ON THE CUBE FREE NUMBER SEQUENCES

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ABSTRACT. The main purpose of this paper is to study the asymptotic property of the cube free numbers, and obtain some interesting asymptotic formulas.

1. INTRODUCTION AND RESULTS

A natural number a is called a cube free number if it can not be divided by any b^3, where $b \geq 2$ is an integer. One can obtain all cube free numbers by the following method: From the set of natural numbers (except 0 and 1)

-take off all multiples of 2^3(i.e. 8, 16, 24, 32, 40, ...).

-take off all multiples of 3^3.

-take off all multiples of 5^3.

...and so on (take off all multiples of all cube primes).

Now the cube free number sequences is $2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, \cdots$. In reference [1], Professor F. Smarandache asked us to study the properties of the cube free number sequences. About this problem, it seems that none had studied it before. In this paper, we use the analytic method to study the asymptotic properties of this sequences, and obtain some interesting asymptotic formulas. That is, we shall prove the following three Theorems.

Theorem 1. *Let A denotes the set of all cube free numbers. Then we have the asymptotic formula*

$$\sum_{\substack{a \in A \\ a \leq x}} a = \frac{x^2}{2\zeta(3)} + O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where ε denotes any fixed positive number, $\zeta(s)$ is the Riemann zeta-function.

Theorem 2. *Let A denotes the set of all cube free numbers, $\varphi(n)$ is the Euler function. Then we have the asymptotic formula*

$$\sum_{\substack{a \in A \\ a \leq x}} \varphi(a) = \frac{x^2}{2\zeta(3)} \prod_p \left(1 - \frac{p+1}{p^3+p^2+1}\right) + O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

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Theorem 3. *Let A denotes the set of all cube free numbers, $d(n)$ is the Dirichlet divisor function. Then we have the asymptotic formula*

$$\sum_{\substack{a\in A\\ a\leq x}} d(a) = \frac{36x}{\pi^4}\prod_p \frac{p^2+2p+3}{(1+p)^2}\left(\ln x + (2\gamma-1) - \frac{24\zeta'(2)}{\pi^2}\right.$$
$$\left. -4\sum_p \frac{p\ln p}{(p^2+2p+3)(1+p)}\right) + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $\zeta'(2) = -\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$, $\sum_p$ *denotes the summation over all primes.*

2. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. For conveniently we define a new number theory function $a(n)$ as follows:

$$a(n) = \begin{cases} 0, & \text{if } n=1;\\ n, & \text{if } k^3 \nmid n, n>1, k\geq 2\\ 0, & \text{if } k^3 \mid n, n>1, k\geq 2\end{cases}$$

It is clear that

$$\sum_{\substack{a\in A\\ a\leq x}} a = \sum_{n\leq x} a(n).$$

Let

$$f(s) = 1 + \sum_{n=1}^{\infty}\frac{a(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $a(n)$ we have

$$f(s) = \prod_p\left(1+\frac{a(p)}{p^s}+\frac{a(p^2)}{p^{2s}}\right) = \prod_p\left(1+\frac{1}{p^{s-1}}+\frac{1}{p^{2(s-1)}}\right) = \frac{\zeta(s-1)}{\zeta(3(s-1))}.$$

By Perron formula [3] we have

$$\sum_{n\leq x}\frac{a(n)}{n^{s_0}} = \frac{1}{2i\pi}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}ds + O\left(\frac{x^b B(b+\sigma_0)}{T}\right)$$
$$+O\left(x^{1-\sigma_0}H(2x)\min(1,\frac{\log x}{T})\right) + O\left(x^{-\sigma_0}H(N)\min(1,\frac{x}{||x||})\right).$$

Taking $s_0=0$, $b=3$, $T=x^{\frac{3}{2}}$, $H(x)=x$, $B(\sigma)=\frac{1}{\sigma-2}$, in the above formula , then we have

$$\sum_{n\leq x} a(n) = \frac{1}{2i\pi}\int_{3-iT}^{3+iT}\frac{\zeta(s-1)}{\zeta(3(s-1))}\frac{x^s}{s}ds + O(x^{\frac{3}{2}+\epsilon}).$$

