ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by **Stanley Rabinowitz**

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-766 ***Proposed by R. André-Jeannin, Longwy, France***

Let n be an even positive integer such that $L_n \equiv 2 \pmod{p}$, where p is an odd prime. Prove that

$$L_{n+1} \equiv 1 \pmod{p}.$$

B-767 ***Proposed by James L. Hein, Portland State University, Portland, OR***

Consider the following two mutual recurrences:

$$G_1 = 1; \quad G_n = F_{n+1}G_{n-1} + F_n H_{n-2}, \; n \ge 2$$

and

$$H_0 = 0; \quad H_n = F_{n+1}G_n + F_n H_{n-1}, \; n \ge 1.$$

Prove that H_{n-1} and G_n are consecutive Fibonacci numbers for all $n \ge 1$.

B-768 ***Proposed by Juan Pla, Paris, France***

Let u_n, v_n, and w_n be sequences defined by $u_1 = 1/2$, $v_1 = \sqrt{2}$, and $w_1 = (1/2)\sqrt{3}$; $u_{n+1} = u_n^2 + v_n^2 - w_n^2$, $v_{n+1} = 2u_n v_n$, and $w_{n+1} = 2u_n w_n$. Express u_n, v_n, and w_n in terms of Fibonacci and/or Lucas numbers.

B-769 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy***

Solve the recurrence

$$a_{n+1} = 5a_n^3 - 3a_n, \; n \ge 0$$

with initial condition $a_0 = 1$.

B-770 ***Proposed by Andrew Cusumano, Great Neck, NY***

Let $U(x)$ denote the unit's digit of x when written in base 10. Let H_n be any generalized Fibonacci sequence that satisfies the recurrence $H_n = H_{n-1} + H_{n-2}$. Prove that, for all n,

$$\begin{aligned} U(H_n + H_{n+4}) &= U(H_{n+47}), & U(H_n + H_{n+17}) &= U(H_{n+34}), \\ U(H_n + H_{n+5}) &= U(H_{n+10}), & U(H_n + H_{n+19}) &= U(H_{n+41}), \\ U(H_n + H_{n+7}) &= U(H_{n+53}), & U(H_n + H_{n+20}) &= U(H_{n+55}), \\ U(H_n + H_{n+8}) &= U(H_{n+19}), & U(H_n + H_{n+23}) &= U(H_{n+37}), \\ U(H_n + H_{n+11}) &= U(H_{n+49}), & U(H_n + H_{n+25}) &= U(H_{n+50}), \\ U(H_n + H_{n+13}) &= U(H_{n+26}), & U(H_n + H_{n+28}) &= U(H_{n+59}), \\ U(H_n + H_{n+16}) &= U(H_{n+23}), & U(H_n + H_{n+29}) &= U(H_{n+58}). \end{aligned}$$

B-771 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)F_n}{2^n n(n+1)} = \ln 4$.

SOLUTIONS

Square Root of a Recurrence

B-735 ***Proposed by Curtis Cooper & Robert E. Kennedy, Central Missouri State University, Warrensburg, MO***
(Vol. 31, no. 1, February 1993)

Let the sequence (y_n) be defined by the recurrence

$$\begin{aligned} y_{n+1} = {} & 8y_n + 22y_{n-1} - 190y_{n-2} + 28y_{n-3} + 987y_{n-4} - 700y_{n-5} - 1652y_{n-6} + 1652y_{n-7} \\ & + 700y_{n-8} - 987y_{n-9} - 28y_{n-10} + 190y_{n-11} - 22y_{n-12} - 8y_{n-13} + y_{n-14} \end{aligned}$$

for $n \ge 15$ with initial conditions given by the table on page 185 of the May 1994 issue of this *Quarterly*. Prove that y_n is a perfect square for all positive integers n.

