ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stan@wwa.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

B-866 ***Proposed by the editor***

For n an integer, show that $L_{8n+4} + L_{12n+6}$ is always divisible by 25.

B-867 ***Proposed by the editor***

Find small positive integers a and b so that 1999 is a member of the sequence $\langle u_n \rangle$, defined by $u_0 = 0$, $u_1 = 1$, $u_n = au_{n-1} + bu_{n-2}$ for $n > 1$.

B-868 ***Based on a proposal by Richard André-Jeannin, Longwy, France***

Find an integer $a > 1$ such that, for all integers n, $F_{an} \equiv aF_n \pmod{25}$.

B-869 ***Based on a communication by Larry Taylor, Rego Park, NY***

Find a polynomial $f(x)$ such that, for all integers n, $2^n F_n \equiv f(n) \pmod 5$.

B-870 ***Proposed by Richard André-Jeannin, Longwy, France***

Solve the equation

$$\tan^{-1} y - \tan^{-1} x = \tan^{-1} \frac{1}{x+y}$$

in nonnegative integers x and y, expressing your answer in terms of Fibonacci and/or Lucas numbers.

B-871 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Prove that

$$\sum_{k=0}^{2n} \binom{2n}{k} |n-k|^3 = n^2 \binom{2n}{n}.$$

Notice to proposers: All problems submitted prior to 1999 for consideration for the Elementary Problems Column that have not yet been used are hereby released back to their authors.

SOLUTIONS

Class Identity

B-848 ***Proposed by Russell Euler's Fall 1997 Number Theory Class, Northwest Missouri State University, Maryville, MO***
(Vol. 36, no. 2, May 1998)

Prove that $F_nF_{n+1} - F_{n+6}F_{n-5} = 40(-1)^{n+1}$ for all integers n.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

We shall prove a generalization. For all integers n and k, since $\alpha\beta = -1$, we have

$$\begin{aligned}(\alpha^n - \beta^n)(\alpha^{n+1} - \beta^{n+1}) - (\alpha^{n+k} - \beta^{n+k})(\alpha^{n-k+1} - \beta^{n-k+1})\\ = -\alpha^n\beta^{n+1} - \beta^n\alpha^{n+1} + \alpha^{n+k}\beta^{n-k+1} + \beta^{n+k}\alpha^{n-k+1}\\ = (\alpha\beta)^{n+1}(\alpha+\beta) + (\alpha\beta)^{n-k+1}(\alpha^{2k-1} + \beta^{2k-1})\\ = (\alpha\beta)^{n-k+1}[(\alpha\beta)^k(\alpha+\beta) + \alpha^{2k-1} + \beta^{2k-1}]\\ = (-1)^{n-k+1}[-\alpha^k\beta^{k-1} - \alpha^{k-1}\beta^k + \alpha^{2k-1} + \beta^{2k-1}]\\ = (-1)^{n-k+1}(\alpha^k - \beta^k)(\alpha^{k-1} - \beta^{k-1}).\end{aligned}$$

It follows from the Binet Formula that $F_nF_{n+1} - F_{n+k}F_{n-k+1} = (-1)^{n-k+1}F_kF_{k-1}$. In particular,

$$F_nF_{n+1} - F_{n+6}F_{n-5} = (-1)^{n-5}F_6F_5 = 40(-1)^{n+1}$$

for all integers n.

Several readers found the generalization

$$F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^nF_aF_b,$$

which comes from formula (20a) of [1].

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Solutions also received by Brian Beasley, David M. Bloom, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Herta T. Freitag, Pentti Haukkanen, Russell Jay Hendel, Hans Kappus, Carl Libis, Bob Prielipp, Maitland A. Rose, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Fibonacci Arithmetic Progression

B-849 ***Proposed by Larry Zimmerman & Gilbert Kessler, New York, NY***
(Vol. 36, no. 2, May 1998)

If F_a, F_b, F_c, x forms an increasing arithmetic progression, show that x must be a Lucas number.

