frac{7}{2} + \frac{3}{2}\sin(x_1)\right)x_1 - 4x_2,\\
\dot{x}_2 &= \left(\frac{19}{2} - \frac{21}{2}\sin(x_1)\right)x_1 - 2x_2.
\end{aligned}
$$

The following TS model can be obtained by sector nonlinearity approach (Ohtake et al., 2001) for $|x_1| < \pi/2$:

$$
\begin{aligned}
A_1 &= \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix},\quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix},\\
h_1 &= \frac{1+\sin(x_1)}{2},\quad h_2 = \frac{1-\sin(x_1)}{2},\\
\dot{h}_1 &= -\cos(x_1)\left(\frac{7}{4}x_1 + 2x_2\right) - \frac{3}{4}x_1\sin(x_1)\cos(x_1).
\end{aligned}
$$

Even though the analysis of the phase plane shows that this system is stable no quadratic function can prove stability. Using a grid with resolution 0.05 the maximum value of $\dot{h}_1$ is computed as 3.45 and the minimum as -3.62. Using $\phi_i = 3.7$, Theorem 2 provides the following matrices for $i = 1, 2$:

$$
P_1 = \left[\begin{array}{cc|cc} 0.6737 & 0.3842 & 0.1318 & -0.0128 \\ 0.3842 & 0.8019 & 0.1548 & -0.0058 \\ \hline 0.1318 & 0.1548 & 0.0592 & -0.0034 \\ -0.0128 & -0.0058 & -0.0034 & 0.0032 \end{array}\right],
$$

$$
P_2 = \left[\begin{array}{cc|cc} 1.2728 & 0.2731 & 0.1306 & -0.0050 \\ 0.2731 & 0.5565 & 0.1094 & -0.0041 \\ \hline 0.1306 & 0.1094 & 0.0452 & -0.0028 \\ -0.0050 & -0.0041 & -0.0028 & 0.0018 \end{array}\right].
$$

Figure 1 shows the evolution of the Lyapunov function for different initial conditions $x = [-1\ 1]$ (dashed line) and $x = [1\ 1]$ (solid line). Notice how its decay is monotonic. Figure 2 depicts the level curves of the Lyapunov function and the system trajectories for those initial conditions considered in Figure 1. Notice that the level curves are convex but are not ellipses, as occurs in quadratic stability, and the trajectories go from one contour to another innermost. Finally, Figure 3 shows the phase portrait illustrating that this function is indeed a Lyapunov for this system.

4.2 Example 2

Consider the nonlinear system from (Tanaka et al., 2009)

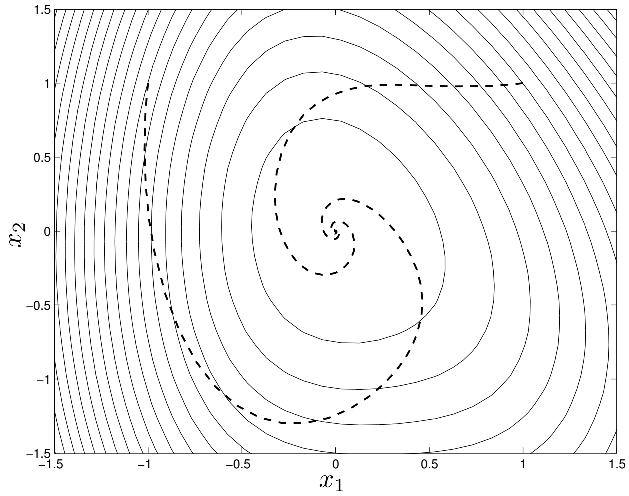

Fig. 2. Some level curves of the Lyapunov function and two trajectories of the system (dashed lines).

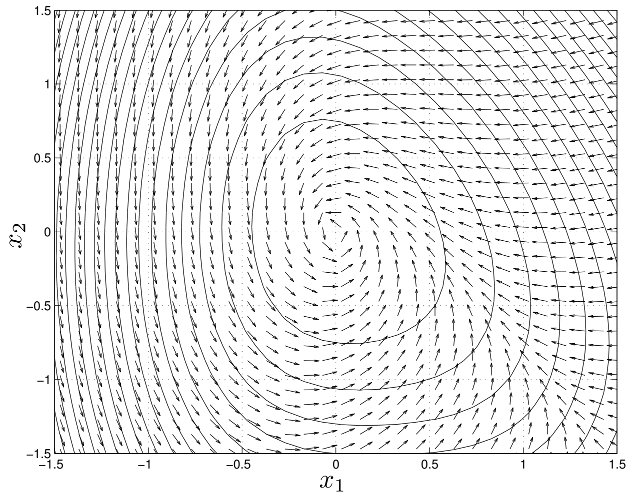

Fig. 3. Some level curves of the Lyapunov function and the phase portrait.