To estimate the main term

$$\frac{1}{2i\pi}\int_{3-iT}^{3+iT}\frac{\zeta(s-1)x^s}{\zeta(3(s-1))s}ds,$$

we move the integral line from $s=3+it$ to $s=\frac{3}{2}+it$. This time, the function

$$f(s)=\frac{\zeta(s-1)x^s}{\zeta(3(s-1))s}$$

have a simple pole point at $s=2$, so we have

$$\frac{1}{2i\pi}\left(\int_{3-iT}^{3+iT}+\int_{3+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{3-iT}\frac{\zeta(s-1)x^s}{\zeta(3(s-1))s}ds\right)=\frac{x^2}{2\zeta(3)}.$$

We can easy get the estimate

$$\left|\frac{1}{2i\pi}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}\frac{\zeta(s-1)x^s}{\zeta(3(s-1))s}ds\right|\ll x^{\frac{3}{2}+\varepsilon};$$

$$\left|\frac{1}{2i\pi}\int_{\frac{3}{2}-iT}^{3+iT}\frac{\zeta(s-1)x^s}{\zeta(3(s-1))s}ds\right|\ll\frac{x^{3+\varepsilon}}{T}$$

and

$$\left|\frac{1}{2i\pi}\int_{\frac{3}{2}+iT}^{3+iT}\frac{\zeta(s-1)x^s}{\zeta(3(s-1))s}ds\right|\ll\frac{x^{3+\varepsilon}}{T}.$$

Taking $T=x^{\frac{3}{2}}$, we have

$$\sum_{\substack{a\in A\\ a\leq x}}a=\sum_{n\leq x}a(n)=\frac{x^2}{2\zeta(3)}+O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

This completes the proof of Theorem 1.

Let $f_1(s)=1+\sum_{n=1}^{\infty}\frac{\varphi(a(n))}{n^s}$ and $f_2(s)=1+\sum_{n=1}^{\infty}\frac{d(a(n))}{n^s}$.

From the Euler product formula [2] and the definition of $a(n)$, we also have

$$\begin{aligned}f_1(s)&=\prod_p\left(1+\frac{\varphi(a(p))}{p^s}+\frac{\varphi(a(p^2))}{p^{2s}}\right)=\prod_p\left(1+\frac{p-1}{p^s}+\frac{p^2-p}{p^{2s}}\right)\\&=\prod_p\left(1+\frac{1}{p^{s-1}}+\frac{1}{p^{2(s-1)}}-\frac{1}{p^s}-\frac{1}{p^{2s-1}}\right)\\&=\frac{\zeta(s-1)}{\zeta(3(s-1))}\prod_p\left(1-\frac{p^{s-1}+1}{p^{2s-1}+p^s+p}\right);\end{aligned}$$

$$f_2(s) = \prod_p \left(1 + \frac{d(a(p))}{p^s} + \frac{d(a(p^2))}{p^{2s}}\right) = \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}}\right)$$

$$= \prod_p \left((1 + \frac{1}{p^s})^2 + \frac{2}{p^{2s}}\right) = \prod_p \left(1 + \frac{1}{p^s}\right)^2 \left(1 + \frac{\frac{2}{p^{2s}}}{(1+\frac{1}{p^s})^2}\right)$$

$$= \frac{\zeta^2(s)}{\zeta^2(2s)} \prod_p \left(1 + \frac{2}{(p^s+1)^2}\right).$$

By Perron formula [3] and the method of proving Theorem 1 we can easy obtain

$$\sum_{\substack{a\in A\\ a\leq x}} \varphi(a) = \frac{x^2}{2\zeta(3)} \prod_p \left(1 - \frac{p+1}{p^3+p^2+p}\right) + O\left(x^{\frac{3}{2}+\varepsilon}\right);$$

$$\sum_{\substack{a\in A\\ a\leq x}} d(a) = \prod_p \frac{p^2+2p+3}{(1+p)^2} \times$$

$$\frac{x}{\zeta^2(2)} \left(\ln x + (2\gamma - 1) - 4\frac{\zeta'(2)}{\zeta(2)} - 4\sum_p \frac{p\ln p}{(p^2+2p+3)(1+p)})\right) + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

This proves the Theorem 2 and Theorem 3.

References

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