Solution by H.-J. Seiffert, Berlin, Germany

In view of the counterexample, F_{-4}, F_0, F_4, 6, we must suppose that $a \geq 0$.

Let a, b, and c be nonnegative integers such that $F_a < F_b < F_c < x$ and $F_b - F_a = F_c - F_b = x - F_c$. If $c-2 \ge b$, then $F_{c-2} \ge F_b$, and from

$$2F_b = F_a + F_c \ge F_c = F_{c-1} + F_{c-2} \ge 2F_b,$$

it follows that we get equality; hence, $F_a = 0$ and $F_{c-1} = F_{c-2} = F_b$. Thus, $a = 0$, $b = 1$, and $c = 3$, so that $x = 2F_c - F_b = 2F_3 - F_1 = 3 = L_2$ is a Lucas number.

Now suppose that $c-2 < b$ or $c-1 \le b$. Since $0 \le F_b < F_c$, we must have $b < c$. Thus, $c-1 \le b < c$ and we must have $b = c-1$. In this case, $x = 2F_c - F_{c-1} = L_{c-1}$ is also a Lucas number.

Solutions also received by Paul S. Bruckman, Aloysius Dorp, Leonard A. G. Dresel, Russell Euler, Russell Jay Hendel, N. J. Kuenzi & Bob Prielipp, Bob Prielipp, Indulis Strazdins, and the proposers.

Unknown Subscripts

B-850 ***Proposed by Al Dorp, Edgemere, NY***
(Vol. 36, no. 2, May 1998)

Find distinct positive integers a, b, and c so that $F_n = 17F_{n-a} + cF_{n-b}$ is an identity.

Solution by Leonard A. G. Dresel, Reading, England

We shall find two solutions, namely $a=6, b=9, c=4$, and $a=12, b=9, c=72$, and show that these are the only solutions.

Putting $n=b$ in the given identity, we have $F_b = 17F_{b-a}$, so that 17 divides F_b, giving $b = 9$, 18, 27, ... In fact, $b=9$ is the only solution since, for $b > 9$, we have $17_{b-6} < F_b < 17_{b-5}$. Letting $b=9$, we have $17F_{9-a} = F_9 = 34 = 17F_3$, which gives $9-a = \pm 3$, so that we have $a=6$ or $a=12$. To determine c, we put $n=10$ in the given identity. Then, with $a=6$, we have $F_{10} = 17F_4 + cF_1$, giving $c = 55-51 = 4$; whereas, with $a=12$, we have $F_{10} = 17F_{-2} + cF_1$, giving $c = 55+17 = 72$. We therefore obtain the two identities $F_n = 17F_{n-6} + 4F_{n-9}$ and $F_n = 17F_{n-12} + 72F_{n-9}$. Each identity is true for $n=9$ and $n=10$, and can therefore be shown to be true for all n by induction.

Most solvers only found one solution.

Solutions also received by Brian Beasley, Paul S. Bruckman, Russell Jay Hendel, Daina A. Krigens, H.-J. Seiffert, Indulis Strazdins, and the proposer. Partial solution by A. Plaza & M. A. Padrón.

Repeating Series

B-851 ***Proposed by Pentti Haukkanen, University of Tampere, Tampere, Finland***
(Vol. 36, no. 2, May 1998)

Consider the repeating sequence $\langle A_n \rangle_{n=0}^{\infty} = 0, 1, -1, 0, 1, -1, 0, 1, -1, \ldots$.

(a) Find a recurrence formula for A_n.

(b) Find an explicit formula for A_n of the form $(a^n - b^n)/(a-b)$.

Solution by H.-J. Seiffert, Berlin, Germany

Since the sum of any three consecutive terms of the sequence is seen to be 0, we have the recurrence $A_{n+2} = -A_{n+1} - A_n$, for $n \ge 0$.