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -2x_1 - x_2 - f(t)x_1. \quad (18)$$

where $f(t) \in [0, k]$ is at least $\mathcal{C}^1$.

The dynamics in (18) can be exactly described by the TS model (1) with 2 rules assuming that

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix},$$

and with the following membership functions

$$h_1 = (k - f(t))/k, \quad h_2 = f(t)/k.$$

Several values of λ (a single upper bound for the absolute value of all time-derivatives of the membership functions) were considered. In each case, the maximum value of k for which the system is stable using Theorem 2 was verified. For further reference Theorem 2 will be indicated by T2. The maximum k was also calculated using different approaches: quadratic stability, using Theorem 2 in Montagner et al. (2009) with $g = d = 5$ [MOP09]; piecewise function, using Theorem 3.1 in Xie et al. (1997) [XFS97]; fuzzy Lyapunov approach that depends on the time-derivative of membership functions, using Theorem 1 in Mozelli et al. (2009a) [MPA:09]; another fuzzy Lyapunov approach that is membership derivative independent, using Theorem 3 in Rhee and Won (2006) [RW06]; a tenth-order polynomial function using Theorem 1 in Tanaka et al. (2009) [TYOW09].

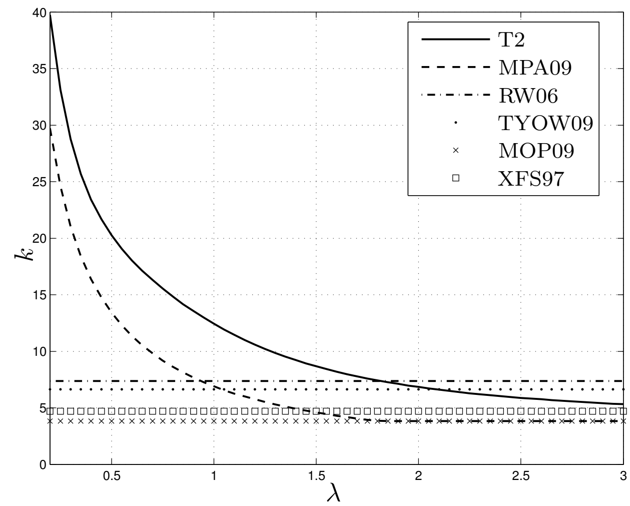

Fig. 4. Maximum values of k for several bounds on the time-derivatives of the membership functions

Figure 4 depicts the values of parameter k for different values λ bounding the time-derivatives of the membership functions. Table 1 sumarizes the extremes in Figure 4. When the approaches that depend on the time-derivative of the membership functions are compared with each other (T2 against MPA09) there is a clear superiority for the proposed approach. Notice that MPA09 converges to the results of quadratic stability (MOP09) as the value of k increases. This result is expect because the fuzzy Lyapunov function includes the quadratic Lyapunov function as special case, see the discussion in Mozelli et al. (2009b). Notice however that the proposed approach outperforms the quadratic stability for all values of λ. Even for values of λ larger than those shown in Figure 4 this is verified. The same conclusions can be made comparing the proposed approach with XFS97. It emphasizes the relevance of the augmented Lyapunov function proposed.

When compared with the approaches that do not depend on the time-derivative of the membership functions the proposed approach is also very competitive. For values of $\lambda > 2$, T2, RW06, and TYOW09 differ at most by 30%, see Table 1. However as λ decreases T2 can be 500% better. This is a great improvement, considering for instance that MPA09 differs at most by 400% from the time-derivative independent approaches.

5. CONCLUSION

In this paper an improved stability analysis condition for Takagi-Sugeno fuzzy systems based on a new fuzzy Lyapunov function is presented. This Lyapunov function includes more information related to the membership function variation in a new fashion. The LMI test conditions obtained from this Lyapunov function are derived using extra null terms and including the time-derivative information using a finite number of vectors. The numerical examples illustrates the potential of this new approach.

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Table 1. Maximum values of the parameter k

Methods	$\lambda = 3$	$\lambda = 0.2$
MOP09	3.82	3.82
XFS97	4.70	4.70
TYOW	6.64	6.64
RW06	7.37	7.37
MPA09	3.82	29.7
T2	5.33	39.7

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