Solution 2 by Leonard A. G. Dresel, Reading, England, and the Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark (independently)

"Though this be madness, yet there is method in it."
—Shakespeare, *Hamlet*, Act 2, Scene 2

Let $x_n = \sqrt{y_n}$. We find, for $n = 1, 2, \ldots, 15$, that y_n is a perfect square, and $x_n = 1, 1, 5, 11, 36, 95, 281, 781, 2245, 6336, 18061, 51205, 145601, \ldots$. We.will show that x_n satisfies the recurrence

$$x_{n+1} = x_n + 5x_{n-1} + x_{n-2} - x_{n-3}. \tag{1}$$

To do this, we consider the sequence (x_n) defined by recurrence (1) with initial conditions $x_1 = x_2 = 1$, $x_3 = 5$, and $x_4 = 11$. Then $x_n^2 = y_n$ for $1 \le n \le 15$, and we need to show that $x_n^2 = y_n$ for all n.

The characteristic polynomial for recurrence (1) is

$$x^4 - x^3 - 5x^2 - x + 1. \tag{2}$$

Writing this as $(x^2 - px + 1)(x^2 - qx + 1)$, we find that

$$x^4 - (p+q)x^3 + (pq+2)x^2 - (p+q)x + 1 = x^4 - x^3 - 5x^2 - x + 1.$$

This is an identity in x if $p+q=1$ and $pq=-7$, i.e., if p and q are the roots of the equation $w^2-w-7=0$. We thus find $p=(1+\sqrt{29})/2$ and $q=(1-\sqrt{29})/2$. The zeros of polynomial (2) are therefore a, b, a^{-1}, and b^{-1}, where

$$a+a^{-1}=\frac{1+\sqrt{29}}{2},\ b+b^{-1}=\frac{1-\sqrt{29}}{2},\ a^2+a^{-2}=\frac{11+\sqrt{29}}{2},\ \text{and } b^2+b^{-2}=\frac{11-\sqrt{29}}{2}.$$

The Binet form for recurrence (1) is

$$x_n=\frac{a^n+a^{-n}-b^n-b^{-n}}{\sqrt{29}}.$$

The recurrence whose elements are x_n^2 will have the form

$$y_n=\frac{(a^n+a^{-n}-b^n-b^{-n})^2}{29} \tag{3}$$

and we need to show that y_n satisfies the recurrence given in the problem statement, i.e., that it satisfies the characteristic polynomial

$$\begin{aligned}y^{15}-8y^{14}-22y^{13}+190y^{12}-28y^{11}-987y^{10}+700y^9+1652y^8\\ -1652y^7-700y^6+987y^5+28y^4-190y^3+22y^2+8y-1.\end{aligned} \tag{4}$$

Expanding equation (3) shows that the characteristic polynomial for y_n is

$$(y-1)(y-a^2)(y-a^{-2})(y-b^2)(y-b^{-2})(y-ab)(y-a^{-1}b)(y-ab^{-1})(y-a^{-1}b^{-1}). \tag{5}$$

Breaking this up into parts, we find

$$\begin{aligned}(y-a^2)(y-a^{-2})(y-b^2)(y-b^{-2})&=\left(y^2-\frac{11+\sqrt{29}}{2}y+1\right)\left(y^2-\frac{11-\sqrt{29}}{2}y+1\right)\\ &=y^4-11y^3+25y^2-11y+1.\end{aligned}$$

Another factor is $(y-ab)(y-a^{-1}b)(y-ab^{-1})(y-a^{-1}b^{-1})$. This polynomial must be symmetrical since its roots are reciprocal in pairs. Since

$$ab+a^{-1}b^{-1}+a^{-1}b+ab^{-1}=(a+a^{-1})(b+b^{-1})=\frac{1+\sqrt{29}}{2}\cdot\frac{1-\sqrt{29}}{2}\doteq-7$$

and

$$\begin{aligned}&ab\cdot a^{-1}b^{-1}+ab\cdot a^{-1}b+ab\cdot ab^{-1}+a^{-1}b^{-1}\cdot a^{-1}b+a^{-1}b^{-1}\cdot ab^{-1}+a^{-1}b\cdot ab^{-1}\\ &=1+b^2+a^2+a^{-2}+b^{-2}+1=2+\frac{11+\sqrt{29}}{2}+\frac{11-\sqrt{29}}{2}=13,\end{aligned}$$

this polynomial must be $y^4+7y^3+13y^2+7y+1$. Thus, the characteristic polynomial (5) is

$$\begin{aligned}&(y-1)(y^4-11y^3+25y^2-11y+1)(y^4+7y^3+13y^2+7y+1)\\ &=y^9-5y^8-35y^7+67y^6+145y^5-145y^4-67y^3+35y^2+5y-1.\end{aligned}$$

Since this polynomial divides the polynomial (4), we see that the squares of the x_n satisfy the original recurrence and hence every element of that recurrence is a perfect square.