Based on the equation

$$\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2},$$

a simple induction argument shows that

$$A_n=\frac{2}{\sqrt{3}}\sin\frac{2n\pi}{3},\quad \text{for } n\ge 0.$$

Using Euler's Relation

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i},$$

from above we get

$$A_n=\frac{e^{2n\pi i/3}-e^{-2n\pi i/3}}{e^{2\pi i/3}-e^{-2\pi i/3}},\quad \text{for } n\ge 0,$$

i.e., $a=e^{2\pi i/3}$ and $b=e^{-2\pi i/3}$ work. Equivalently,

$$a=\frac{-1+i\sqrt{3}}{2}\quad\text{and}\quad b=\frac{-1-i\sqrt{3}}{2}.$$

Cook found the recurrence $A_{n+3}=A_n$. *Libis found the amazing recurrence*

$$A_n=(-1)^{1+A_{n-2}}A_{n-1}+(-1)^{1+A_{n-1}}A_{n-2}.$$

Solutions also received by Brian Beasely, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Russell Jay Hendel, Hans Kappus, Harris Kwong, Carl Libis, A. Plaza & M. A. Padrón, Indulis Strazdins, and the proposer.

The Determinant Vanishes

B-852 ***Proposed by Stanley Rabinowitz, Westford, MA (Vol. 36, no. 2, May 1998)***

Evaluate

$$\begin{vmatrix} F_0 & F_1 & F_2 & F_3 & F_4 \\ F_9 & F_8 & F_7 & F_6 & F_5 \\ F_{10} & F_{11} & F_{12} & F_{13} & F_{14} \\ F_{19} & F_{18} & F_{17} & F_{16} & F_{15} \\ F_{20} & F_{21} & F_{22} & F_{23} & F_{24} \end{vmatrix}.$$

Solution by Indulis Strazdins, Riga Tech. University, Riga, Latvia

Adding the 1st row and the 5th row, we obtain $F_{n+20}+F_n$ in the n^{th} column, $n=0,1,2,3,4$. This expression is equal to $L_{10}F_{n+10}$ by using identity (15a) of [1], which says that

$$F_{n+m}+(-1)^m F_{n-m}=L_mF_n$$

for all integers m and n. The corresponding element of the 3rd row is F_{n+10}, $n=0,1,2,3,4$. These rows are proportional, and hence the value of the determinant is 0.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Generalization by Pentti Haukkanen, University of Tampere, Tampere, Finland

Let $\langle w_n \rangle$ be a recurrence sequence defined by $w_{n+2} = aw_{n+1} + bw_n$, $n \geq 0$. Define the $m \times m$-determinant as

$$D_m = \begin{vmatrix} w_0 & w_1 & \cdots & w_{m-1} \\ w_{2m-1} & w_{2m-2} & \cdots & w_m \\ w_{2m} & w_{2m+1} & \cdots & w_{3m-1} \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix}.$$

The m^{th} row is

$$w_{(m-1)m} \quad w_{(m-1)m+1} \quad \cdots \quad w_{m^2-1}$$

if m is odd, and

$$w_{m^2-1} \quad w_{m^2-2} \quad \cdots \quad w_{(m-1)m}$$

if m is even.

We show that $D_m = 0$ whenever $m \geq 5$.

We add the $(m-1)^{\text{th}}$ column multiplied with a and the $(m-2)^{\text{th}}$ column multiplied with b to the m^{th} column. Then D_m reduces to the form

$$D_m = \begin{vmatrix} w_0 & w_1 & \cdots & w_{m-2} & 0 \\ w_{2m-1} & w_{2m-2} & \cdots & w_{m+1} & * \\ w_{2m} & w_{2m+1} & \cdots & w_{3m-2} & 0 \\ w_{4m-1} & w_{4m-2} & \cdots & w_{3m+1} & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{vmatrix}.$$

Proceeding in a similar way with respect to the $(m-1)^{\text{th}}$, the $(m-2)^{\text{th}}$, ..., the 3^{th} column, the determinant D_m reduces to the form

$$D_m = \begin{vmatrix} w_0 & w_1 & 0 & 0 & \cdots & 0 \\ w_{2m-1} & w_{2m-2} & * & * & \cdots & * \\ w_{2m} & w_{2m+1} & 0 & 0 & \cdots & 0 \\ w_{4m-1} & w_{4m-2} & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix}.$$

Now, it is easy to see that $D_m = 0$ whenever $m \geq 5$, since we have a square matrix of zeros involving more than half of the rows of the whole matrix.