Note: We also see that the original sequence satisfies the simpler recurrence

$$y_{n+1}=5y_n+35y_{n-1}-67y_{n-2}-145y_{n-3}+145y_{n-4}+67y_{n-5}-35y_{n-6}-5y_{n-7}+y_{n-8}.$$

B-736 ***Proposed by Herta T. Freitag, Roanoke, VA***
(Vol. 31, no. 2, May 1993)

Prove that $(2L_n + L_{n-3})/5$ is a Fibonacci number for all n.

Solution by Graham Lord, Mathtech, Inc., Princeton, NJ

If $A_n = (2L_n + L_{n-3})/5$, then $A_1 = 1$ and $A_2 = 1$. Furthermore, $A_{n-1} + A_n = A_{n+1}$ from the recursive property of Lucas numbers. Therefore, $A_n = F_n$.

Haukkanen found the corresponding result for Fibonacci numbers which is that $2F_n + F_{n-3}$ is a Lucas number for all n.

Also solved by Miguel Amengual Covas, Charles Ashbacher, M. A. Ballieu, Seung-Jin Bang, Margherita Barile, Glenn Bookhout, Scott H. Brown, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, Jane Friedman, Pentti Haukkanen, Russell Jay Hendel, Joe Howard, John Ivie, Joseph J. Kostal, Carl Libis, Bob Prielipp, H.-J. Seiffert, A. G. Shannon, Sahib Singh, Lawrence Somer, J. Suck, Ralph Thomas, and the proposer.

Golden Radii

B-737 ***Proposed by Herta T. Freitag, Roanoke, VA***
(Vol. 31, no. 2, May 1993)

A right triangle, one of whose legs is twice as long as the other leg, has a hypotenuse that is one unit longer than the longer leg. Let r be the inradius of this triangle (radius of inscribed circle) and let r_a, r_b, r_c be the exradii (radii of circles outside the triangle that are tangent to all three sides). Express $r, r_a, r_b,$ and r_c in terms of the golden ratio, α.

Solution by Sahib Singh, Clarion University, Clarion, PA

Let the three sides of the right triangle be $x, 2x,$ and $2x+1$. Thus, x is the positive root of $(2x+1)^2 = x^2 + 4x^2$ by the Pythagorean Theorem. This yields $x = 2 + \sqrt{5} = 2\alpha + 1$. Consequently, the sides of the triangle are $2\alpha+1, 4\alpha+2,$ and $4\alpha+3$. If A is the area of the triangle, we have $A = (2\alpha+1)^2 = 4\alpha^2 + 4\alpha + 1 = 8\alpha + 5$. The semiperimeter, s, of the triangle is $(a+b+c)/2 = 5\alpha + 3$.

Using well-known formulas for the inradius and exradii [1], we find:

$$r = \frac{A}{s} = \frac{8\alpha+5}{5\alpha+3} = \alpha;$$
$$r_a = \frac{A}{s-a} = \frac{8\alpha+5}{(5\alpha+3)-(2\alpha+1)} = \frac{8\alpha+5}{3\alpha+2} = \alpha+1 = \alpha^2;$$
$$r_b = \frac{A}{s-b} = \frac{8\alpha+5}{(5\alpha+3)-(4\alpha+2)} = \frac{8\alpha+5}{\alpha+1} = 3\alpha+2 = \alpha^4;$$
$$r_c = \frac{A}{s-c} = \frac{8\alpha+5}{(5\alpha+3)-(4\alpha+3)} = \frac{8\alpha+5}{\alpha} = 5\alpha+3 = \alpha^5.$$

Reference

1. E. W. Hobson. *A Treatise on Plane and Advanced Trigonometry*. New York: Dover, 1957, p. 193.

Also solved by Miguel Amengual Covas, Seung-Jin Bang, Margherita Barile, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Bob Prielipp, H.-J. Seiffert, J. Suck, and the proposer.

A Dozen Identities

B-738 ***Proposed by Daniel C. Fielder & Cecil O. Alford, Georgia Institute of Technology, Atlanta, GA***
(Vol. 31, no. 2, May 1993)

Find a polynomial $f(w, x, y, z)$ such that

$$f(L_n, L_{n+1}, L_{n+2}, L_{n+3}) = 25f(F_n, F_{n+1}, F_{n+2}, F_{n+3})$$

is an identity.

Solutions 1-12 by many readers

Joseph J. Kostal:	$(xy-wz)^2$
A. N. 't Woord:	$(w^2+x^2+y^2+z^2)^2$
H.-J. Seiffert:	$(w^2+x^2)^2+(y^2+z^2)^2$
Leonard A. G. Dresel:	$(w^2+x^2)(y^2+z^2)$
J. Suck:	$(y^2-w^2)(z^2-x^2)$
Margherita Barile:	$[(x+z)^2+(y+w)^2]^2$
Paul S. Bruckman:	$(x^2+xy-y^2)^2$
Herta T. Freitag:	$(wz)^2+4(xy)^2$
Shannon/Hendel/et al.:	$(x^2-wy)(y^2-xz)$
Ralph Thomas:	$wxyz+\frac{3}{5}wz(xz-y^2)$
Paul S. Bruckman:	$w^4+(y+z)^4-4x^4-19y^4-4z^4$
David Zeitlin:	$y^4-wxz(y+z)$

Solution 13 by H.-J. Seiffert, Berlin, Germany

More generally, we show that if p is a natural number, then $f_p(w,x,y,z)=(w^2+x^2)^p+(y^2+z^2)^p$ is a polynomial such that $f_p(L_n,L_{n+1},L_{n+2},L_{n+3})=5^p f_p(F_n,F_{n+1},F_{n+2},F_{n+3})$ is an identity. Using equation (I_{12}) of [1], $L_k^2=5F_k^2+4(-1)^k$, we obtain

$$\begin{aligned}
f_p(L_n,L_{n+1},L_{n+2},L_{n+3}) &= (L_n^2+L_{n+1}^2)^p+(L_{n+2}^2+L_{n+3}^2)^p \\
&= (5F_n^2+4(-1)^n+5F_{n+1}^2+4(-1)^{n+1})^p \\
&\quad +(5F_{n+2}^2+4(-1)^{n+2}+5F_{n+3}^2+4(-1)^{n+3})^p \\
&= 5^p(F_n^2+F_{n+1}^2)^p+5^p(F_{n+2}^2+F_{n+3}^2)^p \\
&= 5^p f_p(F_n,F_{n+1},F_{n+2},F_{n+3}).
\end{aligned}$$

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.

Solution 14 by David Zeitlin, Minneapolis, MN

Let $f(w,x,y,z) = y^4 - wxz(y+z)$. If (H_n) is any sequence that satisfies the recurrence $H_{n+2} = H_{n+1} + H_n$, then

$$f(H_n, H_{n+1}, H_{n+2}, H_{n+3}) = (H_2^4 - H_0H_1H_3H_4)f(F_n, F_{n+1}, F_{n+2}, F_{n+3}).$$

This follows from [1], where it is shown that, for all nonnegative integers n,

$$H_{n+2}^4 - H_nH_{n+1}H_{n+3}H_{n+4} = H_2^4 - H_0H_1H_3H_4.$$

Note that for the Fibonacci sequence, the value of $H_2^4 - H_0H_1H_3H_4$ is 1, and for the Lucas sequence, the value is 25.

See also [2] for related identities.

References

1. David Zeitlin. "Generating Functions for Products of Recursive Sequences." *Transactions of the American Mathematical Society* **116** (1965):300-15.
2. David Zeitlin. "Power Identities for Sequences Defined by $W_{n+2} = dW_{n+1} - cW_n$." *The Fibonacci Quarterly* **3.3** (1965):241-56.

Thankfully, no solver submitted the "trivial" solution: $f(w,x,y,z) = x + y - z$. Zeitlin points out that the solutions given are not independent. If f_1 and f_2 are correct solutions, then so are $f_1 + f_2$ and $f_1 - f_2$. Thus, for example, Seiffert's solution plus 2 times Dresel's solution yields Woord's solution. The identities $w + x = y$ and $x + y = z$ can also be used to transform one solution into another valid solution.

❖❖❖