Comment by the proposer: Let $\langle w_n \rangle$ be any second-order linear recurrence defined by the recurrence $w_n = Pw_{n-1} - Qw_{n-2}$. Consider the determinant

$$\begin{vmatrix} w_0 & w_1 & w_2 & w_3 & w_4 & \cdots \\ * & * & * & * & * & \cdots \\ w_a & w_{a+1} & w_{a+2} & w_{a+3} & w_{a+4} & \cdots \\ * & * & * & * & * & \cdots \\ w_{2a} & w_{2a+1} & w_{2a+2} & w_{2a+3} & w_{2a+4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix},$$

where the asterisks can be any values whatsoever. All the rows after the fifth one can have any values as well. Then the value of this determinant is 0 because the 1^{st}, 3^{rd}, and 5^{th} rows are linearly dependent. This follows from the identity $w_{an+b} = v_a w_{a(n-1)+b} - Q^a w_{a(n-2)+b}$, which is straightforward to verify by using algorithm `LucasSimplify` from [1].

Reference

1. Stanley Rabinowitz. "Algorithmic Manipulation of Fibonacci Identities." In *Applications of Fibonacci Numbers* **6**:389-408. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.

Solutions also received by Charles Ashbacher, Brian Beasley, David M. Bloom, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Pentti Haukkanen, Russell Jay Hendel, Hans Kappus, H.-J. Seiffert, and the proposer. One incorrect solution was received.

A Deranged Sequence

B-853 ***Proposed by Gene Ward Smith, Brunswick, ME (Vol. 36, no. 2, May 1998)***

Consider the recurrence $f(n+1)=n(f(n)+f(n-1))$ with initial conditions $f(0)=1$ and $f(1)=0$. Find a closed form for the sum

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}f(k).$$

Solution by Hans Kappus, Rodersdorf, Switzerland

We claim that $S(n)=n!$.

Proof:

$$\begin{aligned}
S(n+1)-S(n) &= \sum_{k=0}^{n}\left[\binom{n+1}{k}-\binom{n}{k}\right]f(k)+f(n+1)\\
&= \sum_{k=1}^{n}\binom{n}{k-1}f(k)+f(n+1)\\
&= \sum_{k=0}^{n}\binom{n}{k}f(k+1) \qquad (*)\\
&= n\sum_{k=1}^{n}\binom{n-1}{k-1}[f(k)+f(k-1)]\\
&= n\left[\sum_{k=0}^{n-1}\binom{n-1}{k}f(k+1)+\sum_{k=0}^{n-1}\binom{n-1}{k}f(k)\right]\\
&= n[S(n)-S(n-1)+S(n-1)] \quad \text{because of } (*)\\
&= nS(n).
\end{aligned}$$

Hence, $S(n+1)=(n+1)S(n)$. Since $S(0)=1$, the proof is complete.

Cook observes that S satisfies the same recurrence as f with different initial conditions. Many readers pointed out that the recurrence is well known for the number $f(n)$ of derangements (permutations with no fixed points) of the set $\{1,2,3,\dots,n\}$. See J. Riordan's Introduction to Combinatorial Analysis *(New York: Wiley, 1958) for more information about the derangement number.*

Solutions also received by David M. Bloom, Paul S. Bruckman, Charles K. Cook, Carl Libis, H.-J. Seiffert, Indulis Strazdins, and the proposer.

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